VARIETIES WHOSE HYPERPLANE SECTIONS ARE $\mathbb{P}^k_{\mathbb{C}}$ BUNDLES

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In this article we study the following problem.

Problem: Let X be a projective variety. Let L be an ample line bundle on X that is spanned at all points of X by global sections. Assume that some irreducible $A \in |L|$ is a $\mathbb{P}^k_{\mathbb{C}}$ bundle $f : A \to Y$ over a projective variety Y. Describe X.

The second author studied this earlier in [So1] where he showed (as a consequence of a very general extension theorem of his) that if A is a smooth ample divisor on a smooth projective X and $k \geq 2$ then f extends holomorphically to a $\mathbb{P}^{k+1}$ bundle $\overline{f} : X \longrightarrow Y$ with L restricted to a general fibre isomorphic to $\mathcal{O}_{\mathbb{P}^{k+1}}(1)$. Some technical improvements were made in this result by Fujita [Fu1, Fu2] and Silva [Si]. We include a quite general extension theorem subsuming all these results in a short appendix. This paper is concerned with the much more subtle case when the fibre of $f : A \longrightarrow Y$ is $\mathbb{P}^1$.

The key to analyzing X is to show that the map $f : A \longrightarrow Y$ extends to a holomorphic map $\overline{f} : X \longrightarrow Y$. This is not always true-examples with $Y = \mathbb{P}^n$ for some $n \geq 1$ are easy to construct. We rule this sort of example out by assuming that Y has a nontrivial top degree holomorphic form.

Theorem. Let L be an ample line bundle on a normal projective variety X. Assume that L is spanned at all points by global sections and that there is a smooth $A \in |L|$ which is a holomorphic $\mathbb{P}^1$ bundle $f : A \longrightarrow Y$ over a connected projective variety Y. If $h^\circ(K_Y) \neq 0$ then f extends to a meromorphic map $\overline{f} : X \longrightarrow Y$ holomorphic in a neighborhood of A; if X is a local complete intersection then $\overline{f}$ is holomorphic.

If $\overline{f}$ is holomorphic it is an easy consequence of an earlier result of the second author [So1] that $\dim Y \leq 2$ and in the case $\dim Y = 2$, $\overline{f} : X \longrightarrow Y$ is a holomorphic $\mathbb{P}^2$ bundle with L restricted to a fibre of $\overline{f}$ isomorphic to $\mathcal{O}_{\mathbb{P}^2}(1)$. The case when $\dim Y = 1$ is classical and leads to "Quadric bundles" besides $\mathbb{P}^2$ bundles.

The above theorem is proved as a consequence of a considerably more powerful meromorphic extension theorem.

One form of it is the following.

Theorem. Let L be an ample line bundle on a normal projective variety X. Assume that L is spanned at all points by global sections and that there is a normal $A \in |L|$ which fibres holomorphically $f : A \longrightarrow Y$ over a normal projective variety Y. Assume that:

a) X is a local complete intersection whose locus of non-rational singularities is at most dimension 1,

b) the general fibre of f is $\mathbb{P}^1$ and both A and Y have at most rational singularities,

c) there is a desingularization $\overline{Y}$ of Y with $h^0(K_{\overline{Y}}) \neq 0$.

Then f extends to a meromorphic map $\overline{f} : X \longrightarrow Y$ which is holomorphic in a neighborhood of the open set $U \subseteq A_{reg}$ such that $f_U : U \longrightarrow f(U)$ is a $\mathbb{P}^1$ bundle.

The most natural approach to such extension theorems is to choose a very ample line bundle E on Y, show that f^*E extends to a line bundle $\mathcal{E}$ on X, and show that a "lot" of sections of f^*E extend to $\mathcal{E}$. This was the approach in [So1] (cf. the appendix to this paper) but it works if $\dim A - \dim Y = 1$ only in very special cases (cf. [B] for the case of A a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$).

The second approach is to attempt to construct $\bar{f}$ geometrically. The idea is to take a general fibre ℓ of f and look at the closure F of all deformations ℓ' of ℓ such that $\ell \cap \ell' \neq \phi$. F should be the general fibre of $\bar{f}$. The main trouble in this approach is showing that $\dim F = 2$. A counterexample with $Y = \mathbb{P}^n$ shows that F can equal X. A modified form of the above approach does work. We want to use the non-trivial holomorphic form on the desingularization of Y to guarantee that $\dim F = 2$. To do this we need control over the parameter space of the set of deformations ℓ' above. For this reason we restrict to deformations ℓ' of ℓ such that $\ell' \cap \ell \neq \phi$ and ℓ' is a fibre of a deformation $f' : A' \longrightarrow Y'$ of $f : A \longrightarrow Y$ where $A' \in |L|$. This requires us to first show that for most $A' \in |L|$, $f' : A' \longrightarrow Y'$ exists.

The contents of this paper are as follows.

In § 0 we present background material for which there are no good references (especially material on vanishing theorems and extension of line bundles). We also present the classical material when $\dim Y = 1$ and the standard counterexamples to extension.

In § 1 we present various results on holomorphic forms and the groups $H^i(\mathcal{O}_X)$.

In § 2 we prove the meromorphic extension theorem.

In § 3 we use this result to analyze the global structure of X . We also deduce some results on when a modification of a hyperplane section extends to a modification of a projective variety; these results which are in the same vein as [Fa1, Fa2, Fa + So, So2, So3, So4] where one of our main motivations to study the problem stated at the beginning of this introduction.

In § 4 we discuss the proof of the main results and what should be true in general.

In a short appendix we include the strongest version of the extension theorem originally given for manifolds in [So1] for holomorphic surjections $f : A \longrightarrow Y$ with $\dim A - \dim Y \geqq 2$.
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§ 0 BACKGROUND MATERIAL

Our notation is the same as in [So2] and [Fa1]. For the convenience of the reader we review it here.

(0.1) All spaces and manifolds are complex analytic unless otherwise specified; all dimensions are over $\mathbb{C}$. Given an analytic space X , we denote its structure sheaf by $\mathcal{O}_X$. We don't distinguish between a holomorphic vector bundle E and its locally free sheaf of germs of holomorphic sections. Thus when E is tensored with a coherent analytic sheaf $\mathcal{S}$ we mean the tensor product over $\mathcal{O}_X$ of the sheaf of of holomorphic sections of E and $\mathcal{S}$; we denote this $E \otimes \mathcal{S}$.

We denote the sections of a sheaf $\mathcal{S}$ over X by $\Gamma(X,\mathcal{S})$, or $\Gamma(\mathcal{S})$ when no confusion will result.

Similarly we often suppress X and write $H^i(\mathcal{S})$ for the ith cohomology group of $\mathcal{S}$ on X . We write its dimension $h^i(\mathcal{S})$, or $h^i(X,\mathcal{S})$ if there is a posibility of confusion.

Let X be an n dimensional normal irreducible complex analytic space. The canonical sheaf ω_X of X is defined to be the sheaf of holomorphic n forms if X is smooth and the direct image $i_*(\omega_{X_{reg}})$ in general where i is the inclusion of the smooth locus X_{reg} of X into X . A good reference for dualizing sheaves is [Ha]. Let K_X denote the

Grauert-Riemenschneider canonical sheaf of X [Gra+Ri]. This is defined to be $\pi_* \omega_{\overline{X}}$ where $\pi : \overline{X} \longrightarrow X$ is a resolution of the singularities of X; it is independent of the resolution. There is the basic exact sequence:

$$0 \longrightarrow K_X \longrightarrow \omega_X \longrightarrow S \longrightarrow 0 \tag{0.1.1}$$

where the coherent sheaf S is supported on an analytic subset X_{irr} of X_{sing}. It is a theorem of Kemph ([Ke], pg. 50) that the set X_{irr} is the locus of non rational-singularities of X, i.e. the union of the supports of $\{\pi_{(i)}(\mathcal{O}_{\overline{X}}) \mid i \geq 1\}$ where $\pi_{(i)}$ denotes the ith direct image of any resolution $\pi : \overline{X} \longrightarrow X$ (the sheaves $\pi_{(i)}(\mathcal{O}_{\overline{X}})$ are basic invariants of X that are independent of the resolution used to define them). We refer to X_{irr} as the <u>irrational locus</u> of X.

(0.2) Vanishing Theorem of Kawamata - Viehweg - Kodaira-Ramanujan - Grauert-Riemenschneider. <u>Let X be an n dimensional irreducible normal projective variety. Let L be a numerically effective line bundle, i.e. $L \cdot C \geq 0$ for all irreducible curves $C \subseteq X$. If</u> $c_1(L)^{n-t} \cdot H^t > 0$ <u>for some ample divisor H and some $t \geq 0$ then</u> $H^i(X, \omega_X \otimes L) = 0$ <u>for</u> $i > \max\{t, \dim(X_{irr})\}$.

