REAL K3 SURFACES WITH NON-SYMPLECTIC INVOLUTIONS AND APPLICATIONS

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Abstract. Classification of real K3 surfaces X with a non-symplectic involution τ is considered. For some exactly defined and one of the weakest possible type of degeneration (giving the very reach discriminant), we show that the connected component of their moduli is defined by the isomorphism class of the action of τ and the anti-holomorphic involution φ in the homology lattice. (There are very few similar cases known.) For their classification we apply invariants of integral lattice involutions with conditions which were developed by the first author in 1983. As a particular case, we describe connected components of moduli of real non-singular curves $A \in |-2K_V|$ for the classical real surfaces: $V = \mathbb{P}^2$, hyperboloid, ellipsoid, $\mathbb{F}_1$, $\mathbb{F}_4$.

As an application, we describe all real polarized K3 surfaces which are deformations of general real K3 double rational scrolls (surfaces V above). There are very few exceptions. For example, any non-singular real quartic in $\mathbb{P}^3$ can be constructed in this way.

Contents

1. Introduction

In this paper, a K3 surface is a simply-connected Kälerian compact complex surface with the trivial canonical class. Thus, there exists a 2-dimensional holomorphic differential form ω_X with the zero divisor. The form ω_X is unique up to $\lambda\omega_X$ where $\lambda \in \mathbb{C}$ and $\lambda \neq 0$.

A K3 surface may have two very different kinds of holomorphic involutions τ: symplectic when $\tau^*(\omega_X) = \omega_X$ and non-symplectic when $\tau^*(\omega_X) = -\omega_X$.

There exists only one type of symplectic involutions [18] and many types of non-symplectic involutions [20], [22]. The types of non-symplectic involutions are classified by invariants which are all presented in Figure 1 below. They describe irreducible components of moduli of complex K3 surfaces X with non-symplectic involutions τ. See more delicate classification in [1] and [2].

In this paper we consider K3 surfaces with non-symplectic involutions (X, τ). The quotient surface $Y = X/\{1, \tau\}$ is either Enriques surface when τ has no fixed points, or a rational surface when τ has fixed points. The set of fixed points is a non-singular curve $A = X^\tau$, and its image $A \subset Y$ is an element of the linear system $|-2K_Y|$ where K_Y is the canonical system of Y. Then X is the double covering of Y ramified in A and τ the involution of the double covering. Pairs (A, Y) when Y is either Enriques or a rational surface (they are characterized by the condition $q(Y) = 0$ where q is the irregularity) and $A \subset |-2K_Y|$ is a non-singular curve are called right DPN-pairs. Classification of K3 surfaces with non-symplectic involutions (X, τ) is equivalent to classification of right DPN-pairs $A \subset Y$.

In this paper we consider description of connected components of moduli of real K3 surfaces with non-symplectic involutions and real right DPN-pairs. Equivalently, we consider classification of triplets (X, τ, φ) where X is a K3 surface, τ its holomorphic non-symplectic involution and φ its anti-holomorphic involution. The involutions τ and φ must commute: $\varphi\tau = \tau\varphi$. Equivalently, we consider classification of real right DPN-pairs $A \subset Y$ with an anti-holomorphic involution θ where $\theta(A) = A$.

This classification depends on the definition of degeneration of (X, τ, φ). We choose one of *the weakest types of degeneration* (i. e. *giving one of the reachest discriminants*). Geometrically (over $\mathbb{C}$) it is equivalent to appearance of an exceptional curve F with square -2 on the quotient Y. Over $\mathbb{R}$ this curve should be either real: $\theta(F) = F$, or $\theta(F) \cdot F \leq 1$. See Section 2.1.

We mention that our definition of degeneration is weaker (it gives a much reacher discriminant) then the one used in [4] for description of connected components of moduli of real Enriques surfaces. Thus, even for real Enriques surfaces our results are stronger (more exact). Difference of our results from [4] is in the appearance of the group G.

On the one hand, our type of degeneration gives *a reach discriminant.* For example, it is enough to describe connected components of moduli of real bidegree $(4, 4)$ curves on the hyperboloid which was one of the main purposes of this paper. A degeneration of this case gives non-singular curves $A \in |-2K_{\mathbb{F}_2}|$ which we must exclude. Here we denote as $\mathbb{F}_n$, $n \geq 0$ a relatively minimal rational surface with the section s having $s^2 = -n$. See Proposition 4. On the other hand, using Global Torelli Theorem [24] and epimorphicity of Torelli map [13] for K3 surfaces we prove that connected components of moduli are described by sufficiently simple invariants which are *isomorphism classes of integral involutions with conditions.* They are triplets $(L, \varphi, (S, \theta))$. Here $L = H_2(X, \mathbb{Z})$ is the even unimodular homology lattice with the action of φ, $S = L^\tau$ and $\theta = \varphi|S$. The pair (S, θ) determines the type of real K3 surfaces (X, τ, φ) with non-symplectic involution whose connected components of moduli we want to describe. See Theorems 1, 6, 7. All genus invariants of integral involutions with condition were classified in [21]. See Sect. 2.3 about these invariants. They give very universal invariants which work in all cases we consider. This paper can be considered as one of applications of this classification and the invariants which it uses.

We mention that similar arithmetic description of connected components of moduli had been obtained in [19] for real polarized K3 surfaces. Similar result was obtained in [4] for real Enriques surfaces with the much stronger (or with the much smaller discriminant) type of degeneration used in [4]. Thus, this paper adds some more cases to these beautiful series. It seems there are very few geometrically significant cases when similar arithmetic results are valid.

Our results are valid for any type (S, θ) of real K3 surfaces with non-symplectic involutions (or the real right DPN-pairs). In Sections 3 — 7 we specify them in the most simple and general cases (by the dimension $20 - \operatorname{rk} S$ of moduli). We consider all cases when $\operatorname{rk} S \le 2$. They give the following real DPN-pairs and the corresponding real K3 surfaces with non-symplectic involutions as double coverings:

$\mathbb{P}^2$: Real non-singular curves A of degree 6 on the real projective plane $\mathbb{P}^2$; see Sect. 3.

$\mathbb{H}$ (hyperboloid): Real non-singular curves A of bidegree $(4,4)$ on hyperboloid and on $\mathbb{P}^1 \times \mathbb{P}^1$ with all possible spinor (i. e. with empty set of real points) real structures; see Sect. 4.

$\mathbb{E}$ (ellipsoid): Real non-singular curves A of bidegree $(4,4)$ on ellipsoid, see Sect. 5.

$\mathbb{F}_1$: Real non-singular curves $A \in |-2K_{\mathbb{F}_1}|$ on $\mathbb{F}_1$ with real structure; see Sect. 6.

$\mathbb{F}_4$: Real non-singular curves $A \in |-2K_{\mathbb{F}_4}|$ on $\mathbb{F}_4$ with a real structure; see Sect. 7.

In all these cases we get classification of connected components of moduli of the non-singular real curves A on the real surfaces up to the action of the automorphism group of the real surfaces.

The case of $\mathbb{P}^2$ had been considered in [19].

For hyperboloid $\mathbb{H}$, the isotopy classification had been obtained by Gudkov [7]. Zvonilov [35] added description of divideness by $A(\mathbb{R})$ of $A(\mathbb{C})$ and possible complex schemes (i. e. complex orientations of $A(\mathbb{R})$ which are induced by $A(\mathbb{C})$ in the dividing case). We show that these visual invariants: *isotopy type, divideness and complex orientations (in some very few dividing cases), define the connected component of moduli* up to the action of the automorphism group which has 8 connected components for hyperboloid.

For ellipsoid $\mathbb{E}$, the isotopy classification had been obtained by Gudkov and Shustin [8]. Zvonilov [35] added description of divideness by $A(\mathbb{R})$ of $A(\mathbb{C})$ and possible complex schemes which are prescribed by the isotopy type in this case. We show that these visual invariants: *isotopy type and divideness, define the connected component of moduli,* up to the action of the automorphism group of ellipsoid which has 2 connected components. (Similar result is valid for $\mathbb{P}^2$.)

For $\mathbb{F}_1$ we show that also *the connected component of moduli is defined by the isotopy type and by divideness.* The related result had been obtained by Itenberg [9], [10] who gave the classification of connected components of moduli of real curves of degree 6 with one non-degenerate quadratic singular point. In principle, our classification should follow from Itenberg's results, but it would be hard for us to deduce it from [9], [10].

For $\mathbb{F}_4$ we also show that *the connected component of moduli is defined by the isotopy type and by divideness.* This case corresponds to very important class of K3 surfaces which are elliptic K3 surfaces with section. The involution τ is the inverse map of the corresponding elliptic pencil.

In Sect. 8 we consider an important application of the above results to description of real polarized K3 surfaces which are *deformations of general real K3 double rational scrolls.* We mean the following. Let (X, P) be a polarized K3 surface with a primitive ample H where $P^2 = n$. For a *general K3 double rational scroll* the complete linear system $|P|$ or $|2P|$ gives the double covering $\pi : X \to Y = X/\{1, \tau\}$ where Y is one of surfaces $Y = \mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1$ or $\mathbb{F}_4$ above. Moreover, $|kP|$ is the preimage $|kP| = \pi^*|\widetilde{P}|$ of some standard linear system $\widetilde{P}$ on Y. General K3 double rational scrolls give codimension 1 moduli subspace (if $n \ge 4$) of the connected moduli space $\mathcal{M}_n$ of the polarized K3 surfaces. Thus, any polarized K3 surface

(X, P) over $\mathbb{C}$ is a deformation of some general K3 double rational scroll. It is interesting to ask, if the same is valid over $\mathbb{R}$.

Using classification results of Sects. 3 — 7 and the classification results about real polarizes K3 surfaces [19], in Sect. 8 we show that a real polarized K3 surface (X, P) of degree $n = P^2 \geq 6$ is not a deformation of a general real K3 double rational scroll, if the real part $X(\mathbb{R})$ either consists of 10 spheres or it consists of k spheres and $X(\mathbb{R}) \sim P \mod 2$ in $H_2(X, \mathbb{Z})$. "All other" (in some sense) real polarized K3 surfaces (X, P) are deformations of general real K3 double rational scrolls. In [19] all possible genus invariants (for the action of the anti-holomorphic involution φ in homology)

$$(r, a, \delta_P, \delta_\varphi, \delta_{\varphi P})$$

of real polarized K3 surfaces were defined and classified. Excluding few (possibly extremely few) exceptions the genus invariants determine the connected component of moduli of real polarized K3 surfaces. "All other" above means all possible genus invariants different from genus invariants of the shown above exceptions. Additional considerations show that for $P^2 = 2$ (double coverings of $\mathbb{P}^2$) and $P^2 = 4$ (quartics in $\mathbb{P}^3$) any real K3 surface is a deformation of a general real K3 double rational scroll. For $P^2 = 2$ it is obvious. For quartics, $P^2 = 4$, we can additionally use double coverings of ellipsoid $\mathbb{E}$ which one cannot use for $P^2 \neq 4$. This gives a very effective method of construction of all (i. e. representatives of all connected components of moduli) real quartic surfaces. Using results of Shah [26] (over $\mathbb{C}$), one can describe all these deformations very explicitly. It gives an effective method of construction of real quartic surfaces.

The first author wants to thank University of Liverpool, Steklov Mathematical Institute, Max-Planck Institut für Mathematik and Hakodate Campus of Hokkaido University of Education for hospitality.

2. Connected components of moduli of real K3 surfaces with non-symplectic involutions and real DPN-pairs

2.1. **Connected components of moduli of non-degenerate real K3 surfaces with non-symplectic involutions.** We remind that a K3 surface X is a non-singular projective algebraic surface over $\mathbb{C}$ (more generally a compact Kählerian surface) such that it has a holomorphic 2-dimensional differential form $0 \neq \omega_X$ with zero divisor (or the canonical class $K_X = 0$) and X is simply-connected. The form ω_X is defined uniquely up to $\lambda\omega_X$, $\lambda \in \mathbb{C}^*$. E.g. see [29], [24] and [27] about K3 surfaces.

An involution τ of a K3 surface X can be *symplectic,* if $\tau^*\omega_X = \omega_X$, and *non-symplectic,* if $\tau^*\omega_X = -\omega_X$. Symplectic involutions on K3 surfaces were classified in [18], for this case it is natural to consider Kählerian K3 surfaces, and there exists only one type of such involutions. Non-symplectic involutions on K3 surfaces were classified in [20], [22]; all K3 surfaces with non-symplectic involutions are algebraic.

The main invariant of a K3 surface with a non-symplectic involution τ is the isomorphism class of the lattice S which is the fixed part $H_2(X, \mathbb{Z})^\tau = S$. Here and in what follows we denote

$$M^\xi = \{x \in M \mid \xi(x) = x\},$$

and by

$$M_\xi = \{x \in M \mid \xi(x) = -x\}$$

for an action of an involution ξ on a module M. By the Hodge decomposition, the lattice S is hyperbolic (i. e. it has exactly one positive square), it is contained in the Picard lattice

$N(X)$ of X (which is generated by classes of all algebraic curves on X). All possible S are known (up to isomorphisms) and are described in [20] (see also [22]).

We can additionally fix a half-cone (the *light-cone*) $V^+(S)$ of the cone

$$V(S) = \{x \in S \otimes \mathbb{R} \mid x^2 > 0\}.$$

We can also fix a fundamental chamber $\mathcal{M} \subset V^+(S)$ for the group $W^{(-2)}(S)$ generated by reflections in all elements with square (-2) in S. This is equivalent to fixing *a fundamental subdivision* $\Delta(S) = \Delta(S)_+ \cup -\Delta(S)_+$ of all elements with square -2 in S. The $\mathcal{M}$ and $\Delta(S)_+$ define each other by the condition $(\mathcal{M}, \Delta_+) \geq 0$. These additional structures $\mathcal{M} \subset V^+(S)$ of the hyperbolic lattice S are defined uniquely up to the action of the group $\{\pm 1\}W^{(-2)}(S)$.

We can restrict considering K3 surfaces X with a non-symplectic involution τ, i.e. pairs (X, τ), such that $V^+(S)$ contains a hyperplane section of X and the set $\Delta(S)_+$ contains only classes of effective curves with square -2 of X. I. e. $V^+(S)$ and the fundamental subdivision $\Delta(S)_+$ are prescribed by the geometry of the K3 surface X.

If a pair (X, τ) is general, then $S = N(X)$ is the Picard lattice of X (remind that it is generated by classes of algebraic curves) and $\mathcal{M}$ gives the nef cone (or Kählerian cone) of X. The weakest condition of degeneration (i. e. giving the most reach discriminant) is the following condition:

$(\mathcal{D})$: *We say that* (X, τ) *of the type* $H_2(X, \mathbb{Z})^\tau = S$ *is degenerate* if there exists $h \in \mathcal{M}$ such that h is not nef for X. I.e., there are elements of S which are nef in general (when $S = N(X)$), but they are not nef for X. It is not difficult to see [22], that this is equivalent to existence of an exceptional curve with square -2 on the quotient $Y = X/\{1, \tau\}$. This is also equivalent to have an element $\delta \in N(X)$ with $\delta^2 = -2$ such that $\delta = (\delta_1 + \delta_2)/2$ where $\delta_1 \in S$, $\delta_2 \in S^\perp_{N(X)}$ and $\delta_1^2 = \delta_2^2 = -4$. Remark that $(\delta_1, S) \equiv 0 \mod 2$ and $(\delta_2, S^\perp) \equiv 0 \mod 2$. I.e. δ_1 and δ_2 are roots with square -4 for lattices S and $S^\perp$ respectively.

Let us consider examples of lattices S of a small rank, when $\operatorname{rk} S \leq 2$. All these cases will be considered in details in Sect. 2.4 below. We use notations: $\langle A \rangle$ denote a lattice defined by a symmetric integral matrix A. We denote $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For a lattice K, we denote by $K(m)$ a lattice obtained from the lattice K by multiplication of the form of K by $m \in \mathbb{Q}$. By $\oplus$ we denote the orthogonal sum of lattices.

If $\operatorname{rk} S = 1$, then $S \cong \langle 2 \rangle$, $Y = X/\{1, \tau\} = \mathbb{P}^2$ and all X are non-degenerate, because S has no elements with square -4.

If $\operatorname{rk} S = 2$, then the lattice S is isomorphic to one of three lattices: $S \cong U(2)$, $\langle 2 \rangle \oplus \langle -2 \rangle$ or U.

If $S \cong U(2)$, then non-degenerate K3 surfaces (X, τ) give $X/\{1, \tau\} \cong \mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{F}_0$, but degenerate K3 surfaces (X, τ) give $X/\{1, \tau\} \cong \mathbb{F}_2$ where we denote by $\mathbb{F}_n$ a relatively minimal rational surface with a section having the square $-n$, $n \geq 0$.

If $S \cong \langle 2 \rangle \oplus \langle -2 \rangle$, then $Y = X/\{1, \tau\} = \mathbb{F}_1$. All X are non-degenerate because S has no elements with square -4.

If $S \cong U$, then $Y = X/\{1, \tau\} = \mathbb{F}_4$. All X are non-degenerate because S has no elements δ_1 with $\delta_1^2 = -4$ and $(\delta_1, S) \equiv 0 \mod 2$.

We shall consider all these cases in Sect 2.4 below.

Further we fix one of possible lattices S together with $\mathcal{M} \subset V^+(S)$ and consider K3 surfaces with non-symplectic involutions (X, τ) and a fixed identification $S = H_2(X, \mathbb{Z})^\tau$ such that $\mathcal{M} \subset V^+(S)$ are defined by the geometry of X. In complex case we can consider two pairs (X, τ) and (X', τ') to be isomorphic if there exists an isomorphism $f : X \to X'$

of K3 surfaces which is identical on S. Then $f\tau = \tau' f$ (it is sufficient to check this identity for the action on second homology) and f defines the isomorphism of K3 with involutions. The moduli space of pairs (X, τ) is connected, and its subspace of degenerate pairs (X, τ) has codimension one over $\mathbb{C}$ (if it is not empty). Thus, the space of non-degenerate pairs is also connected. This follows from the fact that there exists a unique (up to automorphisms) primitive embedding $S \to L$ where $L \cong H_2(X, \mathbb{Z})$ is an unimodular even lattice of signature $(3, 19)$.

For real K3 surfaces with non-symplectic involutions (X, τ), we should additionally fix the type θ of the action of the anti-holomorphic involution φ of X on S. The θ should satisfy the following properties: $\theta(V^+(S)) = -V^+(S)$ and $\theta(\Delta(S)_+) = -\Delta(S)_+$. It follows that the lattice $S_+ = S^\theta$ is negative definite and it has no elements with square -2. Moreover, the linear subspace $S_- \otimes \mathbb{R}$ where $S_- = S_\theta$ must intersect the interior of the nef cone $\mathcal{M}$. For the fixed type (S, θ) we consider K3 surfaces X with holomorphic involutions τ of the type S and an anti-holomorphic involution φ, such that $\varphi|S = \theta$, in particular, $\varphi(S) = S$. It follows that the anti-holomorphic involution φ commutes with τ. The φ defines then the real structure of the pair (X, τ). The triplets (X, τ, φ) of the type (S, θ) give *real K3 surfaces with non-symplectic involutions of the type* (S, θ).

We consider the following real analogy for real K3 surfaces with non-symplectic involutions of the type (S, θ) of the degeneration $(\mathcal{D})$ above. An element $h \in S$ is called *real* if $\theta(h) = -h$, i.e. $h \in S_-$. For a general real X we have $N(X) = S$, and all real nef elements are elements of $S_- \cap \mathcal{M}$.

$(\mathcal{D}\mathbb{R})$: *A real K3 surface* (X, τ, φ) *with a non-symplectic involution of the type* (S, θ) *is called degenerate* if there exists a real element $h \in S_- \cap \mathcal{M}$ which is not nef for X. I. e. h is nef for general real K3 surfaces with non-symplectic involutions of the type (S, θ), but it is not nef for the triplet (X, τ, φ) itself. This is equivalent to have an element $\delta \in N(X)$ with $\delta^2 = -2$ such that $\delta = (\delta_1 + \delta_2)/2$ where $\delta_1 \in S$, $\delta_2 \in S^\perp_{N(X)}$ and $\delta_1^2 = \delta_2^2 = -4$ (i. e. (X, θ) is degenerate in the sense of $(\mathcal{D})$ as a complex surface). Additionally, δ_1 must be orthogonal to an element $h \in S_- \cap \mathrm{int}(\mathcal{M})$ with $h^2 > 0$. Here $\mathrm{int}(\mathcal{M})$ denote the interior part of $\mathcal{M}$, i. e. the polyhedron $\mathcal{M}$ without its faces.

The condition $(\mathcal{D}\mathbb{R})$ for (X, τ, φ) implies the condition $\mathcal{D}$ for (X, τ). Thus, condition $(\mathcal{D}\mathbb{R})$ is stronger for (X, τ) than $(\mathcal{D})$, and it is not difficult to give examples of possible (S, θ) when $(\mathcal{D})$ does not imply $(\mathcal{D}\mathbb{R})$.

On the other hand, for some types (S, θ) the conditions $(\mathcal{D})$ implies $(\mathcal{D}\mathbb{R})$. E. g. it is obviously true if $S = S_-$. It is easy to see that $(\mathcal{D})$ implies $(\mathcal{D}\mathbb{R})$ for all lattices S of $\mathrm{rk}\, S \le 2$ above and all possible θ for these lattices. Only for $S = U(2)$, it is possible to have $S_- \ne S$ (this is the case when $Y = X/\{1, \theta\}$ is an ellipsoid as a real surface). Then $S_- = \langle 4 \rangle$, $S_+ = \langle -4 \rangle$, and all elements of S with square -4 belong to S_+. Then $(\mathcal{D})$ implies $(\mathcal{D}\mathbb{R})$. But, to formulate a result about connected components of moduli of non-degenerate real K3 surfaces with non-symplectic involutions (Theorem 1) in general (for arbitrary S), we have to consider the type $(\mathcal{D}\mathbb{R})$ of degeneration.

Let $L \cong H_2(X, \mathbb{Z})$ where $H_2(X, \mathbb{Z})$ is the homology lattice of K3 with intersection pairing. Then L is an even unimodular lattice of signature $(3, 19)$. Let us fix a primitive embedding $S \subset L$, it is unique up to automorphisms of L. Let $\Delta(S, L)^{(-4)}$ be the set of all elements δ_1 in S such that $\delta_1^2 = -4$ and there exists $\delta_2 \in S^\perp_L$ such that $(\delta_2)^2 = -4$ and $\delta = (\delta_1 + \delta_2)/2 \in L$. Then $\delta^2 = -2$. All elements $\delta_1 \in \Delta(S, L)^{(-4)}$ are roots of S since $-4 = (\delta_1)^2 | 2(\delta_1, S)$ because $(\delta_1, S) \subset 2\mathbb{Z}$. Let $W^{(-4)}(S, L) \subset O(S)$ be the group generated by reflections in all roots from $\Delta(S, L)^{(-4)}$, and $W^{(-4)}(S, L)_{\mathcal{M}}$ the stabilizer subgroup of $\mathcal{M}$ in $W^{(-4)}(S, L)$. The set $\Delta(S,L)^{(-4)}$ is invariant with respect to $W^{(-2)}(S)$. It follows that the $W^{(-4)}(S,L)_{\mathcal{M}}$ is generated by reflections s_{δ_1} in $\delta_1\in\Delta(S,L)^{(-4)}$ such that the the hyperplane $(\delta_1)^{\perp}_S\otimes\mathbb{R}$ in $S\otimes\mathbb{R}$ intersects the interior of $\mathcal{M}$.

The real analogy of the group $W^{(-4)}(S,L)_{\mathcal{M}}$ is *the subgroup $G\subset W^{(-4)}(S,L)_{\mathcal{M}}$* which is generated by reflections s_{δ_1} in all elements $\delta_1\in\Delta(S,L)^{(-4)}$ which are contained either in S_+ or in S_- (i. e., s_{δ_1} should commute with θ) and such that $s_{\delta_1}(\mathcal{M})=\mathcal{M}$. Obviously, $G\subset W^{(-4)}(S,L)_{\mathcal{M}}$.

We consider *two real K3 surfaces (X,τ,φ) and (X',τ',φ') with non-symplectic involutions of the type (S,θ) to be isomorphic with respect to the group G,* if there exists an isomorphism $f:X\to X'$ such that $f\tau=\tau' f$, $f\varphi=\varphi' f$ and $f|S\in G$.

We correspond to a real K3 surface(X,τ,φ) with a non-symplectic involution of the type (S,θ) *an integral involution (L,φ,S) with the condition (S,θ) on the sublattice S* (see [21]). Here $L=H_2(X;\mathbb{Z})$, φ is the action of φ on L such that $\varphi|S=\theta$. The integral involution (L,φ,S) has the properties (e.g. see [12] or Sect. 3.10 in [21]):

(RSK3) *The lattice L is even unimodular of signature $(3,19)$. The lattice L^{φ} is hyperbolic (it has the signature $(1,t_{(-)})$).*

Clearly, we can consider abstract integral involutions (L,φ,S) of the type (S,θ) satisfying (RSK3). We consider *two integral involutions (L,φ,S) and (L',φ',S) with the condition (S,θ) to be isomorphic for the group G* if there exists an isomorphism $\xi:L\to L'$ of lattices such that $\xi\varphi=\varphi'\xi$ and $\xi|S$ belongs to the group G. We *denote by In(S,θ,G) the set* of isomorphism classes of integral involutions (L,φ,S) satisfying (RSK3) with the condition (S,θ) and the group G.

By monodromy consideration, two triplets (X,φ,τ) and (X',φ',τ') of the type (S,θ) which belong to one connected component of moduli give isomorphic integral involutions with the condition (S,θ). Thus, we have *the natural map from the set of connected components of moduli of triplets (X,φ,τ) to the set In(S,θ,G).*

Using Global Torelli Theorem for K3 surfaces [24] and epimorphicity of Torelli map for K3 surfaces [13], we can prove the main result which is similar to Theorem 3.10.1 from [19] about moduli of real polarized K3 surfaces. This type of statements is very interesting because it reduces the problem of description of connected components of moduli of real algebraic varieties to a purely arithmetic problem.

Theorem 1. *The natural map above gives the one to one correspondence between connected components of moduli of ($\mathcal{D}\mathbb{R}$)-non-degenerate real K3 surfaces (X,τ,φ) with non-symplectic involutions of the type (S,θ) and the set In(S,θ,G) of isomorphism classes for G of integral involutions (L,φ,S) with the condition (S,θ) satisfying (RSK3).*

Proof. Let (L,S,φ) be an isomorphism class from $\mathrm{In}(S,\theta,G)$. We have the orthogonal decomposition up to finite index

$$S_+\oplus S_-\oplus L_\tau^{\varphi}\oplus L_{\tau,\varphi}\subset L \tag{2.1}$$

Here is the key statement (compare with conditions ($\mathcal{D}$) and ($\mathcal{D}\mathbb{R}$) above).

Lemma 2. *Let $\delta\in L$ has $\delta^2=-2$. Assume that $\delta=(\delta_1+\delta_2)/2$ where $\delta_1\in S$, $\delta_2\in L_\tau$ and $(\delta_1)^2=(\delta_2)^2=-4$.*

If additionally $\delta_2\in L_\tau^{\varphi}\cup L_{\tau,\varphi}$ and $(\delta_1,h)=0$ for $h\in S_-$ with $h^2>0$, then $\delta_1\in S_+\cup S_-$.

Proof. We have $(\delta_1)^*=\delta_1/2=(\delta_1)^*_++(\delta_1)^*_-$ where $(\delta_1)^*_\pm\in(S_\pm)^*$. Since $(\delta_1,h)=((\delta_1)^*_-,h)=0$ and the lattice S_- is hyperbolic, then $((\delta_1)^*_-)^2<0$ if $(\delta_1)^*_-\neq 0$.

Assume that $\delta_2 \in L_{\tau,\varphi}$. Then $(\delta_1)^*_- + \delta_2/2 \in (L_\varphi)^*$. The lattice L_φ is 2-elementary (since L is unimodular) which means that $2L^*_\varphi \subset L_\varphi$. It follows, $\widetilde{\delta} = 2(\delta_1)^*_- + \delta_2 \in L_\varphi$ and $\widetilde{\delta}^2 = (2(\delta_1)^*_-)^2 - 4 < -4$ if $(\delta_1)^*_- \neq 0$. Since L is even, it follows that $\widetilde{\delta}^2 \leq -6$. Thus, we have $\delta = (\delta_1)^*_+ + \widetilde{\delta}/2$. We then get $(\delta_1)^*_+ = (\delta_1)_+/2$ where $(\delta_1)_+ \in S_+$. Since S_+ is negative definite, even and does not have elements with square -2, then $((\delta_1)_+)^2 \leq -4$ if $(\delta_1)^*_+ \neq 0$. If both elements $(\delta_1)^*_\pm \neq 0$, we then get $\delta^2 \leq (-4-6)/2 < -2$ which is a contradiction. It follows $\delta_1 \in S_+ \cup S_-$.

If $\delta_2 \in L^\varphi_\tau$, one should replace φ by $\tilde{\varphi} = \tau\varphi$ and argue similarly. We have $L^{\tilde{\varphi}}_\tau = L_{\tau,\varphi}$ and $L_{\tau,\tilde{\varphi}} = L^\varphi_\tau$. $\square$

We shall use the lemma later. The rest considerations are similar to the proof of Theorem 3.10.1 in [19] and are now standard.

Both lattices L^φ_τ and $L_{\tau,\varphi}$ are hyperbolic and define hyperbolic spaces $\mathcal{L}^\varphi_\tau$ and $\mathcal{L}_{\tau,\varphi}$. E.g. to define $\mathcal{L}^\varphi_\tau$, one should choose one of half-cones $V^+(L^\varphi_\tau)$ of the cone $V(L^\varphi_\tau) = \{x \in L^\varphi_\tau \otimes \mathbb{R} \mid x^2 > 0\}$ and consider the set $\mathcal{L}^\varphi_\tau = V^+(L^\varphi_\tau)/\mathbb{R}^+$ of rays in $V^+(L^\varphi_\tau)$. Similarly, $\mathcal{L}_{\tau,\varphi} = V^+(L_{\tau,\varphi})/\mathbb{R}^+$.

By global Torelli Theorem [24], the triplets

$$(\mathcal{D}\mathbb{R})\text{-non-degenerate triplets } (X,\tau,\varphi) \text{ of the type } (S,\theta) \tag{2.2}$$

are described by their periods. Since $S \subset N(X)$, periods should be orthogonal to S. It follows that periods of data (2.2) are pairs

$$(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-) \in \mathcal{L}^\varphi_\tau \times \mathcal{L}_{\tau,\varphi} \tag{2.3}$$

(see Sect. 3.10 in [19] for details). (We identify cohomology with homology using Poincaré duality.)

