Max-Planck-Institut für Mathematik
Bonn

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by

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Max-Planck-Institut für Mathematik
Preprint Series 2018 (57)

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MPIM 18-57

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Abstract

Motivated by the tropical enumeration of plane cuspidal tropical curves given by Y. Ganor and the author and the refined count of plane rational tropical curves with marked vertices of arbitrary valency, we suggest a refined enumeration of plane rational tropical curves with one unmarked vertex of arbitrary valency. Our invariant extends (in the rational case) the Block-Göttsche refined invariant counting trivalent tropical curves and the refined rational descendant tropical invariant suggested by L. Blechman and the author.

MSC-2010: 14N10, 14T05

Introduction

Existence of refined (i.e., depending on a formal parameter) enumerative invariants is one of the most interesting phenomena in the tropical enumerative geometry. They were discovered by F. Block and L. Göttsche [2] in the problem of enumeration of plane trivalent tropical curves with unmarked vertices (see also [7] for the invariance statement). L. Göttsche and F. Schroeter [6] found a refined invariant counting plane rational trivalent tropical curves with marked vertices. The latter invariant was extended by L. Blechman and the author [1] to the case of of plane rational tropical curves with marked vertices of arbitrary valency and all unmarked vertices trivalent. In this note we suggest a refined enumeration of plane rational tropical curves with marked points of arbitrary valency and one unmarked point of arbitrary valency (while other unmarked points are trivalent). We were inspired by the tropical enumeration of plane unicuspidal curves [3], which necessarily leads to plane tropical curves with a four-valent unmarked vertex.

The Block-Göttsche invariants can be explicitly related to the enumeration of complex and real nodal algebraic curves on toric surfaces. In the case of plane

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tropical curves with marked points on edges and all but one vertices trivalent, while one vertex is four-valent, our invariant, evaluated at $y = 1$ gives the number of unicuspidal tropical curves passing through an appropriate generic configuration of points and tropicalizing to the given tropical curve.

Acknowledgements. The author has been supported by the Israel Science Foundation grants no. 176/15 and 501/18. A part of this research was performed during the author's stay at the Max-Planck Institut für Mathematik in August-September 2018. The author thanks the MPIM for support and excellent working conditions.

1 Plane marked rational tropical curves

We shortly recall some basic definitions concerning rational tropical curves adapted to our setting and define the class of tropical curves under consideration (for details, see [4, 5, 8, 9]).

(1) A *plane n-marked rational tropical curve* is a triple $(\Gamma, h, \boldsymbol{p})$, where

- Γ is either isometric to $\mathbb{R}$, or is a finite connected metric tree without vertices of valency ≤ 2, whose set Γ^0 of vertices in nonempty, the set of edges Γ^1 contains a subset $\Gamma^1_\infty \ne \emptyset$ consisting of edges isometric to $[0, \infty)$ (called *ends*), while $\Gamma^1 \setminus \Gamma^1_\infty$ consists of edges isometric to compact segments in $\mathbb{R}$ (called *finite edges*);
- $h : \Gamma \to \mathbb{R}^2$ is a proper continuous map such that h is nonconstant, affine-integral on each edge of Γ in the length coordinate and, at each vertex V of Γ, the balancing condition holds
$$\sum_{E \in \Gamma^1,\ V \in E} \boldsymbol{a}_V(E) = 0 ,$$
where $\boldsymbol{a}_V(E)$ (called the *directing vector* of E centered at V) is the image under the differential $D(h\big|_E)$ of the unit tangent vector to E emanating from its endpoint V;
- $\boldsymbol{p}$ is a sequence of n distinct points of Γ.

The multiset of vectors $\deg(\Gamma, h) := \left\{\boldsymbol{a}_V(E)\ :\ E \in \Gamma^1_\infty\right\} \subset \mathbb{Z}^2 \setminus \{0\}$ is called the *degree* of $(\Gamma, h, \boldsymbol{p})$. Clearly the vectors of $\deg(\Gamma, h)$ sum up to zero (we call such a multiset *balanced*). The degree Δ is called *nondegenerate* if $\dim \operatorname{Span}\{\boldsymbol{a} \in \Delta\} = 2$, and is called *primitive* if all vectors $\boldsymbol{a} \in \Delta$ are primitive integral vectors.