<u>Proof</u>. We will be brief since results like this are discussed in great detail in [Sh+So]. Tensoring (0.1.1) with

L and using the long exact cohomology sequence it follows that the theorem will be proved if we show that $H^i(X, K_X \otimes L) = 0$ for $i > t$. Let $\pi : \overline{X} \longrightarrow X$ be a projective desingularization of X. By the projection formula:

$$*) \qquad \pi_{(i)}(\omega_{\overline{X}} \otimes \pi^* L) = \pi_{(i)}(\omega_{\overline{X}}) \otimes L \ .$$

By this and the definition of K_X it follows that $\pi_* \ (\omega_{\overline{X}} \otimes \pi^* L) = \pi_{(0)}(\omega_{\overline{X}} \otimes \pi^* L)$ is $K_X \otimes L$. The Grauert-Riemenschneider vanishing theorem [Gra+Ri] says that $\pi_{(i)}(\omega_{\overline{X}}) = 0$ for $i > 0$. Therefore by *) and the Leray spectral sequence for π, the proof will follow from $H^i(\overline{X}, \omega_{\overline{X}} \otimes \pi^* L) = 0$ for $i > t$. This is of course the Kawamata-Viehweg vanishing theorem (see [V]; remark (0.2)).

□

We will also need a relative form of the above in one situation. Rather than formulate and prove the general result we merely prove a special case [generalizing So1], by reducing to a result of Fujita [Fu1].

(0.2.1) Theorem. *Let $f : X \longrightarrow Y$ be a holomorphic surjective map from a compact normal irreducible projective variety X to a projective variety Y. Assume that $\dim X - \dim Y \geq 2$. Assume that L is a line bundle on X such that some power L^t for $t > 0$ is spanned at all points by global sections and such that the map associated to $\Gamma(L^t)$ has a $\dim X$ dimensional image. Then given any locally free sheaf $\mathcal{E}$ on Y,* $H^1(X, L^{-k} \otimes f^*\mathcal{E}) = 0$ for $k \geq 1$.

Proof. Let $\pi : \overline{X} \longrightarrow X$ be a projective resolution of singularities of X. It is clear by the Leray spectral sequence that $H^1(X, L^{-k} \otimes f^*\mathcal{E})$ injects into $H^1(\overline{X}, (\pi^*L)^{-k} \otimes (f \circ \pi)^*\mathcal{E})$. Therefore using $\overline{X}$ instead of X, $f \circ \pi$ instead of f and π^*L instead of L we have reduced to the case X is smooth.

Using $\dim X - \dim Y \geq 2$ the result is now clear from [Fu1; Corollary A6].

□

(0.3) We need some information about extension of line bundles.

(0.3.1) Lemma. *Let A be an effective ample divisor on an irreducible projective variety X of dimension* ≥ 4. *Assume that* $A \subseteq X_{reg}$. *Then for any desingularization $\widetilde{X}$ of X the restriction map* $\operatorname{Pic}(\widetilde{X}) \longrightarrow \operatorname{Pic}(A)$ *has finite cokernel.*

<u>Proof</u>. Since $A \subseteq X_{reg}$, X has isolated singularities and we can assume without loss of generality that X is normal.

Let $\pi : \tilde{X} \longrightarrow X$ denote a desingularization of X. Since π is a biholomorphism from $\tilde{X} - \pi^{-1}(\mathrm{Sing}(X)) \longrightarrow X - \mathrm{Sing}(X)$ we identify A and $\pi^{-1}(A)$.

Consider the long exact cohomology sequences associated to the exponential sequences on $\tilde{X}$ and A.

$$\begin{array}{ccccccc}
H^1(\tilde{X},\mathcal{O}_X) & \longrightarrow & \mathrm{Pic}(\tilde{X}) & \longrightarrow & H^2(\tilde{X},\mathbb{Z}) & \longrightarrow & H^2(\tilde{X},\mathcal{O}_{\tilde{X}}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(A,\mathcal{O}_A) & \longrightarrow & \mathrm{Pic}(A) & \longrightarrow & H^2(A,\mathbb{Z}) & \longrightarrow & H^2(A,\mathcal{O}_A)
\end{array}$$

where the vertical maps are restrictions.

As it is well known $H^i(\tilde{X},\mathcal{O}_X) \approx H^i(A,\mathcal{O}_A)$ for $i \leq \dim A-1$. This follows from the Kodaira vanishing theorem (0.2), $H^i(\tilde{X},[A]^{-1}) = 0$ for $i \leq \dim A$.

Therefore we will be done by a diagram chase if we show that the restriction $H^2(\tilde{X},\mathbb{Z}) \longrightarrow H^2(A,\mathbb{Z})$ has finite cokernel. This will follow if we show that $H^2(\tilde{X},\mathbb{Q}) \longrightarrow H^2(A,\mathbb{Q})$ is onto.

Choose $n > 0$ such that $[A]^n$ is very ample and embed X in $\mathbb{P}^N_{\mathbb{C}}$ using $\Gamma([A]^n)$. There is a hyperplane $H' = \mathbb{P}^{N-1}_{\mathbb{C}}$ that meets X in nA. The hypherplanes sufficiently near H' meet X in sets contained in a neighborhood $V \subseteq X_{reg}$ of A

which is a deformation retract of A . The basic result of [So5] shows that for any of these nearby hyperplanes H , the restriction mapping $R_H : H^j(V,\mathbb{Z}) \longrightarrow H^j(H \cap X,\mathbb{Z})$ is an isomorphism for $j \leq \dim X - 2$. Choosing an H near H' so that $A' = H' \cap X$ is smooth we see that $H^2(\tilde{X},\mathbb{Q}) \longrightarrow H^2(A,\mathbb{Q}) \longrightarrow 0$ is equivalent to showing that $H^2(\tilde{X},\mathbb{Q}) \longrightarrow H^2(A',\mathbb{Q}) \longrightarrow 0$. Indeed consider:

$$H^2(\tilde{X},\mathbb{Q}) \longrightarrow H^2(V,\mathbb{Q}) \begin{array}{l} \nearrow \; H^2(A,\mathbb{Q}) \\ \searrow_{\approx} \; H^2(A',\mathbb{Q}) \end{array}$$

By Kronecker duality we are reduced to showing that:

$$0 \longrightarrow H_2(A',\mathbb{Q}) \longrightarrow H_2(\tilde{X},\mathbb{Q}) \ .$$

Since the intersection homology of a manifold is equal to its usual homology [(G+M)3] and since the rational intersection homology of a complex algebraic variety X injects into the rational intersection homology of any desingularization $\tilde{X}$ [(G+M)1] we are reduced to showing that:

$$0 \longrightarrow IH_2(A',\mathbb{Q}) \longrightarrow IH_2(X,\mathbb{Q})$$

where IH_* denotes intersection homology. This last injection follows from the beautiful result [(G+M)3] that for a hpyerplane section of a variety by a hyperplane to all strata of a Morse stratification of the variety (which $A' \subseteq X_{reg}$ certainly is) the usual first Lefschetz theorem holds with intersection homology replacing the usual homology.

□

We need the following result also.

(0.3.2) Lemma. Let A be an ample divisor on an irreducible projective local complete intersection X. Assume that cod $\mathrm{Irr}(X) \geq 3$. Under restriction:

$$\mathrm{Pic}(X) \approx \mathrm{Pic}\,(A) \qquad \text{if} \quad \dim X \geq 4$$

$$0 \longrightarrow \mathrm{Pic}\,(X) \longrightarrow \mathrm{Pic}\,(A) \quad \text{with torsion free cokernel if}$$

$$\dim X = 3.$$

Proof. By the usual argument using the long exact cohomology sequence associated to the exponential sequence of X and A the above result will follow if we show that $\pi_i(X,A,a)$ with $i \leq \dim A$ and any basepoint $a \in A$ and also that $H^i(X,[A]^{-1})=0$ for $i = 1,2$. The former is the Lefschetz theorem of Goresky-MacPherson [(G+M)2] and the latter is just (0.2).

□

In the same spirit as the above results we need information about when we can conclude that there is a non-trivial holomorphic k form on a desingularization of a variety.

(0.3.3) Lemma. Let L be a line bundle on a normal irreducible projective variety, X, of dimension n. Assume that some positive power L^m of L is spanned by global sections at all points of X and that the map associated to $\Gamma(L^m)$ has an n dimensional image, e.g. assume that L is ample. Let $A \in |L|$ be normal with at most rational singularities. Let $\pi_2 : \overline{X} \to X$ be a desingularization of X and let $\pi_1 : \overline{A} \longrightarrow A'$ be a desingularization of the proper transform A' in X of A. Let $\mathcal{R} : H^0(\Lambda^k T^*_{\overline{X}}) \longrightarrow H^0(\Lambda^k T^*_{\overline{A}})$ be the map induced by π_1. Then $\mathcal{R}$ is a surjection for $k < \dim A$.

Proof. By Hodge theory it sufficies to show that the map $\overline{\mathcal{R}} : H^k(\mathcal{O}_{\overline{X}}) \longrightarrow H^k(\mathcal{O}_{\overline{A}})$ induced by π_1 is a surjection for $k < \dim A$. By (0.2) the restriction

$$H^k(\mathcal{O}_{\overline{X}}) \longrightarrow H^k(\mathcal{O}_{\pi_2^{-1}(A)})$$

is an isomorphism for $k < \dim A$.