Let $O(L,S,G,\varphi)$ be the group of all automorphisms $\alpha \in O(L)$ such that $\alpha(S) = S$, $\alpha|S \in G$ and $\alpha\varphi = \varphi\alpha$. The group $O(L,S,G,\varphi)$ acts in $\mathcal{L}^\varphi_\tau \times \mathcal{L}_{\tau,\varphi}$ as follows. For its element β we set $\beta(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-) = (\mathbb{R}^+(\pm\beta(\omega_+)), \mathbb{R}^+(\pm\beta(\omega_-)))$ where $\pm\beta(\omega_+)$ and $\pm\beta(\omega_-)$ are chosen to be in $V^+(L^\varphi_\tau)$ and $V^+(L_{\tau,\varphi})$ respectively. By global Torelli Theorem [24], periods (2.3) define isomorphic data (2.2), if and only if they belong to one orbit of $O(L,S,G,\varphi)$. These periods we shall call equivalent.

Let $W^{(-2)}(L^\varphi_\tau)$ and $W^{(-2)}(L_{\tau,\varphi})$ be the groups generated by all reflections in elements with square -2 of lattices L^φ_τ and $L_{\tau,\varphi}$ respectively. They are subgroups of $O(L,S,G,\varphi)$. Both these groups are discrete in hyperbolic spaces $\mathcal{L}^\varphi_\tau$ and $\mathcal{L}_{\tau,\varphi}$ and have fundamental chambers $\Omega^\varphi_\tau \subset \mathcal{L}^\varphi_\tau$ and $\Omega_{\tau,\varphi} \subset \mathcal{L}_{\tau,\varphi}$ respectively. They are connected. We get all periods of data (2.2) up to equivalence if consider periods

$$(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-) \in \Omega^\varphi_\tau \times \Omega_{\tau,\varphi}. \tag{2.4}$$

Further we consider only periods (2.4).

Consider $(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-) \in \Omega^\varphi_\theta \times \Omega_{\tau,\varphi}$. Consider the set

$$\Delta^{(-2)}(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-) = \{\delta \in L_\tau \mid \delta^2 = -2,\ (\delta,\omega_+) = (\delta,\omega_-) = 0\}. \tag{2.5}$$

A point $(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-)$ in $\Omega^\varphi_\theta \times \Omega_{\tau,\varphi}$ does not correspond to periods of data (2.2) if $\Delta(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-) \neq \emptyset$. Really, assume that $\delta \in \Delta(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-)$. Conditions $(\delta,\omega_+) = (\delta,\omega_-) = 0$ mean that $\delta \in N(X)$. Since $\delta^2 = -2$, by Riemann-Roch Theorem for K3 surfaces δ or $-\delta$ is effective. Since $\delta \in L_\tau$, then $\tau(\delta) = -\delta$ which is impossible for an effective class $\pm\delta$ and a holomorphic involution τ.

Thus, we should delete from the set $\Omega^\varphi_\tau \times \Omega_{\tau,\varphi}$ of possible periods of data (2.2) all points $(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-)$ where ω_+ or ω_- belongs to a face of the polyhedron Ω^φ_τ or $\Omega_{\tau,\varphi}$ since they are orthogonal to elements of the corresponding lattices with square -2 and define elements of $\Delta(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-)$.

Thus, all possible periods of data (2.2) are points

$$(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-) \in (\Omega_\tau^\varphi)' \times (\Omega_{\tau,\varphi}^-)' \tag{2.6}$$

where $'$ means that we deleted all faces of the fundamental chambers (we can call them as *open fundamental chambers*). The set is connected since it is the product of open convex sets. For points $(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-)$ from $(\Omega_\varphi^+)' \times (\Omega_{\tau,\varphi}^-)'$, elements $\delta \in \Delta^{(-2)}(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-)$ cannot belong to L^φ or $L_{\tau,\varphi}$.

Any $\delta \in \Delta^{(-2)}(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-)$ has the property that $\delta = \delta_+ \oplus \delta_-$ where $\delta_+ \in (L_\tau^\varphi)^*$ and $\delta_- \in L_{\tau,\varphi}^*$. Moreover $(\delta_+)^2 < 0$ if $\delta_+ \neq 0$, and $(\delta_-)^2 < 0$ if $\delta_-^2 < 0$ since they are elements of hyperbolic lattices orthogonal to their elements ω_+ and ω_- with positive square. Since for points (2.6) elements of $\Delta^{(-2)}(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-)$ cannot belong to $L_{\tau,\varphi}$ or L^φ, we can also assume that $\delta_+ \neq 0$ and $\delta_- \neq 0$.

Let

$$\begin{aligned} \Delta^{(-2)} = &\{\delta = \delta_+ \oplus \delta_- \in L_\tau \mid \delta_+ \in (L_\tau^\varphi)^*,\ \delta_- \in (L_{\tau,\varphi})^*, \\ &(\delta_+)^2 < 0,\ (\delta_-)^2 < 0,\ \delta^2 = (\delta_+)^2 + (\delta_-)^2 = -2\}. \end{aligned} \tag{2.7}$$

Let $\delta = \delta_+ \oplus \delta_- \in \Delta^{(-2)}$. Let $\mathcal{H}_{\delta_+} = \{\mathbb{R}^+\omega_+ \in \mathcal{L}_\tau^\varphi \mid (\omega_+, \delta_+) = 0\}$ be the hyperplane in $\mathcal{L}_\tau^\varphi$ which is orthogonal to δ_+. Similarly, let $\mathcal{H}_{\delta_-} = \{\mathbb{R}^+\omega_- \in \mathcal{L}_{\tau,\varphi} \mid (\omega_-, \delta_-) = 0\}$ be the hyperplane in $\mathcal{L}_{\tau,\varphi}$ which is orthogonal to δ_-. Points $(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-) \in \mathcal{H}_{\delta_+} \times \mathcal{H}_{\delta_-}$ are exactly the points such that $\Delta(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-)$ contains δ, and they should be deleted from the set of possible periods of data (2.2). Thus, the set of possible periods of data (2.2) is

$$(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-) \in \Omega_1 = (\Omega_\tau^\varphi)' \times (\Omega_{\tau,\varphi})' - \left(\bigcup_{\delta \in \Delta^{(-2)}} \mathcal{H}_{\delta_+} \times \mathcal{H}_{\delta_-}\right). \tag{2.8}$$

The space Ω_1 is connected because $\mathcal{H}_{\delta_+} \times \mathcal{H}_{\delta_-}$ has codimension 2. Moreover, the set of hyperplanes $\mathcal{H}_{\delta_+}$, $\delta \in \Delta^{(-2)}$, is locally finite in $\mathcal{L}_\tau^\varphi$ because all δ_+ belong to a lattice $(L_\tau^\varphi)^*$ and $0 > (\delta_+)^2 > -2$. Hyperplanes $\mathcal{H}_{\delta_-}$, $\delta \in \Delta^{(-2)}$, have the same property in $\mathcal{L}_{\tau,\varphi}$.

The set (2.8) *gives periods of all triplets* (X, τ, φ) *of the type* (S, θ) *without additional condition that they are* ($\mathcal{DR}$)*-non-degenerate (it follows that Theorem 1 is valid if we delete the words* ($\mathcal{DR}$)*-non-degenerate).* Let us add the last condition.

Let us denote by

$$\widetilde{\Delta}^{(-4)} \subset L_\tau \tag{2.9}$$

the set of all elements $\delta_2 \in L_\tau$ such that ${\delta_2}^2 = -4$ and there exists $\delta_1 \in S$ with conditions: ${\delta_1}^2 = -4$, $\delta = (\delta_1 + \delta_2)/2 \in L$, there exists an element $h \in S_- \cap \mathrm{int}(\mathcal{M})$ such that $(\delta_1, h) = 0$.

For $(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-) \in \mathcal{L}_\tau^\varphi \times \mathcal{L}_{\tau,\varphi}$, we denote by $\Omega^{(-4)}(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-)$ the set of all $\delta_2 \in \widetilde{\Delta}^{(-4)}$ such that $(\omega_+, \delta_2) = (\omega_-, \delta_2) = 0$. A triplet (2.2) is ($\mathcal{DR}$)-degenerate, if and only if its periods $(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-)$ have the non-empty set $\Omega^{(-4)}(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-)$. Thus, we should delete from Ω_1 in (2.8) all these points.

Like for $\Delta^{(-2)}$ above, the set of these points has codimension one only if there exists $\delta_2 \in \Omega^{(-4)}(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-)$ such that $\delta_2 \in L_\tau^\varphi \cup L_{\tau,\varphi}$. By Lemma 2, then the corresponding δ_1 belongs to $S_- \cup S_+$ (see definition of $\widetilde{\Delta}^{(-4)}$ in (2.9)). Let $\delta = (\delta_1 + \delta_2)/2 \in L$ and $\delta' = (-\delta_1 + \delta_2)/2 \in L$. Since $\delta^2 = (\delta')^2 = -2$, there exist reflections s_δ, $s_{\delta'}$ in these elements which belong to $O(L)$. Their composition $s = s_\delta s_{\delta'}$ acts in $S = L^\tau$ as the reflection s_{δ_1} in δ_1, and it acts in L_τ as the reflection s_{δ_2} in δ_2. Moreover, the reflection $s_{\delta_1} \in G$ since $\delta_1 \in S_- \cup S_+$ and $(h, \delta_1) = 0$ for $h \in \mathrm{int}(\mathcal{M})$. It follows that $s \in O(L, S, G, \varphi)$.

Thus, we get all equivalent periods of data (2.2), if we replace Ω_τ^φ, $(\Omega_\tau^\varphi)'$ and $\Omega_{\tau,\varphi}$, $(\Omega_{\tau,\varphi})'$ by the corresponding fundamental chambers (and open fundamental chambers) $\widetilde{\Omega}_\tau^\varphi$, $(\widetilde{\Omega}_\tau^\varphi)'$ and $\widetilde{\Omega}_{\tau,\varphi}$, $(\widetilde{\Omega}_{\tau,\varphi})'$ for the groups $W(L_\tau^\varphi)$ and $W(L_{\tau,\varphi})$ generated by reflections in all elements with square -2 and in all elements from $\widetilde{\Delta}^{(-4)}$ which are contained in lattices L_φ^τ and $L_{\tau,\varphi}$ respectively.

Let $\Delta^{(-4)}=\widetilde{\Delta}^{(-4)}-(L_\tau^\varphi\cup L_{\tau,\varphi})$. If $\delta_2\in\Delta^{(-4)}$, then we write $\delta_2=(\delta_2)_++(\delta_2)_-$ where $(\delta_2)_+\in(L_\tau^\varphi)^*$ and $(\delta_2)_-\in(L_{\tau,\varphi})^*$.

The set of all periods of data (2.2) is then

$$(\mathbb{R}^+\omega_+,\mathbb{R}^+\omega_-)\in\Omega=(\widetilde{\Omega}_\varphi^\tau)'\times(\widetilde{\Omega}_{\tau,\varphi})'-\left(\bigcup_{\delta\in\Delta\cup\Delta^{(-4)}}\mathcal{H}_{\delta_+}\times\mathcal{H}_{\delta_-}\right)\tag{2.10}$$

where $(\widetilde{\Omega}_\varphi^\tau)'$ and $(\widetilde{\Omega}_{\tau,\varphi})'$ are open locally-finite polyhedra in hyperbolic spaces; the products $\mathcal{H}_{\delta_+}\times\mathcal{H}_{\delta_-}$ have codimension two and their set is locally finite in $(\widetilde{\Omega}_\varphi^\tau)'\times(\widetilde{\Omega}_{\tau,\varphi})'$. It follows that Ω is connected.

By epimorphicity of Torelli map for K3 surface [13] applied to real K3 surfaces, there exist data (2.2) with any periods from (2.10). Thus, by Global Torelli Theorem [24] and epimorphicity of Torelli map for K3 surfaces [13], the moduli space of data (2.2) is covered by the connected space Ω defined in (2.10). More exactly, the moduli space of data (2.2) is the quotient of Ω by the stabilizer subgroup of Ω in $O(L,S,G,\varphi)$, and it is connected.

This finishes the proof of the Theorem.

Let us add the standard details to that. Assume $(\mathbb{R}^+\omega_+,\mathbb{R}^+\omega_-)\in\Omega$. We can choose ω_+ and ω_- to satisfy conditions $\omega_+^2=\omega_-^2>0$. Let $\omega=\omega_++i\omega_-$. Then $\omega^2=0$ and $(\omega,\overline{\omega})>0$. We have $\tau(\omega)=-\omega$ and $\varphi(\omega)=\overline{\omega}$. Let $H^{2,0}=\mathbb{C}\omega\subset L\otimes\mathbb{C}$. We then have $\tau(H^{2,0})=H^{2,0}$ and $\varphi(H^{2,0})=H^{0,2}=\overline{H^{2,0}}$.

We choose $h\in S_-$ such that h belongs to the interior of the fundamental chamber $\mathcal{M}$. This is possible because of our conditions on θ.

Let $N=\{x\in L\mid(\omega,x)=0\}=\{x\in L\mid(\omega_+,x)=(\omega_-,x)=0\}$. The lattice N is hyperbolic. Let $\delta\in N$ and $\delta^2=-2$. We claim that $(h,\delta)\neq0$.

We can write $\delta=(\delta_1+\delta_2)/2$ where $\delta_1\in S$ and $\delta_2\in L_\tau$. We have ${\delta_2}^2<0$ if $\delta_2\neq0$ because its orthogonal complement in $L_\tau\otimes\mathbb{R}$ contains, $\omega_+\perp\omega_-$ with positive squares and L_τ has exactly two positive squares.

We have $\delta_1\neq0$. Otherwise, $\delta\in L_\tau$ and $\Delta^{(-2)}(\mathbb{R}^+\omega_+,\mathbb{R}^+\omega_-)\neq\emptyset$ which is not the case for points of Ω by our considerations above.

Assume $\delta_1\neq0$ and ${\delta_1}^2\geq0$. Then $(h,\delta)=(h,\delta_1)\neq0$ because S is hyperbolic and $h^2>0$.

Assume $\delta_1\neq0$ and ${\delta_1}^2<0$. Since $-2=\delta^2=({\delta_1}^2+{\delta_2}^2)/4$ and L is even, then we have three possibilities: (0) $\delta_2=0$ and $\delta=\delta_1/2\in S$; (1) ${\delta_1}^2=-2$ and ${\delta_2}^2=-6$. (2) ${\delta_1}^2={\delta_2}^2=-4$. Let us consider all of them.

(0) and (1). Since h is in interior of $\mathcal{M}$ and $\mathcal{M}$ is the fundamental chamber for the group generated by reflections in all elements of S with square -2, we then have $(h,\delta)\neq0$ for (0), and $(h,\delta)=(h,\delta_1)/2\neq0$ for (1).

(2) If $(h,\delta_1)=0$, we get a contradiction with our construction of Ω in (2.10). Thus, $(h,\delta)=(h,\delta_1)/2\neq0$.

It proves the claim: $(h,\delta)\neq0$ for any $\delta\in N$ with $\delta^2=-2$.

By epimorphicity of the period map for K3 surfaces [13], there exists a complex K3 surface X and an isomorphism $\xi:H_2(X,\mathbb{Z})\to L$ such that $\xi^{-1}(H^{2,0})=H^{2,0}(X)$ and $\xi^{-1}(h)$ is a polarization of X. By Global Torelli Theorem for K3 surfaces [24], the choice of (X,ξ) is unique up to isomorphisms. The pair $(X, \tau\xi)$ satisfies the same conditions. Thus, $\xi^{-1}\tau\xi$ is induced by a unique automorphism of X which we denote by the same letter τ (for simplicity). The τ acts as -1 on $H^{2,0}(X)$ and as an involution in $H_2(X,\mathbb{Z})$. It follows that τ is a non-symplectic involution of X of the type S.

The pair $(\overline{X}, \varphi\xi)$ is another pair with the same properties as (X, ξ) where $\overline{X}$ is the surface X with the conjugate complex structure. It follows that $\xi^{-1}\varphi\xi$ is induced by an anti-holomorphic involution of X which we denote by the same letter φ.

Finally we get a unique, up to isomorphisms, a triplet (X, τ, φ) from (2.2) with periods which are equivalent to $(\mathbb{R}^+\omega_+, \mathbb{R}^+\omega_-) \in \Omega$.

□

2.2. **Connected components of moduli of non-degenerate real right DPN-pairs.** Below we consider a description of K3 surfaces with non-symplectic involutions using the corresponding quotients and double coverings. See [20], [22], [1], [2] and also [4] for details.

Let (X, τ) be a K3 surface with a non-symplectic involution τ and $A = X^\tau$ the set of fixed points of τ. Since $\tau^*(\omega_X) = -\omega_X$ and ω_X has zero divisor, it follows that A is a non-singular curve of X. It follows that the surface $Y = X/\{1, \tau\}$ is non-singular, the quotient map $\pi : X \to Y$ is a double covering ramified along a non-singular curve $A \subset Y$. Simple considerations show that Y is Enriques surface if $A = \emptyset$, and Y is a rational surface if $A \neq \emptyset$. The non-singular curve $A \in |-2K_Y|$ and the irregularity $q_Y = \dim H^0(Y, \Omega^1) = 0$. The opposite statement is also valid. If Y is a non-singular surface, $A \in |-2K_Y|$ a non-singular curve, and $q_Y = 0$, then the double covering of Y ramified along A gives a K3 surface X with a non-symplectic involution of the double covering. A pair (Y, A) with these properties is called a *right DPN pair.* The surface Y is called *a right DPN-surface.* We follow the terminology of [22], [1] and [2].

We mention that a general (not necessarily right) DPN-pair is a pair (Y, A) where Y is a non-singular surface with $q_Y = 0$, $A \in |-2K_Y|$ and A has only ADE-singularities; the surface Y is called a DPN-surface. Any DPN-pair gives a right one after a sequence of blow-ups in singular points of A. This sequence is canonical, and studying of general DPN-pairs can be reduced to studying right ones. Any general DPN-pair can be obtained from a right one by a sequence of contractions of exceptional curves of the 1st kind. Further we consider only right DPN-pairs and right DPN-surfaces.

Like K3 surfaces with non-symplectic involutions, *we call a right DPN-pair (Y, A) to be $(\mathcal{D})$-degenerate if the corresponding K3 surface with non-symplectic involution (X, τ) is $(\mathcal{D})$-degenerate.* Similarly, we call *a real (i.e. with an anti holomorphic involution θ) DPN-pair (Y, A) to be $(\mathcal{D}\mathbb{R})$-degenerate if the corresponding real K3 surface with non-symplectic involution (X, τ, φ) is $(\mathcal{D}\mathbb{R})$-degenerate.* Below we discuss that. In particular, we shall see that the property to be degenerate depends only on the quotient $Y = X/\{1, \tau\}$. It does not depend on A.

We follow notations from [1] which are a little different from [22]. We use the following facts about right DPN-pairs (Y, A) and the corresponding K3 surfaces with non-symplectic involutions (X, τ) of the type S.

Any exceptional curve F of Y (i. e. an irreducible algebraic curve F with negative square $F^2 < 0$) is a non-singular rational curve with square -1, -2 or -4. If $F^2 = -4$ *(the type I),* then F is a component of A and its preimage $E = \pi^{-1}(F)$ is a component of X^τ with $E^2 = -2$. If $F^2 = -1$ *(the type II)*, then $(F, A) = 2$. If A intersects F in two different points *(the type IIa),* then $\pi^{-1}(F) = E$ where E is a non-singular rational curve with square -2. If $F^2 = -1$ and A is tangent to F with multiplicity 2 *(the type*

IIb), then $\pi^{-1}(F) = E + \tau(E)$ where E is a non-singular rational curve with square -2 and $(E, \tau(E)) = 1$ (remark that $(E + \tau(E))^2 = -2$ like for E^2 of the type IIa). If $F^2 = -2$ *(the type III),* then $\pi^{-1}(F) = E + \tau(E)$ is disjoint union of two non-singular rational curves of X with square -2.

We have the following description of exceptional curves on Y in terms of X. Let $\Delta(S)^{(-2)} = \{\delta \in S \mid \delta^2 = -2\}$. Let $\Delta(S, X)^{(-4)}$ be the set of all $\delta_1 \in S$ such that ${\delta_1}^2 = -4$ and there exists $\delta_2 \in S^{\perp}_{N(X)}$ such that ${\delta_2}^2 = -4$ and $\delta = (\delta_1 + \delta_2)/2 \in N(X)$ (then $\delta^2 = -2$). All elements of $\Delta(S, X)^{(-4)}$ are roots of S with square -4. Let $W(S, X) \subset O(S)$ be the group generated by reflections in all elements of $\Delta(S, X) = \Delta(S)^{(-2)} \cup \Delta(S, X)^{(-4)}$. The set $\Delta(S, X)$ is invariant with respect to $W(S, X)$. The group $W(S, X)$ is a discrete reflection group in $V^+(S)$ and in the corresponding hyperbolic space $V^+(S)/\mathbb{R}^+$. All reflections from $W(S, X)$ are reflections with respect to elements from $\Delta(S, X)$. All faces of a fundamental chamber of $W(S, X)$ are orthogonal to $\Delta(S, X)$. Let $\mathcal{K}(X) \subset N(X) \otimes \mathbb{R}$ be the Kählerian (or the *nef* cone) of X. The crucial fact is that $\mathcal{M}(S, X) = (S \otimes \mathbb{R}) \cap \mathcal{K}(X)$ *is a fundamental chamber for the group $W(S, X)$. Its faces are orthogonal exactly to preimages $\pi^*(F)$ of exceptional curves F of Y.*

Let us take a face of $\mathcal{M}(S, X)$. Then it is orthogonal to $\delta \in \Delta(S, X)$. We have:
$\delta = \pi^*(F)$ where $F^2 = -1$, if $\delta^2 = -2$ and δ is not the class of a component of X^τ;
$2\delta = \pi^*(F)$ where $F^2 = -4$, if $\delta^2 = -2$ and δ is the class of a component of X^τ.
$\delta = \pi^*(F)$ where $F^2 = -2$, if $\delta^2 = -4$, i.e. $\delta \in \Delta(S, X)^{(-4)}$.

By the projection formula, we also have $\mathcal{M}(S, X) = \pi^*(\mathcal{M}(Y))$ where $\mathcal{M}(Y)$ is the Kählerian cone of Y.

Using these results, we get

Proposition 3. *A right DPN-pair (Y, A) (or the corresponding non-symplectic involution (X, τ)) is $(\mathcal{D})$-degenerate, if and only if $Y = X/\{1, \tau\}$ has an exceptional curve with square -2.*

Proof. We have $\mathcal{M}(S, X) \subset \mathcal{M}$ where $\mathcal{M}$ is the fundamental chamber for the group $W^{(-2)}(S)$ generated by reflections in $\Delta^{(-2)}(S)$.

If X is $(\mathcal{D})$-degenerate, then $\mathcal{M}(S, X) \neq \mathcal{M}$. It follows that at least one face of $\mathcal{M}(S, X)$ is not orthogonal to an element $\delta \in S$ with $\delta^2 = -2$. Then it is orthogonal to $\delta \in \Delta(S, X)^{(-4)}$ where $\delta^2 = -4$. It follows that $\delta = \pi^*(F)$ where F is an exceptional curve of Y with $F^2 = -2$.

Let F be an exceptional curve on Y with $F^2 = -2$. Then $\delta = \pi^*(F)) \in S$ where $\delta^2 = 2F^2 = -4$. It follows that at least one face of $\mathcal{M}(S, X)$ is orthogonal to an element of S with square -4. Then it cannot be orthogonal to an element of S with square -2. Otherwise these two elements are $\mathbb{Q}$-proportional which is impossible for an integral lattice. It follows that $\mathcal{M}(S, X) \neq \mathcal{M}$ and Y is $(\mathcal{D})$-degenerate. $\square$

As an example, let us apply cited above theory of DPN-surfaces to $S = U(2)$. The lattice S has no elements with square -2. Then Y has no exceptional curves with square -1. Moreover, $\operatorname{rk} H^2(Y, \mathbb{R}) = \operatorname{rk} S = 2$. Since Y can be only a rational or Enriques surface, it follows that Y is a relatively minimal rational surface $Y \cong \mathbb{F}_n$ where $n = 0$, 2 or 4 since a right DPN-surface may have only exceptional curves with squares -1, -2 or -4. For $S = U(2)$, the set of fixed points $X^\tau = C_9 \in |-K_Y|$ is a non-singular irreducible curve of genus 9 (see the formula (2.12) for X^τ below). This is only possible for $\mathbb{F}_0$ or $\mathbb{F}_2$ since any curve $C \in |-2K_{\mathbb{F}_4}|$ contains the exceptional section (rational) of $\mathbb{F}_4$ (see Sect. 7 below). The case $Y = \mathbb{F}_2$ gives $(\mathcal{D})$-degenerate case because the exceptional section of $\mathbb{F}_2$ gives an exceptional curve with square (-2). In the $(\mathcal{D})$-non-degenerate case, $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$. Thus, we get

Proposition 4. *For a non-symplectic involution τ of a K3 surface X with the lattice $S = U(2)$ the surface $Y = X/\{1,\tau\}$ is $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$ in the $\mathcal{D}$-non-degenerate case, and $Y \cong \mathbb{F}_2$ in the $\mathcal{D}$-degenerate case.*

We have a similar to Proposition 3 statement over $\mathbb{R}$.

Proposition 5. *A right real DPN-pair (Y, A, θ) (or the corresponding real non-symplectic involution (X, τ, φ) where $\theta = \varphi \mod \{1, \tau\}$) is ($\mathcal{D}\mathbb{R}$)-degenerate, if and only if $Y = X/\{1, \tau\}$ has an exceptional curve F with square -2 such that $(F, \theta(F)) \le 1$ (equivalently, either $\theta(F) = F$ or the intersection matrix of F and $\theta(F)$ is negative definite).*

Proof. If $\overline{Y}$ is the complex conjugate surface to Y, then $\overline{Y}$ has the same exceptional curves as Y with the complex structures on them changed by the conjugate. It follows that for any exceptional curve F of Y we have $\theta(F) = F_1$ where F_1 is another exceptional curve of Y (called θ-conjugate) and $\theta^*([F]) = -[F_1]$ where $[C]$ denotes the homology class (or cohomology class since we always identify homology and cohomology) of a holomorphic curve C.

The pair (Y, A, θ) is ($\mathcal{D}\mathbb{R}$)-degenerate, if and only if the inclusion

$$\mathcal{M}(S, X) \cap (S_- \otimes \mathbb{R}) \subset \mathcal{M} \cap (S_- \otimes \mathbb{R})$$

is strict. By considerations for the proof of Proposition 3, it is equivalent to existence of an exceptional curve F of Y with $F^2 = -4$ such that $\pi^*[F] = \delta \in \Delta(S, X)^{(-4)}$ and the inequality $(x, \delta) \ge 0$ considered in $\mathcal{M} \cap (S_- \otimes \mathbb{R})$ gives its proper subset. Let $\delta = (\delta_- + \delta_+)/2$ where $\delta_- \in {S_-}^*$ and $\delta_+ \in {S_+}^*$.

The inequality $(x, \delta) \ge 0$ is equivalent to $(x, \delta_-) \ge 0$ for $x \in S_- \otimes \mathbb{R}$. If ${\delta_-}^2 \ge 0$, the inequality is trivial on $\mathcal{M} \cap (S_- \otimes \mathbb{R})$ since S_- is hyperbolic and $\mathcal{M}$ has only elements with non-negative square.

Thus, we can assume that ${\delta_-}^2 < 0$ if the pair (Y, A, θ) is ($\mathcal{D}\mathbb{R}$)-degenerate. Let $\widetilde{F}$ be one of components of $\pi^{-1}(F)$. We have

$$\delta = [\widetilde{F}] + [\tau(\widetilde{F})],$$

$$\delta_- = [\widetilde{F}] + [\varphi(\widetilde{F})] + [\tau(\widetilde{F})] + [\tau(\varphi(\widetilde{F})],$$

$$\delta_+ = [\widetilde{F}] - [\varphi(\widetilde{F})] + [\tau(\widetilde{F})] - [\tau(\varphi(\widetilde{F})].$$

It follows, $0 > {\delta_-}^2 = (\pi^*(F) + \pi^*(\theta(F)))^2 = 2(F + \theta(F))^2 = 2(F^2 + \theta(F)^2 + 2(F, \theta(F)) = 2(-4 + 2(F, \theta(F))$. Then $(F, \theta(F)) \le 1$.

Let us prove the opposite statement. Let F be an exceptional curve of Y with $F^2 = -2$ and $(F, \theta(F)) \le 1$. If $(F, \theta(F)) < 0$, then $\theta(F) = F$ because both curves F and $\theta(F)$ are irreducible. If $(F, \theta(F)) = 0$ or 1, then $\theta(F) \ne F$, and the intersection matrix of the curves F, $\theta(F)$ is negative definite. It follows that F and $\theta(F)$ (where $\theta(F) = F$ for the first case) define a face of the *nef* cone $\mathcal{M}(Y)$ of Y: there exists a *nef* element H of Y with $H^2 > 0$ such that $(H, F) = (H, \theta(F)) = 0$ and $(H, E) > 0$ for any other exceptional curve E of Y which is different from F and $\theta(F)$. The elements $(-\theta^*(H))$ and $h = H - \theta^*(H)$ are also *nef*. We have $\theta^*(h) = -h$, thus h is a real *nef* element. Since $\theta^*[F] = -[\theta(F)]$, it follows that $(h, F) = (H, F) - (\theta^*(H), F) = (H, F) - (H, \theta^*[F]) = (H, F) + (H, \theta(F)) = 0$. Similarly, $(h, \theta(F)) = 0$. Moreover, $(h, E) = (H, E) + (-\theta^*(H), E) \ge (H, E) > 0$ for any exceptional curve E of Y which is different from the curves F and $\theta(F)$. It follows that the real *nef* element h belongs to a face of $\mathcal{M}(Y)$ which is orthogonal to F and $\theta(F)$, and h does not belong to a face of $\mathcal{M}(Y)$ which is orthogonal to any other exceptional curve E of Y different from F and $\theta(F)$.

We have $\mathcal{M}(X,S)=\pi^*(\mathcal{M}(Y))$ and $\pi^*(h)\in\mathcal{M}(X,S)$ is a real nef element which belongs to one or two faces of $\mathcal{M}(X,S)$ which are orthogonal to elements $\pi^*[F]$ and $\pi^*[\theta(F)]$ with square -4 of S, and $\pi^*(h)$ does not belong to any other face of $\mathcal{M}(X,S)$. It follows that $\pi^*(h)$ is inside of $\mathcal{M}$ because faces of $\mathcal{M}$ are orthogonal to elements with square -2. It finishes the proof of the statement.

$\square$

Let (Y,θ,A) be *a real right DPN-pair,* i. e. (Y,θ) is a non-singular projective algebraic surface over $\mathbb{C}$ with an anti-holomorphic involution θ (i. e. a real projective algebraic surface $Y/\mathbb{R}$), and $A\in|-2K_Y|$ is a non-singular curve such that $\theta(A)=(A)$ (i. e. a real curve $A/\mathbb{R}$). Later we shall also use notation $(Y,A)/\mathbb{R}$ (or just (Y,A)) for a pair defined over $\mathbb{R}$. It is natural to consider two such pairs (Y,A,θ) and (Y',A',θ') to be isomorphic if there exists an isomorphism $f:Y\to Y'$ of algebraic surfaces over $\mathbb{C}$ such that $f(A)=A'$ and $f\theta=\theta' f$. The corresponding moduli space should parameterize isomorphism classes of such pairs. But we consider a more delicate isomorphism relation between real right DPN-pairs. Studying connected components of moduli for this more delicate isomorphism relation gives an additional information about monodromy.