Denote by $\mathbb{Z}^\infty_+$ the set of sequences of nonnegative integers $(k_i)_{i \ge 0}$ such that $\sum_i k_i < \infty$, and by $\mathbb{Z}^{\infty,*}_+ \subset \mathbb{Z}^\infty_+$ the set of sequences with the vanishing initial member. Let $\overline{m} = (m_i)_{i \ge 0} \in \mathbb{Z}^{\infty,*}_+$, $\overline{n} = (n_i)_{i \ge 0} \in \mathbb{Z}^\infty_+$. We say that $(\Gamma, h, \boldsymbol{p})$ is of V-type $(\overline{m}, \overline{n})$, if exactly m_i vertices in $\Gamma^0 \setminus \boldsymbol{p}$ have valency $i + 2$ for all $i \ge 1$, exactly n_0

points of $\boldsymbol{p}$ lie in $\Gamma \setminus \Gamma^0$, exactly n_i vertices in $\Gamma^0 \cap \boldsymbol{p}$ have valency $i+2$ for all $i \geq 1$. It is easy to see that

$$|\Gamma^0| = \sum_{i\geq 1}(m_i + n_i), \quad |\Gamma^1| = \sum_{i\geq 1}(i+1)(m_i+n_i)+1, \quad |\Gamma^1_\infty| = \sum_{i\geq 1} i(m_i+n_i)+2\ . \quad (1)$$

(2) Two plane n-marked rational tropical curves $(\Gamma, h, \boldsymbol{p})$ and $(\Gamma', h', \boldsymbol{p}')$ are called isomorphic, if there exists an isometry $\varphi : \Gamma \to \Gamma'$ such that $h = h' \circ \varphi$ and $\varphi(p_i) = p'_i$ for all $p_i \in \boldsymbol{p}, p'_i \in \boldsymbol{p}'$, $i = 1, ..., n$. Assuming that a finite balanced multiset $\Delta \subset \mathbb{Z}^2 \setminus \{0\}$ and sequences of nonnegative integers $\overline{m} \in \mathbb{Z}^{\infty,*}_+$, $\overline{n} \in \mathbb{Z}^\infty_+$, satisfy

$$|\Delta| = \sum_{i\geq 1} i(m_i+n_i)+2 \quad (2)$$

(cf. (1)), we consider the moduli space $\mathcal{M}_{0,\overline{n},\overline{m}}(\mathbb{R}^2, \Delta)$ parameterizing isomorphism classes $[(\Gamma, h, \boldsymbol{p})]$ of plane $n = \sum_{i\geq 0} n_i$-marked rational tropical curves of V-type $(\overline{m}, \overline{n})$ and degree Δ.

Lemma 1.1 *The space $\mathcal{M}_{0,\overline{n},\overline{m}}(\mathbb{R}^2, \Delta)$ can be identified with a finite union of open convex polyhedral cones of pure dimension $\sum_{i\geq 0}(m_i+n_i)+1$.*

Proof. Given a combinatorial type of the pair $(\Gamma, \boldsymbol{p})$ and the distribution of the directing vectors $\boldsymbol{a}_V(E) \in \mathbb{Z}^2 \setminus \{0\}$ for all edges $E \in \Gamma^1$, the lengths of the finite edges, the distances from marked points in $\Gamma \setminus \Gamma^0$ to chosen vertices of the corresponding edges, and the freely chosen image $h(V)$ of a fixed vertex $V \in \Gamma^0$ give $N = \sum_{i\geq 0}(m_i+n_i)+1$ independent coordinates in the positive orthant of $\mathbb{R}^N$, from which one should get rid suitable diagonals in case when more than one marked points occur in the interior of the same edge of Γ. $\square$

By $\widehat{\mathcal{M}}_{0,\overline{n},\overline{m}}(\mathbb{R}^2, \Delta)$ we denote the polyhedral fan obtained by extending $\mathcal{M}_{0,\overline{n},\overline{m}}(\mathbb{R}^2, \Delta)$ with classes corresponding either to contraction of some finite edges (i.e., vanishing of their lengths), or to arrival of marked points from edges to vertices of Γ, or to collision of marked points.