Using this and considering the commutative diagram:

$$\begin{array}{ccc}
H^k(\mathcal{O}_{\overline{A}}) & \overset{\mathcal{R}}{\longleftarrow} & \\
\uparrow & & \\
H^k(\mathcal{O}_{A'}) & & H^k(\mathcal{O}_{\overline{X}}) \\
\uparrow & \swarrow & \\
H^k(\mathcal{O}_{\pi_2^{-1}(A)}) & & \\
\uparrow & & \\
H^k(\mathcal{O}_{A}) & &
\end{array}$$

it sufficies to show that the map

$$\pi^* : H^k(\mathcal{O}_A) \longrightarrow H^k(\mathcal{O}_{\bar{A}}) ,$$

induced by the composition $\pi : \bar{A} \longrightarrow A$ of π_1 and $A' \longrightarrow A$ is an isomorphism. Using the Leray spectral sequence for π and the fact that A having only rational singularities is equivalent to $\pi_{(i)}(\mathcal{O}_{\bar{A}}) = 0$ for $i > 0$, this is clear.

□

(0.4) Lemma. Let $\phi : Z \longrightarrow \mathbb{P}_{\mathbb{C}}$ be a holomorphic map of an irreducible projective variety Z with $\dim \phi(Z) \geq 2$. Given a general hyperplane H on $\mathbb{P}_{\mathbb{C}}$, $\phi^{-1}(H)$ is irreducible. Given any hyperplane H on $\mathbb{P}_{\mathbb{C}}$, $\phi^{-1}(H)$ is connected.

Proof. This is a standard fact, e.g. [Sh+So; theorem (3.42)].

□

(0.5) We give here a few standard counterexamples to the extension problem discussed in the introduction. The most obvious is $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$. This can be generalized slightly. Let $H_d \subseteq \mathbb{P}^3$ be a smooth degree d hypersurface in $\mathbb{P}^3$ that contains a line ℓ, e.g. let

$$H_d = \{(Z_0,\ldots,Z_3) \mid \sum_{i=0}^{3} Z_i^d = 0 \}.$$

Then $L = \mathcal{O}_{\mathbb{P}^3}(1)|_{H_d} \otimes [\ell]^{-1}$ is spanned by global sections and gives a holomorphic surjection $f: H_d \longrightarrow \mathbb{P}^1$ with general fibre biholomorphic to a curve of degree $d - 1$ in $\mathbb{P}^2$ see [So1] for more or this type of fibration. Clearly f can't extend holomorphically to $\mathbb{P}^3$.

Many examples of non-extendible maps with $\dim \overline{Y} = 1$ can be given. We know of only one example of a $\mathbb{P}1$ bundle A over a Y with $\dim Y > 1$, where X is not a $\mathbb{P}^2$ bundle. The following simple argument was given to us by E. Sato.

Let $Y = \mathbb{P}^n$ with $n > 1$. Let γ be a non trivial element of $H^1(\mathcal{O}_{\mathbb{P}^1}(-2))$ and let $F^* = \underbrace{\oplus\, \mathcal{O}_{\mathbb{P}^1}(-2)}_{n \text{ copies}}$. Let E^* be the unique extension

$$0 \longrightarrow F^* \longrightarrow E^* \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0$$

such that $1 \in H^0(\mathcal{O}_{\mathbb{P}^1})$ goes to $\underbrace{\gamma \oplus \dots \oplus \gamma}_{n \text{ copies}} \in H^1(F^*)$. Note that $\mathbb{P}(F)$ is a very ample divisor on $\mathbb{P}(E)$. To see this it must just be noted that E is ample. By dualizing the above exact sequence we can easily check that E is spanned by global sections. We must only check that E doesn't contain a trivial summand.

If it did then E^* would have a nowhere vanishing section. Since F^* has no section, the image of this section would split the above exact sequence contradicting the non-triviality of γ. These $\mathbb{P}(F)$ is a very ample divisor of $\mathbb{P}(E)$.

Note that $\mathbb{P}(F) = \mathbb{P}^1 \times \mathbb{P}^n$.

Since there are no non trivial map $\mathbb{P}^{n+1} \longrightarrow \mathbb{P}^n$ the map $\mathbb{P}(F) \longrightarrow \mathbb{P}^n$ cannot extend to a map from the $\mathbb{P}^{n+1}$ bundle $\mathbb{P}(E)$ to $\mathbb{P}^n$.

(0.6) We give here a summary of the solution to the problem posed in the introduction when Y is a curve of genus $g > 0$. This result was more a less known a half century ago (cf. [Ro], Ch. 4, § 11,12), a short proof can be found in [Fa+So].

(0.6.1) Theorem. Let A be a smooth ample divisor on an irreducible projective local complete intersection threefold X. Assume that there is a holomorphic map $f:A \longrightarrow R$ with generic fibre $\mathbb{P}^1$ onto a Riemann surface R of genus $g \geq 1$. Then f extends to a holomorphic map $\bar{f} : X \longrightarrow R$. Either

a) f is a $\mathbb{P}^1$ bundle and $\bar{f}$ is a $\mathbb{P}^2$ bundle with the restriction of $[A]$ to a general fibre isomorphic to $\mathbb{P}^1$ or

b) $\bar{f}$ has $\mathbb{P}^1 \times \mathbb{P}^1$ as general fibre and the restriction of $[A]$ to $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to $[\Delta]$ where Δ is the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$.

(0.7) Lemma. Let L be an ample line bundle on a normal projective local complete intersection X. Assume that L is spanned at all points by global sections and that the locus of

irrational singularities is of codimension ≥ 3.
Assume that there is an $A \in |L|$ and a surjective holomorphic map $f:A \longrightarrow Y$ into a normal projective variety Y. If f extends to a meromorphic map $\bar{f}:X \longrightarrow Y$ holomorphic in a neighborhood of A and $\dim A > \dim Y$ then $\bar{f}$ is holomorphic.

Proof. Let E be a very ample line bundle on Y and let $\mathcal{E}$ be the extension of f^*E to X that exists by lemma (0.3.3). If we knew that pullbacks under f of sections of E extended to sections of $\mathcal{E}$ we would be done by an argument of [So1] when X is smooth that was nicely generalized to arbitrary X in [Fu1]. Indeed $\dim Y + 1$ sections span E. Thus $\dim Y + 1$ sections span $\mathcal{E}$ off an analytic set $\mathcal{A} \subseteq X - A$. Thus $\mathcal{A}$ is empty or $\dim \mathcal{A} \geq \dim X - \dim Y - 1 > 0$. But since $\mathcal{A} \subseteq X - A$, $\dim \mathcal{A} = 0$. Thus since $\mathcal{E}$ is spanned by $\dim Y + 1$ sections, the map associated to pullbacks of sections has a $\dim Y$ dimensional image. It is easy to see this must be Y.

The natural supposition is that if we take $D \in |E|$ then $\bar{f}^*D \in |\mathcal{E}|$. If this is true we are done by the above reasoning. If we knew that $\bar{f}^*D$ were Cartier this would be clear since $0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(A)$. Unforunately this is not immediately obvious.

Let $\pi:\bar{X} \longrightarrow X$ be a desingularization of the graph of $\bar{f}$. Choose an $A' \in |L|$ such that $\bar{A} = \pi^{-1}(A')$ is smooth and $\bar{f}$ is holomorphic in a neighborhood of A'. This is possible since $\overline{f}$ is holomorphic in a neighborhood of A.

Let $f':\overline{X} \longrightarrow \overline{Y}$ be the holomorphic map induced by $\overline{f}$. Let E and $\mathcal{E}$ be us before and let $\mathcal{M} = [f'^*D]$ for a general divisor $D \in |E|$ such that $f'^{-1}(D)$ is irreducible. If we show that $\mathcal{M} \approx \pi^*\mathcal{E}$ we will be done. Consider:

$$0 \longrightarrow \pi^*(\mathcal{E} \otimes L^{-1}) \otimes \mathcal{M}^{-1} \longrightarrow \pi^*\mathcal{E} \otimes \mathcal{M}^{-1} \longrightarrow (\pi^*\mathcal{E} \otimes \mathcal{M}^{-1})_{\overline{A}} \longrightarrow 0$$

Since $\pi^*\mathcal{E} \otimes \mathcal{M}^{-1} \approx \mathcal{O}_{\overline{A}}$ it sufficies to show that $H^1(\pi^*(\mathcal{E} \otimes L^{-1}) \otimes \mathcal{M}^{-1}) = 0$. Since the map associated to $\Gamma((\pi^*L)_{\overline{A}})$ has a dim A dimensional image, it follows that

$$H^1(((\pi^*\mathcal{E}) \otimes \mathcal{M}^{-1} \otimes \pi^*L^{-1})_{\overline{A}}) = H^1((\pi^*L^{-t})_{\overline{A}}) = 0$$

for $t > 0$. Therefore by tensoring the above exact sequence with π^*L^{-t} for $t = 1,2,3 \ldots$ and using the associated long exact cohomology sequence we reduce to showing that

$$H^1(\overline{X}, \pi^*(\mathcal{E} \otimes L^{-t}) \otimes \mathcal{M}^{-1}) = 0 \quad \text{for some} \quad t > 0.$$

By Serre duality and the Leray spectral sequence we reduce to showing that:

$$H^i(X, L^t \otimes \mathcal{E}^* \otimes \pi_{(j)}(\omega_{\overline{X}} \otimes \mathcal{M})) = 0$$

for $i + j = \dim A$ and $t >> 0$. Since $\mathcal{M}$ is spanned it follows from [Gr + Ri] that $\pi_{(j)}(\omega_{\overline{X}} \otimes \mathcal{M}) = 0$ for $j > 0$. Since L is ample $H^{\dim A}(X, L^t \otimes \mathcal{E}^* \otimes \pi_* \omega_X \otimes \mathcal{M})) = 0$ for $t >> 0$.