There exist two real double coverings of Y ramified along A which give two real K3 surfaces with non-symplectic involutions (X,τ,φ) and $(X,\tau,\widetilde{\varphi})$ where $\widetilde{\varphi}=\tau\varphi$. They define the same real right DPN-pair ($Y=X/\{1,\tau\}$, $A=X^\tau$, $\theta=\varphi \mod \{1,\tau\}$). *The real K3 surfaces with non-symplectic involutions (X,τ,φ) and $(X,\tau,\widetilde{\varphi}=\tau\varphi)$ are called related.* A real right DPN-pair (Y,A,θ) together with a choice of one, between two, real double coverings of Y ramified along A (equivalently together with a choice of one, between two, real K3 surfaces with non-symplectic involutions (X,τ,φ) such that its quotient by τ gives (Y,θ,A)) is called *a positive real right DPN-pair.* We shall denote a positive DPN-pair as $(Y,\theta,A)^+$ (or $(Y,A)^+$ for $(Y,A)/\mathbb{R}$). Then a *related DPN-pair* will be denoted by $(Y,\theta,A)^-$. If $(Y,\theta,A)^+$ is given by (X,τ,φ), then $(Y,\theta,A)^-$ is given by $(X,\tau,\widetilde{\varphi}=\tau\varphi)$. Obviously, considering of real right DPN-pairs (Y,θ,A) is equivalent to considering of sets of related positive real right DPN-pairs $\{(Y,\theta,A)^+,(Y,\theta,A)^-\}$. *An isomorphism of two positive real right DPN-pairs* is equivalent to an isomorphism of the corresponding real K3 surfaces with non-symplectic involutions. Thus, we can define *positive real right DPN-pairs of the type* (S,θ) and *isomorphism of real right DPN-pairs with respect to the group G defined by* (S,θ). Obviously, an isomorphism of real non-symplectic involutions (X,τ,φ) and (X',τ',φ') defines the corresponding isomorphism of the related non-symplectic involutions $(X,\tau,\widetilde{\varphi})$ and $(X',\tau',\widetilde{\varphi'})$. Moreover, the type (S,θ) and the group G don't change for related real non-symplectic involutions. Moreover, by Proposition 5 related pairs are $(\mathcal{D}\mathbb{R})$-degenerate simultaneously.

Since positive real right DPN-pairs are equivalent to real non-symplectic involutions of K3, we get from Theorem 1 an equivalent

Theorem 6. *The natural map gives the one to one correspondence between connected components of moduli of ($\mathcal{D}\mathbb{R}$)-non-degenerate positive real right DPN-pairs $(Y,\theta,A)^+$ of the type (S,θ) and the set In(S,θ,G) of isomorphism classes for G of integral involutions (L,φ,S) with the condition (S,θ) satisfying (RSK3).*

Integral involutions (L,φ,S) and $(L,\widetilde{\varphi},S)$ corresponding to related pairs $(Y,\theta,A)^+$ and $(Y,\theta,A)^-$ are related as $\widetilde{\varphi}=\tau\varphi$. We remind that τ acts as $+1$ on S and as -1 on $S_L^\perp$. Naturally, *integral involutions (L,φ,S) and $(L,\widetilde{\varphi},S)$ are called related.* We denote the involution $(L,\varphi,S)\to(L,\widetilde{\varphi},S)$ in $\mathrm{In}(S,\theta,G)$ as τ too. From Theorem 6 we get

Theorem 7. *The natural map gives the one to one correspondence between connected components of moduli of ($\mathcal{D}\mathbb{R}$)-non-degenerate real right DPN-pairs (Y,θ,A) of the type (S,θ) and*

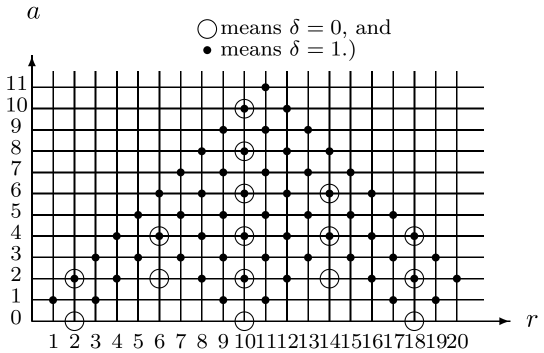

FIGURE 1. All possible $(r, a, \delta) = (r(S), a(S), \delta(S)) = (r(\tau), a(\tau), \delta(\tau))$

the set $In(S, \theta, G)/\{1, \tau\}$ of isomorphism classes for G of pairs $\{(L, \varphi, S), (L, \widetilde{\varphi} = \tau\varphi, S)\}$ of related integral involutions with the condition (S, θ) satisfying (RSK3).

2.3. **General classification results about real K3 surfaces with non-symplectic involutions and real right DPN-pairs.** The first, the most important invariant of a K3 surface with a non-symplectic involution (X, τ) is the isomorphism class of the lattice $S = H_2(X, \mathbb{Z})^\tau$. *The lattice S defines the type of (X, τ) and of the corresponding right DPN-pair $(Y = X/\{1, \tau\}, A = X^\tau)$.* The lattice S is any hyperbolic (i. e. of signature $(1, t_{(-)}(S))$ and of the rank $r(S) = 1 + t_{(-)}(S))$ even lattice having a primitive embedding $S \subset L$ to the even unimodular lattice $L \cong H_2(X, \mathbb{Z})$ of signature $(3, 19)$ and such that there exists an involution τ of L with $L^\tau = S$. The last property is equivalent for S to be 2-elementary, i. e. $S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^{a(S)}$ where $a(S) \geq 0$ is an invariant of S. Another invariant $\delta(S)$ of S is equal to 0 or 1. The $\delta(S) = 0$ if and only if $(x^*)^2 \in \mathbb{Z}$ for any $x^* \in S^*$; equivalently, $(z, \tau(z)) \equiv 0 \mod 2$ for any $z \in L$. (Namely, $\delta(S)$ means $\delta(\tau)$ like $\delta(\varphi)$ below.) The triplet $(r(S), a(S), \delta(S))$ is the complete list of invariants of the isomorphism class of S. All possible triplets $(r, a, \delta) = (r(S), a(S), \delta(S))$ were classified in [19], see also [20] and [22]. They are presented in Figure 1 (it is the same as Figure 2 in [22]).

We have

(2.11)
$$\begin{aligned} X^\tau =& \emptyset, \text{ if } \ (r(S), a(S), \delta(S)) = (10, 10, 0); \\ X^\tau =& C_1 + C_1{}' \text{ if } (r(S), a(S), \delta(S)) = (10, 8, 0); \\ X^\tau =& C_{g(S)} + E_1 + \cdots + E_{k(S)}, \quad g(S) = (22 - r(S) - a(S))/2, \ k(S) = (r(S) - a(S))/2. \end{aligned}$$

Here C_g denote a curve of genus g and $E_i \cong \mathbb{P}^1$. We have

$$X^\tau \sim 0 \mod 2 \text{ in } H_2(X, \mathbb{Z}) \tag{2.12}$$

if and only if $\delta(S) = 0$. The dimension of moduli of pairs (X, τ) and the corresponding DPN-pairs (Y, A) is equal to $20 - r(S)$. See [20] and [22].

The type (S, θ) of a real K3 surface with a non-symplectic involution (X, τ, φ) is defined by the isomorphism class θ of the action of φ on S. Any θ with the conditions *S^θ is negative definite and does not have elements $x \in S^\theta$ with $x^2 = -2$* can be taken as a type. Equivalently, $(-\theta) \in O^+(S)$ and $(-\theta)$ gives an involution of a fundamental chamber $\mathcal{M}$ of $W^{(-2)}(S)$. Finding of all possible types (S, θ) is a purely arithmetic and not difficult problem related with the automorphism group of the hyperbolic lattice S and its subgroup $W^{(-2)}(S)$. In this paper, we shall consider lattices S with $r(S) \leq 2$ when the problem of finding of possible types (S, θ) is very simple. It is why we don't consider this problem in this paper any more. See [4] about some results in this direction.

Assume that the type (S, θ) is fixed. We denote $S_+ = S^\theta$ and $S_- = S_\theta$. We shall use invariants

$$s = \operatorname{rk} S,\ p = \operatorname{rk} S_+ \tag{2.13}$$

Then S has the signature $(s_{(+)}, s_{(-)}) = (1, s-1)$ and S_+ has the signature $(p_{(+)}, p_{(-)}) = (0, p)$.

First we calculate the group G which we had uses in Theorems 1, 6 and 7. Let $\alpha \in S$ and $\alpha^2 = -4$. If $\alpha \in \Delta(S, L)^{(-4)}$, then α is a root of S, i. e. $(\alpha, S) \equiv 0 \mod 2$. Let $\langle -4 \rangle$ be a lattice generated by β with $\beta^2 = -4$. Let $S \oplus \langle -4 \rangle$ be the orthogonal sum of S and $\langle -4 \rangle$. Let $S \oplus \langle -4 \rangle \subset S(\alpha)$ be its overlattice of index two which is generated by $(\alpha + \beta)/2$.

Lemma 8. *The element $\alpha \in \Delta(S, L)^{(-4)}$ if and only if there exists a primitive embedding $S(\alpha) \subset L$.*

In particular, this is true if $r(S) + a(S) < 20$. It is also valid if $r(S) < 10$. Thus under one of these conditions, $\Delta(S, L)^{(-4)}$ consists of all roots $\alpha \in S$ with $\alpha^2 = -4$.

Proof. The first statement is trivial.

We have $\operatorname{rk} S(\alpha) = r(S) + 1$, the minimal number $l(A_{S(\alpha)})$ of generators of $A_{S(\alpha)} = S(\alpha)^*/S(\alpha)$ is not more than $a(S) + 1$. Thus, $\operatorname{rk} S(\alpha) + l(A_{S(\alpha)}) \le r(S) + 1 + a(S) + 1 < 22 = \operatorname{rk} L$. By Corollary 1.12.3 in [19], there exists a primitive embedding $S(\alpha) \subset L$.

The last statement follows from the obviouse inequality $a(S) \le r(S)$.

$\square$

Using results of [19] more heavily, one can give a necessary and sufficient condition of $\alpha \in \Delta(S, L)^{(-4)}$. We have restricted here the simple sufficient condition which is enough for this paper.

Let us introduce

$$\Delta(S_\pm, S)^{(-4)} = \{\alpha \in S_\pm \mid \alpha^2 = -4,\ (\alpha, S) \equiv 0 \mod 2\}.$$

Let $W^{(-4)}(S_\pm, S)$ be the group generated by reflections in all elements of $\Delta(S_\pm, S)^{(-4)}$. From Lemma 8, we get

Lemma 9. *If $a(S) + r(S) < 20$ (in particurlar, if $r(S) < 10$), then the group $G = W^{(-4)}(S_+, S) \times W^{(-4)}(S_-, S)$.*

In general, the group G is a subgroup of $W^{(-4)}(S_+, S) \times W^{(-4)}(S_-, S)$. In Theorems 1, 6 and 7 one can replace G by the bigger group $W^{(-4)}(S_+, S) \times W^{(-4)}(S_-, S)$ to simplify considerations. Of course, the results can be weaker then.

We say that *two integral involutions (L, φ, S) and (L', φ', S) of the type (S, θ) have the same genus with respect to the group G of (S, θ)* if there exists an automorphism $\xi : S \to S$ from G which can be continued to an isomorphism $(L, \varphi, S) \otimes \mathbb{R} \to (L', \varphi', S) \otimes \mathbb{R}$ over $\mathbb{R}$, and an isomorphism $(L, \varphi, S) \otimes \mathbb{Z}_p \to (L', \varphi', S) \otimes \mathbb{Z}_p$ over the ring $\mathbb{Z}_p$ of p-adic integers for any prime p. Other speaking, the integral involutions (L, φ, S) and (L', φ', S) cannot be distinguished considering over $\mathbb{R}$ and modulo any integer N using isomorphisms on S which belong to the group G. All the genus invariants (for an arbitrary even lattice S with an involution θ) were found in [21] together with necessary and sufficient conditions of existence. In many cases the genus has only one isomorphism class. Then the genus invariants give complete invariants of the isomorphism class. Below we remind these results adapting them to our case.

We assume that the integral involution (L,φ,S) satisfies the condition (RSK3). I. e. the lattice L is even unimodular of signature $(l_{(+)},l_{(-)})=(3,19)$. The lattice L^φ is hyperbolic of the signature $(t_{(+)},t_{(-)})=(1,t_{(-)})$. Then the only real invariant of (L,φ,S) is

$$r=\operatorname{rk}L^\varphi=1+t_{(-)}. \tag{2.14}$$

Below we describe *the genus invariants* of the integral evolution (L,φ,S) of the type (S,θ). To simplify notations, we temporarily denote $L_+=L^\varphi$ and $L_-=L_\varphi$.

Since L is unimodular, we have

$$A_{L_\pm}={L_\pm}^*/L_\pm\cong L/(L_+\oplus L_-)\cong(\mathbb{Z}/2\mathbb{Z})^a \tag{2.15}$$

where $a\ge 0$ is an integer. It is one of the most important genus invariants.

We also have *genus invariants*

$$\delta_\varphi=\begin{cases}0 & \text{if } (x,\varphi(x))\equiv 0 \mod 2\ \forall x\in L\\ 1 & \text{otherwise}\end{cases}, \tag{2.16}$$

and

$$\delta_{\varphi S}=\begin{cases}0 & \text{if } (x,\varphi(x))\equiv (x,s_\varphi) \mod 2\ \forall x\in L\\ & \text{for some element } s_\varphi \text{ in } S\\ 1 & \text{otherwise}\end{cases}. \tag{2.17}$$

If $\delta_{\varphi S}=0$, then the element s_φ occurring in the definition of $\delta_{\varphi S}$ is uniquely defined modulo $2S$. It is called *the characteristic element of the involution* φ.

Depending on these invariants, we divide (L,φ,S) into the following 3 *types*.

Type 0: $\delta_{\varphi S}=0$ and $\delta_\varphi=0$;
Type Ia: $\delta_{\varphi S}=0$ and $\delta_\varphi=1$;
Type Ib: $\delta_{\varphi S}=1$.

For $x_\pm\in S_\pm$ we put ,

$$\delta_{x_\pm}=\begin{cases}0 & \text{if } (x_\pm,L_\pm)\equiv 0 \mod 2,\\ 1 & \text{otherwise.}\end{cases} \tag{2.18}$$

Equivalently, $\delta_{x_\pm}=0$ iff $\frac{1}{2}x_\pm\in 2L_\pm^*$.

Since L is unimodular,

$$\delta_{x_\pm}=0\Leftrightarrow\exists x'_\pm\in L_\mp\ :\ \frac{1}{2}(x_\pm+x'_\pm)\in L. \tag{2.19}$$

The elements $x'_\pm$ are defined by the elements $x_\pm$ uniquely modulo $2L_\mp$; this enables us, for elements $x_+\in S_+$ and $x_-\in S_-$ for which $\delta_{x_+}=\delta_{x_-}=0$, to define the invariant

$$\rho_{x_+x_-}=\frac{1}{2}(x_+,x'_-)\mod 2=-\frac{1}{2}(x'_+,x_-)\mod 2\in\mathbb{Z}/2\mathbb{Z}.$$

Thus, we get a list of genus invariants

$$(r,a;\delta_{x_+},\delta_{x_-},\rho_{x_+x_-};\delta_\varphi,\delta_{\varphi S},s_\varphi) \tag{2.20}$$

where they are defined. E. g. the element $s_\varphi\in S\mod 2S$ is defined only if $\delta_{\varphi S}=0$; the invariant $\rho_{x_+x_-}$ is defined only if $\delta_{x_+}=\delta_{x_-}=0$. From [21], it follows that:

Two integral involutions with the condition (S,θ) have the same genus with respect to G, if and only if the invariants (2.20) *are conjugate by the group G of the condition (S,θ).*

To formulate conditions of existence of an integral involution with the genus invariants (2.20), we need to reformulate the invariants (2.20).

We have the function $\delta_\pm:S_\pm\to\mathbb{Z}/2\mathbb{Z}$ where $x_\pm\mapsto\delta_{x_\pm}$. We set

$$H_\pm=\delta_\pm^{-1}(0)/2S_\pm\subset S_\pm/2S_\pm.$$

The subgroups are equivalent to the invariants $\delta_{x_\pm}$. We have a more exact range for the subgroups $H_\pm$:

$$\Gamma_\pm = (2S)_\pm/2S_\pm \subset H_\pm \subset 2({S_\pm}^* \cap (\frac{1}{2}S_\pm))/2S_\pm \cong ({S_\pm}^* \cap (\frac{1}{2}S_\pm))/S_\pm = A_{S_\pm}^{(2)} \subset A_{S_\pm}.$$

Here $(2S)_\pm$ are the orthogonal projections of $2S \subset S_+ \oplus S_-$ to $S_\pm$ respectively. This projections also give the graph Γ of the isomorphism γ of the groups Γ_+ and Γ_-. The $A_{S_\pm}^{(2)}$ denote the subgroup of $A_{S_\pm}$ generated by all elements of order two. Let

$$H = H_+ \oplus_\gamma H_- = (H_+ \oplus H_-)/\Gamma.$$

For simplicity we identify $H_\pm = H_\pm \mod \Gamma \subset H$. We can define *a finite quadratic form* $q_\rho : H \to \frac{1}{2}\mathbb{Z}/2\mathbb{Z}$ as follows

$$q_\rho|H_+ = q_{S_+}|H_+;\ q_\rho|H_- = -q_{S_-}|H_-;\ q_\rho(x_+,x_-) = \frac{\rho_{x_+x_-}}{2} \mod \mathbb{Z} \text{ for } x_\pm \in H_\pm.$$

The finite quadratic form q_ρ is defined correctly and is equivalent to the invariants $\rho_{x_+x_-}$. In $H_+ \times \Gamma_-$ and $\Gamma_+ \times H_-$ the invariant $\rho_{x_+x_-}$ is defined by the discriminant quadratic forms q_{S_+} and q_{S_-} of lattices S_+ and S_- respectively. We shall also denote by

$$\rho : H_+ \times H_- \to \mathbb{Z}/2\mathbb{Z}$$

the pairing $\rho(x_+,x_-) = \rho_{x_+x_-}$. Thus $q_\rho(x_+,x_-) = \rho(x_+,x_-)/2 \mod 1$.

Moreover (see [21]), *the characteristic element*

$$v = s_\varphi \in H = H_+ \oplus_\gamma H_- \subset (S_+ \oplus S_-)/2S, \text{ if } \delta_{\varphi S} = 0.$$

In particular, $v = 0$ if $\delta_\varphi = 0$. The element v should be *characteristic for the quadratic form q_ρ on H* which means that $q_\rho(x,v) \equiv q_\rho(x,x) \pmod 1$ for any $x \in H$. The element v is zero, if $\delta_\varphi = 0$, and v is not zero, if $\delta_\varphi = 1$.

Thus, *the genus invariants* (2.20) *are equivalent to the described above genus invariants*

$$(r, a; H_+, H_-, q_\rho; \delta_\varphi, \delta_{\varphi S}, v). \tag{2.21}$$

Now we introduce some numerical invariants of the data (2.21) which are important for their existence.

Let a_M be the rank over $(\mathbb{Z}/2\mathbb{Z})$ of a 2-elementary group M. Thus, we have *the numerical invariants $a_{H_\pm}$, $a_{\Gamma_\pm} = a_{\Gamma_+} = a_{\Gamma_-}$ and $a_H = a_{H_+} + a_{H_-} - a_{\Gamma_\pm}$.*

Any finite 2-elementary quadratic form is an orthogonal sum $\oplus$ of elementary forms. The elementary forms are:
z on $\mathbb{Z}/2\mathbb{Z}\xi$ with $z(\xi) = 0 \mod 2$;
w on $\mathbb{Z}/2\mathbb{Z}\xi$ with $w(\xi) = 1 \mod 2$;
$u_+(2)$ on $\mathbb{Z}/2\xi_1 + \mathbb{Z}/2\mathbb{Z}\xi_2$ with $u_+(\xi_1) = u_+(\xi_2) = 0 \mod 2$, $u_+(\xi_1,\xi_2) = \frac{1}{2} \mod 1$, $\sigma(u_+(2)) \equiv 0 \mod 8$;
$v_+(2)$ on $\mathbb{Z}/2\xi_1 + \mathbb{Z}/2\mathbb{Z}\xi_2$ with $v_+(\xi_1) = v_+(\xi_2) = 1 \mod 2$, $v_+(\xi_1,\xi_2) = \frac{1}{2} \mod 1$; $\sigma(v_+(2)) \equiv 4 \mod 8$;
$q_\alpha(2)$, $\alpha = \pm 1 \mod 4$, on $\mathbb{Z}\xi$ with $q_\alpha(2)(\xi) = \alpha/2 \mod 2$. By definition, $\sigma(q_\alpha(2)) \equiv \pm 1 \mod 8$.

Any finite 2-elementary quadratic form f with a non-degenerate bilinear form is an orthogonal sum of $u_+(2)$, $v_+(2)$ and $q_\alpha(2)$ and has the invariant $\sigma \mod 8$ which is equal to the sum of σ for all its summands. It has an additional invariant δ. The $\delta = 0$ if f is orthogonal sum of only $u_+(2)$ and $v_+(2)$ (equivalently, f takes values in $\mathbb{Z}$). Otherwise, $\delta = 1$. Up to isomorphisms, the f is defined by its rank k, $\sigma \mod 8$ and $\delta \in \{0,1\}$. Then we denote f as $f = q(k,\delta,\sigma)$.

The quadratic form q_ρ on H is one of forms of the rank $a_H = a_{H_+} + a_{H_-} - a_{\Gamma_\pm}$ with the invariants $\delta_H = 0$ or 1; $\mu_\rho = 0$ or 1:

$$\delta_H = 0: \ q_\rho = z^{k_\rho - \mu_r} \oplus w^{\mu_\rho} \oplus v_+(2)^{\sigma_\rho/4} \oplus u_+(2)^{(a_H - k_\rho)/2 - \sigma_\rho/4},$$

where $\mu_\rho = 0$ or 1, $\sigma_\rho \equiv 0$ or $4 \mod 8$ and $\sigma_\rho \geq 0$, and $\mu_\rho + \sigma_\rho/4 \leq 1$;

$$\delta_H = 1, \ \mu_\rho = 0: \ q_\rho = z^{k_\rho} \oplus q(a_H - k_\rho, 1, \sigma_\rho),$$
$$\delta_H = 1, \ \mu_\rho = 1: \ q_\rho = z^{k_\rho - 1} \oplus w \oplus q_1(2)^{a_H - k_\rho}$$

where $a_H > k_\rho$.

The invariant $\delta_H = 0$ iff $q_\rho(x) \in \mathbb{Z}$ for any $x \in H$. We similarly introduce *the invariants* $\delta_{H_\pm}$ for the form $q_\rho | H_\pm$. We have

$$\delta_H = \max\{\delta_{H_+}, \delta_{H_-}\}.$$

If $\delta_{\varphi S} = 0$ and $v = s_\varphi \in H$ is the characteristic element of φ, we get *the invariant*

$$c_v \mod 4 \ \textit{where}\ c_v/2 = q_\rho(v) \mod 2.$$

Thus, we get invariants

$$(a_{H_\pm}, a_{\Gamma_\pm}, \delta_{H_\pm}; k_\rho, \mu_\rho, \sigma_\rho, c_v) \tag{2.22}$$

of the data (2.21).

We need some more invariants of data (2.21). An element $\tilde{v} \in A_{S_\pm}^{(2)}$ is called *characteristic* if $q_{S_\pm}(x) \equiv q_{S_\pm}(\tilde{v}, x) \mod 1$ for any $x \in A_{S_\pm}^{(2)}$. We remind that $A_{S_\pm}^{(2)}$ is generated by all elements of order 2 in $A_{S_\pm}$.

We say that $\delta_{\varphi S_\pm} = 0$ if $\delta_{\varphi S} = 0$, the element $v = s_\varphi \in H_\pm$ and v is equal to a characteristic element of $A_{S_\pm}^{(2)}$. Otherwise $\delta_{\varphi S_\pm} = 1$.

Assume that $\delta_{\varphi S_\pm} = 0$. We consider a non-degenerate finite quadratic form

$$\gamma_\pm = \begin{cases} 0, & \text{if } \delta_\varphi = 0, \text{ equivalently } v = 0, \\ q_1(2) \oplus q_{-1}(2), & \text{if } \delta_\varphi = 1 \text{ and } c_v \equiv 0 \mod 4, \\ q_1(2)^2, & \text{if } c_v \equiv 2 \mod 4, \\ q_{\pm 1}(2), & \text{if } c_v \equiv \pm 1 \mod 4. \end{cases}$$

Let $v_{\gamma_\pm}$ be the characteristic element of $\gamma_\pm$ (it is unique). We have $\gamma_\pm(v_{\gamma_\pm}) = c_v/2 \mod 2$. We denote $(q_{S_\pm})_v = (v \oplus v_{\gamma_\pm})^\perp_{q_{S_\pm} \oplus (\mp \gamma_\pm)} / [v \oplus v_{\gamma_\pm}]$. There exists a unique even 2-adic lattice $K((q_{S_\pm})_v)$ having the discriminant quadratic form $(q_{S_\pm})_v$ and the same rank as the form $(q_{S_\pm})_v$ (i. e. the minimal number of generators of the 2-group where the form is defined). We denote by A_q the group where a finite form q is defined and by $|A_q|$ its order. *The invariant*

$$\varepsilon_{v_\pm} \in (\mathbb{Z}/2\mathbb{Z})$$

is defined by

$$5^{\varepsilon_{v_\pm}} \equiv \frac{\det K((q_{S_\pm})_v)}{|A_{S_\pm}| \cdot |A_{\gamma_\pm}|} \pmod{\pm \mathbb{Q}_2^{*2}}$$

where $\mathbb{Q}_p$ is the field of p-adic integers.

Finally we get invariants

$$(\delta_{\varphi S_+}, \delta_{\varphi S_-}, \varepsilon_{v_+}, \varepsilon_{v_-}) \tag{2.23}$$

By our construction, the finite quadratic form q_ρ on H has an embedding to the discriminant quadratic form q_{L_+} which is 2-elementary, and the characteristic element v (if $\delta_{\varphi S} = 0$) goes to the characteristic element of q_{L_+}. The following conditions from [21] are sufficient and necessary for its existence (we numerate these condition as in [21]). They use invariants (2.22) and invariants r, a, $\delta_{\varphi S}$, and δ_φ.

CONDITIONS 1.8.1 (from [21]).
Type 0 *($\delta_\varphi=\delta_{\varphi S}=0$):*

Conditions on $H_\pm$. $\delta_{H_+}=\delta_{H_-}=0$.
Inequality. $a\ge a_{H_+}+a_{H_-}-a_{\Gamma_\pm}+k_\rho$.
Congruences. 1) $a+r\equiv 0\pmod 2$; *2)* $2-r\equiv 0\pmod 4$.
Boundary condition. If $(a=a_{H_+}+a_{H_-}-a_{\Gamma_\pm}+k_\rho,\ \mu_\rho=0)$, *then* $2-r\equiv\sigma_\rho\pmod 8$.
Type Ia *($\delta_\varphi=1$, $\delta_{\varphi S}=0$):*
Conditions on H, ρ and c_v. 1) $c_v\equiv a_{H_+}+a_{H_-}-a_{\Gamma_\pm}-k_\rho\pmod 2$. *2) If* $\mu_\rho=0$, *then* $c_v\equiv\sigma_\rho\pmod 4$. *3)* $a_{H_+}+a_{H_-}-a_{\Gamma_\pm}>0$. *4) If* $\delta_{H_+}=\delta_{H_-}=0$, *then* $k_\rho\ge 1$. *5) If* $(\delta_{H_+}=\delta_{H_-}=0,\ k_\rho=\mu_\rho=1)$, *then* $c_v\equiv 2\pmod 4$.
Inequality. $a\ge a_{H_+}+a_{H_-}-a_{\Gamma_\pm}+k_\rho$.
Congruences. 1) $a+r\equiv 0\pmod 2$; *2)* $2-r\equiv c_v\pmod 4$.
Boundary condition. If $(a=a_{H_+}+a_{H_-}-a_{\Gamma_\pm}+k_\rho,\ \mu_\rho=0)$, *then* $2-r\equiv\sigma_\rho\pmod 8$.
Type Ib *($\delta_\varphi=1$, $\delta_{\varphi S}=1$):*
Inequality. $a\ge a_{H_+}+a_{H_-}-a_{\Gamma_\pm}+k_\rho+1$.
Congruences. $a+r\equiv 0\pmod 2$.
Boundary condition. 1) If $(a=a_{H_+}+a_{H_-}-a_{\Gamma_\pm}+k_\rho+1,\ \mu_\rho=0)$, *then* $2-r\equiv\sigma_\rho\pm 1\pmod 8$. *2) If* $(a=a_{H_+}+a_{H_-}-a_{\Gamma_\pm}+k_\rho+2,\ \mu_\rho=0)$, *then* $2-r\not\equiv\sigma_\rho+4\pmod 8$.

The following conditions from [21] are necessary and sufficient for existence of the lattices $K_\pm=(S_\pm)^\perp_{L_\pm}$ with the local invariants prescribed by the data (2.20) or (2.21). They also use the invariants (2.13) and (2.23). The $l(A)$ denotes the minimal number of generators of a finite Abelian group A. We numerate the conditions as in [21] and drop unnecessary (for our case) conditions.

CONDITION 1.8.2 (from [21]).
Conditions on the invariants $\delta_{\varphi S_\pm}$. If $\delta_{\varphi S}=1$, *then* $\delta_{\varphi S_+}=\delta_{\varphi S_-}=1$.
Inequalities. 1) $p+1\le r\le 21-s+p$; *2)* $r-a\ge -2a_{H_+}+p+l(A_{S_+})$; *3)* $r+a\le 2a_{H_-}+p-s-l(A_{S_-})+22$.

Boundary conditions. 1) If $\delta_{\varphi S_+}=0$ *and* $r-a=-2a_{H_+}+p+l(A_{S_+})$, *then* $2-r\equiv 4\varepsilon_{v_+}+c_v\pmod 8$.
2) If $\delta_{\varphi S_-}=0$ *and* $r+a=2a_{H_-}+p-s-l(A_{S_-})+22$, *then* $2-r\equiv 4\varepsilon_{v_-}+c_v\pmod 8$.