Assume that $2n = \sum_{i\geq 0}(m_i+n_i)+1$, or, equivalently,

$$n = \sum_{i\geq 1} m_i + 1\ . \quad (3)$$

Then the evaluation map

$$\mathrm{Ev} : \widehat{\mathcal{M}}_{0,\overline{n},\overline{m}}(\mathbb{R}^2, \Delta) \to \mathbb{R}^{2n}, \quad \mathrm{Ev}[(\Gamma, h, \boldsymbol{p})] = h(\boldsymbol{p}) \in \mathbb{R}^{2n}\ ,$$

relates spaces of the same dimension $\sum_{i\geq 0}(m_i+n_i)+1 = 2n$.

Definition 1.2 *Let a balanced, nondegenerate multiset $\Delta \subset \mathbb{Z}^2 \setminus \{0\}$ and sequences $\overline{m} \in \mathbb{Z}^{\infty,*}_+$, $\overline{n} \in \mathbb{Z}^\infty_+$ satisfy (2) and (3).*

(1) We say that a class $[(\Gamma, h, \boldsymbol{p})] \in \mathcal{M}_{0,\overline{n},\overline{m}}(\mathbb{R}^2, \Delta)$ is regular, if each connected component K of $\Gamma \setminus \boldsymbol{p}$ is unbounded, and its closure $\overline{K} \subset \Gamma$ possesses a unique orientation of its edges (called regular orientation) such that

- *all marked points in $\overline{K}$ are sources, all ends of $\overline{K}$ are oriented towards infinity;*
- *for each vertex $V \in K \cap \Gamma^0$ exactly two of its incident edges are incoming, and, moreover, the h-images of these edges are not collinear.*

(2) A cell of $\mathcal{M}_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$ is called enumeratively essential, if Ev *injectively takes it to $\mathbb{R}^{2n}$. Denote by $\mathcal{M}^e_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$ the union of the enumeratively essential cells of $\mathcal{M}_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$, by $\widehat{\mathcal{M}}^e_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$ the closure of $\mathcal{M}^e_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$ in $\widehat{\mathcal{M}}_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$, and by* Ev^e *the restriction of* Ev *to $\widehat{\mathcal{M}}^e_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$.*

Lemma 1.3 *Let $\Delta \subset \mathbb{Z}^2 \setminus \{0\}$ be a balanced nondegenerate multiset, $\overline{m} \in \mathbb{Z}^{\infty,*}_{+}$, $\overline{n} \in \mathbb{Z}^{\infty}_{+}$. Suppose that (2) and (3) hold. Then $\mathcal{M}^e_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta) \neq \emptyset$ and each cell of $\mathcal{M}^e_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$ consists of regular classes.*

Proof. Suppose that $[(\Gamma,h,\boldsymbol{p})] \in \mathcal{M}_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$ is a regular class. Then it belongs to $\mathcal{M}^e_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$. Indeed, whenever we fix the position of $h(\boldsymbol{p})$, the image of $h : \Gamma \to \mathbb{R}^2$ is fixed as well (recall that the combinatorial type of $(\Gamma,\boldsymbol{p})$ and the differentials of h on the edges of Γ are a priori fixed), and then we recover the lengths of compact edges of Γ. On the other hand, it immediately follows from the regularity that any small variation of $h(\boldsymbol{p})$ induces a (unique) small variation of $(\Gamma,h,\boldsymbol{p})$ in its combinatorial class.