□

§ 1 Some Results on Holomorphic Forms

(1.1) *Theorem*. *Let* $f : X \longrightarrow Y$ *be a holomorphic surjection with connected fibres between normal irreducible projective varieties* X *and* Y*. Assume that there is a non-empty Zariski open set* $V \subseteq Y$ *such that* V *and* $f^{-1}(V)$ *are smooth and* $f : f^{-1}(V) \longrightarrow V$ *is of maximal rank. Assume that* X *and* Y *have at most rational singularities. If* $h^i(\mathcal{O}_F) = 0$ *for* $0 < i \leq q$ *where* F *is a general fibre of* f *then*

$$f_{(i)}(\mathcal{O}_X) = 0 \quad \textit{for} \quad 0 < i \leq q.$$

In particular if $h^i(\mathcal{O}_F) = 0$ *for* $i > 0$ *then*

$$f_{(i)}(\mathcal{O}_X) = 0 \quad \textit{for} \quad i > 0.$$

Proof. It can be assumed without loss of generality that X and Y are smooth. To see this let $g:\tilde{Y} \longrightarrow Y$ be a desingularization of Y and let $\tilde{X}$ be a desingularization of the irreducible component of the fibre product $X \times_Y \tilde{Y}$ which surjects onto both $\tilde{Y}$ and X under the natural projections. We have the commutative square:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & X \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \tilde{Y} & \xrightarrow[g]{} & Y \end{array}$$

The horizontal maps are birational morphisms and since the singularities of X and Y are rational:

$$*)\qquad \tilde{g}_{(i)}(\mathcal{O}_{\tilde{X}}) = 0 = g_{(i)}(\mathcal{O}_{\tilde{Y}}) \quad \text{for } i > 0$$

and since X and Y are normal:

$$**)\qquad \tilde{g}_*(\mathcal{O}_{\tilde{X}}) = \mathcal{O}_X \qquad g_*(\mathcal{O}_{\tilde{Y}}) = \mathcal{O}_Y .$$

The condition on the general fibre of f and the fact that $g : g^{-1}(V) \longrightarrow V$ and $\tilde{g} : \tilde{g}^{-1}(f^{-1}(V)) \longrightarrow f^{-1}(V)$ are biholomorphisms imply that $h^i(\mathcal{O}_{\tilde{F}}) = 0$ for $0 \leq i \leq q$ where $\tilde{F}$ is a general fibre of $\tilde{f}$. If the theorem is true for $\tilde{f}$ then using *) and **) and the Leray spectral sequence for $g \circ \tilde{f}$ we see that

$$(g \circ \tilde{f})_{(i)} = 0 \quad \text{for } 0 < i \leq q \quad \text{and} \quad (g \circ \tilde{f})_*(\mathcal{O}_{\tilde{X}}) = \mathcal{O}_Y .$$

Using this, $f \circ \tilde{g} = g \circ \tilde{f}$, *), **), and the Leray spectral sequence for $f \circ \tilde{g}$ we see that:

$$0 = f_{(i)}(\tilde{g}_* \mathcal{O}_{\tilde{X}}) = f_{(i)}(\mathcal{O}_X) \quad \text{for } 0 < i \leq q.$$

Therefore assume that X and Y are smooth.

We need a lemma.

(1.1.1) Lemma. Let $f : X \longrightarrow Y$ be a surjective holomorphic map with connected fibres between projective manifolds X and Y. Assume that $h^i(\mathcal{O}_F) = 0$ for $0 < i \leq q$ where F is a generic fibre of f. Then

$$(1.1.1.1) \qquad f^* : H^i(\mathcal{O}_Y) \longrightarrow H^i(\mathcal{O}_X)$$

is an isomorphims for $0 < i \leq q$.

Proof. By standard Hodge theory the map in (1.1.1.1) is an injection for all i, e.g. [W]. We must only show that the map is surjective. By conjugation and the Hodge theory anti-isomorphism of $H^i(\mathcal{O}_X)$ with $H^0(\Lambda^i T^*_X)$ and of $H^i(\mathcal{O}_Y)$ with $H^0(\Lambda^i T^*_Y)$ this is equivalent to showing that every holomorphic i form η on X with $0 < i \leq q$ is of the form $f^*\mu$ for a holomorphic i form on Y.

This is certainly true over the dense Zariski open set $V \subseteq Y$ such that $f : f^{-1}(V) \longrightarrow V$ is of maximal rank. Indeed let $V' = f^{-1}(V)$ consider the exact sequence:

$$0 \longrightarrow f^*T^*_V \longrightarrow T^*_{V'} \longrightarrow T^*_{V'/V} \longrightarrow 0.$$

We get a filtration.

$$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_i$$

where $F_j = (\Lambda^{i-j}f^*T^*_V) \wedge (\Lambda^j T^*_{V'})$. The quotients are $F_j/F_{j-1} \approx (\Lambda^{i-j}f^*T^*_V) \otimes \Lambda^j T^*_{V'/V}$. Let $\eta_{V'}$ denote the restriction of η to V'. Since by shrinking V it is easy to see that $\eta_{V'} = f^*\omega_V$ for some holomorphic i form if $\Lambda^j T^*_{V'/V}|_F \approx \Lambda^j T^*_F$ has no holomorphic sections for $0 < j \leq i$. By the Hodge theory isomorphism $H^i(\mathcal{O}_F) = H^0(\Lambda^i T^*_F)$ and our hypothesis this is clear. We must only show that ω_V has a holomorphic extension to Y. Assume otherwise. Since by Hartogs theorem holomorphic sections of vector bundles extend over codimension 2 sets it follows that ω_V extends to a holomorphic i form ω' on $Y - Z$ where Z is a set of pure codimension 1. Choosing $\dim X - \dim Y$ general hyperplane sections of X and intersecting we get a submanifold X' of X such that $f_{X'}$ is generically finite to one Further the pullback of ω' to X' extends holomorphically since it agrees with the restriction of η on a dense open set. Choose a smooth point x of Z such that $f_{X'}$ is finite to one over a neighborhood of x. An easy calculation shows that ω' has at worst poles on Z and extends holomorphically if it has no poles. Slicing Y with sufficiently ample hyperplane sections through x we can choose an i-dimensional submanifold $Y' \subseteq Y$ such that the restriction ω'' of ω' to $Y' - Z \cap Y'$ has poles along $Z \cap Y'$ if ω' has poles along Z. Further desingularizing an irreducible component of $f^{-1}_{X'}(Y')$ we get a projective i-dimensional manifold X'' and a gerically finite to one surjective $f'':X'' \longrightarrow Y$ such that the pullback of ω'' to X'' extends to all of X'' holomorphically. But this implies

$\int \omega'' \wedge \overline{\omega''}$ is finite since

$$\deg(f'') \int \omega'' \wedge \overline{\omega''} = \int (f''^*\omega'') \wedge (\overline{f''^*\omega''}).$$

If $\int \omega'' \wedge \overline{\omega''}$ is finite an easy calculation shows that ω' has no poles along $Z \cap Y'$. Thus ω_V has a holomorphic extension to Y.

□

We need a general slicing lemma also.

(1.1.2) Slicing Lemma. Let $f : X \longrightarrow Y$ be a holomorphic surjection between projective manifolds. If H is a general hyperplane section of Y then

a) H and $H' = f^{-1}(H)$ are smooth,

b) dim support $(f_{(i)}(\mathcal{O}_X))$ = dim support $((f_{H'})_{(i)}(\mathcal{O}_{H'}))$ +1 whenever $f_{(i)}(\mathcal{O}_X)$ is non trivial (here we adopt the convention that the empty set has dimension -1) and $(f_{H'})_{(i)}(\mathcal{O}_{H'})$ is non-trival if dim support $f_{(i)}(\mathcal{O}_X) = 1$.

Proof. a) is true by Bertini's theorem. We have the exact sequence:

$$0 \longrightarrow [H']^{-1} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{H'} \longrightarrow 0$$

The long exact sequence of direct image sheaves gives:

$$\longrightarrow f_{(i)}(\mathcal{O}_X) \otimes [H]^{-1} \longrightarrow f_i(\mathcal{O}_X) \longrightarrow f_{(i)}(\mathcal{O}_{H'}) \longrightarrow$$

If S is any coherent sheaf in a manifold Y, then a general hyperplane section will not contain the support of any subsheaf of S. Thus

$$0 \longrightarrow S \otimes [H]^{-1} \longrightarrow S.$$

From this and the long exact sequence of direct image sheaves above, we get the lemma.