By [21] both these conditions are necessary and sufficient for existence of the integral involution. Thus, finally we get

Theorem 10. *([21]). The invariants (2.20) or (2.21) give the complete genus invariant of integral involutions (L,φ,S) of the type (S,θ) from the set $In(S,\theta,G)$. The conditions 1.8.1 and 1.8.2 above are necessary and sufficient for existence of the involution from $In(S,\theta,G)$ with the genus invariants.*

Obviously, the genus of the integral involution (L,φ,S) defines the genus of the related involution $(L,\widetilde{\varphi}=\tau\varphi,S)$. Thus, invariants (2.20) or (2.21) of (L,φ,S) define similar invariants of the related involution. We denote by

$$\big(r(\varphi),a(\varphi);H(\varphi)_+,H(\varphi)_-,q_{\rho(\varphi)};\delta_\varphi,\delta_{\varphi S},v(\varphi)\big) \tag{2.24}$$

and

$$\big(r(\tau\varphi),a(\tau\varphi);H(\tau\varphi)_+,H(\tau\varphi)_-,q_{\rho(\tau\varphi)};\delta_{\tau\varphi},\delta_{\tau\varphi S},v(\tau\varphi)\big) \tag{2.25}$$

the genus invariants (2.21) for φ and $\widetilde{\varphi}=\tau\varphi$. Similarly we mark by φ or $\tau\varphi$ the equivalent genus invariants (2.20).

Theorem 11. *We have the following relations between genus invariants of the related involutions (L,φ,S) and $(L,\widetilde{\varphi}=\tau\varphi,S)$:*

$$r(\varphi)+r(\tau\varphi)=22-s+2p, \tag{2.26}$$

$$a(\tau\varphi)-a(\varphi)=a(S)-2a_{H(\varphi)}+2\operatorname{rk}\rho(\varphi), \tag{2.27}$$

$$s_\varphi+s_{\tau\varphi}\equiv s_\theta\in S \mod 2L, \tag{2.28}$$

where $\operatorname{rk}\rho$ *means the rank of the matrix which gives* ρ *in some bases of* H_+ *and* H_- *over the field* $\mathbb{Z}/2\mathbb{Z}$*, and* $s_\theta\in S$ *is the characteristic element of* (S,θ) *defined by the property* $2(x,\theta(x))\equiv(x,s_\theta)\mod 2$ *for any* $x\in S^*$ *(the* s_θ *is defined* $\mod 2S$*). In particular,*

$$\delta_{\varphi S}=\delta_{(\tau\varphi)S}. \tag{2.29}$$

The $H(\varphi)_\pm$ *and* $H(\tau\varphi)_\pm$ *are orthogonal with respect to the discriminant bilinear form* $b_{S_\pm}$ *on* $A_{S_\pm}$ *respectively.*

Moreover,

$$l(A_{S_+})+a(\varphi)-2a_{H(\varphi)_+}=l(A_{S_-})+a(\tau\varphi)-2a_{H(\tau\varphi)_-}. \tag{2.30}$$

Proof. We have the orthogonal decomposition up to a finite index

$$S_+\oplus S_-\oplus L^\varphi_\tau\oplus L_{\tau,\varphi}\subset L. \tag{2.31}$$

We have $r(\tau\varphi)=\operatorname{rk}S_++\operatorname{rk}L_{\tau,\varphi}=\operatorname{rk}S_++(\operatorname{rk}L-\operatorname{rk}L^\varphi-\operatorname{rk}S_-)=p+22-r(\varphi)-(s-p)$. It follows the first relation (2.26).

The proof of second relation (2.27) follows the proof of Lemma 3.3.2 in [23]. There was shown that

$$2^{a(\tau\varphi)}=\#A_{L^\tau_\varphi}\cdot\#A_{L^{\tau,\varphi}}\cdot\#A_{L_\varphi}/(2^{2\operatorname{rk}H(\varphi)_+}\cdot2^{2\operatorname{rk}(H(\varphi)_+{}^\perp\cap H(\varphi)_-)})$$

where $H(\varphi)_+{}^\perp\subset H(\varphi)_-$ is the orthogonal complement to $H(\varphi)_+$ with respect to the pairing $\rho(\varphi)$. We have $\operatorname{rk}(H(\varphi)_+{}^\perp\cap H(\varphi)_-)=\operatorname{rk}H(\varphi)_--\operatorname{rk}\rho(\varphi)$. Moreover, $\#A_{L_\varphi}=2^{a(\varphi)}$, $\#A_{L^\tau_\varphi}\cdot\#A_{L^{\tau,\varphi}}=\#A_{L^\tau}\cdot2^{\operatorname{rk}\Gamma_\pm}=2^{a(S)+2\operatorname{rk}\Gamma_\pm}$. Thus, we get

$$a(\tau\varphi)=a(\varphi)+a(S)+2\operatorname{rk}\Gamma_\pm-2\operatorname{rk}H(\varphi)_+-2\operatorname{rk}H(\varphi)_-+2\operatorname{rk}\rho(\varphi).$$

It follows (2.27).

We can write up $x\in L$ as $x=s_+{}^*\oplus s_-{}^*\oplus x^{\varphi*}_\tau\oplus x_{\tau,\varphi}{}^*$ where $s_+{}^*\in S_+{}^*$, $s_-{}^*\in S_-{}^*$, $x^{\varphi*}_\tau\in L^{\varphi*}_\tau$, $x_{\tau,\varphi}{}^*\in L_{\tau,\varphi}{}^*$. We have

$$(x,\varphi(x))=(s_+{}^*)^2-(s_-{}^*)^2+(x^{\varphi*}_\tau)^2-(x_{\tau,\varphi}{}^*)^2\equiv(x,s_\varphi)\mod 2;$$
$$(x,\tau\varphi(x))=(s_+{}^*)^2-(s_-{}^*)^2-(x^{\varphi*}_\tau)^2+(x_{\tau,\varphi}{}^*)^2\equiv(x,s_{\tau\varphi})\mod 2.$$

Taking sum, we get

$$(x,s_\varphi)+(x,s_{\tau\varphi})\equiv2\left((s_+{}^*)^2-(s_-{}^*)^2\right)\equiv(p_S(x),s_\theta)\mod 2.$$

where $p:L\to S^*$ the orthogonal projection. Here $p(L)=S^*$ since L is unimodular. Since L is unimodular, it follows that $s_\varphi+s_{\tau\varphi}\equiv s_\theta\mod 2L$. It proves the third relation (2.28).

Let $u\in S_-$ has the invariant $\delta_u=0$ for φ and $v\in S_-$ has the invariant $\delta_v=0$ for $\tau\varphi$. Then (see [21], (7.6)) there exist $u_1\in L^\varphi$ and $v_1\in L^{\tau\varphi}=L_{\tau,\varphi}$ such that $u/2+u_1/2\in L$ and

$v/2+v_1/2\in L$. We then have $(u/2,v/2)=(u/2+u_1/2,v/2+v_1/2)\in \mathbb{Z}$. Thus $u/2\in A_{S_-}$ and $v/2\in A_{S_-}$ are orthogonal with respect to the discriminant bilinear form b_{S_-} on A_{S_-}. It proves that the groups $H(\varphi)_-$ and $H(\tau\varphi)_-$ are orthogonal to each other. The proof for H_+ is similar.

By the relation (8.9) from [21], we have

$$l(A_{L_\tau^\varphi})=l(A_{S_+})+a(\varphi)-2a_{H(\varphi)_+};$$
$$l(A_{L_{\tau,\varphi}})=l(A_{S_-})+a(\varphi)-2a_{H(\varphi)_-}.$$

Similar relation for $\tau\varphi$ gives

$$l(A_{L_\tau^\varphi})=l(A_{S_-})+a(\tau\varphi)-2a_{H(\tau\varphi)_-};$$
$$l(A_{L_{\tau,\varphi}})=l(A_{S_+})+a(\tau\varphi)-2a_{H(\tau\varphi)_+}.$$

It follows the last relation (2.30) of the theorem. $\square$

Theorem 11 permits to choose one from two related involutions by some conditions on its invariants. It helps to classify the pairs of related involutions which is important for a classification of the corresponding DPN-pairs.

One should follow [21] *to give the exact construction of the genus invariants* (2.25) *of the related involution* $\tau\varphi$. The idea is as follows. Using construction of [21], we can calculate the genus of the integral involution from the invariants (2.24) of φ. From the genus, we can find the invariants (2.25) of $\tau\varphi$. We don't know a simpler procedure in general. Below we give this construction in details.

Let $(A_q,q)=(A_{q(\varphi)},q(\varphi))$ be the discriminant quadratic form of the lattice L^φ. It is defined by the invariants $(r(\varphi),a(\varphi),\delta(\varphi))$. Let $\xi^{(+)}:H=H_+\oplus_\gamma H_-\to A_q$ be an embedding of (H,q_ρ) to (A_q,q) such that $\xi^{(+)}(v(\varphi))$ is the characteristic element of q, if $\delta_{\varphi S}=0$, and $\xi^{(+)}(H)$ does not contain a characteristic element of q if $\delta_{\varphi S}\neq 0$. Such an embedding is defined uniquely up to automorphisms of q.

We denote by $(A_{-q},-q)$ the form (A_q,q) with the the opposite sign and by $\Delta_q\subset A_q\oplus A_{-q}$ the corresponding isotropic diagonal for $q\oplus(-q)$. The $\xi^{(+)}$ gives $\xi^{(-)}:(H,-q_\rho)\to(A_{-q},-q)$ where $\xi^{(-)}=\xi^{(+)}$ on the groups. Let $\Gamma(H_+)\subset A_{S_+}\oplus A_{-q}$ be the graph of $\xi^{(-)}|H_+$. We then get the non-degenerate quadratic form

$$(A_{k_+}=\Gamma(H_+)^\perp_{q_{S_+}\oplus(-q)}/\Gamma(H_+),k_+=-q_{S_+}\oplus q|A_{k_+}) \tag{2.32}$$

with the corresponding projection $\pi_{k_+}:\Gamma(H_+)^\perp_{q_{S_+}\oplus(-q)}\to A_{k_+}$ and the graph $\Gamma_{\pi_{k_+}}\subset A_{S_+}\oplus A_{-q}\oplus A_{k_+}$. The form k_+ can be identified with the discriminant form of the lattice L_τ^φ. As a result, we get the quadratic form

$$(A_{S_+}\oplus A_{-q}\oplus A_{k_+},q_{S_+}\oplus A_{-q}\oplus k_+)$$

with the isotropic subgroup $\Gamma_{(S_+,-q,k_+)}=\Gamma(H_+)\oplus\{0\}+\Gamma_{\pi_{k_+}}$. Further we identify A_{S_+}, A_{-q}, A_{k_+} with the corresponding subgroups in $A_{S_+}\oplus A_{-q}\oplus A_{k_+}$. By the construction, $\Gamma^\perp_{(S_+,-q,k_+)}=\Gamma_{(S_+,-q,k_+)}$ is the maximal isotropic subgroup. Moreover, $\Gamma_{(S_+,-q,k_+)}\cap A_{S_+}=\Gamma_{(S_+,-q,k_+)}\cap A_{-q}=\Gamma_{(S_+,-q,k_+)}\cap A_{k_+}=\{0\}$, $\Gamma_{(S_+,-q)}=\Gamma_{(S_+,-q,k_+)}\cap(A_{S_+}\oplus A_{-q})=\Gamma(H_+)$. Let $\Gamma_{(S_+,k_+)}=\Gamma_{(S_+,-q,k_+)}\cap(A_{S_+}\oplus A_{k_+})$. By our construction, we then have the natural identification

$$\left(A_q=(\Gamma_{(S_+,k_+)})^\perp_{q_{S_+}\oplus k_+}/\Gamma_{(S_+,k_+)},\ q=q_{S_+}\oplus k_+|A_q\right) \tag{2.33}$$

where $\Gamma_{(S_+,-q,k_+)}/\Gamma_{(S_+,k_+)}$ gives the graph of $\Delta_q\subset A_q\oplus A_{-q}$ above. Let $\pi_q:(\Gamma_{(S_+,k_+)})^\perp_{q_{S_+}\oplus k_+}\to A_q$ be the corresponding projection. Let $\Gamma(H_-)\subset A_q\oplus A_{S_-}$ be the graph of $\xi^{(+)}|H_-$ and $\Gamma_{(S_+,S_-,k_+)}=\widetilde{\Gamma(H_-)}=(\pi_q\oplus \mathrm{id}(A_{S_-}))^{-1}(\Gamma(H_-))\subset A_{S_+}\oplus A_{S_-}\oplus A_{k_+}$. Let

(2.34)
$$\left(A_{k_-}=(\Gamma_{(S_+,S_-,k_+)})^{\perp}_{q_{S_+}\oplus q_{S_-}\oplus q_{k_+}}/\Gamma_{(S_+,S_-,k_+)},\ k_-=(-q_{S_+})\oplus(-q_{S_-})\oplus(-q_{k_+})|A_{k_-}\right).$$

The form k_- can be identified with the discriminant form of the lattice $L_{\tau,\varphi}$. We remark that one gets the same form as

$$(A_{k_-}=\Gamma(H_-)^{\perp}_{q_{S_-}\oplus q}/\Gamma(H_-), k_-=-q_{S_-}\oplus(-q)|A_{k_-}). \tag{2.35}$$

Thus, finally we get the form

$$\left(A_{S_+}\oplus A_{S_-}\oplus A_{k_+}\oplus A_{k_-}, q_{S_+}\oplus q_{S_-}\oplus k_+\oplus k_-\right) \tag{2.36}$$

which corresponds to the orthogonal decomposition (2.31). Further we use this form. Let

$$\Gamma=\Gamma_{(S_+,S_-,k_+,k_-)}\subset A_{S_+}\oplus A_{S_-}\oplus A_{k_+}\oplus A_{k_-} \tag{2.37}$$

be the graph of the projection $\pi_{k_-}:\Gamma_{(S_+,S_-,k_+)})^{\perp}_{q_{S_+}\oplus q_{S_-}\oplus q_{k_+}}\to A_{k_-}$. By our construction, the group $\Gamma\subset A_{S_+}\oplus A_{S_-}\oplus A_{k_+}\oplus A_{k_-}$ is the maximal isotropic subgroup: $\Gamma^{\perp}=\Gamma$. We have

$$\Gamma=L/(S_+\oplus S_-\oplus L_\tau^{\varphi}\oplus L_{\tau,\varphi})$$

for the orthogonal decomposition (2.31). The equality $\Gamma^\perp=\Gamma$ is equivalent for the lattice L to be unimodular.

The orthogonal decomposition (2.36) and the isotropic subgroup Γ (together with the corresponding real invariants) are equivalent to the genus of the integral involution. Thus, we have calculated the genus from the invariants (2.24) of φ. Using this genus, now we can calculate the invariants (2.25) of $\tau\varphi$. For $\widetilde{\varphi}=\tau\varphi$, the lattices L_τ^{φ} and $L_{\tau,\varphi}$ change places: $L_\tau^{\widetilde{\varphi}}=L_{\tau,\varphi}$ and $L_{\tau,\widetilde{\varphi}}=L_\tau^{\varphi}$.

Let $\alpha_i\in\{S_+,S_-,k_+,k_-\}$. We denote $\Gamma_{(\alpha_1,\dots,\alpha_n)}=\Gamma\cap(A_{\alpha_1}\oplus\cdots\oplus A_{\alpha_n})$ and by $p_{(\alpha_1,\dots,\alpha_n)}$ the corresponding orthogonal projection into the group $A_{\alpha_1}\oplus\cdots\oplus A_{\alpha_n}$.

By definition of the invariants (2.24) and (2.25), we have:

$$H(\varphi)_+=p_{S_+}(\Gamma_{(S_+,S_-,k_-)}) \tag{2.38}$$

with the natural embedding $H(\varphi)_+\subset A_q$ (use (2.33)) since $H(\varphi)_+\perp\Gamma_{(S_+,k_+)}$. Similarly, we have

$$H(\varphi)_-=p_{S_-}(\Gamma_{(S_+,S_-,k_+)}) \tag{2.39}$$

with the natural embedding $H(\varphi)_-\subset A_q$ defined by the graph $\Gamma_{(S_+,S_-,k_+)}/\Gamma_{(S_+,k_+)}$. The form q on A_q defines then the pairing $\rho(\varphi)$ between $H(\varphi)_+$ and $H(\varphi)_-$.

For $\tau\varphi$ we similarly get:

$$H(\tau\varphi)_+=p_{S_+}(\Gamma_{(S_+,S_-,k_+)}) \tag{2.40}$$

with the natural embedding into

$$\left(A_{q(\tau\varphi)}=(\Gamma_{(S_+,k_-)})^{\perp}_{q_{S_+}\oplus k_-}/\Gamma_{(S_+,k_-)},\ q(\tau\varphi)=q_{S_+}\oplus k_-|A_{q(\tau\varphi)}\right); \tag{2.41}$$

$$H(\tau\varphi)_-=p_{S_-}(\Gamma_{(S_+,S_-,k_-)}) \tag{2.42}$$

with the natural embedding into $A_{q(\tau\varphi)}$ defined by the graph $\Gamma_{(S_+,S_-,k_-)}/\Gamma_{(S_+,k_-)}$. The form $q(\tau\varphi)$ defines then the pairing $\rho(\tau\varphi)$ in between $H(\tau\varphi)_+$, $H(\tau\varphi)_-$.

This calculates the genus invariants (2.25). For these calculations, the trivial (from the construction above) and crucial observation is:

Theorem 12. *Assume that $q = q(\varphi) = q_1 \oplus q'$ and $\xi^{(+)}$ gives an embedding of q_ρ into $q_1 \oplus \{0\}$. Then the calculations above of the genus invariants $H(\tau\varphi)_+$, $H(\tau\varphi)_-$ and $\rho(\tau\varphi)$ give the same result if one replaces $q = q(\varphi)$ by q_1.*

The Theorem 12 reduces calculations of the invariants $H(\tau\varphi)_\pm$ and $\rho(\tau\varphi)$ to some finite and simple calculations.

There is a case when these calculations are very simple

Corollary 13. *Assume that $S_+ = 0$, $S_- = S$. Then $H(\varphi)_+ = H(\tau\varphi)_+ = 0$, $\rho(\varphi) = \rho(\tau\varphi) = 0$ and $H(\varphi)_+ \perp H(\tau\varphi)_+$ are orthogonal complements to each other with respect to $q_S = q_{S_-}$.*

In many cases the genus invariants (2.21) have a unique isomorphism class of integral involutions. By Theorems 1, 6 and 7, then the genus invariants define the connected component of moduli of real K3 surfaces with a non-symplectic involution or the connected component of moduli of the corresponding real right DPN-pairs. We shall use the invariant $\kappa(A) = m$ of a finite Abelian group $A = (\mathbb{Z}/2)^m \oplus (\mathbb{Z}/4\mathbb{Z})^n$. Using results of [21] and [18], [19], we get the following statement.

Theorem 14. *The genus invariants* (2.21) *have a unique isomorphism class from $In(S, \theta, G)$ of integral involutions (L, φ, S) of the type (S, θ) satisfying (RSK3) (in particular, by Theorems 1, 6 and 7, the moduli of real K3 surfaces with a non-symplectic involution, the moduli of the corresponding positive real right DPN-pairs, the moduli of the corresponding real right DPN-pairs having these invariants are connected), if both conditions $(a_\pm)$ below are satisfied:*

(a_+) an even lattices K_+ with the discriminant quadratic form k_+ (see (2.32)*) and the signature $(1, r-p-1)$ is unique up to isomorphisms, and the canonical homomorphism $O(K_+) \to O(k_+)$ is epimorphic;*
(a_-) an even lattice K_- with the discriminant quadratic form k_- (see (2.35)*) and the signature $(1, 21-r-s+p)$ is unique up to isomorphisms, and the canonical homomorphism $O(K_-) \to O(k_-)$ is epimorphic.*

The condition $(a_\pm)$ is valid if the corresponding condition $(b_\pm)$ below satisfies:

(b_+) either $r - a > -2a_{H_+} + p + l(A_{S_+})$, or ($r - a = -2a_{H_+} + p + l(A_{S_+})$, $a \geq 2a_{H_+} - \kappa(A_{S_+}) + 3$), or ($r - a = -2a_{H_+} + p + l(A_{S_+})$, $a = 2a_{H_+} - \kappa(A_{S_+}) + 2$, $\delta_{\varphi S_+} = 0$);
(b_-) either $r + a < 2a_{H_-} + p - s - l(A_{S_-}) + 22$, or ($r + a = 2a_{H_-} + p - s - l(A_{S_-}) + 22$, $a \geq 2a_{H_-} - \kappa(A_{S_-}) + 3$), or ($r + a = 2a_{H_-} + p - s - l(A_{S_-}) + 22$, $a = 2a_{H_-} - \kappa(A_{S_-}) + 2$, $\delta_{\varphi S_-} = 0$).

Proof.
The first statements follows from Theorem 1.3.1 in [21] (see also Remark 1.6.2 there).
The lattice K_+ is indefinite and even, its discriminant form k_+ is a 2-form. Moreover, the discriminant group $A_{k_+} \cong (\mathbb{Z}/2\mathbb{Z})^m \oplus (\mathbb{Z}/4\mathbb{Z})^n$ since the lattice S is 2-elementary and the discriminant groups $A_{S_\pm}$ are annulated by 4. Applying Theorem 1.14.2 from [19] to this case, we see that (a_+) is valid if either $\operatorname{rk} K_+ > l(A_{k_+})$ or $\operatorname{rk} K_+ = l(A_{k_+})$, but the discriminant form $k_+ \cong u_+(2) \oplus k_1$, $v_+(2) \oplus k_1$.
The relation (8.9) in [21] gives that

$$l(A_{k_+}) = l(A_{S_+}) + a - 2a_{H_+}.$$

Thus, the condition $\operatorname{rk} K_+ > l(A_{k_+})$ means $r - p > l(A_{S_+}) + a - 2a_{H_+}$. Equivalently, $r - a > -2a_{H_+} + p + l(A_{S_+})$.

We have $A_{k_+}\cong (\mathbb{Z}/2\mathbb{Z})^m\oplus(\mathbb{Z}/4\mathbb{Z})^n$. Let us calculate m. Assume $A_{S_+}\cong(\mathbb{Z}/2\mathbb{Z})^\alpha\oplus(\mathbb{Z}/4\mathbb{Z})^\beta$ where $\alpha=\kappa(A_{S_+})$. By (2.32), $\#A_{k_+}=2^{m+2n}=\#A_{S_+}\cdot\#A_q/(\#H_+)^2=2^{\alpha+2\beta+a-2a_{H_+}}$. It follows, $m+2n=\alpha+2\beta+a-2a_{H_+}$. Moreover, the relation $l(A_{k_+})=l(A_{S_+})+a-2a_{H_+}$ used above gives $m+n=l(A_{k_+})=l(A_{S_+})+a-2a_{H_+}=\alpha+\beta+a-2a_{H_+}$. It follows, $m=\alpha+a-2a_{H_+}=\kappa(A_{S_+})+a-2a_{H_+}$.

From classification of finite quadratic forms, we get that $k_+\cong u_+(2)\oplus k_1$ or $v_+(2)\oplus k_2$, if $m\ge 3$. Moreover, it is shown in ([21], page 116), that $k_+\cong u_+(2)\oplus q_1$, $v_+(2)\oplus q_2$, if $m=2$ and $\delta_{\varphi S_+}=0$. Thus, cited above conditions of Theorem 1.14.2 from [19] are equivalent to (b_+). Thus, (b_+) implies (a_+).

The proof of (b_-) implies (a_-) is the same.

We mention the useful formulae which follow from the proof:

$$A_{k_\pm}\cong(\mathbb{Z}/2\mathbb{Z})^{m_\pm+a-2a_{H_\pm}}\oplus(\mathbb{Z}/4\mathbb{Z})^{n_\pm},\ \text{if}\ A_{S_\pm}\cong(\mathbb{Z}/2\mathbb{Z})^{m_\pm}\oplus(\mathbb{Z}/4\mathbb{Z})^{n_\pm}. \tag{2.43}$$

We also mention the following useful statement which follows from (Theorem 3.6.3 in [19]):

Proposition 15. *The condition $(a_\pm)$ of Theorem 14 is valid, if $A_{k_\pm}$ is 2-elementary. In particular, this is true (by (2.43)), if the lattice $S_\pm$ is 2-elementary.*

At last, we mention the main geometric interpretation of the invariants $(r(\varphi),a(\varphi),\delta(\varphi))=(r,a,\delta)$. We denote by S_g an oriented surface of the genus g.

We have (see Theorem 3.10.6 in [19]) for the real part of $X_\varphi(\mathbb{R})=X^\varphi$ of X with the real structure defined by φ the same result as for the holomorphic non-symplectic involution τ:

$$X_\varphi(\mathbb{R})=\begin{cases}\emptyset, & \text{if } (r,a,\delta)=(10,10,0)\\ T_1\amalg T_1, & \text{if } (r,a,\delta)=(10,8,0)\\ T_g\amalg (T_0)^k, & \text{otherwise, where}\\ & g=(22-r-a)/2,\ k=(r-a)/2\end{cases}; \tag{2.44}$$

and

$$X_\varphi(\mathbb{R})\sim s_\varphi \mod 2 \text{ in } H_2(X,\mathbb{Z}). \tag{2.45}$$

We have the same formula for $X_{\tau\varphi}(\mathbb{R})$ using invariants $(r(\tau\varphi),a(\tau\varphi),\delta(\tau\varphi))$ of $\tau\varphi$.

2.4. **Possible types of (S,θ) with $\operatorname{rk} S\le 2$.** By classification in [19] and also in [20], [22] (see the beginning of Sect. 2.3 and Figure 1), there are the following and only the following possibilities for S with $r(S)=\operatorname{rk} S\le 2$. We have

$$(r(S),a(S),\delta(S))=(1,1,1),\ (2,2,0),\ (2,2,1),\ (2,0,0).$$

We order these cases according to the natural ordering of the corresponding quotients $Y=X/\{1,\tau\}$, see below.

We consider all possible θ for these cases.

The case: $(r(S),a(S),\delta(S))=(1,1,1)$. Then $S\cong\langle 2\rangle$. Since S_- should be hyperbolic, $S_-=S$ and $\theta=-1$ on S. Then $S_+=\{0\}$. We consider this case in Sect. 3 below. For this case $Y=X/\{1,\tau\}=\mathbb{P}^2$.

The case $(r(S),a(S),\delta(S))=(2,2,0)$. Then $S\cong U(2)$. We consider this case in Sects. 4 and 5. By Proposition 4, we have $Y=\mathbb{P}^1\times\mathbb{P}^1$ in the non-degenerate case, which we are considering in this paper (in the degenerate case $Y=\mathbb{F}_2$). Let us consider possible θ. Dividing form of S by 2, we get the unimodular lattice U. It follows that if $\operatorname{rk} S_-=1$, then $S_-\cong\langle 4\rangle$ and $S_+\cong\langle -4\rangle$. If $\operatorname{rk} S_-=2$, then $\theta=-1$ on S. Thus, we get two cases

The case $S \cong U(2)$, the involution θ is -1 on S. We consider this case in Sect. 4. Then $Y = X/\{1,\tau\} = \mathbb{P}^1 \times \mathbb{P}^1$ in the non-degenerate case which we consider. Moreover, $Y = X/\{1,\tau\} = \mathbb{P}^1 \times \mathbb{P}^1$ over $\mathbb{R}$ is a hyperboloid, if $Y(\mathbb{R}) \neq \emptyset$.

The case $S \cong U(2)$, $S_- \cong \langle 4\rangle$, $S_+ \cong \langle -4\rangle$. We consider this case in Sect. 5. For this case, $Y = X/\{1,\tau\} = \mathbb{P}^1 \times \mathbb{P}^1$ and Y over $\mathbb{R}$ is an ellipsoid. Really, if $Y = \mathbb{F}_2$, then any anti-holomorphic involution of Y acts as -1 in $H^2(Y,\mathbb{R}) = \mathbb{R}^2$, and then $\theta = -1$ in $S \otimes \mathbb{R}$.

The case $(r(S), a(S), \delta(S)) = (2, 2, 1)$. Then $S \cong \langle 2\rangle \oplus \langle -2\rangle$. Assume that $\operatorname{rk} S_- = 1$. Since the lattice $S(2^{-1}) \cong \langle 1\rangle \oplus \langle -1\rangle$ is unimodular and odd, it follows that $S_- = \langle 2\rangle$ and $S_+ \cong \langle -2\rangle$. Again the lattice S_+ has elements with square -2 which is impossible for θ. Thus, $\theta = -1$ on S and $S_+ = \{0\}$. We consider this case in Sect. 6. For this case $Y = X/\{1,\tau\} = \mathbb{F}_1$.

The case $(r(S), a(S), \delta(S)) = (2, 0, 0)$. Then $S \cong U$ where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since S_- is hyperbolic, $\operatorname{rk} S_- = 1$ or 2. Let $\operatorname{rk} S_- = 1$. Since S is unimodular and even, $S_- \cong \langle 2\rangle$ and $S_+ \cong \langle -2\rangle$. Then S_+ has elements with square -2 which is impossible for θ. Thus, $\operatorname{rk} S_- = 2$ and $\theta = -1$ on S. Then $S_- = S$ and $S_+ = \{0\}$. We consider this case in Sect. 7 below. For this case, $Y = X/\{1,\tau\} = \mathbb{F}_4$.

3. Connected components of moduli of real non-singular curves of degree 6 in $\mathbb{P}^2$

Here we consider the case $(r(S), a(S), \delta(S)) = (1, 1, 1)$. Then $S \cong \langle 2\rangle$. The involution $\theta = -1$ on S. Since S has no elements with square -4, the group G is trivial.

For this case, $Y = X/\{1,\tau\}$ is a rational surface with Picard number $\operatorname{rk} S = 1$. It follows that $Y \cong \mathbb{P}^2$. Then $|-2K_Y|$ consists of curves of degree 6. Thus, X is a double covering of $\mathbb{P}^2$ ramified in a non-singular curve of degree 6 and τ is the involution of the double covering. The lattice $S = \mathbb{Z}h$ where h is the preimage of a line in $\mathbb{P}^2$, evidently $h^2 = 2$.

The $\mathbb{P}^2$ has only one real structure which is defined by a homogeneous real coordinates $(x_0 : x_1 : x_2)$ in $\mathbb{P}^2$. Then $\mathbb{P}^2(\mathbb{R}) = \mathbb{RP}^2$ is the real projective plane. A curve A is the zero set $P = 0$ of a real homogeneous polynomial $P(x_0, x_1, x_2)$ of degree 6. A positive curve A^+ is equivalent to a half $P \geq 0$ of the $\mathbb{RP}^2$. It defines the real double covering (X, τ, φ) of $\mathbb{P}^2$ ramified along A by the condition that A^+ is the image of the real part $X_\varphi(\mathbb{R})$. A curve A is defined by the polynomial P up to λP where $\lambda \in \mathbb{R}^*$. A positive curve A^+ is defined by the polynomial P up to λP where $\lambda \in \mathbb{R}_{++}$. Here $\mathbb{R}_{++}$ is the set of positive real numbers.

Classification of non-singular A and A^+ of degree 6 is well-known. The isotopy classification was obtained by Gudkov [6]. Classification of connected components of moduli was obtained in [19]. See also Arnol'd [3] and Rokhlin [25] for the geometric interpretation of invariants. The last classification (for positive curves) is equivalent to description of connected components of

$$\left(\left(\mathbb{R}^{28} - Discr\right)/\mathbb{R}_{++}\right)/PGL(3,\mathbb{R})$$

where $\mathbb{R}^{28}$ is the space of real homogeneous polynomials $P(x_0, x_1, x_2)$ of degree 6, and $Discr$ is the subspace of singular polynomials, defining singular curves $P = 0$. Since both groups $\mathbb{R}_{++}$ and $PGL(3,\mathbb{R})$ are connected, this classification is equivalent to the classification of connected components of $\mathbb{R}^{28} - Discr$ which is called *the rigid isotopy classification* due to V.A. Rokhlin [25].