For the proof of the existence of a regular class, we make the following elementary observation (left to the reader as an exercise):

(O) Let $|\Delta| > 3$ and let Δ by cyclically ordered by rotation in the positive direction. Then, for any $\boldsymbol{a}_i \in \Delta$, which is not simultaneously collinear to $\boldsymbol{a}_{i-1}$ and $\boldsymbol{a}_{i+1}$, and any $1 \le j \le |\Delta|-2$, there exist a sequence $\boldsymbol{a}_k,...,\boldsymbol{a}_{k+j} \in \Delta$, including $\boldsymbol{a}_i$, such that $\dim \mathrm{Span}\{\boldsymbol{a}_k,...,\boldsymbol{a}_{k+j}\} = 2$, $\sum_{s=k}^{k+j} \boldsymbol{a}_s \neq 0$, and the multiset $\Delta' = (\Delta \setminus \{\boldsymbol{a}_k,...,\boldsymbol{a}_{k+j}\}) \cup \{\boldsymbol{a}'\}$, where $\boldsymbol{a}' = \boldsymbol{a}_k + ... + \boldsymbol{a}_{k+j}$, is balanced and nondegenerate.

Then we proceed in the same way as the existence statement in [1, Lemma 1.4]. We remind here this argument referring to [1] for the details. First, we construct the (convex lattice) Newton polygon $P(\Delta)$, whose boundary can be represented as the union of cyclically ordered integral segments $[v_k,v_{k+1}]$, $k = 1,...,|\Delta|$, $v_{|\Delta|+1} = v_1$, obtained by rotating the ordered as above vectors $\boldsymbol{a}_k \in \Delta$, $k = 1,...,|\Delta|$, by $\frac{\pi}{2}$ clockwise (we call $\boldsymbol{a}_k$ and $[v_k,v_{k+1}]$ dual to each other). The set $\mathcal{V} = \{v_1,...,v_{|\Delta|}\}$ includes all the vertices of $P(\Delta)$.

We proceed by induction on n. If $n = 1$, the curve Γ is a fan with the center at the unique (marked) vertex, and $(\Gamma,h,\boldsymbol{p})$ is regular. Suppose that $n > 1$. Then (cf. (3)) there are $n_i > 0$ and $m_j > 0$. If $j = |\Delta| - 2$ and respectively $n = n_0 = 2$, then γ again has a unique (unmarked) vertex, and we pick two marked points on two ends with non-collinear directing vectors, obtaining a regular curve $(\Gamma,h,\boldsymbol{p})$. Suppose that $j \le |\Delta| - 3$. If $i = 0$, we choose $\boldsymbol{a}_k$, which is not parallel both to $\boldsymbol{a}_{k-1}$ and $\boldsymbol{a}_{k+1}$, then find a sequence $\boldsymbol{a}_s,...,\boldsymbol{a}_{s+j-1}$ as in observation (O), and, finally draw the chord in $P(\Delta)$ joining the points v_s and v_{s+j}. It follows that the interior of this chord is disjoint from $\partial P(\Delta)$. The chord cuts $P(\Delta)$ into a polygon containing $j+2$