□

Now assume the theorem is false. Let i be the smallest integers $0 < i \leq q$ such that $f_{(i)}(\mathcal{O}_X) \neq 0$. If $f_{(i)}(\mathcal{O}_X)$ is supported in a finite set then by the Leray spectral sequence and lemma (1.1.1) we have a contradiction. If $f_{(i)}(\mathcal{O}_X)$ is supported on a $k \geq 1$ dimensional set then by lemma (1.1.2) we can slice with k hyperplane sections on Y and reduce to a situation where we get the same contradiction as the last sentence.

□

The following lemmas will be convenient.

(1.2) <u>Lemma.</u> <u>Let</u> $f : X \longrightarrow Y$ <u>be a holomorphic surjective map of irreducible projective varieties. If there is a nontrivial holomorphic</u> k <u>form on a desingularization of</u> Y, then there is a non-trivial holomorphic k form on a desingularization of X.

Proof. Let $\pi_1 : \overline{X} \longrightarrow X$ and $\pi_2 : \overline{Y} \longrightarrow Y$ be desingularizations of X and Y. Since holomorphic forms pullback to holomorphic forms under meromorphic maps the lemma follows by considering $\pi_2^{-1} \circ f \circ \pi_1$.

□

(1.3) Lemma. Let $f : X \longrightarrow Y$ be a meromorphic surjective map between irreducible projective varieties. Assume that there is an open set $V \subseteq Y_{reg}$ $f : f^{-1}(V) \longrightarrow V$ is of maximal rank and

a) $f^{-1}(V) \subseteq X_{reg}$,

b) on $f^{-1}(V)$, f has connected fibres and is maximal rank,

c) given a generic fibre F of f on $f^{-1}(V)$, $h^i(\mathcal{O}_F) = 0 \quad 0 < i \leq k$.

If a desingularization of X has a non-trivial k form then a desingularization of Y has a non-trivial k form.

Proof. Let X' denote the graph of f and $f' : X' \longrightarrow Y$ the induced map. Let $\pi_2 : \overline{Y} \longrightarrow Y$ be a desingularization of Y. Let Z be the irreducible component of $X' \times_Y \overline{Y}$ that surjects onto both X and $\overline{Y}$ under the induced map. Let $\pi_1 : \overline{X} \longrightarrow Z$ be a desingularization of Z. We have a commutative diagram:

$$\begin{array}{ccc} \bar{X} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \bar{Y} & \longrightarrow & Y \end{array}$$

where the horizontal maps are birational. The hyptheses of lemma (1.1.1) are satisfied and therefore our lemma follows.

□

§ 2 The Meromorphic Extension Theorem

(2.0) Let L be a line bundle on a compact irreducible normal complex analytic space X. Assume that L is spanned at all points by global sections and that X is Cohen-Macaulay, i.e. that the localrings of X are all Cohen-Macaulay local rings. Let

$$e : X \times \Gamma(L) \longrightarrow L$$

denote the evaluation map on sections. Since $\Gamma(L)$ spans L at all points it follows that e is onto and the kernel K is a vector bundle on X. We denote $\mathbb{P}(K^*)$ by $\mathcal{A}$ and note that $\mathcal{A} \subseteq X \times |L|$ is the family of pairs (x,A) with $x \in A \in |L|$. Let $p : \mathcal{A} \longrightarrow X$ and $q : \mathcal{A} \longrightarrow |L|$ denote the maps induced by the product projections and note that p is the natural projection of $\mathbb{P}(K^*) \longrightarrow X$.

Since $\mathcal{A}$ is a fibre bundle with smooth fibre over a Cohen-Macaulay variety it follows that $\mathcal{A}$ is Cohen-Macaulay. Since q has equal dimensional fibres, $\mathcal{A}$ is Cohen-Macaulay, and $|L|$ is smooth, it follows that:

(2.0.1) q is flat.

(2.1.) Lemma. <u>Let</u> $X,L,\mathcal{A}$ <u>and</u> q <u>be as above. Assume that there is an irreducible normal</u> $A \in |L|$ <u>that fibres holomorphically</u> $f : A \longrightarrow Y$ <u>where</u> Y <u>is a normal irreducible</u>

analytic space and where f has connected fibres. Assume further that there is a smooth Zarisky open set $V \subseteq Y$ such that $U = f^{-1}(V)$ is smooth and such that f is of maximal rank on U. Assume that there is an ample line bundle E on Y such that f^*E extends to a line bundle $\mathcal{E}$ on X. Assume that $f_{(i)}(\mathcal{O}_A) = 0$ for all odd i. Then there is a compact normal analytic space $\mathcal{Y}$, a holomorphic surjection $g : \mathcal{Y} \longrightarrow |L|$ and a meromorphic surjection $F : \mathcal{A} \longrightarrow \mathcal{Y}$ such that :

a)
$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{Y} \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle g} & \\ & |L| & \end{array}$$
commutes

b) F is holomorphic on a Zariski open set containing $q^{-1}(A)$, $g^{-1}(A)$ is biholomorphic to Y and $F_{q^{-1}(A)} = f$,

c) g is equal dimensional in a neighborhood of $g^{-1}(A)$,

d) there is a smooth Zariski open set $\mathcal{V} \subseteq \mathcal{Y}$ such that F is of maximal rank in the set $\mathcal{U} = F^{-1}\mathcal{V}$ which is smooth and such that $\mathcal{V} \cap g^{-1}(A) = V$.

Proof. Choose n large enough so that E^n is very ample and by Serre's theorem $H^j(Y, f_{(i)}(f^*E^n)) = H^j(Y, f_{(i)}(\mathcal{O}_A) \otimes E^n)$ is zero for $j > 0$ and all i. By the Leray spectral sequence for f and f^*E^n and the hypothesis that $f_{(i)}(\mathcal{O}_A) = 0$ for odd i, it follows that $H^j(A, f^*E^n) = 0$ for odd $j > 0$. By the flatness (2.0.1) of q it follows that $\chi(E^n_{A'})$ is independent of $A' \in |L|$. From this and the upper semi-continuity of dimensions of cohomology groups it follows that $h^0(A', E^n_{A'})$ is constant for a Zariski open set of $|L|$ that contains A. This and the flatness of q imply by a theorem of Grauert that the coherent sheaf:

$$S = q_*(p^*E^n)$$

is locally free of rank $h^0(E^n)$ in a neighborhood of A in $|L|$. Since sections of f^*E^n therefore extend to give sections of S it follows that (p^*E^n), is spanned by global section for A' in a Zarisky open set $\mathcal{O}$ containing A in $|L|$. Therefore we have a meromorphic map F' from A into $\mathbb{P}roj(S)$ which is holomorphic in a neighborhood of $q^{-1}(A)$. Let Y denote the normalization of the image of F' and let F denote the induced meromorphic map. Note that $\dim F(q^{-1}(A'))$ is independent of $A' \in \mathcal{O}$. Indeed if $E^n_{A'}$ is spanned, then its image is of dimension:

$$\max\{k \mid \underbrace{E \cdots E}_{k \text{ times}} \cdot L \text{ is non-trivial in } H^{2k+2}(X,\mathbb{Q})\}.$$

This implies c) where $g : Y \longrightarrow |L|$ is the induced map.

The assertion d) is straighforward and left to the reader.

□

(2.3) Meromorphic Extension Theorem. Let X be an n dimensional Cohen Macaulay compact irreducible normal complex analytic space. Assume that $h^0(\Lambda^{n-2}T^*_{\tilde{X}}) \neq 0$ where $\tilde{X}$ is a desingularization of X. Assume that L is a line bundle spanned at all points of X by global sections and that $C_1(L)^{\dim X} > 0$, i.e., the map associated to $\Gamma(L)$ has image of dimension $\dim X$. Assume that there is an irreducible normal $A \in |L|$ such that there is a holomorphic surjection $f : A \longrightarrow Y$ with generic fibre $\mathbb{P}^1$ onto a compact normal complex analytic space Y. Assume that there is an ample line bundle E on Y such that f^*E extends to a holomorphic line bundle $\mathcal{E}$ on X. If either f is flat or A and Y have only rational singularties (if any) then f extends to a meromorphic map:

$$\bar{f} : X \longrightarrow Y$$

holomorphic in a neighborhood of the open set $U \subseteq A_{reg}$ such that $f_U : U \longrightarrow f(U)$ is a $\mathbb{P}^1$ bundle.

Proof. Lemma (2.1) applies. Let

$$\begin{array}{ccccc} X & \xleftarrow{p} & A & \xrightarrow{F} & \mathcal{Y} \\ & & {\scriptstyle q}\searrow & \swarrow{\scriptstyle g} & \\ & & & |L| & \end{array}$$

be as in that lemma.