For completeness, we outline general arguments of Sect. 2, in this case.

We have $S_+ = 0$, $S_- = S$, $H_+ = 0$, $\rho = 0$. The invariants $s = 1$, $p = 0$, $l(A_{S_+}) = 0$, $l(A_{S_-}) = 1$. The group $H = H_- \subset \mathbb{Z}h/2\mathbb{Z}h$. We get two possibilities for the group H.

(1) $H=0$ (equivalently, $\delta_h=1$). Then $a_H=0$, $\delta_H=0$, $k_\rho=0$, $\mu_\rho=0$, $\sigma_\rho\equiv 0 \pmod 8$. If $\delta_{\varphi S}=0$, then $v=0$ (equivalently, $\delta_\varphi=0$), $c_v\equiv 0 \pmod 4$, $\delta_{\varphi S_+}=0$, $\delta_{\varphi S_-}=1$, $\varepsilon_{v_+}\equiv 0 \pmod 2$.

(2) $H=[h]=\{0,h\} \mod 2S$ (equivalently, $\delta_h=0$). Then $a_H=1$, $\delta_H=1$, $k_\rho=0$, $\mu_\rho=0$, $\sigma_\rho\equiv -1 \pmod 8$. If $\delta_{\varphi S}=0$, then $v=h \mod 2S$ (equivalently, $\delta_{\varphi h}=0$, see [19]), $c_v\equiv -1 \pmod 4$, $\delta_{\varphi S_+}=1$, $\delta_{\varphi S_-}=0$, $\varepsilon_{v_-}\equiv 0 \pmod 2$.

In particular, from these relations, $a_{H_+}=0$, $a_H=a_{H_-}=1-\delta_h$.

Since $S_\pm$ are 2-elementary, by Proposition 15, Theorem 14 and results of Sect. 1, we get that *connected components of moduli of (X,τ,φ) (or A^+) are in one-to-one correspondence with the genus invariants*

$$(r,a,\delta_h,\delta_{\varphi S},v)=(r(\varphi),a(\varphi),\delta(\varphi)_h,\delta_{\varphi S},v(\varphi)). \tag{3.1}$$

The complete list of the invariants (where $r,a(\geq 0)\in\mathbb{Z}$, $\delta_h,\delta_{\varphi S}\in\{0,1\}$, the v is defined only if $\delta_{\varphi S}=0$) is given by the conditions:

Inequalities and general relations:
$1\leq r\leq 20$; $1-\delta_h+\delta_{\varphi S}\leq a$; $a\leq r$; $r+a\leq 22-2\delta_h$.
$r+a\equiv 0 \pmod 2$;
if $\delta_h=1$ and $\delta_{\varphi S}=0$, then $v=0 \pmod{2S}$ and $2-r\equiv 0 \pmod 4$;
if $\delta_h=0$ and $\delta_{\varphi S}=0$, then $v=h \pmod{2S}$ and $2-r\equiv -1 \pmod 4$.
Boundary conditions:
If $a=1-\delta_h$ and $v=0 \pmod{2S}$, then $2-r\equiv 0 \pmod 8$;
If $a=1-\delta_h$ and $v=h \pmod{2S}$, then $2-r\equiv -1 \pmod 8$;
If $a=2-\delta_h$ and $\delta_{\varphi S}=1$, then $2-r\equiv -1+\delta_h\pm 1 \pmod 8$;
If $a=3-\delta_h$ and $\delta_{\varphi S}=1$, then $2-r\not\equiv -1+\delta_h+4 \pmod 8$.
If $a=r$, $\delta_{\varphi S}=0$ and $v=0$, then $2-r\equiv 0 \mod 8$.
If $r+a=22-2\delta_h$, $\delta_{\varphi S}=0$ and $v=h \pmod{2S}$, then $2-r\equiv -1 \pmod 8$.

All possible invariants (3.1) satisfying these conditions are given in Figures 2 and 3.

The relations between related involutions are:

$$r(\varphi)+r(\tau\varphi)=21;\ a(\tau\varphi)-a(\varphi)=2\delta(\varphi)_h-1;\ \delta(\varphi)_h+\delta(\tau\varphi)_h=1; \tag{3.2}$$

$$\delta_{\varphi S}=\delta_{\tau\varphi S};\ v(\varphi)+v(\tau\varphi)\equiv h \pmod{2S},\ \text{if } \delta_{\varphi S}=\delta_{\tau\varphi S}=0. \tag{3.3}$$

Because of the relation $\delta(\varphi)_h+\delta(\tau\varphi)_h=1$, it is sufficient to describe only involutions with $\delta(\varphi)_h=0$ or with $\delta(\varphi)_h=1$.

It follows that there are 49 connected components of moduli of positive curves with $\delta_{\varphi S}=1$, and 15 with $\delta_{\varphi S}=0$ (for the fixed $\delta_h=1$ or 0). Thus, there are 128 connected components of moduli of positive curves (or involutions (X,τ,φ)) and 64 connected components of moduli of real non-singular curves of degree 6 on the real projective plane.

According to Arnol'd [3], the invariant $\delta_h=0$, if $A^+\neq\emptyset$ and the positive curve A^+ is orientable. Otherwise, $\delta_h=1$. Using this, Bezout Theorem and (2.44), (2.45), (3.2), (3.3), one can draw the corresponding pictures. It is enough to draw them for $\delta_h=0$ when the positive curve is orientable. We have

If $(\delta_h;r,a,\delta_{\varphi S})=(0;11,11,0)$, the positive curve is the whole $\mathbb{RP}^2$.

If $(\delta_h;r,a,\delta_{\varphi S})=(0;11,9,0)$ the positive curve is drawn in Figure 4.

If $\delta_h=0$ and $(r,a,\delta_{\varphi S})$ is different from $(11,11,0)$ and $(11,9,0)$ the positive curve is drawn in Figure 5 where $g=(22-r-a)/2$ and $k=(r-a)/2$.

The invariant $\delta_{\varphi S}=0$, if and only if the real curve $A(\mathbb{R})$ divides $A(\mathbb{C})$. See [3] and [25].

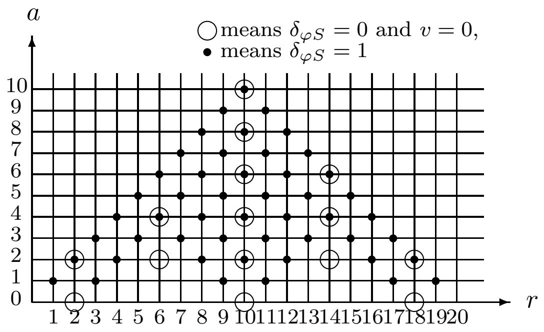

FIGURE 2. $\mathbb{P}^2$: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = 0$ (i. e. $\delta_h = 1$)

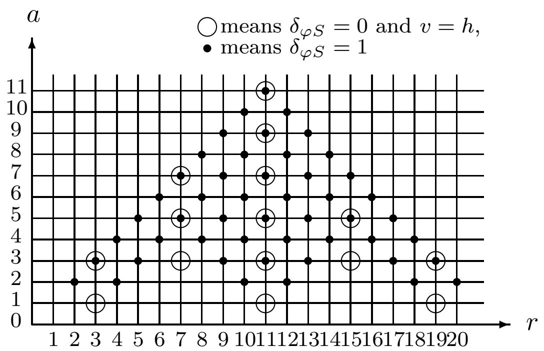

FIGURE 3. $\mathbb{P}^2$: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = [h]$ (i. e. $\delta_h = 0$)

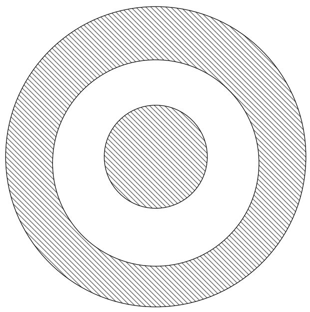

FIGURE 4. $\mathbb{P}^2$: A^+ for $H = [h]$ and $(r, a, \delta_{\varphi S}, v) = (11, 9, 0, h)$

More exactly, by [3] the invariant $\delta_{\varphi S} = 0$, if the curve $A(\mathbb{R})$ divides $A(\mathbb{C})$. By the results above, connected component of moduli is defined by the isotopy type and by the invariant $\delta_{\varphi S}$. Thus, to prove the opposite statement, it is sufficient to construct a dividing curve on the plane for each isotopy type with $\delta_{\varphi S} = 0$. This construction is explained in [25].

The connected component of moduli of a positive non-singular real curve A of the degree 6 on the real projective plane is defined by its isotopy type and by dividiness of $A(\mathbb{C})$ by $A(\mathbb{R})$. All possibilities are described above.

Applying results of Sect. 2, below we get similar results for all lattices S of $\operatorname{rk} S = 2$.

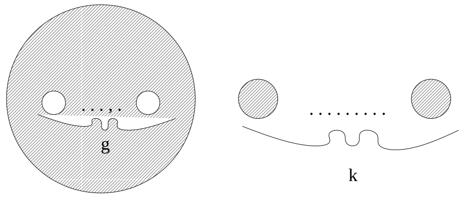

FIGURE 5. $\mathbb{P}^2$: A^+ for $H=[h]$ and $(r,a,\delta_{\varphi S},v)\neq(11,9,0,h)$, $(11,11,0,h)$.

4. Connected components of moduli of real non-singular curves of bidegree $(4,4)$ on hyperboloid

Here we consider the case $S\cong U(2)$ and $\theta=-1$ on S. Then $Y=X/\{1,\tau\}=\mathbb{P}^1\times\mathbb{P}^1$ in the non-degenerate case which we only consider, and $A=X^\tau$ is a non-singular curve of bidegree $(4,4)$ in Y.

Let $S=\mathbb{Z}e_1+\mathbb{Z}e_2$ where $e_1^2=e_2^2=0$ and $(e_1,e_2)=2$. The generators e_1 and e_2 of $S=H_2(X;\mathbb{Z})^\tau$ are classes of preimages of pt $\times\,\mathbb{P}^1$ and $\mathbb{P}^1\times$ pt respectively.

Since $\theta=-1$ on S, then the real structure on Y (defined by the anti-holomorphic involution φ of X) is one of the four structures: (usual, usual) = *hyperboloid*, (usual, spin), (spin, usual) and (spin, spin), where *usual* means the usual real structure on $\mathbb{P}^1$, which is given by the anti-holomoprhic involution $(z_0:z_1)\mapsto(\overline{z_0}:\overline{z_1})$ in some homogeneous coordinates $(z_0:z_1)$ on $\mathbb{P}^1$, while *spin* is given by $(z_0:z_1)\mapsto(\overline{z_1}:-\overline{z_0})$. In hyperpoloid case $Y(\mathbb{R})=\mathbb{RP}^1\times\mathbb{RP}^1$ is a 2-torus. In all three spin cases $Y(\mathbb{R})$ and $X(\mathbb{R})$ are empty, and $A(\mathbb{R})$ is also empty.

If Y has the usual real structure (i.e. it is a hyperboloid), then A is the zero set of a real bi-homogeneous polynomial $P(x_0:x_1;y_0:y_1)$ of bidegree $(4,4)$. For a positive curve (A,φ), equivalently for $A^+=\pi(X_\varphi(\mathbb{R}))$, like for real non-singular curves of degree 6 in $\mathbb{P}^2$, we can choose P by the condition that $A=\{P=0\}$ in $\mathbb{P}^1\times\mathbb{P}^1$ and $A^+=\{P\geq 0\}$ on $\mathbb{RP}^1\times\mathbb{RP}^1$. The polynomial P is defined up to $\mathbb{R}_{++}\times((PGL(2,\mathbb{R})\times PGL(2,\mathbb{R}))\rtimes\mathbb{Z}/2\mathbb{Z})$. Here $\mathbb{R}_{++}$ denote all positive real numbers. Thus, classification of connected components of moduli of positive curves A^+ on a hyperboloid is equivalent to the description of connected components of

$$\left((\mathbb{R}^{25}-Discr)/\mathbb{R}_{++}\right)/\left((PGL(2,\mathbb{R})\times PGL(2,\mathbb{R}))\rtimes\mathbb{Z}/2\mathbb{Z}\right)$$

where the discriminant $Discr$ is the set of all polynomials giving singular (over $\mathbb{C}$) curves. The group $(PGL(2,\mathbb{R})\times PGL(2,\mathbb{R}))\rtimes\mathbb{Z}/2\mathbb{Z}$ has 8 connected components and has dimension 6 over $\mathbb{R}$.

If Y has one of spin structures, then $Discr$ has codimension two. Hence the moduli of positive curves and the moduli of curves are connected. (We will also give another proof of that below.)

Now we outline general arguments of Subsect. 2.3 in the $(S,\theta)=(U(2),-1)$ case (for finding connected components of moduli).

Since $\Delta(S_+,S)^{(-4)}=\emptyset$ and $\Delta(S_-,S)^{(-4)}=\{\pm(e_1-e_2)\}$, the group $W^{(-4)}(S_+,S)$ is trivial, and the group $W^{(-4)}(S_-,S)$ consists of identity and g where $g(e_1)=e_2$ and $g(e_2)=e_1$. By Lemma 9, we have $G=\{\text{identity},\ g\}\cong\mathbb{Z}/2\mathbb{Z}$.

We have $s=2$, $p=0$. $A_{S_-}=S_-^*/S_-=S^*/S$ is generated by $e_1^*=\frac{1}{2}e_2$ and $e_2^*=\frac{1}{2}e_1$, and hence it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and $l(A_{S_-})=2$. We have $A_{S_+}=0$, $H_+=0$, $\rho=0$. Since $\Gamma_-=0$, $H=H_-$ $(\subset S/2S)$ is one of the following 5 subgroups;

$$0,\ [e_1],\ [e_2],\ [h]=[e_1+e_2],\ S/2S=[e_1,e_2],$$

where we set $h = e_1 + e_2$ and consider e_1, e_2 and h mod $2S$. Since $H_+ = 0$, we have $q_\rho = (-q_{S_-})|H_-$. Since $\delta(S) = 0$, we always have $\delta_H = 0$ and $\delta_{\varphi S_\pm} = \delta_\varphi$. If $\delta_\varphi = 0$, then $\varepsilon_{v_\pm} \equiv 0 \pmod 2$.

Thus, we have the following cases:

(1) $H = 0$: Then $a_H = 0$, $k_\rho = 0$, $\mu_\rho = 0$, $\sigma_\rho \equiv 0 \pmod 8$. If $\delta_{\varphi S} = 0$, then $v = 0$, equivalently, $\delta_\varphi = 0$. We have $c_v \equiv 0 \pmod 4$, $\delta_{\varphi S_\pm} = \delta_\varphi = 0$.

(2) $H = [e_1]$: Then $a_H = 1$, $q_\rho \cong z$, $k_\rho = 1$, $\mu_\rho = 0$, $\sigma_\rho \equiv 0 \pmod 8$.

(3) $H = [e_2]$: Then $a_H = 1$, $q_\rho \cong z$, $k_\rho = 1$, $\mu_\rho = 0$, $\sigma_\rho \equiv 0 \pmod 8$.

(4) $H = [h] = [e_1 + e_2]$: Then $a_H = 1$, $q_\rho \cong w$, $k_\rho = 1$, $\mu_\rho = 1$.

(5) $H = S/2S = [e_1, e_2]$: Then $a_H = 2$, $q_\rho \cong u_+(2)$, $k_\rho = 0$, $\mu_\rho = 0$, $\sigma_\rho \equiv 0 \pmod 8$. If $\delta_{\varphi S} = 0$, then $v = 0$, and $\delta_{\varphi S} = 0$ is equivalent to $\delta_\varphi = 0$.

From above, $\mu_\rho = 1$ if and only if $H = [h]$. If $\mu_\rho = 0$, then $\sigma_\rho \equiv 0 \pmod 8$.

We see that genus (hence, by Theorem 14 and Proposition 15, the isomorphism class) of an integral involutions (L, φ, S) of the type $(U(2), -1)$ satisfying (RSK3) is determined by the data $(r, a, H, \delta_{\varphi S}, v)$, where the v is defined only if $\delta_{\varphi S} = 0$.

By Theorem 10, the complete list of the above invariants is given by the conditions:

Type 0 $(\delta_\varphi = 0)$
$a_H + k_\rho \le a \le r$ and $a \equiv 0 \pmod 2$.
$2 \le r \le 18$ and $2 - r \equiv 0 \pmod 4$.
$r + a \le 2a_H + 18$.
If $a = a_H + k_\rho,\ \mu_\rho = 0$, then $2 - r \equiv 0 \pmod 8$.
If $a = r$, then $2 - r \equiv 0 \pmod 8$.
If $r + a = 2a_H + 18$, then $2 - r \equiv 0 \pmod 8$.

Type Ia $(\delta_{\varphi S} = 0,\ \delta_\varphi = 1)$
$a_H = 1$, $k_\rho = 1$.
$c_v \equiv 0 \pmod 4$ if $\mu_\rho = 0$, and $c_v \equiv 2 \pmod 4$ if $\mu_\rho = 1$.
$2 \le a \le r$ and $a \equiv 0 \pmod 2$.
$2 \le r \le 18$ and $2 - r \equiv c_v \pmod 4$.
$r + a \le 20$.
If $a = 2,\ \mu_\rho = 0$, then $2 - r \equiv 0 \pmod 8$.

Type Ib $(\delta_{\varphi S} = 1)$
$a_H + k_\rho + 1 \le a \le r$,
$1 \le r \le 19$,
$r + a \equiv 0 \pmod 2$, $r + a \le 2a_H + 18$.
If $a = a_H + k_\rho + 1,\ \mu_\rho = 0$, then $2 - r \equiv \pm 1 \pmod 8$.
If $a = a_H + k_\rho + 2,\ \mu_\rho = 0$, then $2 - r \not\equiv 4 \pmod 8$.

We see $a_H = 1$ in Type Ia case. Then H is generated by the unique nonzero element, hence it is nothing but v. Hence we can use the invariant δ_φ instead of v in the collection of invariants above. Thus the isomorphism class is determined by the data

$$(r, a, H, \delta_{\varphi S}, \delta_\varphi).$$

Since $G = \{\text{identity},\ g\}$, the triplets $(r, a, [e_1])$ and $(r, a, [e_2])$ represent the same isomorphism class for each Type (0, Ia and Ib). The other different triplets represent different isomorphism classes.

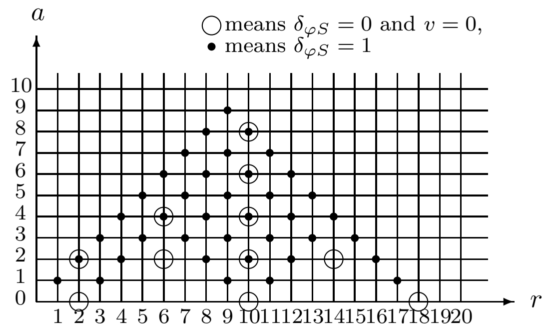

FIGURE 6. $\mathbb{H}$: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = 0$

All possible data

$$(r, a, H, \delta_{\varphi S}, v) \tag{4.1}$$

(which are equivalent to $(r, a, H, \delta_{\varphi S}, \delta_\varphi)$) are given in Figures 6 — 9. We have $\delta_\varphi = 0$, if and only if $\delta_{\varphi S} = v = 0$.

The relations between related involutions are as follows. Since $\delta(S) = 0$ and $\theta = -1$ on S, then $s_\theta = 0 \mod 2L$. Hence, by Theorem 11, we have:

$$r(\varphi) + r(\tau\varphi) = 20; \quad a_{H(\varphi)} + a_{H(\tau\varphi)} = 2; \quad a(\varphi) - a_{H(\varphi)} = a(\tau\varphi) - a_{H(\tau\varphi)}; \tag{4.2}$$

$$H(\tau\varphi) = H(\varphi)^\perp \text{ w. r. t. } b_{S_-} \text{ on } A_{S_-}; \quad \delta_{\varphi S} = \delta_{\tau\varphi S}; \quad s_\varphi \equiv s_{\tau\varphi} \mod 2L. \tag{4.3}$$

Hence, involutions of Type 0 (resp. Type Ib) with $H = 0$ (11 (resp. 39) classes) are related to involutions of Type 0 (resp. Type Ib) with $H = S/2S$. Involutions of Type 0 (resp. Type Ia, Type Ib) with $H = [e_i]$ (9 (resp. 11, 30) classes) are related to involutions of Type 0 (resp. Type Ia, Type Ib) with $H = [e_i]$. (More precisely, the class (r, a) is related to the class $(20 - r, a)$.) Involutions of Type 0 (resp. Type Ia, Type Ib) with $H = [h]$ (11 (resp. 12, 36) classes) are related to involutions of Type 0 (resp. Type Ia, Type Ib) with $H = [h]$. (More precisely, the class (r, a) is related to the class with $(20 - r, a)$, too.)

Thus there are $42(= 11 \times 2 + 9 + 11)$ classes (i.e., connected components of moduli of positives curves of bidegree $(4, 4)$) of Type 0, $23(= 11 + 12)$ classes of Type Ia, and $144(= 39 \times 2 + 30 + 36)$ classes of Type Ib.

Moreover, if we identify related involutions, there are $26(= 11 + 7 + 8)$ classes (i.e., connected components of moduli of real non-singular curves of bidegree $(4, 4)$) of Type 0, $14(= 8 + 6)$ classes of Type Ia, and $76(= 39 + 17 + 20)$ classes of Type Ib.

If two positive curves are in one connected component of moduli or are related, the real structure on Y stays the same. By (2.44) and (2.45), the A^+ is empty, if and only if $(r, a, \delta_\varphi) = (10, 10, 0)$. It follows that the component $(r, a, H, \delta_{\varphi S}, \delta_\varphi) = (10, 10, [h], 0, 0)$ corresponds to the real structure (spin, spin), the component $(10, 10, [e_i], 0, 0)$ corresponds to the real structure (usual, spin) (or (spin, usual)). And all the remaining components correspond to the real structure (usual, usual) (namely, hyperboloid). The component $(10, 10, S/2S, 0, 0)$ consists of empty A^+ on hyperboloid.

The isotopy classification of real non-singular curves of bidegree $(4, 4)$ on a hyperboloid was obtained by Gudkov [7]. Zvonilov [35] clarified all the *complex schemes* of curves of bidegree $(4, 4)$ on hyperboloid and ellipsoid, where *complex schemes* mean real schemes (i.e. real isotopy types) with divideness and their complex orientations (if dividing). Curves on hyperboloid and ellipsoid were studied in [33], [34], [15], [16], [17]. In these articles the

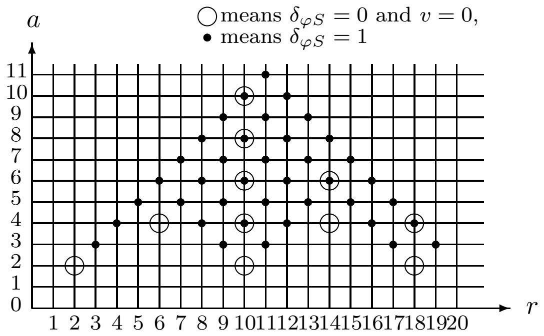

FIGURE 7. $\mathbb{H}$: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = [e_1, e_2] = S/2S$

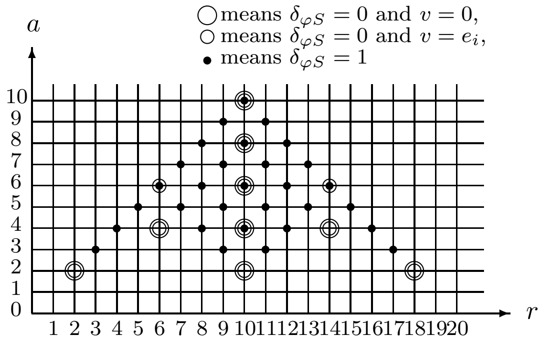

FIGURE 8. $\mathbb{H}$: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = [e_i]$ $(i = 1 \text{ or } 2)$

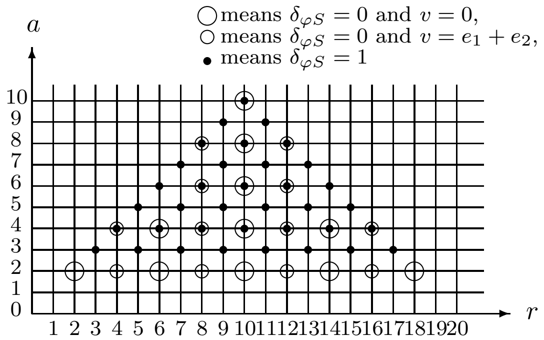

FIGURE 9. $\mathbb{H}$: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = [e_1 + e_2] = [h]$

notions: *torsion* (s, t) $(\in \mathbb{Z} \times \mathbb{Z})$ of a connected component of $A(\mathbb{R})$, *oval*, *non-oval*, *odd branch* and *even branch*, are well-known. We quote the isotopy classification of non-singular curves of bidegree $(4, 4)$ on hyperboloid from [35] in TABLE 1 where we use notations due to Viro [31] and Zvonilov [35], and l_1 and l_2 denote non-ovals with torsions $(1, 0)$ and $(0, 1)$ respectively (e. g. see Figure 15).

If a curve $A(\mathbb{R})$ on hyperboloid has only ovals or $A(\mathbb{R}) = \emptyset$, then one of A^+ and A^- contains the outermost component. Thus we divide isotopy types of positive curves A^+ into the following 4 cases:

$\langle 1\langle m\rangle \sqcup n\rangle$, where $(m,n)=(9,0),\ (5,4),\ (1,8),\ (8,0),\ (5,3),\ (4,4),\ (1,7)$; or $m\geq 1,\ n\geq 0$ and $m+n\leq 7$.
$\langle m\rangle$, where $0\leq m\leq 9$.
$\langle 1\langle 1\rangle \sqcup 1\langle 1\rangle\rangle$
$\langle l_1+l_2,\ m,\ l_1+l_2,\ n\rangle$, where $0\leq m\leq n$ and $m+n\leq 8$.
$\langle 4(l_1+l_2)\rangle$
$\langle l_1,\ m,\ l_1,\ n\rangle$, where $(m,n)=(0,8),\ (4,4),\ (0,7),\ (3,4)$; or $0\leq m\leq n$ and $m+n\leq 6$.
$\langle 4(l_1)\rangle$
$\langle 2(l_1+2l_2)\rangle$

TABLE 1. The list of isotopy types of real non-singular curves of bidegree $(4,4)$ on a hyperboloid.

(i): $A(\mathbb{R})$ has only ovals or $A(\mathbb{R})=\emptyset$, and A^+ contains the outermost component. (In this case we say A^+ is *outer*.)
(ii): $A(\mathbb{R})$ has only ovals or $A(\mathbb{R})=\emptyset$, and A^+ does not contain the outermost component. (In this case we say A^+ is *inner*.) E. g. see Figures 10, 11.
(iii): $A(\mathbb{R})$ has even branches. E. g. see Figures 12, 13
(iv): $A(\mathbb{R})$ has odd branches. E. g. see Figures 14, 15.

The subgroup $H=H_-$ is determined by the invariants δ_{e_1}, δ_{e_2} and δ_h. We have the following interpretation of the invariant δ_{x_-}.

Theorem 16. *Let X be a compact Kähler surface with an anti-holomorphic involution φ, and $X(\mathbb{R})$ be the fixed point set of φ. We assume that $H_1(X;\mathbb{Z})=0$ and $X(\mathbb{R})\neq\emptyset$. We set $L=H_2(X;\mathbb{Z})$ and $L_\varphi=\{x\in L \mid \varphi_*(x)=-x\}$. Let C be a 1-dimensional complex submanifold of X with $\varphi(C)=C$, and $C(\mathbb{R})$ be the fixed point set of φ on C. Let $[C]$ and $[C(\mathbb{R})]$ denote the homology classes represented by C and $C(\mathbb{R})$ respectively. We set $x_-:=[C]\ \ (\in L_\varphi)$. Then we have*

$$\delta_{x_-}=0 \quad\Longleftrightarrow\quad [C(\mathbb{R})]=0 \ \text{ in } H_1(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z}),$$

where we define the symbol δ_{x_-} in the same way as δ_{x_-} in Subsect. 2.3.

Proof. Suppose that $[C(\mathbb{R})]\neq 0$ in $H_1(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z})$. Then there exists an embedded circle D on $X(\mathbb{R})$ such that $[C(\mathbb{R})]\cdot[D]\neq 0$, where $[C(\mathbb{R})]\cdot[D]$ means the $\mathbb{Z}/2\mathbb{Z}$-intersection number in $X(\mathbb{R})$. We set $E_-=\{x\in H^2(X;\mathbb{Z}) \mid \varphi^*(x)=-x\}$. E_- is the Poincaré dual to L_φ. By Théorème 2.4 in [14], there exists a surjective canonical morphism

$$\varphi_X: E_-\to H^1(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z})$$

such that

$$(\varphi_X(\gamma),\varphi_X(\gamma'))\equiv Q(\gamma,\gamma') \pmod 2\ \ \forall\gamma,\gamma'\in E_-,$$

where (,) and Q are the forms induced by the cup products on $H^1(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z})$ and $H^2(X;\mathbb{Z})$. Moreover, by Théorème 2.5 in [14], the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Pic}(X)^G & \xrightarrow{c_1} & E_- \\ \alpha \downarrow & & \varphi_X \downarrow \\ H^1_{\mathrm{alg}}(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z}) & \xrightarrow{i} & H^1(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z}), \end{array}$$

where α is defined as in [14],p.562 (see also below), c_1 is the first Chern class map, and i is the inclusion map. In the sequel, A^P denotes the Poincaré dual element to a (co)homology class A. Since φ_X is surjective, there exists γ ($\in E_-$) such that $\varphi_X(\gamma) = [D]^P$ ($\in H^1(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z})$). We consider the *divisor class* $[C]$ in $\mathrm{Pic}(X)^G$. Then, as is well known, its first Chern class $c_1([C])$ is the Poincaré dual to x_-. By the definition (see [14]) of α, we see

$$\alpha([C]) = \eta(C) = [C(\mathbb{R})]^P \ (\in H^1_{\mathrm{alg}}(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z})).$$

On the other hand, by the above commutative diagram, we see

$$\alpha([C]) = \varphi_X(c_1([C])) = \varphi_X(x_-^P).$$

Hence, we have $[C(\mathbb{R})] \cdot [D] = ([C(\mathbb{R})]^P, [D]^P) = (\varphi_X(x_-^P), \varphi_X(\gamma)) \equiv Q(x_-^P, \gamma) \pmod 2 = (x_-, \gamma^P)$. Thus we have $\delta_{x_-} = 1$. This completes the proof of the implication $\Longrightarrow$. The converse assertion can be proved by the same argument as the proof of Lemma 2 in [17]. Suppose that $[C(\mathbb{R})] = 0$ in $H_1(X(\mathbb{R});\mathbb{Z}/2\mathbb{Z})$. Then $X(\mathbb{R})$ and $C(\mathbb{R})$ satisfy the conditions a) and b) of Remark 2.2 in [11]. By that remark, Lemma 2.3 is applicable to the involution $\varphi : X \to X$ and C. Hence, we see $(x_-)_{\mathrm{mod}\ 2}$ ($\in H_2(X;\mathbb{Z}/2\mathbb{Z})$) is orthogonal to $\mathrm{Im}\alpha_2$. Since $H_1(X;\mathbb{Z}) = 0$, as in the proof of Lemma 3.7 in [12], we have $\mathrm{Im}\alpha_2 = \{x \in H_2(X;\mathbb{Z}/2\mathbb{Z}) \mid \varphi_*(x) = x\}$. Thus we have $\delta_{x_-} = 0$. $\square$

By Theorem 16, we have $H = 0$ if and only if A^+ is outer, $H = S/2S$ if and only if A^+ is inner, $H = [h]$ if and only if $A(\mathbb{R})$ has even branches, $H = [e_1]$ if and only if $A(\mathbb{R})$ has odd branches with odd s, and $H = [e_2]$ if and only if $A(\mathbb{R})$ has odd branches with odd t.