points of $\mathcal{V}$ and the remaining polygon $P(\Delta')$, where $\Delta' = (\Delta \setminus \{\boldsymbol{a}_s, ..., \boldsymbol{a}_{s+j-1}\}) \cup \{\boldsymbol{a}'\}$, $\boldsymbol{a}' = \boldsymbol{a}_s + ... + \boldsymbol{a}_{s+j-1}$. The former polygon (with the corresponding part of $\mathcal{V}$) is dual to a tropical curve with the unique (unmarked) vertex of valency $j+2$, a marked point on the end directed by the vector $\boldsymbol{a}_k$, and the end directed by the vector $-\boldsymbol{a}'$, to which we attach the remaining part of the constructed curve existing due to the induction assumption applied to Δ' and $\overline{n}'$, $\overline{m}'$, obtained by reducing n_i and m_j by one. The regularity of the constructed tropical curve is evident. Suppose that $n_0 = 0$ and $i > 0$. Then $i + j \le |\Delta| - 3$. We, first, choose a sequence $\boldsymbol{a}_k, ..., \boldsymbol{a}_{k+i}$ as in observation (O), join the points $v_k, v_{k+i} \in \mathcal{V}$ by a chord, whose interior must be disjoint from $\partial P(\Delta$. It cuts off $P(\Delta)$ an polygon P_1 that will be dual to a marked point of valency $i+2$ incident to $i+1$ ends directed by $\boldsymbol{a}_k, ..., \boldsymbol{a}_{k+i}$ and to one more edge dual to the chord. Set $\Delta' = (\Delta \setminus \{\boldsymbol{a}_k, ..., \boldsymbol{a}_{k_i}\}) \cup \{\boldsymbol{a}'\}$, $\boldsymbol{a}' = \boldsymbol{a}_k + ..., \boldsymbol{a}_{k+i}$. Since the chord is not collinear with the neighboring sides of $P(\Delta)$, we apply observation (O) to Δ' and obtain a sequence of $j+1$ vectors of Δ' (including $\boldsymbol{a}'$), whose dual segments form a connected part of $\partial P(\Delta')$, and the extreme points of this part are joined by a chord which intersects $\partial P(\Delta')$ only in its endpoints. Thus, we cut off $P(\Delta')$ a polygon P_2 which will be dual to an unmarked vertex of valency $j+2$incident to j ends, an edge dual to the first constructed chord, and one more edge dual to the second chord. So, we attach the two constructed fragment by gluing along the edges dual to the first chord, and, finally, apply the induction assumption to Δ'' that is formed by the vectors $\boldsymbol{a}_s$ dual to the remaining segments $[v_r, v_{r+1}]$ and by the vector $\boldsymbol{a}''$ equal to the sum of all removed vectors $\boldsymbol{a}_s$ (and dual to the second chord), while $\overline{n}$ and $\overline{m}$ turn into $\overline{n}', \overline{m}'$ by reducing 1 from n_i and m_j.

Suppose now that $[(\Gamma, h, \boldsymbol{p})] \in \mathcal{M}^e_{0,\overline{n},\overline{m}}(\mathbb{R}^2, \Delta)$. This means that $\boldsymbol{x} = h(\boldsymbol{p})$ is a general position in $\mathbb{R}^2$. Using induction on $|\Gamma^0|$, we show that $(\Gamma, h, \boldsymbol{p})$ is regular. If $|\Gamma^0| \le 1$, the claim is evident. Assume that $|\Gamma^0| > 1$. Since one can consider all components of $\Gamma \setminus \boldsymbol{p}$ separately and independently, we are left with the case when all marked points belong to ends of Γ, when no two points lie on the same end or on collinear ends. The relation $n = |\Gamma^0| + 1$ (cf. (3)) yields that there are two ends with marked points incident to the same vertex $V \in \Gamma^0$. Note that no any other end with a marked point is incident to V due to the general position of $\boldsymbol{x}$. So, we orient the segments on the chosen above two ends of Γ, which join the marked points with V, towards V, while all other edges of Γ incident to V are oriented outwards. Thus, we reduce the considered case to the study of the connected components of $\Gamma \setminus \{V\}$, and hence derive the required regularity by the induction assumption. ◻

Remark 1.4 *As shown in [1, Proof of Lemma 1.4], the subdivision of $P(\Delta)$ constructed in the proof of Lemma 1.3 can be further refined by extra chords between the points of $\mathcal{V}$ so that the final subdivision will consist of $|\Delta| - 2$ nondegenerate triangles with vertices in $\mathcal{V}$.*

Set

$$Y^{2n-1} = \mathrm{Ev}\Big(\widehat{\mathcal{M}}^e_{0;\overline{m}}(\mathbb{R}^2, \Delta) \setminus \mathcal{M}^e_{0;\overline{m}}(\mathbb{R}^2, \Delta)\Big)\,. \tag{4}$$

This is a polyhedral complex of dimension $\le 2n-1$ in $\mathbb{R}^{2n}$. Denote by X^{2n-1} the union (maybe empty) of open $(2n-1)$-dimensional cells of Y^{2n-1}. Then $X^{2n-2} := Y^{2n-1} \setminus X^{2n-1}$ is a finite polyhedral complex of dimension $\le 2n-2$.