Since A is normal and since a generic fibre of $f : A \longrightarrow Y$ is $\mathbb{P}^1$, it follows that $f(\mathrm{Sing}(A))$ is a proper analytic subset of Y and therefore that f is a holomorphic $\mathbb{P}^1$ bundle over a Zariski open set of Y. This property is clearly inherited by the maps $F_{A'}: A' \longrightarrow F(A')$ given by lemma (1.1) for A' near A in $|L|$. Thus:

(2.3.1) F is a $\mathbb{P}^1$ bundle over a smooth Zariski open set $\mathcal{V} \subseteq \mathcal{Y}$ which meets Y non-trivially in a Zariski open set V (here we identify Y with $g^{-1}(A)$).

Let $B = p^{-1}(A)$ and let B' be the image of B in $\mathcal{A} \times A$ under the map (i,p) where $i : B \longrightarrow \mathcal{A}$ is the inclusion. Let $F': \mathcal{A} \times A \longrightarrow \mathcal{Y} \times A$ be the map (F, id_A).

Note that B is irreducible since it is a fibre bundle over A. Thus $F'(B')$ is irreducible. Since F' is a $\mathbb{P}^1$ bundle over $\mathcal{V} \times A$ where $\mathcal{V}$ is as in (2.3.1) it follows that the closure Z of $F'^{-1}(F'(B') \cap \mathcal{V} \times A)$ in $F'^{-1}(F'(B'))$ is an irreducible set.

(2.3.2) <u>Lemma</u>. <u>The meromorphic map</u> F' <u>from</u> B' <u>to</u> $F'(B')$ is one to one on $F'^{-1}(\mathcal{V} \times A) \cap B'$.

<u>Proof.</u> To see this note that if $(v,x) \in \mathcal{V} \times A$ then $F'^{-1}(v,x) =$

$$\{(w,x) \mid F(w) = v\}.$$

Note that $\{w \in \mathcal{A} \mid F(w) = v\} = \{(z,A') \in X\times|L| \mid g(v) = A',$

$$z \in A', F_{A'}(z) = v\}.$$

Thus $F'^{-1}(v,x) \cap B' = (x,A',x) \in X \times |L| \times \mathcal{A} \mid g(v) = A', x \in A',$

$$F_{A'}(x) = v\}$$

□

Let $h : Z \longrightarrow X$ denote the map onto X induced by the composition of the product projection $\mathcal{A} \times A \longrightarrow \mathcal{A}$ and p. Let $k : Z \longrightarrow Y$ denote the surjection induced by the composition of the product projection $\mathcal{A} \times A \longrightarrow A$ and $f : A \longrightarrow Y$. Let $c : Z \longrightarrow Y \times X$ denote the map (k,h). Let $Z' = c(Z)$ and let $k' : Z' \longrightarrow Y$ and $h' : Z' \longrightarrow X$ be the maps induced by the product projections.

Choose a general element $H \in |E^N|$ where N is chosen so that E^N is very ample. By lemma (0.4), $H' = k^{-1}(H)$ is irreducible since Z is irreducible.

(2.3.3) Lemma. $h(H') \neq X$.

Proof. Assume that $h(H') = X$. Since a desingularization of X has a non-trivial holomorphic $n-2$ form on it it follows from lemma (1.2) that a desingularization $\overline{H'}$ of H' has a non-trivial holomorphic $n-2$ form on it. Since

$F'_{H'} : H' \longrightarrow F'(H')$ is a $\mathbb{P}^1$ bundle over a dense open set of $F'(H')$, it follows from lemma (1.3) that a desingularisation of $F'(H')$ has a non-trivial holomorphic $n-2$ form on it. Using lemma (2.3.2) it is clear that $F'(H')$ is birational to $p^{-1}(f^{-1}(H'))$. Since this is a projective bundle over $f^{-1}(H')$ it follows from lemma (1.3) that the desingularisation of $f^{-1}(H')$ has a non-trivial holomorphic $n-2$ form on it. Since $f^{-1}(H')$ maps onto H' with generic fibre $\mathbb{P}^1$ it follows from lemma (1.3) that the desingularisation of H' has a non-trivial holomorphic $n-2$ form on it. But since $\dim H' = n-3$, this is absurd.

□

We are now in a position to show that Z' is the graph of meromorphic map from X to Y. First note that the dimension of a generic fibre of $h' : Z' \longrightarrow X$ is 0 dimensional. Indeed if it was not then given a general very ample divisor H on Y, it follows that $h'(k'^{-1}(H)) = X$. Since $h'(k'^{-1}(H)) = h(k^{-1}(H))$, this is ruled out by lemma (2.3.3).

Therefore since $h'(Z') = h(Z) = X$ it follows that $\dim Y + \dim$ (generic fibre of k') $= \dim X$ or

(2.3.4) $\qquad$ dim (generic fibre of k') = 2.

(2.3.5) Choose a $y \in Y$ that is general in the sense that:

a) $k^{-1}(y)$ is irreducible and $\dim h(k^{-1}(y)) = 2$,

b) the curve $\ell = f^{-1}(y)$ is a smooth $\mathbb{P}^1 \subseteq A_{reg}$

and f is of maximal rank in a neighborhood of ℓ.

(2.3.6). Choose an $A' \in |L|$ that is general in the following senses:

a) A' is irreducible and smooth away from X_{sing},
b) $A' \cap A$ is irreducible and A' meets A transversely on A_{reg},
c) A' does not contain a point $x \in \ell$ selected in advance of the choice of A',
d) F is holomorphic in a neighborhood of $q^{-1}(A')$.

Note the fact that generically A' and $A' \cap A$ are irreducible follow from lemma (0.4) and the fact that the map associated to $\Gamma(L)$ has an image of dimension $\dim X$.

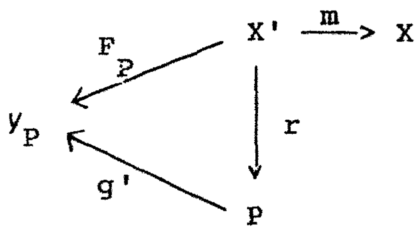
Let $P \subseteq |L|$ denote the pencil joining A and A'. Let Γ denote the graph of the $F_{q^{-1}(P)}$. Let X' denote the irreducible component of Γ such that:

$$p(\text{projection of } X' \text{ on } q^{-1}(P)) = X.$$

Let $m : X' \longrightarrow X$ be the map composed of p and projection of X' to $q^{-1}(P)$. Note that

(2.3.7) m is a birational map.

Let $\mathcal{Y}_P$ denote the irreducible component of $g^{-1}(P)$ such that $g(\mathcal{Y}_P) = P$ and the map $F_P : X' \longrightarrow \mathcal{Y}_P$ induced by F is onto. We have:



where g' and r are the maps induced by g and q respectively.

By (2.3.6) there is a dense open set $\Omega \subseteq X$ which contains ℓ and such that $m_{m^{-1}(\Omega)} : m^{-1}(\Omega) \longrightarrow \Omega$ is Ω with $A \cap A' \cap \Omega$ blown up.

By $a \in \ell \cap A'$, $r^{-1}(r(m^{-1}(a))$ contains a unique irreducible 2 dimensional component $\mathcal{W}_a$ that contains ℓ. To see this note that for most $w \in r(m^{-1}(a))$ $m(r^{-1}(w))$ is a smooth $\mathbb{P}^1$ on an $A'' \in P$ which is also a fibre of $F_{A''}$. From this we see also that $m(\mathcal{W}_a) \subseteq h(k^{-1}(y))$. Since $\dim h(k^{-1}(y)) = 2$ by (2.3.5a) we conclude $m(\mathcal{W}_a) = h(k^{-1}(y))$. This set which we call $\mathcal{W}$ is therefore independent of $a \in \ell \cap A'$ and the general A' chosen subject to (2.3.6).

Let $s_{A'} \in \Gamma(L)$ be a section defining A'. There is a short exact sequence:

$$0 \longrightarrow N_{\ell \setminus A} \longrightarrow N_{A_{reg}}|_{\ell} \longrightarrow L_{\ell} \longrightarrow 0$$

of normal bundles. The infinitesimal deformation of ℓ corresponding to the family $m(r^{-1}(w))$ for w near $r(a)$ has as image in L_{ℓ} the restriction $s_{A'}|_{\ell}$. Since $s_{A'}(x) \neq 0$ by (2.3.6c) we see that $\mathcal{W}$ is smooth near x and $\mathcal{W}$ is transverse to A near x. Since $x \in \ell$ was arbitrary we conclude that $\mathcal{W}$ is smooth in a neighborhood of ℓ and along ℓ $\mathcal{W}$ intersects A transversely.

(2.3.8) Lemma. $\mathcal{W} \cap A = \ell$.

Proof. Since y is general and the map $\phi : X \longrightarrow \mathbb{P}_{\mathbb{C}}$ has an image of dimension equal to $\dim X$, it follows that $\dim \phi(\mathcal{W}) = 2$. By (0.4) $\mathcal{W} \cap A$ is connected. Since $\mathcal{W}$ meets A transversely in ℓ it follows that $\mathcal{W} \cap A = \ell$.

□

The above shows that $\mathcal{W}$ determines y by $f(\mathcal{W} \cap A)$. Thus Z' is the graph of a meromorphic map $\overline{f} : X \longrightarrow Y$.