When A is a dividing curve on hyperboloid with non-ovals, we define the number $\hat{l}$ ($\in \mathbb{Z}/2\mathbb{Z}$) as follows (see [15]): Fix a complex orientation of $A(\mathbb{R})$. When the number of non-ovals of $A(\mathbb{R})$ is 2, we define $\hat{l} = 0$ if the complex orientations of the 2 non-ovals are different in $\mathbb{RP}^1 \times \mathbb{RP}^1$, $\hat{l} = 1$ if otherwise. When the number of non-ovals of $A(\mathbb{R})$ is 4, we fix a non-oval E and define $\hat{l} = 0$ if the number of non-ovals whose complex orientations are the same as E in $\mathbb{RP}^1 \times \mathbb{RP}^1$ is even, $\hat{l} = 1$ if odd.

We quote the following lemma.

Lemma 17 ([15])**.** *For a positive curve (A, φ) of bidegree $(4,4)$ on hyperboloid, we have the following:*
(1) $[X_\varphi(\mathbb{R})] = [X_{\tau\varphi}(\mathbb{R})]$ in $H_2(X;\mathbb{Z}/2\mathbb{Z})$
(2) If A is dividing, then

$$[X_\varphi(\mathbb{R})] \ (= [X_{\tau\varphi}(\mathbb{R})]) = \begin{cases} 0 & (if\ \ A(\mathbb{R})\ has\ only\ ovals) \\ \hat{l}h & (if\ \ A(\mathbb{R})\ has\ even\ branches) \\ \hat{l}e_1 & (if\ \ A(\mathbb{R})\ has\ odd\ branches\ with\ odd\ s) \\ \hat{l}e_2 & (if\ \ A(\mathbb{R})\ has\ odd\ branches\ with\ odd\ t) \end{cases}$$

in $H_2(X;\mathbb{Z}/2\mathbb{Z})$. $\square$

Then we have the following:

Proposition 18. *Let A be a non-singular real curve of bidegree $(4,4)$ on hyperboloid. If A is dividing, then the positive curves (A, φ) satisfies $\delta_{\varphi S} = 0$.*

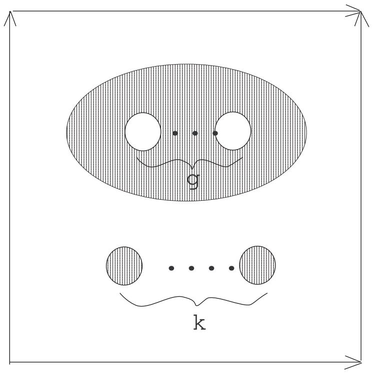

FIGURE 10. $\mathbb{H}$: A^+ for $H = S/2S = [e_1, e_2]$ with $(r, a, \delta_{\varphi S}, v) \neq (10, 10, 0, 0)$ nor $(10, 8, 0, 0)$.

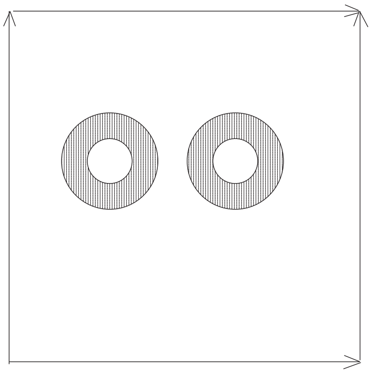

FIGURE 11. $\mathbb{H}$: A^+ for $H = S/2S = [e_1, e_2]$ with $(r, a, \delta_{\varphi S}, v) = (10, 8, 0, 0)$.

Proof. If A is dividing, then by (2) of Lemma 17, we have $[X_\varphi(\mathbb{R})] \equiv s_\varphi \pmod{2H_2(X;\mathbb{Z})}$ for some $s_\varphi \in S$. Hence we have $\delta_{\varphi S} = 0$. $\square$

Moreover, we have the following interpretation of v when A is a dividing curve with non-ovals on hyperboloid:
$v = 0$ if and only if $\hat{l} = 0$,
$v = h \pmod{2S}$ if and only if $A(\mathbb{R})$ has even branches and $\hat{l} = 1$,
$v = e_1 \pmod{2S}$ if and only if $A(\mathbb{R})$ has odd branches with odd s and $\hat{l} = 1$,
$v = e_2 \pmod{2S}$ if and only if $A(\mathbb{R})$ has odd branches with odd t and $\hat{l} = 1$.

By the above geometric interpretations and Zvonilov's classification [35], from the invariants (4.1) of the connected component of moduli of a positive curve A^+ on hyperboloid, we can uniquely determine the complex scheme of A. In particular, one can draw the picture of the positive curve A^+. By (4.2) and (4.3), it is enough to draw them for $H = S/2S$, $[h]$ and $[e_1]$. See Figures 10 — 15. Remind that $A(\mathbb{R})$ is an empty curve on hyperboloid if $H = S/2S$ and $(r, a, \delta_{\varphi S}, v) = (10, 10, 0, 0)$, Y has the (spin,spin) structure if $H = [h]$ and $(r, a, \delta_{\varphi S}, v) = (10, 10, 0, 0)$, and Y has the (usual,spin) (or (spin,usual)) structure if $H = [e_i]$ and $(r, a, \delta_{\varphi S}, v) = (10, 10, 0, 0)$.

Moreover, there is known (due to Zvonilov [35]) a dividing curve on hyperboloid for each connected component with $\delta_{\varphi S} = 0$. Hence, the opposite statement of Proposition 18 is true:

Theorem 19. *Let A be a non-singular real curve of bidegree $(4, 4)$ on hyperboloid. Then, A is dividing or $A(\mathbb{R}) = \emptyset$, if and only if the positive curves (A, φ) (or A^+) has $\delta_{\varphi S} = 0$.* $\square$

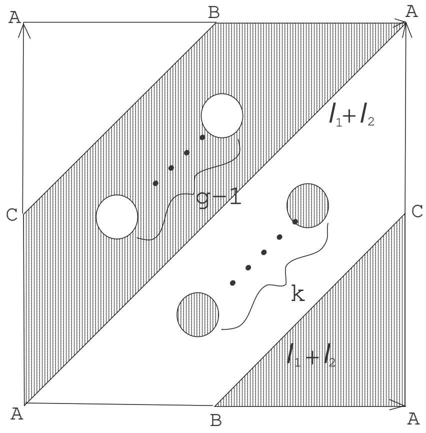

FIGURE 12. $\mathbb{H}$: A^+ for $H = [h] = [e_1 + e_2]$ with $(r, a, \delta_{\varphi S}, v) \neq (10, 10, 0, 0)$ nor $(10, 8, 0, 0)$.

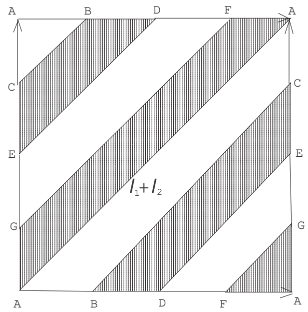

FIGURE 13. $\mathbb{H}$: A^+ for $H = [h] = [e_1 + e_2]$ with $(r, a, \delta_{\varphi S}, v) = (10, 8, 0, 0)$.

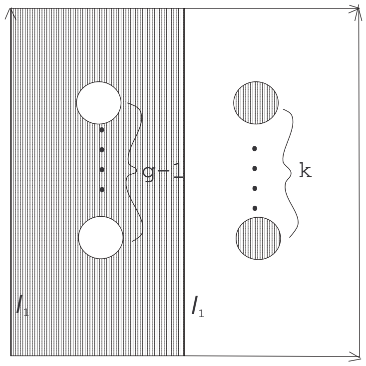

FIGURE 14. $\mathbb{H}$: A^+ for $H = [e_1]$ with $(r, a, \delta_{\varphi S}, v) \neq (10, 10, 0, 0)$ nor $(10, 8, 0, 0)$.

It turns out that if $\delta_{\varphi S} = 0$ and $H = [h]$, then (r, a) determines v. But when $\delta_{\varphi S} = 0$ and $H = [e_i]$, (r, a) does not always determine v. As stated above, when $\delta_{\varphi S} = 0$, $H = [e_i]$ and $(r, a, v) \neq (10, 10, 0)$, $v = 0$ if and only if $\hat{l} = 0$.

Thus, we finally get

Theorem 20. *A connected component of moduli of a positive real non-singular curve A^+ of bidegree $(4, 4)$ on hyperboloid is defined by the isotopy type of $A^+ \subset \mathbb{RP}^1 \times \mathbb{RP}^1$ (up to the action of $(PGL(2, \mathbb{R}) \times PGL(2, \mathbb{R})) \rtimes \mathbb{Z}/2\mathbb{Z}$), divideness of $A(\mathbb{C})$ by $A(\mathbb{R})$, and by*

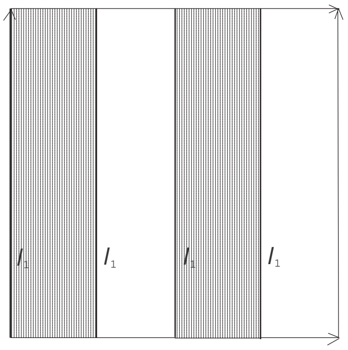

FIGURE 15. $\mathbb{H}$: A^+ for $H = [e_1]$ with $(r, a, \delta_{\varphi S}, v) = (10, 8, 0, 0)$.

the invariant $\hat{l} \mod 2$ *defined by the complex orientation (if* $A(\mathbb{R})$ *has odd branches and is dividing).*

5. Connected components of moduli of real non-singular curves of bidegree (4, 4) on ellipsoid

Here we consider the case $S \cong U(2)$ and $\operatorname{rk} S_- = 1$ (ellipsoid cases). Like for hyperboloid, $Y = \mathbb{P}^1 \times \mathbb{P}^1$, and we use the same notations for S. Obviously, $\theta(e_1) = -e_2$ and $\theta(e_2) = -e_1$, we have $S_+ = \mathbb{Z}(e_1 - e_2)$ and $S_- = \mathbb{Z}(e_1 + e_2)$. Thus $S_+ \cong \langle -4 \rangle$ and $S_- \cong \langle 4 \rangle$.

Since $\Delta(S_+, S)^{(-4)} = \{\pm(e_1 - e_2)\}$ and $\Delta(S_-, S)^{(-4)} = \emptyset$, $W^{(-4)}(S_+, S)$ consists of identity and g where $g(e_1) = e_2$ and $g(e_2) = e_1$, and $W^{(-4)}(S_-, S)$ is trivial. By Lemma 9, we have $G = \{\text{identity},\ g\} \cong \mathbb{Z}/2\mathbb{Z}$.

We call these data (S, θ, G) as *the ellipsoid case.*

For the ellipsoid case we have $s = 2$, $p = 1$. $A_{S_+} \cong \mathbb{Z}/4\mathbb{Z}$ is generated by $\frac{1}{4}(e_1 - e_2)$, and $A_{S_-} \cong \mathbb{Z}/4\mathbb{Z}$ is generated by $\frac{1}{4}(e_1 + e_2)$. We see $\Gamma_+ = [e_1 - e_2] = 2({S_+}^* \cap (\frac{1}{2}S_+))/2S_+$ and $\Gamma_- = [e_1 + e_2] = 2({S_-}^* \cap (\frac{1}{2}S_-))/2S_-$. Hence we have $H_\pm = \Gamma_\pm$. Since $(e_1 - e_2, 0) = (0, e_1 + e_2)$ in $H = H_+ \oplus_\gamma H_-$, we see

$$H = \{0,\ (e_1 - e_2, 0) = (0, e_1 + e_2)\} \cong \mathbb{Z}/2\mathbb{Z},$$

$a_H = 1$ and $q_\rho \cong w$. Hence $\delta_H = 0$, $k_\rho = \mu_\rho = 1$.

If $\delta_\varphi = 0$ (Type 0), then $c_v \equiv 0 \pmod 4$. If $\delta_{\varphi S} = 0$ and $\delta_\varphi = 1$ (Type Ia), then $v = (e_1 - e_2, 0) = (0, e_1 + e_2)$ and $c_v \equiv 2 \pmod 4$.

We have $\delta_{\varphi S} = \delta_{\varphi S_\pm}$. Assume that $\delta_{\varphi S} = 0$. Then

$$\gamma_\pm = \begin{cases} 0 & \text{if } \delta_\varphi = 0, \\ q_1(2)^2 & \text{if } \delta_{\varphi S} = 0 \text{ and } \delta_\varphi = 1. \end{cases}$$

If $\delta_\varphi = 0$, then $(q_{S_\pm})_v = q_{S_\pm}$. Hence we have $\varepsilon_{v_\pm} = 0$. If $\delta_{\varphi S} = 0$ and $\delta_\varphi = 1$, then $(q_{S_\pm})_v$ is defined on $\mathbb{Z}/4\mathbb{Z}$, and the square of a generator is $\pm\frac{5}{4} \pmod{2\mathbb{Z}}$. Hence $(q_{S_\pm})_v$ is the discriminant quadratic form of the 2-adic lattice $K^{(2)}_{\pm 5}(2^2) = \langle \pm 5 \cdot 2^2 \rangle$ ([19]). Thus we have $\varepsilon_{v_\pm} = 1$.

We see that the genus of an integral involutions (L, φ, S) is determined by the data

$$(r, a, \delta_{\varphi S}, \delta_\varphi).$$

By Theorem 10, the complete list of the above invariants for ellipsoid is given by the conditions:

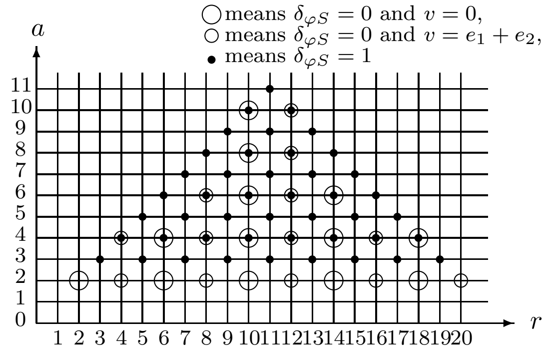

FIGURE 16. $\mathbb{E}$: All possible $(r, a, \delta_{\varphi S}, v)$ (here $H = [e_1 + e_2]$)

Type 0 ($\delta_\varphi = 0$)
$2 \leq a \leq r$ and a is even.
$r = 2, 6, 10, 14, 18$.
$r + a \leq 22$.
If $a = r$, then $(r, a) = (2, 2),\ (10, 10)$.
If $r + a = 22$, then $(r, a) = (18, 4)$.

Type Ia ($\delta_{\varphi S} = 0,\ \delta_\varphi = 1$)
$2 \leq a \leq r$ and a is even.
$r = 4, 8, 12, 16, 20$.
$r + a \leq 22$.
If $a = r$, then $(r, a) = (4, 4)$.
If $r + a = 22$, then $(r, a) = (20, 2),\ (12, 10)$.

Type Ib ($\delta_{\varphi S} = 1$)
$3 \leq a \leq r$.
$r + a$ is even.
$2 \leq r \leq 20$.
$r + a \leq 22$.

All possible data

$$(r, a, \delta_{\varphi S}, v) \tag{5.1}$$

which are equivalent to $(r, a, \delta_{\varphi S}, \delta_\varphi)$ are given in Figure 16. We have $\delta_\varphi = 0$, if and only if $\delta_{\varphi S} = v = 0$. Thus there are 13 cases of Type 0, and 13 cases of Type Ia, and 45 cases of Type Ib.

The relations between related involutions are:

$$r(\varphi) + r(\tau\varphi) = 22, \quad a(\varphi) = a(\tau\varphi), \tag{5.2}$$

$$\delta_{\varphi S} = \delta_{\tau\varphi S}, \quad s_\varphi + s_{\tau\varphi} \equiv e_1 + e_2 \mod 2L. \tag{5.3}$$

Hence, if a positive curve A^+ satisfies $\delta_\varphi = 0$, then the related positive curve A^- satisfies $(\delta_{\tau\varphi S}, \delta_{\tau\varphi}) = (0, 1)$. If a positive curve A^+ satisfies $\delta_{\varphi S} = 1$ and $r(\varphi) \leq 11$, then the related positive curve A^- satisfies $\delta_{\tau\varphi S} = 1$ and $r(\tau\varphi) \geq 11$.

For the ellipsoid case, conditions (b_+) and (b_-) of Theorem 14 are not valid in all cases. We have to check conditions (a_+) and (a_-) in the rest cases directly. As the result, we have

Theorem 21. *For the ellipsoid case, genus always defines the class of an integral involution. In particular (by Theorems 1, 6 and 10), the genus invariants*

$$(r, a, \delta_{\varphi S}, \delta_\varphi)$$

define the connected component of muduli of positive non-singular curves of bidegree $(4,4)$ *on ellipsoid up to the action of the automorphism group of ellipsoid. There are exactly* 13 *connected components of moduli with* $\delta_\varphi = 0$*, exactly* 13 *connected components of moduli with* $(\delta_{\varphi S}, \delta_\varphi) = (0,1)$*, and exactly* 45 *connected components of moduli with* $\delta_{\varphi S} = 1$*.*

Identifying related positive curves, we get that there are exactly 38 *connected components of moduli of real non-singular curves of bidegree* $(4,4)$ *on ellipsoid up to the action of the automorphism group of ellipsoid. There are* 13 *of them with* $\delta_\varphi = 0$ *(equivalently,* $(\delta_{\varphi S}, \delta_\varphi) = (0,1)$*), and* 25 *connected components with* $\delta_{\varphi S} = 1$*: they are distinguished from* 45 *by the inequality* $r(\varphi) \le 11$ *(equivalently, by* $r(\varphi) \ge 11$*).*

Proof. For the ellipsoid case, $a_{H_+} = 1$, $l(A_{S_+}) = 1$, $p = 1$. Moreover, $\kappa(A_{S_+}) = 0$. Thus, the condition (b_+) of Theorem 14 means that either $r - a > 0$, or $r - a = 0$ and $a \ge 5$, or $r - a = 0$, $a = 4$ and $\delta_{\varphi S_+} = 0$ (for our case, $\delta_{\varphi S_+} = \delta_{\varphi S}$). From the list of possible invariants above, we see that this condition is not satisfied only in the following cases:

(a) $\delta_\varphi = 0$ and $r = a = 2$;
(b) $\delta_{\varphi S} = 1$ and $r = a = 3$;
(c) $\delta_{\varphi S} = 1$ and $r = a = 4$.

Let us consider all these cases.

Case (a): In this case, the lattice $L^\varphi \cong U(2)$, and the lattice K_+ is the orthogonal complement to the primitive sublattice $S_+ \subset U(2)$ where $S_+ \cong \langle -4 \rangle$. It follows that $K_+ \cong \langle 4 \rangle$. Obviously, the condition (a_+) of Theorem 14 is valid in this case.

In cases (b) and (c), the invariant $\delta_\varphi = 1$, and the lattice $L^\varphi \cong \langle 2 \rangle \oplus (r-1)\langle -2 \rangle$. The lattice K_+ is the orthogonal complement to the primitive sublattice $S_+ \cong \langle -4 \rangle \subset L^\varphi$. Equivalently, K_+ is the orthogonal complement $\delta^\perp$ to a primitive element $\delta \in L^\varphi$ with $\delta^2 = -4$. Since $\delta_{\varphi S} = \delta_{\varphi S_+} = 1$, the discriminant form q_{K_+} must be isomorphic to $q_{\pm 1}(2) \oplus q'$ in this case. Here $q_{\pm 1}(2)$ is a non-degenerate quadratic form on the group of order 2. Equivalently, there exists $x \in A_{K_+}$ of order 2 such that $q_{K_+}(x) \not\equiv 0 \mod 1$. See ([21], page 116).

Considering the odd unimodular lattice $L = L^\varphi(1/2) \cong \langle 1 \rangle \oplus (r-1)\langle -1 \rangle$ instead of L^φ, we should prove the following for $r = 3, 4$: For all elements $\delta \in L$ with $\delta^2 = -2$ and with odd $K = \delta^\perp$, the lattices $K = \delta^\perp$ are isomorphic, and the canonical homomorphism

$$\xi : O(K) \to O(q_{K(2)}) \tag{5.4}$$

is epimorphic. Now let us consider cases (b) and (c) separately.

Case (b). In this case, by simple calculations, the discriminant form $q_{K(2)} \cong q_1(2) \oplus q_{-1}(4)$. Its automorphism group $O(q_{K(2)}) = \pm 1$ is the image of the automorphism subgroup ± 1 of $O(K)$. Thus, (5.4) is valid. To prove that all lattices $K = \delta^\perp$ are isomorphic, one should argue similarly to the case (c) below which is more complicated.

Case (c). In this case, $q_{K(2)} \cong q_1(2) \oplus q_{-1}(2) \oplus q_{-1}(4)$ (this is clear, but we shall give the proof in considerations below). By simple calculations, $O(q_{K(2)}) \cong (\mathbb{Z}/2)^2$.

Let us consider the group $W(L)$ generated by reflections in all elements of L with squares -1 and -2. The Dynkin diagram of the fundamental chamber of $W(L)$ is well-known (e. g. see [30]). It is (with usual notations)

(5.5)

$$\bullet - \bullet - \circ - \star$$

Here the two $\bullet$ correspond to elements $\delta \in L$ with $\delta^2 = -2$ and with odd orthogonal complement $K = \delta^\perp$. They (i. e. their $O(L)$-orbits) give all elements of L we are looking for. In particular, there exist no more than two possibilities for K up to isomorphisms. The $\circ$ corresponds to elements $e \in L$ with $e^2 = -1$. The $\star$ corresponds to elements $v \in L$ with $v^2 = -2$ and with even $v^\perp$ in L.

Let e_0, e_1, e_2, e_3 be the standard basis of L with the diagonal matrix $diag(1, -1, -1, -1)$. The diagram (5.5) is given (from the left) by $\delta_2 = -e_2 + e_3$, $\delta_1 = -e_1 + e_2$, $e = e_1$ and $v = e_0 - e_1 - e_2 - e_3$. The permutation $(e_0, e_1, e_2, e_3) \to (e_0, e_2, e_3, e_1)$ gives the automorphism of L such that $\delta_1 \mapsto \delta_2$. Thus the lattices K corresponding to $K = \delta_1^\perp$ and $K = \delta_2^\perp$ are isomorphic. Any element $\delta \in K$ with $\delta^2 = -2$ and with odd $\delta^\perp$ belongs to $O(L)$-orbit of δ_1 or δ_2.

The lattice $K = \delta_2^\perp$ has the basis e_0, e_1, $\delta = e_2 + e_3$ with the diagonal matrix $diag(1, -1, -4)$. It follows that $q_{K(2)} \cong q_1(2) \oplus q_{-1}(2) \oplus q_{-1}(4)$ with the corresponding bases $e_0/2$, $e_1/2$, $\delta/4$ mod K. As we have mentioned, it is easy to see that $O(q_{K(2)}) \cong (\mathbb{Z}/2)^2$. The automorphisms ± 1 of the lattice K give the automorphisms ± 1 of $q_{K(2)}$. To prove (5.4), it is sufficient to construct an automorphism of K which does not give the automorphism ± 1 in $O(q_{K(2)})$.

This is the automorphism f: $e_0 \mapsto 2e_0 + e_1 + \delta$, $e_1 \mapsto e_0 + \delta$, $\delta \mapsto 2e_0 + 2e_1 + \delta$ of the lattice K. One can easily check that this is an automorphism of K. We have that $f : \delta/4 \mapsto e_0/2 + e_1/2 + \delta/4$. It follows that f does not give ± 1 in $O(q_{K(2)})$. Thus, (5.4) is epimorphic.

In the case (b), the Dynkin diagram of $W(L)$ is

(5.6)

$$\bullet - \circ - \circ\ .$$

Replacing ϕ by $\tau\phi$, we get the statement (a_-) or Theorem 13.

It follows the theorem.

Now we consider the geometric interpretation of the above calculations. Using (2.44), (2.45) for both positive curves A^+, A^- and (5.2), (5.3), we can easily draw the picture of A^+ on ellipsoid for all invariants (5.1). See Figures 17 and 18. As usual, we set $g = (22-r-a)/2$ and $k = (r-a)/2$. Thus we get the isotopy classification of real non-singular curves of bidegree $(4,4)$ on ellipsoid. We can see that the isotopy type is defined by the topology type of A^+. The isotopy types of A are:

a nest of depth 4; $\langle 0\rangle$ (or $\emptyset$); $\langle 1\langle m\rangle \sqcup n\rangle$ where $m + n \le 9$, $m \le n$.

This isotopy classification was first obtained by Gudkov and Shustin [8]. As we have mentioned in Sect. 4, Zvonilov [35] classified complex schemes of these isotopy types. Below we partly repeat his results.

We have similar statement to Arnold [3]

Proposition 22. *Let (A, φ) (or A^+) be a positive curve of bidegree $(4,4)$ on an ellipsoid. If A is dividing, then*

$$[X_\varphi(\mathbb{R})] = c(e_1 + e_2)$$

in $H_2(X; \mathbb{Z}/2\mathbb{Z})$ for some c $(\in \mathbb{Z}/2\mathbb{Z})$. In particular, we have $\delta_{\varphi S} = 0$.

Proof. We use arguments like Wilson [32], Lemma 6.7. Let C^+ and C^- be the connected components of $A \setminus A(\mathbb{R})$. $C^\pm$ have the orientations induced from A. A^+ and A^- are the halfs of $Y(\mathbb{R}) \setminus A(\mathbb{R})$. $A^+ = \pi(X_\varphi(\mathbb{R}))$ and $A^- = \pi(X_{\tau\varphi}(\mathbb{R}))$. $Y(\mathbb{R})$ (ellipsoid) is homeomorphic

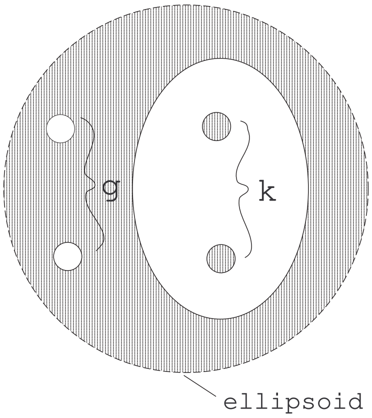

FIGURE 17. $\mathbb{E}$: A^+ for $(r, a, \delta_{\varphi S}) \neq (10, 10, 0)$, $(12, 10, 0)$, $(10, 8, 0)$, $(12, 8, 0)$.

to the 2-sphere S^2, and orientable. We give $A^\pm$ the orientations induced from an orientation of $Y(\mathbb{R})$. We regard $C^\pm$ and $A^\pm$ as $\mathbb{Z}$-chain. Then we have $\partial(C^+ + A^+) = 2\partial D$ where D is the sum of some 2-disks on $Y(\mathbb{R})$. Hence $C^+ + A^+ - 2D$ is $\mathbb{Z}$-cycle. We consider the anti-holomorphic involution $\theta = \varphi \mod \{1, \tau\}$ on Y. Then $\theta(\text{pt} \times \mathbb{P}^1) = -(\mathbb{P}^1 \times \text{pt})$, $\theta(\mathbb{P}^1 \times \text{pt}) = -(\text{pt} \times \mathbb{P}^1)$, $\theta(C^+) = -C^-$ and θ fix $\mathbb{Z}$-chains on $Y(\mathbb{R})$.

For any integer m, n, we have

$$(1 + \theta)(m(\text{pt} \times \mathbb{P}^1) + n(\mathbb{P}^1 \times \text{pt})) = (m - n)(\text{pt} \times \mathbb{P}^1 - \mathbb{P}^1 \times \text{pt}).$$

Hence $(1 + \theta)(C^+ + A^+ - 2D) \sim c(\text{pt} \times \mathbb{P}^1 - \mathbb{P}^1 \times \text{pt})$ for some integer c, where $\sim$ means $\mathbb{Z}$-homologous. Since A is of bidegree $(4, 4)$, $C^+ + C^- \sim 4(\text{pt} \times \mathbb{P}^1 + \mathbb{P}^1 \times \text{pt})$. Thus we have

$$2[C^+ + A^+ - 2D - 2(\text{pt} \times \mathbb{P}^1 + \mathbb{P}^1 \times \text{pt})] = c[\text{pt} \times \mathbb{P}^1 - \mathbb{P}^1 \times \text{pt}]$$

in $H_2(\mathbb{P}^1 \times \mathbb{P}^1; \mathbb{Z})$. Since $H_2(\mathbb{P}^1 \times \mathbb{P}^1; \mathbb{Z})$ is free, we have $[C^+ + A^+ - 2D - 2(\text{pt} \times \mathbb{P}^1 + \mathbb{P}^1 \times \text{pt})] = c[\text{pt} \times \mathbb{P}^1 - \mathbb{P}^1 \times \text{pt}]$ for some integer c. Hence we have

$$C^+ + A^+ \sim c(\text{pt} \times \mathbb{P}^1 + \mathbb{P}^1 \times \text{pt})$$

as $\mathbb{Z}/2\mathbb{Z}$-cycles for some c $(\in \mathbb{Z}/2\mathbb{Z})$. Taking the transfer (like [32], Lemma 6.7) gives $[X_\varphi(\mathbb{R})] = c(e_1 + e_2)$ in $H_2(X; \mathbb{Z}/2\mathbb{Z})$. $\square$

Suppose that A is dividing. Then $\delta_{\varphi S} = 0$ by Proposition 22. Since $s_\varphi \equiv [X_\varphi(\mathbb{R})] \mod 2H_2(X; \mathbb{Z})$ and $s_\varphi + s_{\tau\varphi} \equiv (e_1 - e_2) \mod 2H_2(X; \mathbb{Z})$ (see above), we have

$$[X_\varphi(\mathbb{R})] + [X_{\tau\varphi}(\mathbb{R})] = (e_1 + e_2)$$

in $H_2(X; \mathbb{Z}/2\mathbb{Z})$. Hence, by Proposition 22, we see $[X_{\tau\varphi}(\mathbb{R})] = (e_1 + e_2)$ if $[X_\varphi(\mathbb{R})] = 0$, and $[X_{\tau\varphi}(\mathbb{R})] = 0$ if $[X_\varphi(\mathbb{R})] = (e_1 + e_2)$ in $H_2(X; \mathbb{Z}/2\mathbb{Z})$.