Lemma 1.5 *Under the hypotheses of Lemma 1.3, suppose that $X^{2n-1} \ne \emptyset$. Then, for each $\boldsymbol{x} \in X^{2n-1}$, the preimage $(\mathrm{Ev}^e)^{-1}(\boldsymbol{x})$ consists of regular classes, or classes $[(\Gamma, h, \boldsymbol{p})]$ such that*

(a) $(\Gamma, h, \boldsymbol{p})$ is of V-type $(\overline{m}', \overline{n}')$, where $m'_i = m_i$ for all $i \ge 1$ except for $m'_{i_1} = m_{i_1} - 1$, and $n'_i = n_i$ for all $i \ge 0$ except for $n'_{i_2} = n_{i_2} - 1$ and $n'_{i_1+i_2} = n_{i_1+i_2} + 1$; furthermore, exactly one connected component of $\Gamma \setminus \boldsymbol{p}$ is not regular;

(b) Γ is of V-type $(\overline{m}', \overline{n})$, where $m'_i = m_i$ for all $i \ge 1$ except either for $m'_{i_1} = m_{i_1} - 2$, $m'_{2i_1-2} = m_{2i_1-2} + 1$ with some $i_1 \ge 1$, or for $m'_{i_1} = m_{i_1} - 1$, $m'_{i_2} = m_{i_2} - 1$, $m'_{i_1+i_2-2} = m_{i_1+i_2-2} + 1$ with some $i_2 > i_1 \ge 1$; furthermore, exactly one connected component of $\Gamma \setminus \boldsymbol{p}$ is not regular.

Proof. By construction, a non-regular element $[(\Gamma, h, \boldsymbol{p})] \in (\mathrm{Ev}^e)^{-1}(\boldsymbol{x})$ is a limit of regular classes $[(\Gamma_t, h_t, \boldsymbol{p}_t)]$, $0 < t < \varepsilon$, and is obtained by vanishing of exactly of the parameters in the corresponding cell of $\mathcal{M}^e_{0,\overline{n},\overline{m}}(\mathbb{R}^2, \Delta)$. If the vanishing parameter is the length of a segment joining a marked point $p_{k,t}$ and a vertex $V_t \in \Gamma^0_t \setminus \boldsymbol{p}_t$, then we get to the case (a). The only non-regular component of $\Gamma \setminus \boldsymbol{p}$ is the component which contains the limit of the edge of $\Gamma_t \setminus \boldsymbol{p}_t$, which is not incident to $p_{k,t}$ and is regularly oriented towards V_t. If the vanishing parameter is the length of the edge joining two vertices of $\Gamma^0_t \setminus \boldsymbol{p}_t$ and not containing points of $\boldsymbol{p}_t$, then we get to the case (b). The only non-regular component of $\Gamma \setminus \boldsymbol{p}$ is that with the vertex appeared in the collision of two vertices of $\Gamma^0_t \setminus \boldsymbol{p}_t$: the regularity fails, since the new vertex is incident to three incoming edges. Note that no two of these three edges have collinear directing vectors, since otherwise the dimension of the corresponding cell of Y^{2n-1} would not exceed $2n-2$. $\square$

2 The refined cuspidal invariant

2.1 Preparation

Throughout this section, we fix a standard basis in $\mathbb{R}^2$, and for any $\boldsymbol{a} = (a_1, a_2)$, $\boldsymbol{b} = (b_1, b_2) \in \mathbb{R}^2$, set $\boldsymbol{a} \wedge \boldsymbol{b} = \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$.

For any $\alpha \in \mathbb{R}$ and a formal parameter y, define

$$[\alpha]^-_y = \frac{y^{\alpha/2} - y^{-\alpha/2}}{y^{1/2} - y^{-1/2}}, \quad [\alpha]^+_y = \frac{y^{\alpha/2} + y^{-\alpha/2}}{y^{1/2} + y^{-1/2}} \ . \tag{5}$$

Let us be given a balanced, nondegenerate multiset $\Delta \subset \mathbb{Z}^2 \setminus \{0\}$ and sequences $\overline{n} \in \mathbb{Z}^\infty_+$, $\overline{m} \in \mathbb{Z}^{\infty,*}_+$ satisfying (2) and (3).