Finally let $\ell \subseteq A_{reg}$ be a fibre of f in a neighborhood of which f is of maximal rank. Choose A' subject to (2.3.6). Choose a local holomorphic section.

$$\sigma : \mathcal{N} \longrightarrow A$$

where $\mathcal{N}$ is a neighborhood of $f(\ell)$ and $\sigma(\mathcal{N}) \subseteq A \cap A'$.

For a small enough N and for y in a small enough neighborhood of x in X, there is a well defined holomorphic map which sends y to $f(a)$ where $a \in \sigma(N)$ and $m^{-1}(y) \in W_a$. This map agrees with $\overline{f}$ on an open set and gives the desired extension.

□

(2.4) Corollary. Let X be a normal irreducible projective variety and let L be an ample line bundle on X spanned at all points of X. Assume there is an irreducible $A \in |L|$ such that:

a) $A \subseteq X_{reg}$ and A has only rational singularities ,

b) there is a holomorphic surjection $f : A \longrightarrow Y$ onto a normal projective variety and the generic fibre of f is $\mathbb{P}^1$,

c) that Y has at worst rational singularities.

Then f extends to a meromorphic map $\overline{f} : X \longrightarrow Y$ holomorphic in a neighborhood of the open set $U \subseteq A_{reg}$ such that $f_U : U \longrightarrow f(U)$ is a $\mathbb{P}^1$ bundle.

Proof. Let $\pi : \overline{X} \longrightarrow X$ be a desingularization of X. Since $A \subseteq X_{reg}$ we have π giving a biholomorphism of A and $\pi^{-1}(A)$. Let E be an ample line bundle on Y. By lemma (0.3.1) f^*E^n extends to $\overline{X}$ for some $n > 0$. By (1.2) and (0.3.3) $h^0(\Lambda^{\dim Y} T^*_{\overline{X}}) \neq 0$. Thus $(\overline{X}, \pi^*L, \pi^{-1}(A), E^n)$ satisfies the hypothesis on (X,L,A,E) in theorem (2.3). Therefore there is a meromorphic extension $\overline{f} : \overline{X} \longrightarrow Y$. The composition $\overline{f} \circ \pi^{-1} : X \longrightarrow Y$

is the desired extension.

□

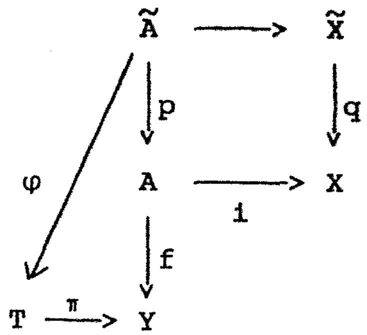
(2.5) Corollary. Assume the same hypothesis as (2.4) except that $A \subseteq X_{reg}$ is replaced by the assumption that X is a local complete intersection with the locus of irrational singularities having codimension ≥ 3 . Then the same conclusion as in (2.4) holds.

Proof. Use (0.3.4) instead of (0.3.3).

□

(2.6) Theorem Let X be a projective variety which is a local complete intersection. Assume that L is an ample line bundle spanned at all points of X by global sections. Assume that there exists a smooth $A \in |L|$ which is a $\mathbb{P}^1$ bundle $f : A \longrightarrow Y$ over a projective manifold Y. Assume that there exists an unramified cover $\pi : T \longrightarrow Y$ with $h^{\dim T,0}(T) \neq 0$. Then f extends to a holomorphic map $\overline{f} : X \longrightarrow Y$.

Proof. We can assume without loss of generality that T is a regular covering of Y. Suppressing base points for simplicity we note that $\pi_1(T) \overset{f_*^{-1}}{\simeq} \pi_1(A) \overset{i_*}{\simeq} \pi_1(X)$ where $i : A \hookrightarrow X$ is the inclusion map. Let $H_1 = \pi_*(\pi_1(T))$, $H_2 = f_*^{-1}(H_1)$ and $H_3 = i_*(H_2)$. Denote by $\tilde{A}$ and $\tilde{X}$ the covering spaces of A and X corresponding to H_2 and H_3 , subgroups of $\pi_1(A)$ and $\pi_1(X)$ respectively. Thus we have the following diagram



Note that $f \circ p$ and $i \circ p$ lift to a map from $\tilde{A}$ to T and from $\tilde{A}$ to $\tilde{X}$ respectively since

$$(f \circ p)_*(\pi_1(\tilde{A})) = H_1 = \pi_*(\pi_1(T)) \quad \text{and}$$

$$(i \circ p)_*(\pi_1(\tilde{A})) = H_3 = q_*(\pi_1(\tilde{X})).$$

It is easy to see that $\tilde{A}$ is an ample divisor on $\tilde{X}$ and that $\tilde{A} \xrightarrow{\varphi} T$ is $\mathbb{P}^1$ bundle over T. Using (0.7) and (2.4) we conclude that the map φ extends to a holomorphic map $\tilde{\varphi} : \tilde{X} \longrightarrow T$. The group of the deck transformations of $\tilde{X}, \tilde{A}$ and T are all isomorphic to one another by construction. Denote such group by G. Note that everything descends. Therefore we get a holomorphic map $\tilde{f} : X \longrightarrow Y$, where $\tilde{f}$ is obtained from $\tilde{\varphi} : \tilde{X} \longrightarrow T$ after we have considered the action of G on $\tilde{X}$ and T. Clearly $\tilde{f}$ is holomorphic and is an extension of our given f.

□

(2.7) Corollary Let X,L and A be as in (2.6). Assume that $K_Y^t = \mathcal{O}_Y$, with t minimal. Then the same conclusion as in (2.6) holds.

Proof. Let $\pi : T \longrightarrow Y$ be the t-cyclic unramified cover of Y determined by the torsion line bundle K_Y. Note that $h^{\dim T,0}(T) \neq 0$. Therefore (2.6) applies.

□

§ 3 $\mathbb{P}^1$ Bundles as Hyperplane Sections

(3.0) *Theorem. Let* X *be a projective local complete intersection. Assume that* L *is an ample line bundle on* X *spanned at all points by global sections. Assume that there is a smooth* $A \in |L|$ *which is a* $\mathbb{P}^1$ *bundle* $f : A \longrightarrow Y$ *over a projective* Y. *Then if* $h^0(K_Y) \neq 0$ f *extends to a holomorphic map* $\overline{f} : X \longrightarrow Y$. $\dim Y \leq 2$ *and if* $\dim Y = 2$, $\overline{f} : X \longrightarrow Y$ *is a* $\mathbb{P}^2$ *bundle with the restriction of* L *to a fibre of* $\overline{f}$ *isomorphic to* $\mathcal{O}_{\mathbb{P}^2}(1)$.

Proof. By lemma (0.7) and (2.4) the holomorphic extension $\overline{f} : X \longrightarrow Y$ exists. By Prop. V of [So1] it follows that if f is a $\mathbb{P}^1$ bundle then $\dim Y \leq 2$.

We can therefore by theorem (0.6.1) assume that $\dim Y = 2$. Let $\mathcal{A}$ be the union of the singular set of X and the set where $\overline{f}$ is not of maximal rank. Since $\mathcal{A} \subseteq X - A$, the set $\mathcal{A}$ is finite. Choose a smooth connected curve $C \subseteq Y$ such that $\overline{f}(\mathcal{A}) \cap C = \emptyset$. Let $f' = f_{f^{-1}(C)}$, $\overline{f'} = \overline{f}_{\overline{f}^{-1}(C)}$, and let F denote a general fibre of $\overline{f}'$. Suppressing basepoints for simplicity we have the long exact sequences of homotopy groups of fibre bundles:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \uparrow & & \\
 & \nearrow & \pi_2(F) & \longrightarrow & \pi_2(\overline{f}^{-1}(C)) & \searrow & \\
\pi_3(C) & & \uparrow a & & \uparrow b & & \pi_2(C) \\
 & \searrow & \pi_2(\mathbb{P}^1) & \longrightarrow & \pi_2(f^{-1}(C)) & \nearrow &
\end{array}$$

Note that b is surjective by the first Lefschetz theorem on hyperplane sections. A diagram chase shows that a is surjective. Since $\mathbb{P}^1$ is a hyperplane section of F it is very well known [e.g. So2, (0.6.1)] that F is either $\mathbb{P}^2$ or a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$. Since a is surjective F is $\mathbb{P}^2$ and $[\mathbb{P}^1]_F = L_F$ is $\mathcal{O}_{\mathbb{P}^2}(1)$.

We are done except for the possibility of a singular fibre F of $\bar{f} : X \longrightarrow Y$. Dim $F \leq \dim f^{-1}(\bar{f}(D)) + 1 = 2$. By the above it is clear that F is irreducible (since $L \cdot L \cdot F = 1$). An easy arugment using [So2 (0.6.1)] shows that $F \simeq \mathbb{P}^2$.

To finish the argument note that since $\bar{f}^{-1}\bar{f}(F) = \mathbb{P}^2$ and since $\bar{f}$ is flat (fibres are equal dimensional, X is a local complete intersection and Y is smooth) it follows that $\bar{f}$ is of maximal rank in a neighborhood of F.