By Proposition 22, if $\delta_{\varphi S} = 1$, then the curves are not dividing. Hence, the following 6 isotopy types $\langle 7 \sqcup 1\langle 0 \rangle\rangle$, $\langle 6 \sqcup 1\langle 1 \rangle\rangle$, $\langle 5 \sqcup 1\langle 2 \rangle\rangle$, $\langle 4 \sqcup 1\langle 3 \rangle\rangle$, $\langle 4 \sqcup 1\langle 1 \rangle\rangle$ and $\langle 3 \sqcup 1\langle 2 \rangle\rangle$ with $\delta_{\varphi S} = 1$ are not dividing. By Zvonilov [35], these 6 isotopy types are of *indeterminate type*, i.e., there exist dividing curves of these isotopy types. Then these dividing curves must realize some connected components with $\delta_{\varphi S} = 0$. As a result, it turns out that if $\delta_{\varphi S} = 0$ and $A(\mathbb{R}) \neq \emptyset$, then A is dividing. Thus we have the following characterization of the invariant $\delta_{\varphi S}$.

Theorem 23. *Let A^+ be a positive non-singular real curve of bidegree $(4, 4)$ on ellipsoid. Then, A is dividing or $A(\mathbb{R}) = \emptyset$, if and only if $\delta_{\varphi S} = 0$.* $\square$

Finally we get

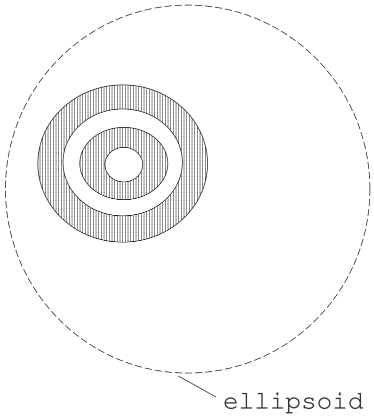

FIGURE 18. $\mathbb{E}$: A^+ for $(r, a, \delta_{\varphi S}) = (10, 8, 0)$ ($A^+ = \emptyset$ for $(r, a, \delta_{\varphi S}) = (10, 10, 0)$).

Theorem 24. *A connected component of moduli of a positive real non-singular curve A^+ of bidegree $(4, 4)$ on ellipsoid is defined by the topology type (or isotopy type) of A^+ and by divideness of $A(\mathbb{C})$ by $A(\mathbb{R})$ (or the invariant $\delta_{\varphi S}$). All these possibilities are presented in figures $\mathbb{E}$ above.*

6. Connected components of moduli of real non-singular curves in $|-2K_{\mathbb{F}_1}|$

Here we consider the case $S \cong \langle 2\rangle \oplus \langle -2\rangle$ and $\theta = -1$ on S.

Let us show that $Y = X/\{1, \tau\} = \mathbb{F}_1$ in this case. We apply general results about right DPN-pairs which we have cited in Sect. 2.2. By (2.11), (2.12), $A = X^\tau$ is a non-singular irreducible curve of genus 9. Since $A \neq \emptyset$, Y is a rational surface with the Picard number 2. Then $Y \cong \mathbb{F}_n$ with $n \geq 0$. Since $A \subset Y$ is a right DPN-pair, exceptional curves of Y are rational with square -1, -2 or -4. Thus, $Y \cong \mathbb{F}_n$ with $n = 0$, 1, 2 or 4. If $n = 0$, then $Y = \mathbb{P}^1 \times \mathbb{P}^1$, and we have seen that $S = U(2)$ in this case. If $n = 2$, then preimage of the exceptional section gives an element of S with square -4 (it is also a root of S), but S does not have elements with square -4. If $n = 4$, then the exceptional rational curve with square -4 must be a component of A, but A is irreducible of genus 9. Thus, $Y = \mathbb{F}_1$.

This case is $(\mathcal{D})$-non-degenerate because S has no elements with square -4.

For the exceptional section of $\mathbb{F}_1$ with square -1, we denote by e its preimage on X. Then $e^2 = -2$. We consider the contraction $\mathbb{F}_1 \to \mathbb{P}^2$ of the exceptional section and denote by h the preimage on X of a line $l \subset \mathbb{P}^2$. Then $h^2 = 2$, $h \cdot e = 0$. We have $S = \mathbb{Z}h + \mathbb{Z}e$.

Since $\theta = -1$, we have $S_+ = \{0\}$. Obviously, $s = 2$, $p = 0$.

Since $\Delta(S_+, S)^{(-4)} = \Delta(S_-, S)^{(-4)} = \emptyset$, by Lemma 9, G is trivial.

The group A_{S_-} is generated by $h^* = \frac{1}{2}h$ and $e^* = -\frac{1}{2}e$, and hence $A_{S_-} \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $l(A_{S_-}) = 2$. The characteristic element of q_{S_-} is $h + e \pmod{2S}$.

We have $A_{S_+} = 0$, $H_+ = 0$, $\rho = 0$. Since $\Gamma_- = 0$, $H = H_-$ ($\subset S/2S$) is one of the following 5 subgroups:

$$0,\ [h],\ [e],\ [h+e],\ S/2S = [h, e],$$

where we consider h, e mod $2S$. Since $H_+ = 0$, we have $q_\rho = (-q_{S_-})|H_-$. For the groups H we have:

(1) $H = 0$. Then $a_H = 0$, $q_\rho = 0$, $\delta_H = 0$, $k_\rho = 0$, $\mu_\rho = 0$, $\sigma_\rho \equiv 0 \pmod 8$. If $\delta_{\varphi S} = 0$, then $v = 0$, equivalently, $\delta_\varphi = 0$, and $c_v \equiv 0 \pmod 4$.

(2) $H = [h]$. Then $a_H = 1$, $q_\rho \cong q_{-1}(2)$, $\delta_H = 1$, $k_\rho = 0$, $\mu_\rho = 0$, $\sigma_\rho \equiv -1 \pmod 8$. If $\delta_{\varphi S} = 0$, then $v = h$, and hence $c_v \equiv -1 \pmod 4$.

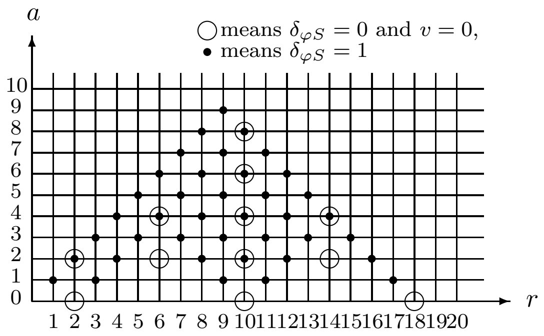

FIGURE 19. $\mathbb{F}_1$: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = 0$

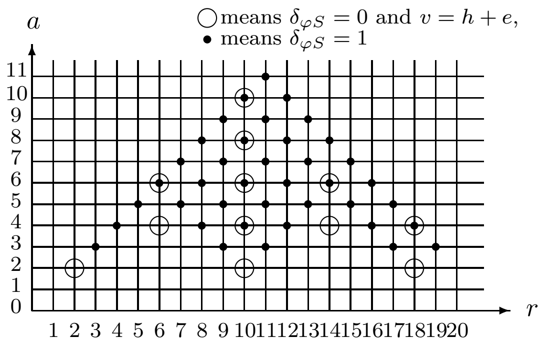

FIGURE 20. $\mathbb{F}_1$: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = S/2S = [h, e]$

(3) $H = [e]$. Then $a_H = 1$, $q_\rho \cong q_1(2)$, $\delta_H = 1$, $k_\rho = 0$, $\mu_\rho = 0$, $\sigma_\rho \equiv 1 \pmod 8$. If $\delta_{\varphi S} = 0$, then $v = e$, and hence $c_v \equiv 1 \pmod 4$.

(4) $H = [h+e]$. Then $a_H = 1$, $q_\rho \cong z$, $\delta_H = 0$, $k_\rho = 1$, $\mu_\rho = 0$, $\sigma_\rho \equiv 0 \pmod 8$. If $\delta_{\varphi S} = 0$, then $v = 0$ or $h + e$, and hence $c_v \equiv 0 \pmod 4$.

(5) $H = S/2S$. Then $a_H = 2$, $q_\rho \cong q_1(2) \oplus q_{-1}(2)$, $\delta_H = 1$, $k_\rho = 0$, $\mu_\rho = 0$, $\sigma_\rho \equiv 0 \pmod 8$. If $\delta_\varphi = 0$, then $c_v \equiv 0 \pmod 4$. If $\delta_{\varphi S} = 0$ and $\delta_\varphi = 1$ (Type Ia), then we also have $c_v \equiv 0 \pmod 4$ as we mention below.

From above we always have $\mu_\rho = 0$. We see $\delta_{\varphi S_+} = \delta_\varphi$. We also see $\delta_{\varphi S_-} = 0$ if and only if $\delta_{\varphi S} = 0$, $v = h + e$ and $H = [h + e]$ or $S/2S$. If $\delta_{\varphi S_+} = \delta_\varphi = 0$, then we have $\varepsilon_{v_+} \equiv 0 \pmod 2$. If $\delta_{\varphi S_-} = 0$, then we have $c_v \equiv 0 \pmod 4$ (see (4) and (5) above), and hence $\gamma_- = q_1(2) \oplus q_{-1}(2)$. We see $(q_{S_-})_v = u_+(2)$, and hence $\varepsilon_{v_-} \equiv 0 \pmod 2$.

We see that the genus (hence, by Theorem 14 and Proposition 15, isomorphism class) of an integral involutions (L, φ, S) of the type $(\langle 2\rangle \oplus \langle -2\rangle, -1)$ satisfying (RSK3) is determined by the data

$$(r, a, H, \delta_{\varphi S}, v), \tag{6.1}$$

where the v ($\in H$) is defined only if $\delta_{\varphi S} = 0$. We remind that $\delta_\varphi = 0$ if and only if $\delta_{\varphi S} = v = 0$.

By Theorem 10, the complete list of the above invariants is given by the conditions:
Type 0 ($\delta_\varphi = 0$)
$H = 0$ or $[h + e]$.

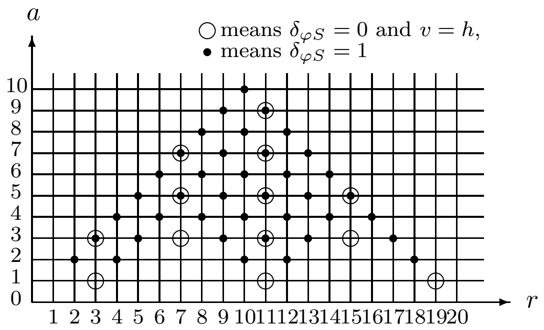

FIGURE 21. $\mathbb{F}_1$: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = [h]$

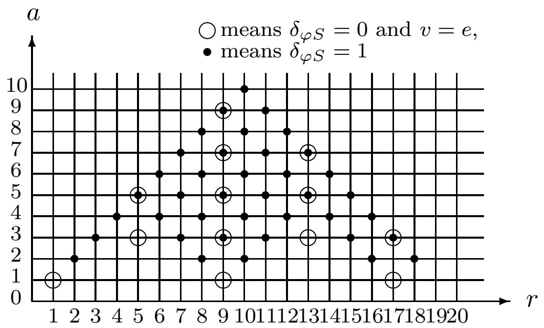

FIGURE 22. $\mathbb{F}_1$: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = [e]$

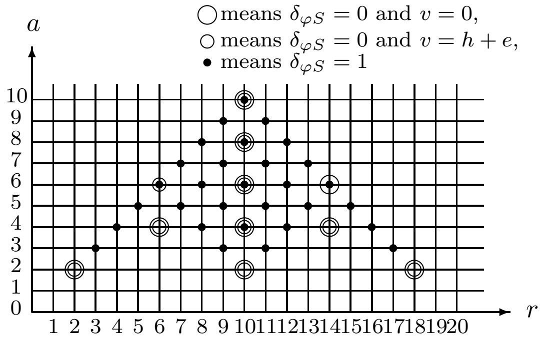

FIGURE 23. $\mathbb{F}_1$: All possible $(r, a, \delta_{\varphi S}, v)$ with $H = [h + e]$

$r = 2,\ 6,\ 10,\ 14,\ 18.$
$a \equiv 0 \pmod 2.$
$a \leq r.$
$r + a \leq 2a_H + 18.$
If $H = 0$, then $a \geq 0$.
If $H = [h + e]$, then $a \geq 2$.
If $H = 0$ and $a = 0$, then $2 - r \equiv 0 \pmod 8$.
If $H = [h + e]$ and $a = 2$, then $2 - r \equiv 0 \pmod 8$.
If $a = r$, then $2 - r \equiv 0 \pmod 8$.

Type Ia ($\delta_{\varphi S}=0,\ \delta_\varphi=1$)
$a_H\geq 1$.
$c_v\equiv\sigma_\rho \pmod 4$. Hence, $c_v\equiv 0 \pmod 4$ if $H=S/2S$.
If $H=[h]$, then $a\geq 1$, $r=3,\ 7,\ 11,\ 15$ and $a\equiv 1 \pmod 2$.
If $H=[e]$, then $a\geq 1$, $r=1,\ 5,\ 9,\ 13,\ 17$ and $a\equiv 1 \pmod 2$.
If $H=[h+e]$, then $a\geq 2$, $r=2,\ 6,\ 10,\ 14,\ 18$ and $a\equiv 0 \pmod 2$.
If $H=S/2S$, then $a\geq 2$, $r=2,\ 6,\ 10,\ 14,\ 18$ and $a\equiv 0 \pmod 2$.
If $H=[h]$ or $H=[e]$ and $a=1$, then $2-r\equiv\sigma_\rho \pmod 8$.
If $H=[h+e]$ or $H=S/2S$ and $a=2$, then $2-r\equiv 0 \pmod 8$.
$a\leq r,\ \ r+a\leq 2a_H+18$.
If $\delta_{\varphi S}=0$, $v=h+e$, $H=[h+e]$ or $S/2S$ and $r+a=2a_H+18$, then $2-r\equiv 0 \pmod 8$.

Type Ib ($\delta_{\varphi S}=1$)
$a\geq a_H+k_\rho+1$.
$r+a\equiv 0 \pmod 2$.
If $a=a_H+k_\rho+1$, then $2-r\equiv\sigma_\rho\pm 1 \pmod 8$.
If $a=a_H+k_\rho+2$, then $2-r\not\equiv\sigma_\rho+4 \pmod 8$.
$1\leq r\leq 19,\ \ a\leq r,\ \ r+a\leq 2a_H+18$.
If $\delta_{\varphi S}=0$, $v=h+e$, $H=[h+e]$ or $S/2S$ and $r+a=2a_H+18$, then $2-r\equiv 0 \pmod 8$.

In Type Ia case, if $H=S/2S$, then $c_v\equiv 0 \pmod 4$. Thus $v=h+e$. Hence, H determines v in Type Ia case. Thus, by the above, the isomorphism class of an integral involutions (L,φ,S) of the type $(\langle 2\rangle\oplus\langle -2\rangle,-1)$ satisfying (RSK3) is determined by the data
$$(r,a,H,\delta_{\varphi S},\delta_\varphi).$$
We give all these possible data in Figures 19 — 23.

By Theorem 11 about related involutions, we have
$$\begin{aligned}&r(\varphi)+r(\tau\varphi)=20,\quad a(\varphi)-a_{H(\varphi)}=a(\tau\varphi)-a_{H(\tau\varphi)},\\ &a_{H(\varphi)}+a_{H(\tau\varphi)}=2,\quad H(\tau\varphi)=H(\varphi)^\perp \text{ w.r.t. } b_{S_-},\\ &\delta_{\varphi S}=\delta_{(\tau\varphi)S},\quad s_\varphi+s_{\tau\varphi}=h+e \mod 2L.\end{aligned}\tag{6.2}$$
Thus, integral involutions of Type 0 with $H=0$ (12 classes) are related to involutions of Type Ia with $H=S/2S$. Involutions of Type 0 with $H=[h+e]$ (10 classes) are related to involutions of Type Ia with $H=[h+e]$. Involutions of Type Ia with $H=[h]$ (13 classes) are related to involutions of Type Ia with $H=[e]$. Involutions of Type Ib with $H=0$ (39 classes) are related to involutions of Type Ib with $H=S/2S$. Involutions of Type Ib with $H=[h]$ (39 classes) are related to involutions of Type Ib with $H=[e]$. Finally, the class $(r,a,H=[h+e],\text{Type Ib})$ is related to $(20-r,a,H=[h+e],\text{Type Ib})$. (There are 30 classes of Type Ib with $H=[h+e]$.)

Moreover, if we identify related involutions, there are $35(=12+10+13)$ classes with $\delta_{\varphi S}=0$ and $95(=39+39+17)$ classes with $\delta_{\varphi S}=1$.

Let us consider the geometric interpretation of the above calculations. We denote by s the exceptional section of $\mathbb{F}_1$ with $s^2=-1$ and by c the fiber of the natural fibration $\pi:\mathbb{F}_1\to\mathbb{P}^1$. The contraction of s (as an exceptional curve of the first kind) gives the natural morphism $p:\mathbb{F}_1\to\mathbb{P}^2$, and we denote $P=p(s)$. A non-singular curve $A\in|-2K_{\mathbb{F}_1}|$ gives then a curve $A_1=p(A)$ of degree 6 in $\mathbb{P}^2$ with only one singular point P which is a quadratic singular

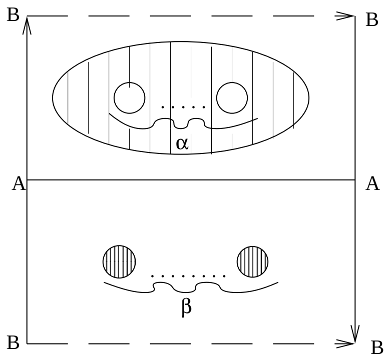

FIGURE 24. $\mathbb{F}_1$: A^+ for $H = [h, e]$ with $(r, a, \delta_{\varphi S}, v) \neq (10, 10, 0, h + e)$; here $\alpha = g$, $\beta = k$.

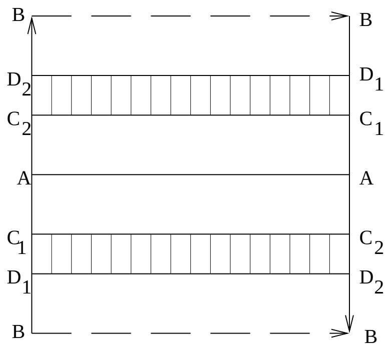

FIGURE 25. $\mathbb{F}_1$: A^+ for $H = [h, e]$ with $(r, a, \delta_{\varphi S}, v) = (10, 10, 0, h + e)$.

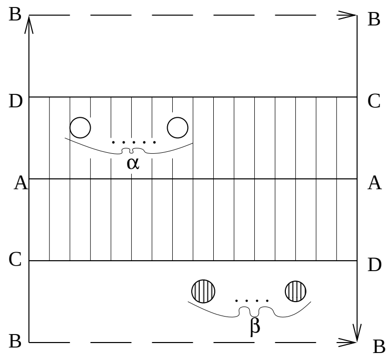

FIGURE 26. $\mathbb{F}_1$: A^+ for $H = [h]$; here $\alpha = g - 1$, $\beta = k$.

point resolving by one blow-up. A small deformation of A or A_1 (in the same connected component of moduli) makes P non-degenerate. Using Bezoute theorem, one can easily draw all in principle possible pictures of A_1. For example, one can find these pictures in Figure 1 of [10]. Lifting these pictures to $\mathbb{F}_1$ and using (2.44), (2.45) and Theorem 16 applied to both positive curves A^+ and A^-, we get from Figures 19 — 23 all pictures of A^+ on $\mathbb{F}_1$ up to isotopy. They are all presented in Figures 24—29 where we denote $g = (22 - r - a)/2$ and $k = (r - a)/2$, as usual. The interval AA denotes $s(\mathbb{R})$, the real part of the exceptional section s, and the interval BB denotes the $p^{-1}(l(\mathbb{R}))$ of a real projective line $l \subset \mathbb{P}^2$ which does not contain P.

We have the following geometric interpretation of the invariants δ_h and δ_e (see (2.18)) of A^+ if $A^+ \neq \emptyset$. We have $\delta_h = 0$ (equivalently, $h \mod 2 \in H$), if and only if homotopically

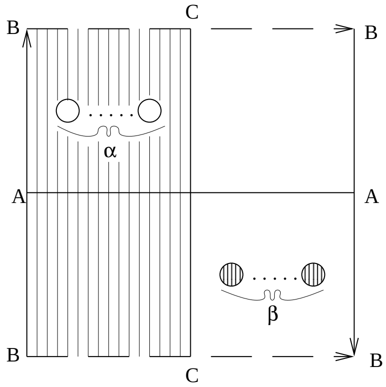

FIGURE 27. $\mathbb{F}_1$: A^+ for $H = [h+e]$ with $(r, a, \delta_{\varphi S}) \neq (10, 8, 0),\ (10, 10, 0)$; here $\alpha = g - 1$, $\beta = k$.

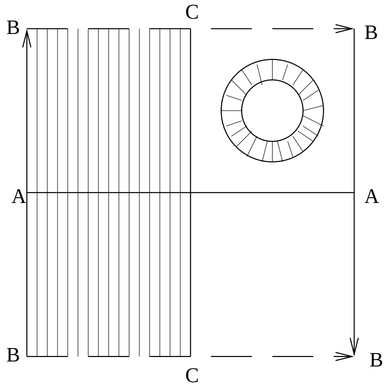

FIGURE 28. $\mathbb{F}_1$: A^+ for $H = [h+e]$ with $(r, a, \delta_{\varphi S}, v) = (10, 8, 0, 0)$.

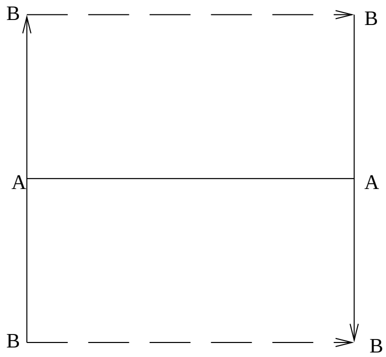

FIGURE 29. $\mathbb{F}_1$: $A^+ = \emptyset$ for $H = [h+e]$ with $(r, a, \delta_{\varphi S}, v) = (10, 10, 0, 0)$.

$l(\mathbb{R}) \subset A^-$ (i. e. some deformation of $l(\mathbb{R})$ is contained in A^-). Similarly, $\delta_e = 0$, if and only if homotopically $s(\mathbb{R}) \subset A^-$. The invariants δ_h and δ_e for both positive curves A^+ and A^- are sufficient to find the group H.

Thus, we get the isotopy classification of real non-singular curves $A \in |-2K_{\mathbb{F}_1}|$. In this classification we don't care about position of $A(\mathbb{R})$ with respect to the real part $s(\mathbb{R})$ of the exceptional section (one can see that more delicate classification in [10]).

We have the following interpretation of the invariant $\delta_{\varphi S}$: *one has $\delta_{\varphi S} = 0$, if and only if the curve A is dividing: $A(\mathbb{R})$ divides $A(\mathbb{C})$ in two connected parts or $A(\mathbb{R}) = \emptyset$. Equivalently, $A(\mathbb{R}) = 0$ in $H_1(A(\mathbb{C}), \mathbb{Z}/2)$.* Like in previous cases (because we know that invariants (6.1) define the connected component of moduli), it would be enough to construct a dividing curve A on $\mathbb{F}_1$ in each case when $\delta_{\varphi S}=0$, and it should follow from known results about real curves of degree 6 (e. g. see [9] and [10]). Another general argument is to apply Donaldson's trick [5], like in [4]. We can consider φ as holomorphic involution and τ as anti-holomorphic. Then the statement follows from Theorem 16. The same argument works in all previous cases, but we did not need it. Thus, we finally get

Theorem 25. *A connected component of moduli of a positive real non-singular curve $A\in|-2K_{\mathbb{F}_1}|$ is defined by the isotopy type of $A^+\subset\mathbb{F}_1(\mathbb{R})$ and by the divideness by $A(\mathbb{R})$ of $A(\mathbb{C})$ (equivalently, by the invariant $\delta_{\varphi S}$). All these possibilities are presented in in the Figures $\mathbb{F}_1$ above and in* (6.2).

7. Connected components of moduli of real non-singular curves in $|-2K_{\mathbb{F}_4}|$

Here we consider the case $S\cong U$ and $\theta=-1$.

Like in previous cases, we can prove that $Y=X/\{1,\tau\}=\mathbb{F}_4$ in this case. This case is $(\mathcal{D})$-non-degenerate because S has no roots with square -4.

We have $S_+=\{0\}$ and $S_-=S$, $s=2$, $p=0$.

Since $\Delta(S_+,S)^{(-4)}=\Delta(S_-,S)^{(-4)}=\emptyset$, by Lemma 9 the group G is trivial.

Since U is unimodular, $A_{S_-}=\{0\}$. Obviously, $A_{S_+}=\{0\}$. Hence, $H=H_+=H_-=0$, $\rho=0$ and $\delta_{\varphi S}=\delta_\varphi$.

By Theorem 14 and Proposition 15, the genus defines the isomorphism class of an integral involutions (L,φ,S) of the type $(U,-1)$ satisfying (RSK3), and it defines the connected component of moduli. It is determined by the data

$$(r,a,\delta_{\varphi S}=\delta_\varphi,v) \tag{7.1}$$

where $v=0$ if $\delta_{\varphi S}=0$ (otherwise, v is not defined). The complete list of the invariants (7.1) is given by the conditions:

$\delta_\varphi=0$ case

$r=2,6,10,14,18,\ a\equiv 0\pmod 2,\ a\le r,\ r+a\le 20.$

If $a=0$, $a=r$ or $r+a=20$, then $2-r\equiv 0\pmod 8$.

$\delta_\varphi=1$ case

$a\ge 1,\ r+a\equiv 0\pmod 2,\ 1\le r\le 19,\ a\le r,\ r+a\le 20.$

If $a=1$, then $2-r\equiv\pm 1\pmod 8$.

If $a=2$, then $2-r\not\equiv 4\pmod 8$.

All possible data (7.1) are given on Figure 30. There are 14 isomorphism classes with $\delta_\varphi=0$ and 49 isomorphism classes with $\delta_\varphi=1$.

By Theorem 11, we have for related involutions

$$r(\varphi)+r(\tau\varphi)=20,\ \ a(\varphi)=a(\tau\varphi)\ \text{ and }\ \delta_{\varphi S}=\delta_{\tau\varphi S},\ s_\varphi=s_{\tau\varphi}. \tag{7.2}$$

It corresponds to the symmetry with respect to the line $r=10$ of Figure 30. Thus, if we identify related involutions, there are 10 isomorphism classes with $\delta_\varphi=0$ and 27 isomorphism classes with $\delta_\varphi=1$.

Now let us consider the geometric interpretation of the above results. Denote by s the exceptional rational section with $s^2=-4$ of $\mathbb{F}=\mathbb{F}_4$, and by c the fiber of the natural

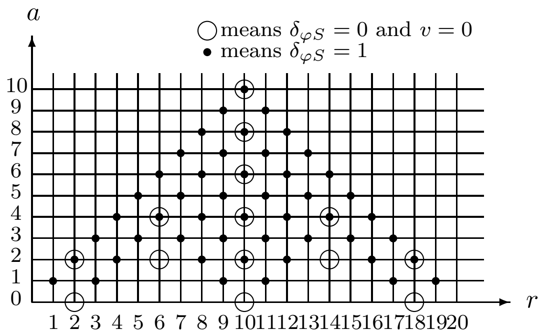

FIGURE 30. $\mathbb{F}_4$: All possible $(r, a, \delta_{\varphi S}, v)$ (here $H = 0$)

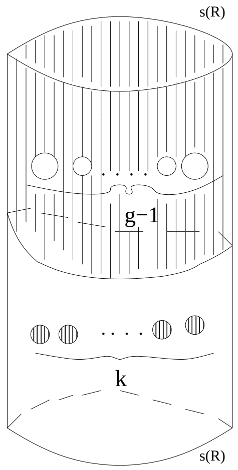

FIGURE 31. $\mathbb{F}_4$: A^+ with $(r, a, \delta_{\varphi S}, v) \neq (10, 8, 0, 0)$, $(10, 10, 0, 0)$.

fibration $f : \mathbb{F}_4 \to s$. One has $c^2 = 0$ and $p_g(c) = 0$. We have $-2K_{\mathbb{F}} = 12c + 4s$. Thus, for $A \in |-2K_{\mathbb{F}}|$, one has $A \cdot s = -4$. It follows $A = s + A_1$ where $A_1 \in |12c + 3c|$. We have $c \cdot A_1 = 3$ and $s \cdot A_1 = 0$. It follows that a non-singular $A \in |-2K_{\mathbb{F}}|$ has two irreducible components: $A = s + A_1$. Thus, we are describing connected components of moduli of non-singular curves A_1 in the linear system $|12c + 3s|$. Any such A_1 gives a non-singular $A = s + A_1$ from $|-2K_{\mathbb{F}}|$.

We mention that the lattice $S \cong U$ is $\mathbb{Z}C + \mathbb{Z}E$ where $C = \pi^*(c)$ and $E = \pi^*(s)/2$. We have $C^2 = 0$, $E^2 = -2$ and $C \cdot E = 1$. As usual, we denote $\pi : X \to Y = \mathbb{F}_4$ the quotient morphism.

If $\mathbb{F}(\mathbb{R})$ is not empty, then $\mathbb{F}(\mathbb{R})$ is a torus and $s(\mathbb{R})$ is a circle giving a generator of the torus. The curves $c(\mathbb{R})$ where $f(c) \in s(\mathbb{R})$ give another generator of the torus. It follows that a real curve $A_1(\mathbb{R})$ belongs to the open cylinder $\mathbb{F}(\mathbb{R}) - s(\mathbb{R})$ with its infinity identified with two copies of $s(\mathbb{R})$.

Using $c \cdot A_1 = 3$ and (2.44), (2.45) for both involutions φ and $\tau\varphi$, we get at once that a positive curve A^+ with the invariants (7.1) has the isotopy type given in Figures 31 and 32. In Figures 31 and 32 we assume that $(r, a, \delta_\varphi) \neq (10, 10, 0)$. Then $\mathbb{F}(\mathbb{R})$ is a torus. If $(r, a, \delta_\varphi) = (10, 10, 0)$, then $\mathbb{F}(\mathbb{R})$ is empty. As usual, $g = (22 - r - a)/2$ and $k = (r - a)/2$.

We have the following interpretation of the invariant δ_φ: *one has $\delta_\varphi = 0$, if and only if the curve A is dividing: $A(\mathbb{R})$ divides $A(\mathbb{C})$ in two connected parts or $A(\mathbb{R}) = \emptyset$. Equivalently, $A(\mathbb{R}) = 0$ in $H_1(A(\mathbb{C}), \mathbb{Z}/2)$.* Like in previous cases (because we know that invariants (7.1)

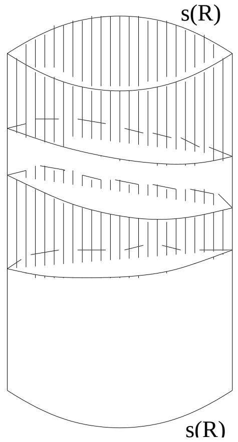

FIGURE 32. $\mathbb{F}_4$: A^+ with $(r, a, \delta_{\varphi S}, v) = (10, 8, 0, 0)$ ($\mathbb{F}_4(\mathbb{R}) = \emptyset$ if $(r, a, \delta_{\varphi S}, v) = (10, 10, 0, 0)$).

define the connected component of moduli), it would be enough to construct a dividing curve A on $\mathbb{F}_4$ in each case when $\delta_\varphi = 0$. Unfortunately, this construction is not known (surely, it is not a hard problem). Thus, we have to find a general argument for divideness. Applying Donaldson's trick [5], like in [4], we can consider φ as holomorphic involution and τ as anti-holomorphic. Then the statement follows from Theorem 16.

Thus, finally, we get

Theorem 26. *The connected component of moduli of positive real non-singular curves A^+, $A \in |-2K_{\mathbb{F}_4}|$ up to the action of the automorphism group of $\mathbb{F}_4$ is defined by the isotopy type and by divideness of $A(\mathbb{C})$ by $A(\mathbb{R})$. All these possibilities are presented in Figures $\mathbb{F}_4$ above.*

We remark that the corresponding real K3 surfaces (X, τ, φ) give an important class of K3 surfaces. They are elliptic K3 surfaces (preimages of c) with section (the preimage of s). The holomorphic involution τ gives the inverse map of the elliptic pencil.

8. APPLICATION: REAL POLARIZED K3 SURFACES AS DEFORMATIONS OF GENERAL K3 DOUBLE RATIONAL SCROLLS

Consider a non-degenerate real K3 surfaces (X, τ, φ) with a non-symplectic involution of the type (S, θ) where (S, θ) is of one of 5 types with $\operatorname{rk} S \le 2$ described in Sect. 2.4. Assume that $P \in S$ is a primitive *nef* element with $P^2 = n$, $n \in \mathbb{N}$ is even, and such that $\theta(P) = -P$. Moreover, assume that the complete linear system $|P|$ or $|2P|$ gives the quotient map $\pi : X \to Y = X/\{1, \tau\}$. By standard results about linear systems on K3 surfaces, e. g. see [27], it is valid in the following and only the following cases:

The Case of $\mathbb{P}^2$, i. e. $S = \langle 2 \rangle$ and $\theta = -1$: Then $P = h$ and $P^2 = n = 2$, both $|P|$ and $|2P|$ give the quotient map $\pi : X \to Y = \mathbb{P}^2$. Thus, $n = 2$.

The Case of ellipsoid $\mathbb{E}$, i. e. $S = U(2)$ and $\theta(e_1) = -e_2$: Then $P = e_1 + e_2$, $P^2 = n = 4$ and $|P|$ gives the quotient map π to the ellipsoid. Thus, $n = 4$.

The Case of hyperboloid $\mathbb{H}$, i. e. $S = U(2)$ and $\theta = -1$: Then either $P = n_1 e_1 + e_2$ (we denote this case as $\mathbb{H}_{(1)}$) or $P = e_1 + n_1 e_2$ (we denote this case as $\mathbb{H}_{(2)}$), and $P^2 = n = 4n_1$, $n_1 \ge 1$. Then $|P|$ gives the quotient map π to the hyperboloid. Thus, $n = 4n_1 \equiv 0 \mod 4$ and $n \ge 4$.

The Case of $\mathbb{F}_1$*, i. e.* $S = \langle 2\rangle \oplus \langle -2\rangle$ *and* $\theta = -1$. We have $P = n_1c + e = n_1h + e(1 - n_1)$ where $c = h - e$ and $P^2 = n = 4n_1 - 2$, $n_1 \geq 2$. Then $|P|$ gives the quotient map to $\mathbb{F}_1$. We have $n \equiv 2 \mod 4$ and $n \geq 6$.

The case of $\mathbb{F}_4$*, i. e.* $S = U$ *and* $\theta = -1$. We denote $C = \pi^*(c)$ and $E = \pi^*(s)/2$ where $C^2 = 0$, $E^2 = -2$ and $C \cdot E = 1$. We have two possibilities:

The case $(\mathbb{F}_4)^{(1)}$*:* $P = n_1C + E$ where $n_1 \geq 3$ and $P^2 = n = 2n_1 - 2$, then $|2P|$ gives the quotient map and $|P|$ gives the composition of π with the natural fibration $\mathbb{F}_4 \to \mathbb{P}^1$. We have $n \equiv 0 \mod 2$ and $n \geq 4$ for this case

The case $(\mathbb{F}_4)^{(2)}$*:* $P = n_1C + 2E$ where $n_1 \geq 5$ is odd and $P^2 = n = 4n_1 - 8$, then $|P|$ gives the quotient map π. We have $n \equiv 4 \mod 8$ and $n \geq 12$ for this case.

For each of these types:

$$\mathcal{T} = \{\mathbb{P}^2,\ \mathbb{E},\ \mathbb{H}_{(1)},\ \mathbb{H}_{(2)},\ \mathbb{F}_1,\ (\mathbb{F}_4)^{(1)},\ (\mathbb{F}_4)^{(2)}\} \tag{8.1}$$

the element P is uniquely defined by $n = P^2$, if it exists. If for $t \in \mathcal{T}$ such P does exist we write $n \in t$. For $t \in \mathcal{T}$ we denote by $\pi_0(\mathcal{M}_t)$ the set of connected components of moduli of non-degenerate (X, τ, ϕ) of the type $t \in \mathcal{T}$ which we had described in Sects 3 — 7.

Since the branch curve is non-singular in all these cases, the P is ample. In all these cases $|3P|$ gives an embedding of X to a projective space. If $n \geq 4$, a small deformation of the pair (X, P) gives a K3 surface with the very ample P, i. e. $|P| : X \subset \mathbb{P}^{n/2+1}$.

We denote by $\mathcal{M}_{n,k}$ where n, $k \in \mathbb{N}$ and n is even, the moduli space of real polarized K3 surfaces (X, kP) with the polarization kP where P is primitive and $P^2 = n$, and we require that the linear system $|kP|$ gives an embedding $|kP| : X \subset \mathbb{P}^{k^2n/2+1}$, i. e. kP is very ample. Thus, in all cases above, for $k \geq 3$ the correspondence $(X, \tau, \phi) \mapsto (X, kP)$ gives an embedding of the moduli of (X, τ, φ) to the moduli $\mathcal{M}_{n,k}$, $k \geq 3$, of real polarized K3 surfaces (X, kP) with the polarization kP of the degree nk^2 where P is primitive. Thus, for $k \geq 3$ we get the natural *deformation map*

$$D_n : \bigcup_{t \in \mathcal{T}:\ n \in t} \pi_0(\mathcal{M}_t) \to \pi_0(\mathcal{M}_{n,k}). \tag{8.2}$$

A polarized K3 surface (X, kP) which belongs to a connected component of moduli from the image of D can be obtained by a deformation inside $\mathcal{M}_{n,k}$ from a real K3 surface which is a general double covering (i. e. with a non-singular branch curve) of a *rational scroll.* All these scrolls and the types of their double coverings are given by the types $t \in \mathcal{T}$ such that $n \in t$. We call such K3 surfaces as *general K3 double rational scrolls.*

One of consequences of Theorem 3.10.1 in [19] (we shall discuss it in details later) is that the natural embedding

$$\mathcal{M}_n = \mathcal{M}_{n,1} \to \mathcal{M}_{n,k},\quad (X, P) \mapsto (X, kP), \tag{8.3}$$

gives the isomorphism of the set of connected components of $\mathcal{M}_n = \mathcal{M}_{n,1}$ and the set of connected components of $\mathcal{M}_{n,k}$ for $k \geq 2$. A difference between $\mathcal{M}_n$ and $\mathcal{M}_{n,k}$ for $k \geq 2$ is only on the boundary of $\mathcal{M}_n$ in $\mathcal{M}_{n,k}$, and this boundary (in spite of it has the real codimension one in $\mathcal{M}_{n,k}$) does not divide connected components of $\mathcal{M}_{n,k}$, $k \geq 2$.

Thus, the deformation map D_n in (8.2) is defined for all $k \geq 1$. *The image of D gives connected components of moduli of real polarized K3 surfaces (X, kP) for any $k \geq 1$, which can be obtained as a deformation of a general real K3 double rational scroll.* Using results of previous sections and [19], the deformation map D_n can be computed explicitly and gives all *real polarized K3 surfaces (X, kP), $k \geq 1$, which are deformations of general real K3 double rational scrolls. Moreover, we can even enumerate (classify) all types of these deformations.*

A polarized real K3 surface (X, cong, kP), where P is primitive and $k \geq 1$, defines an integral involution with a condition $(L = H_2(X, \mathbb{Z}), \varphi = \mathrm{cong}_*, S_n = [P])$ where $S_n = [P] =$

$\mathbb{Z}P \cong \langle n\rangle$ and $\theta = \phi|S_n = -1$. Theorem 3.10.1 in [19] gives that the connected component of moduli of (X, cong, kP) is defined by the isomorphism class of its integral involution (L, φ, S_n) with the condition $(S_n, -1)$. All invariants of the genus of (L, φ, S_n) are given by the invariants

$$(n; r, a, \delta_P, \delta_\varphi, \delta_{\varphi P}) \tag{8.4}$$

which are the same as in (2.20). All possibilities for the genus invariants (8.4) are given in Theorem 3.4.3 from [19] where one should put $l_{(+)} = 3$, $l_{(-)} = 19$, $t_{(+)} = 1$ and $t_{(-)} = r - 1$. The genus (8.4) defines the class (Theorem 3.3.1 in [19]), if $r \le 18 - \delta_{P\varphi}$ (actually, similarly to Theorem 14 one can give much better sufficient conditions).

Assume that the real K3 surface (X, τ, φ) has the integral involution (L, φ, S) of the type (S, θ), and (S, θ) and $P \in S$ is of one of types $t \in \mathcal{T}$ from (8.1) such that $n \in t$. Assume that (L, φ, S) has the invariants (2.21) which are

$$(r, a, H_+, H_-, \delta_{\varphi S}, v) \quad \text{where} \quad \delta_\varphi = \begin{cases} 0 & \text{if } (\delta_{\varphi S}, v) = (0,0) \\ 1 & \text{otherwise} \end{cases} \tag{8.5}$$

(we don't need q_ρ, if $\operatorname{rk} S \le 2$). Then the integral involution of the polarized K3 (X, P) is equal to $(L, \varphi, S_n = \mathbb{Z}P)$ where $S_n = \mathbb{Z}P \subset S$ is the sublattice generated by P. It follows that the invariants (r, a, δ_φ) of (X, P) in (8.4) are same as in (8.5). We have

$$\delta_P = \begin{cases} 0 & \text{if } P \mod 2S_- \in H_- \\ 1 & \text{otherwise} \end{cases}, \tag{8.6}$$

and

$$\delta_{\varphi P} = \begin{cases} 0 & \text{if } P \equiv v \mod 2S_- \\ 1 & \text{otherwise} \end{cases}. \tag{8.7}$$

Symbolically we write in this case

$$D_n : (t; r, a, H_+, H_-, \delta_{\varphi S}, v) \Longrightarrow (n; r, a, \delta_P, \delta_\varphi, \delta_{\varphi P}). \tag{8.8}$$

Applying results of previous Sections (especially Theorem 1), we get

Theorem 27. *Let us fix a type $t \in \mathcal{T}$ of a rational scroll. There exists a real polarized K3 surface (X, kP) with the invariants*

$$(n; r, a, \delta_P, \delta_\varphi, \delta_{\varphi P})$$

which is a deformation of a general real K3 double rational scroll t, if and only if $n \in t$ and

$$D_n : (t; r, a, H_+, H_-, \delta_{\varphi S}, v) \Longrightarrow (n; r, a, \delta_P, \delta_\varphi, \delta_{\varphi P}).$$

All these polarized K3 surfaces (X, kP) belong to one connected component of moduli of real polarized K3 surfaces.

Thus, to find all real polarized K3 surfaces which are deformations of general real K3 double rational scrolls, we need to find out when one has (8.8). These calculations are routine, and we only formulate the result. Using notations of Sects. 3 — 7, we get:

(8.9) **Types of deformations of general real K3 double rational scrolls:**

The Case of $\mathbb{P}^2$*:* $n \in \mathbb{P}^2$, if and only if $n = 2$; $P = h$;

$$D_2 : (\mathbb{P}^2; r, a, H = 0, \delta_{\varphi S}, v) \Longrightarrow (2; r, a, \delta_P = 1, \delta_\varphi = \delta_{\varphi S}, \delta_{\varphi P} = 1);$$

$$D_2 : (\mathbb{P}^2; r, a, H = [h], \delta_{\varphi S}, v) \Longrightarrow (2; r, a, \delta_P = 0, \delta_\varphi = 1, \delta_{\varphi P} = \delta_{\varphi S}).$$

See Figures 2 and 3 for the list of all possible invariants $(\mathbb{P}^2; r, a, H, \delta_{\varphi S}, v)$.

The Case of ellipsoid $\mathbb{E}$*:* $n \in \mathbb{E}$, if and inly if $n = 4$; $P = e_1 + e_2$;

$$D_4 : (\mathbb{E}; r, a, H = [e_1 + e_2], \delta_{\varphi S}, v) \Longrightarrow (4; r, a, \delta_P = 0, \delta_\varphi, \delta_{\varphi P}) \text{ where}$$

$$\delta_{\varphi P} = \begin{cases} 0 & \text{if } (\delta_{\varphi S}, v) = (0, e_1 + e_2) \\ 1 & \text{otherwise} \end{cases} .$$

See Figure 16 for the list of all possible invariants $(\mathbb{E}; r, a, H = [e_1 + e_2], \delta_{\varphi S}, v)$.

The Case of hyperboloid $\mathbb{H}_{(1)}$*.* $n \in \mathbb{H}_{(1)}$, if and only if $n \equiv 0 \mod 4$ and $n \geq 4$; $P = (n/4)e_1 + e_2$;

$$D_n : (\mathbb{H}_{(1)}; r, a, H = 0, \delta_{\varphi S}, v) \Longrightarrow (n; r, a, \delta_P = 1, \delta_\varphi = \delta_{\varphi S}, \delta_{\varphi P} = 1);$$

$$D_n : (\mathbb{H}_{(1)}; r, a, H = [e_1, e_2], \delta_{\varphi S}, v) \Longrightarrow (n; r, a, \delta_P = 0, \delta_\varphi = \delta_{\varphi S}, \delta_{\varphi P} = 1);$$

$$D_n : (\mathbb{H}_{(1)}; r, a, H = [e_1], \delta_{\varphi S}, v) \Longrightarrow (n; r, a, \delta_P = 1, \delta_\varphi, \delta_{\varphi P} = 1);$$

$$D_n : (\mathbb{H}_{(1)}; r, a, H = [e_2], \delta_{\varphi S}, v) \Longrightarrow (n; r, a, \delta_P, \delta_\varphi, \delta_{\varphi P}) \text{ where}$$

$$\delta_P = \begin{cases} 0 & \text{if } n \equiv 0 \mod 8 \\ 1 & \text{otherwise} \end{cases}, \quad \delta_{\varphi P} = \begin{cases} 0 & \text{if } (n \mod 8, \delta_{\varphi S}, v) = (0 \mod 8, 0, e_2) \\ 1 & \text{otherwise} \end{cases};$$

$$D_n : (\mathbb{H}_{(1)}; r, a, H = [e_1 + e_2] \mod 2, \delta_{\varphi S}, v) \Longrightarrow (n; r, a, \delta_P, \delta_\varphi, \delta_{\varphi P}) \text{ where}$$

$$\delta_P = \begin{cases} 0 & \text{if } n \equiv 4 \mod 8 \\ 1 & \text{otherwise} \end{cases}, \quad \delta_{\varphi P} = \begin{cases} 0 & \text{if } (n \mod 8, \delta_{\varphi S}, v) = (4 \mod 8, 0, e_1 + e_2) \\ 1 & \text{otherwise} \end{cases}.$$

See figures 6 — 9 for the list of all possible invariants $(\mathbb{H}; r, a, H, \delta_{\varphi S}, v)$.

The Case of hyperboloid $\mathbb{H}_{(2)}$*:* Change e_1 and e_2 places in the previous case.

The Case of $\mathbb{F}_1$*:* $n \in \mathbb{F}_1$, if and only if $n \equiv 2 \mod 4$ and $n \geq 6$; $P = ((n+2)/4)\, h + ((2-n)/4)\, e$;

$$D_n : (\mathbb{F}_1; r, a, H = 0, \delta_{\varphi S}, v) \Longrightarrow (n; r, a, \delta_P = 1, \delta_\varphi = \delta_{\varphi S}, \delta_{\varphi P} = 1);$$

$$D_n : (\mathbb{F}_1; r, a, H = [h, e], \delta_{\varphi S}, v) \Longrightarrow (n; r, a, \delta_P = 0, \delta_\varphi = 1, \delta_{\varphi P} = 1);$$

$$D_n : (\mathbb{F}_1; r, a, H = [h], \delta_{\varphi S}, v) \Longrightarrow (n; r, a, \delta_P, \delta_\varphi = 1, \delta_{\varphi P}) \text{ where}$$

$$\delta_P = \begin{cases} 0 & \text{if } n \equiv 2 \mod 8 \\ 1 & \text{otherwise} \end{cases}, \quad \delta_{\varphi P} = \begin{cases} 0 & \text{if } (n \mod 8, \delta_{\varphi S}) = (2 \mod 8, 0) \\ 1 & \text{otherwise} \end{cases};$$

$$D_n : (\mathbb{F}_1; r, a, H = [e], \delta_{\varphi S}, v) \Longrightarrow (n; r, a, \delta_P, \delta_\varphi = 1, \delta_{\varphi P}) \text{ where}$$

$$\delta_P = \begin{cases} 0 & \text{if } n \equiv -2 \mod 8 \\ 1 & \text{otherwise} \end{cases}, \quad \delta_{\varphi P} = \begin{cases} 0 & \text{if } (n \mod 8, \delta_{\varphi S}) = (-2 \mod 8, 0) \\ 1 & \text{otherwise} \end{cases};$$

$$D_n : (\mathbb{F}_1; r, a, H = [h + e], \delta_{\varphi S}, v) \Longrightarrow (n; r, a, \delta_P = 1, \delta_\varphi, \delta_{\varphi P} = 1).$$

See the list of all possible invariants $(\mathbb{F}_1; r, a, H, \delta_{\varphi S}, v)$ in figures 19 — 23.

The Case of $(\mathbb{F}_4)^{(1)}$*:* $n \in (\mathbb{F}_4)^{(1)}$, if and only if $n \equiv 0 \mod 2$ and $n \geq 4$; $P = (n/2+1)C + E$;

$$D_n : ((\mathbb{F}_4)^{(1)}; r, a, H = 0, \delta_{\varphi S}, v) \Longrightarrow (n; r, a, \delta_P = 1, \delta_\varphi = \delta_{\varphi S}, \delta_{\varphi P} = 1).$$

The Case of $(\mathbb{F}_4)^{(2)}$*:* $n \in (\mathbb{F}_4)^{(2)}$, if and only if $n \equiv 4 \mod 8$ and $n \geq 12$; $P = (n/4+2)C + 2E$;

$$D_n : ((\mathbb{F}_4)^{(2)}; r, a, H = 0, \delta_{\varphi S}, v) \Longrightarrow (n; r, a, \delta_P = 1, \delta_\varphi = \delta_{\varphi S}, \delta_{\varphi P} = 1).$$

See the figure 30 for the list of all possible invariants $(\mathbb{F}_4; r, a, H, \delta_{\varphi S}, v)$.

We can apply this calculations to answer the following interesting question: What are real K3 surfaces which are deformations of general real K3 double rational scrolls?

We have

Proposition 28. *A real polarized K3 surface (X, kP) where P is primitive and $n = P^2 \geq 6$ cannot be a deformation of a general real K3 double rational scroll, if either $(r, a) = (20, 2)$ (equivalently $X(\mathbb{R}) = (T_0)^{10}$) or $r + a = 22$ and $\delta_{\varphi P} = 0$ (equivalently, $X(\mathbb{R}) = (T_0)^m$ and $X(\mathbb{R}) \sim P \mod 2$ in $H_2(X, \mathbb{Z})$).*

Proof. If $r = 20$, then $L_\varphi = L_-$ is positive definite or the rank 2. It cannot contain S_- of the rank two because S_- is hyperbolic. Thus, $r = 20$ may happen only for $\mathbb{P}^2$ when $n = 2$, and for ellipsoid when $n = 4$. From the list of possible invariants (r, a, δ_φ) of real K3 surfaces (see Figure 1) we get that $a = 2$ and $\delta_\varphi = 1$, if $r = 20$. By (2.44), this is equivalent to $X(\mathbb{R}) = T_0^{10}$.

Assume that $r + a = 22$ and $n \geq 6$. This is possible only for $(\mathbb{H}; H = [e_1, e_2])$ and $(\mathbb{F}_1; H = [h, e])$, but all these cases give $\delta_{\varphi P} = 1$ by deformations (8.9). By (2.44), $r + a = 22$ is equivalent to $X(\mathbb{R}) = (T_0)^m$, and $\delta_{\varphi P} = 0$ is equivalent to $X(\mathbb{R}) \sim P \mod 2$ in $H_2(X, \mathbb{Z})$, by (2.45).

This proves the statement

We have

Theorem 29. *A genus invariant $(n; r, a, \delta_P, \delta_\varphi, \delta_{\varphi P})$ of a real polarized K3 surface (X, kP) can be obtained by the deformation D_n of some general real K3 double rational scroll $t \in \mathcal{T}$ from (8.1), if and only if the following condition is valid:*

$$(8.10) \qquad n \leq 4, \text{ if either } (r, a) = (20, 2) \text{ or } r + a = 22 \text{ and } \delta_{\varphi P} = 0$$

(we just exclude cases of Proposition 28). All possible types of these deformations are given in (8.9).

Proof. Using Theorem 3.4.3 from [19], we can get the list of all possible invariants $(n; r, a, \delta_\varphi, \delta_{\varphi P})$ of real polarized K3 when $(r, a) \neq (20, 2)$ and $r + a < 22$, if $\delta_{\varphi P} = 0$. They depend on $n \mod 8$ and are presented in figures 33 — 39 (when $r + a = 22$ and $\delta_{\varphi P} = 0$, the result of [19] is more complicated: it depends on $n \mod 16$ when $(r, a) \neq (20, 2)$; if $(r, a) = (20, 2)$, it depends even on prime decomposition of n). By inspection (which requires some time and patience), one can see that all these invariants can be obtained applying all possible deformations (8.9) from $\mathbb{P}^2$, $\mathbb{H}$, $\mathbb{F}_1$ and $\mathbb{F}_4$. Again applying Theorem 3.4.3 from [19] when $n \leq 4$ and either $(r, a) = (20, 2)$ or $r + a = 22$ and $\delta_{\varphi P} = 0$, we get all of them as deformations of $\mathbb{P}^2$ and $\mathbb{E}$. The list of all possible invariants for $n = 4$ is given in figures 36 and 41.

It proves the statement.

As a particular result, we get that any non-singular real quartic in $\mathbb{P}^3$ can be obtained as a deformation of a general real double K3 cover of ellipsoid, hyperboloid or $\mathbb{F}_4$. These deformations can be given explicitly [26] and give an effective method of construction of all (i. e. representatives of all connected components of moduli) examples of real quartics in $\mathbb{P}^3$.

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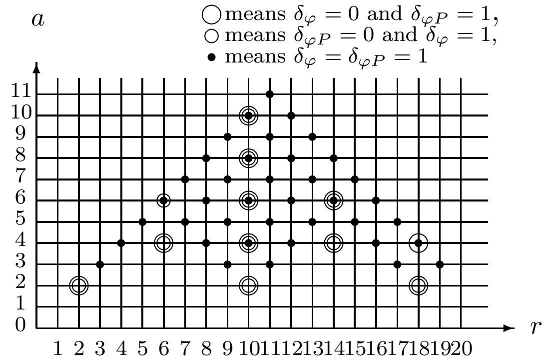

FIGURE 33. Polarized K3 when $n \equiv 0 \mod 8$ and $\delta_P = 0$ without $(r, a) = (20, 2)$ and $(r + a = 22, \delta_{\varphi P} = 0)$

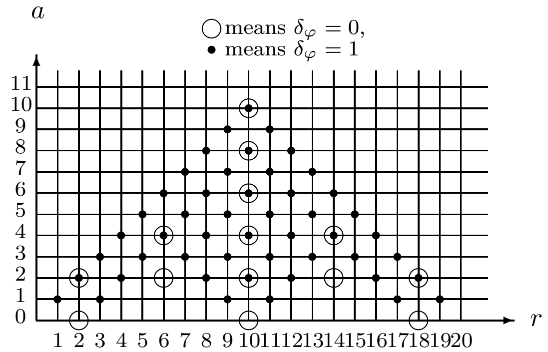

FIGURE 34. Polarized K3 when $n \equiv 0 \mod 8$ and $\delta_P = 1$ (then $\delta_{\varphi P} = 1$)

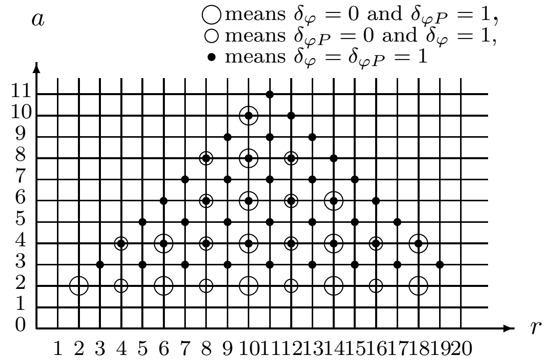

FIGURE 35. Polarized K3 when $n \equiv 4 \mod 8$ and $\delta_P = 0$ without $(r, a) = (20, 2)$ and $(r + a = 22, \delta_{\varphi P} = 0)$

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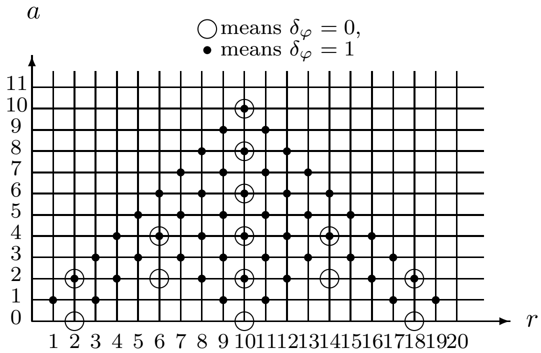

FIGURE 36. Polarized K3 when $n \equiv 4 \mod 8$ and $\delta_P = 1$ (then $\delta_{\varphi P} = 1$)

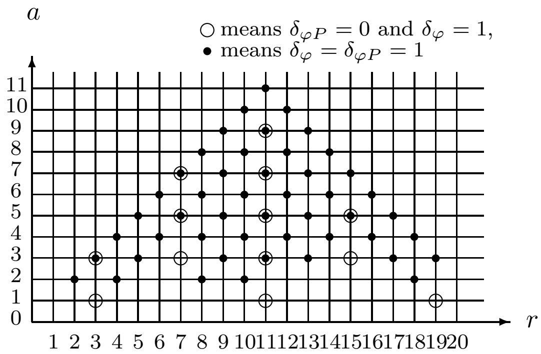

FIGURE 37. Polarized K3 when $n \equiv 2 \mod 8$ and $\delta_P = 0$ (then $\delta_\varphi = 1$) without $(r, a) = (20, 2)$ and $(r + a = 22, \delta_{\varphi P} = 0)$

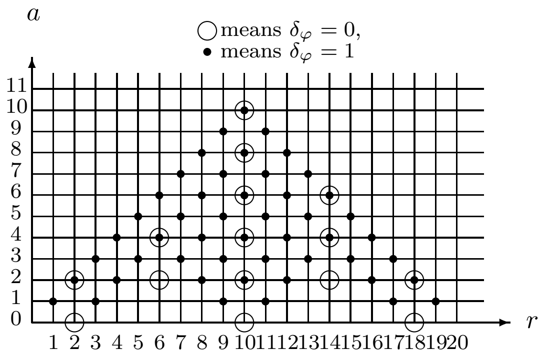

FIGURE 38. Polarized K3 when $n \equiv 2 \mod 8$ and $\delta_P = 1$ (then $\delta_{\varphi P} = 1$)

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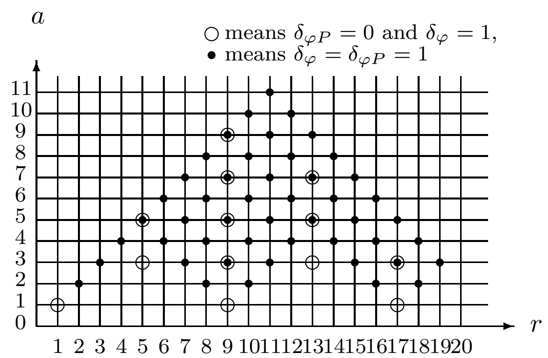

FIGURE 39. Polarized K3 when $n \equiv -2 \mod 8$ and $\delta_P = 0$ (then $\delta_\varphi = 1$) without $(r, a) = (20, 2)$ and $(r + a = 22, \delta_{\varphi P} = 0)$

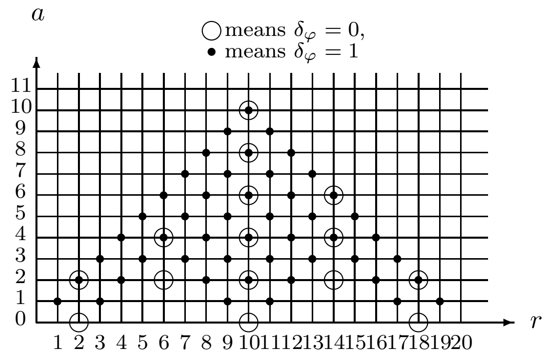

FIGURE 40. Polarized K3 when $n \equiv -2 \mod 8$ and $\delta_P = 1$ (then $\delta_{\varphi P} = 1$)

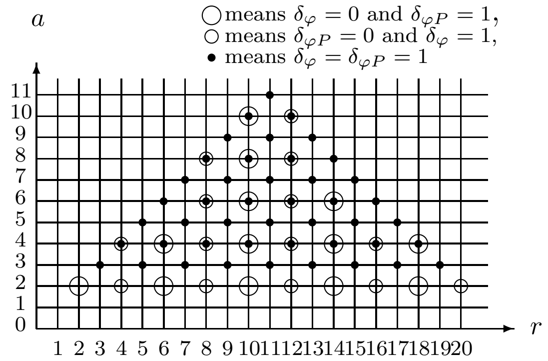

FIGURE 41. Polarized K3 when $n = 4$ and $\delta_P = 0$

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