$\square$

(3.1) <u>Theorem</u> <u>Let</u> L <u>be an ample line bundle on a local complete intersection</u> X <u>assume that:</u>

a) L <u>is spanned by global sections and</u> $h^0((K_X \otimes L)^N) \neq 0$ <u>for some</u> $N > 0$,

b) <u>the singular set of</u> X <u>have codimension</u> ≥ 4.

Let $H \in |L|$ and assume that there is a holomorphic surjection $f : H \longrightarrow H'$ which expresses H as a projective variety with a codimension 2 submanifold $A' \subseteq H'_{reg}$ blown up. Assume that $h^0(K_{A'}) \neq 0$. Then $\dim A' \leq 2$ and f extends to a holomorphic map $\overline{f} : X \longrightarrow X'$ such that

$$\begin{array}{ccc} H & \xrightarrow{\ i\ } & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \overline{f}} \\ H' & \xrightarrow[\ i'\ ]{} & X' \end{array}$$

commutes where i and i' are inclusions. The map $\overline{f}$ expresses X as X' with the smooth subvariety $i'(A') \subseteq X'_{reg}$ blown up.

Proof. By the same argument as in [So2] or [Fa2] it can be shown that there exists a normal Cartier divisor D on X which meets H transversaly along E, the exceptional divisor of f over A'. By (3.0) the map $f_E : E \longrightarrow A'$ extends to a holomorphic map $\overline{f} : D \longrightarrow A'$ and $\dim A' \leq 2$. The case $\dim A' = 1$ has been done, see [Fa2]. So the only case left out is $\dim A' = 2$. In such case $\overline{f} : D \longrightarrow A'$ is a $\mathbb{P}^2$ bundle with the restriction of L_D to the general fibre of $\overline{f}$ isomorphic to $\mathcal{O}_{\mathbb{P}^2}(1)$. It is then clear that the line bundle $[D]$ restricted to the general fibre of $\overline{f}$ is $\mathcal{O}_{\mathbb{P}^2}(-1)$. Therefore by Nakano's theorem we can smoothly blow down D. Thus there exists a variety X and a holomorphic map $\overline{f} : X \longrightarrow X'$ such that the following diagramm

$$\begin{array}{ccc} H & \xrightarrow{i} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ H' & \longrightarrow & X' \end{array}$$

commutes. Clearly $i'(A') \subset X'_{reg}$ since H' is a Cartier divisor on X'. And the map $\overline{f}$ expresses X as X' with $i'(A')$ blown up.

□

§4 Concluding Remarks

(4.0) Conjecture. *Let $f : A \longrightarrow Y$ be a $\mathbb{P}^1$ bundle over a smooth connected projective manifold Y. Assume that Y has non-negative Kodaira dimension. If A is an ample divisor on a projective local complete intersection X, then f extends to a holomorphic surjection $\bar{f} : X \longrightarrow Y$, $\dim Y \leq 2$, and $\dim Y = 2$ implies $\bar{f}$ is a $\mathbb{P}^2$ bundle.*

How do we approach (4.0)? First let us consider the Kodaira dimension condition. It would be natural to have a higher order first Lefschetz theorem. For example there is the following question.

(4.1) Question. *Let A be a smooth ample (or even very ample) divisor on a connected projective manifold X. Is*

$$H^0((\Lambda^i T_X^*)^{(s)}) \approx H^0((\Lambda^i T_A^*)^{(s)})$$

for $i < \dim A$ and all sufficiently large s where $E^{(s)}$ denotes the s th symmetric power of a vector bundle E.

We can weaken the condition $h^0(K_Y) \neq 0$ when $\dim Y = 2$. Indeed if Y is a general type surface then most fibres of $\mathcal{Y}_P \longrightarrow P$ in the proof of (2.3) are general type surfaces. The intersection $A \cap A'$ is also of general type and surjects generically finite to one onto most fibres of $\mathcal{Y}_P \longrightarrow P$. By the 2 dimensional de Francis theorem ([D+M],[M]) most of the maps

$A \cap A'$ to a fibre of $\mathcal{Y}_P \longrightarrow P$ are the same except for some blowing up and down. Assuming the fibre degree of these maps is t then we get a meromorphic map $\mathcal{Y}_P \longrightarrow (A \cap A')^{(t)}$ where $W^{(t)} = W^t/S_t$ where S_t acts by permutations in the t factors. The image of $\mathcal{Y}_P$ in $(A \cap A')^t$ is birational to Y and the composition of $X \longleftarrow X' \longrightarrow \mathcal{Y}_P$ with this gives the desired birational map.

Next how should we remove the condition [A] be spanned? Assume $h^0(K_Y) \neq 0$. Let ω be a non-trivial section of $\Lambda^{\dim Y} T_X^*$ obtained by lifting ω to $\Lambda^{\dim Y} T_A^*$ and extending by the first Lefschetz theorem. We have a sequence:

$$T_X \xrightarrow{\ i_\omega\ } \Lambda^{\dim Y - 1} T_X^*$$

where i_ω is interior multiplication. Let $\mathcal{F}$ be the subsheaf which is the kernel of i_ω. $\mathcal{F}$ should define a foliation on a dense set of X whose sheets are the fibres of the desired meromorphic extension $X \longrightarrow Y$. A careful study of $\mathcal{F}$ will very possibly (at the expense of a more technical proof) remove the need to assume spanning of [A].

Finally it would be interesting to know how much the Kodaira dimension condition on Y can be relaxed.

Appendix Extension Theorems for Maps of Fibre dimension at least 2

(A.1) Theorem. *Let A be an ample divisor on an irreducible normal projective variety X. Assume that A is normal and that there is a holomorphic surjection $f : A \longrightarrow Y$ of A onto a projective variety such that $\dim A - \dim Y \geq 2$. If there is an ample line bundle L on Y such that f^*L extends to a holomorphic line bundle $\mathcal{L}$ on X, then f extends to a holomorphic map $\overline{f} : X \longrightarrow Y$. In particular extension takes place if X is a local complete intersection with the locus of non rational singularities having codimension ≥ 3.*

Proof. The proof follows that of [So1] very closely; we incorporate the improvements of [Fu1] and [Fu2] . By raising L to a sufficiently high positive power we get a very ample line bundle whose pullback extends to X. Thus we can assume that L is very ample without loss of generality. Consider:

$$0 \longrightarrow \mathcal{L} \otimes [A]^{-1} \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}_A \longrightarrow 0$$

If the sections of $\mathcal{L}_A$ extend to sections of $\mathcal{L}$ we will get an extension of f to a meromorphic $\overline{f} : X \longrightarrow \mathbb{P}_{\mathbb{C}}$ by using $\Gamma(\mathcal{L})$ as sketched in (0.7) (cf. [Fu1] also).

To show the sections of $\mathcal{L}_A$ extend to sections of $\mathcal{L}$ it sufficies to show that $H^1(X, \mathcal{L} \otimes [A]^{-1}) = 0$.

Considering the above exact sequence tensored with $[A]^{-r}$ for $r = 1,2,3 \ldots$ we see that $H^1(A, L_A \otimes [A]_A^{-r}) = 0$ for $r = 1,2,3\ldots$ would imply that:

$$h^1(L \otimes [A]^{-1}) \leq h^1(L \otimes [A]^{-2}) \leq \ldots .$$

Since X is normal, $H^1(X, L \otimes [A]^{-k}) = 0$ for $k >> 0$, [Ha Ch. III, Cor. 7.8]. Therefore we have reduced to showing that $h^1(L_A \otimes [A]_A^{-r}) = 0$ for $r \geq 1$ which follows from (0.2.1).

Note that under the local complete intersection condition extension of f^*L occur by (0.3.2).

□

(A.2) Theorem. Let X be a normal irreducible projective variety with isolated singularities. Let A be an ample divisor on X which is normal and such that $A \subseteq X_{reg}$. If there is a holomorphic surjection $f : A \longrightarrow Y$ onto a projective variety with $\dim A - \dim Y \geq 2$ then f extends to a meromorphic map $\bar{f} : X \longrightarrow Y$ which is holomorphic on X_{reg}.

Proof. Let L be an ample line bundle on Y. Let $\pi : \tilde{X} \longrightarrow X$ be a desingularization of X. By lemma (0.3.1), f^*L^m extends to a holomorphic line bundle L on $\tilde{X}$ for some $m > 0$. The proof of the last result and a standard Hartog's theorem argument would prove this result if we show that for some neighborhood U of A

$$H^1(U, L \otimes [A]^{-r}) = 0 \quad \text{for} \quad r >> 0.$$

This is true by a result of Griffiths ([Gri], see also [LP]); we have followed the idea of [Si]. Instead of using Griffiths' theorem we could work on the formal completion of A in X as done by Fujita [Fu2].

□

A consequence of theorem (A.1) using [So1] is that if A and X are as in theorem (A.1) so that the holomorphic extension $\bar{f} : X \longrightarrow Y$ exists and if $f : A \longrightarrow Y$ is a $\mathbb{P}^k$ bundle with $k \geq 2$ then $\bar{f}$ is a $\mathbb{P}^{k+1}$ bundle.

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