We will use also labeled tropical curves. A labeling of a tropical curve $(\Gamma, h, \boldsymbol{p})$ is a linear order on the set of the ends Γ^1_∞. To simplify notations, we use the same symbol Γ when it is clearly indicated whether the curve is labeled or not, and we write $\Gamma^{\rm lab}$, where it is not the case. Denote by $\widehat{\mathcal{M}}^{e,{\rm lab}}_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$ the moduli space of labeled n-marked plane rational tropical curves that project to $\widehat{\mathcal{M}}^{e}_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$.

Lemma 2.1 *The projection forgetting labels*

$$\pi^e_{0,\overline{n},\overline{m}}:\widehat{\mathcal{M}}^{e,{\rm lab}}_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)\to\widehat{\mathcal{M}}^{e}_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$$

is a finite, surjective map, and, for any element $[(\Gamma,h,\boldsymbol{p})]\in\widehat{\mathcal{M}}^{e}_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$, *we have*

$$\left|(\pi^e_{0,\overline{n},\overline{m}})^{-1}[(\Gamma,h,\boldsymbol{p})]\right|=\frac{|\Delta|!}{|{\rm Aut}(\Gamma,h,\boldsymbol{p})|}\ . \tag{6}$$

Proof. We explain only formula (6). The group of permutations of $|\Delta|$ elements transitively acts on $(\pi^e_{0,\overline{n},\overline{m}})^{-1}[(\Gamma,h,\boldsymbol{p})]$. Then it remains to notice that a permutation belongs to a stabilizer of an element of the above preimage if and only if it is induced by an automorphism of $(\Gamma,h,\boldsymbol{p})$. $\square$

2.2 Refined multiplicity of a regular plane rational marked tropical curve

Now we introduce an additional restriction:

$$\sum_{i\ge2}m_i\le1\ . \tag{7}$$

It means that all but at most one unmarked vertices are trivalent.

Remark 2.2 *The refined multiplicity of plane tropical curves which we give below naturally extends to arbitrary $\overline{m}$ and $\overline{n}$ satisfying (3). However the invariance statement holds only under restriction (7). We do not know how to correct the refined multiplicity in order to obtain an invariant in the general case.*

Let $[(\Gamma,h,\boldsymbol{p})]\in\mathcal{M}^{e}_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$, and let $(\Gamma^{\rm lab},h,\boldsymbol{p})$ be one of the labelings of $(\Gamma,h,\boldsymbol{p})$. We start with defining a refined cuspidal multiplicity $RCM_y(\Gamma,h,\boldsymbol{p},V)$ (depending on a formal parameter y) for each vertex $V\in\Gamma^0$.

(1) Refined cuspidal multiplicity of a trivalent unmarked vertex. Suppose that $V\in\Gamma^0$ is trivalent and the regularly oriented edges $E_1,E_2\in\Gamma^1$ incident to V are incoming. Define the *Mikhalkin's multiplicity* of the vertex V by (cf. [8, Definition 2.16])

$$\mu(\Gamma,h,\boldsymbol{p},V)=|\boldsymbol{a}_1\wedge\boldsymbol{a}_2|,\quad\text{where}\quad\boldsymbol{a}_i=D(h\big|_{E_i})(\boldsymbol{a}_V(E_i)),\ i=1,2\ .$$

Following [2], we put

$$RCM_y(\Gamma, h, \boldsymbol{p}, V) = [\mu(\Gamma, h, \boldsymbol{p}, V)]_y^- . \tag{8}$$

(2) The function $\mu_y^+(A)$. We recall here the definition of the function $\mu_y^+(A)$ for any balanced sequence $A = (\boldsymbol{a}_i)_{i=1,...,r}$, $r \ge 2$, $\boldsymbol{a}_i \in \mathbb{R}^2$, $i = 1, ..., r$, as given in [1, Section 2.1, item (2)]. If $r = 2$, we set $\mu_y^+(A) = 1$. If $r = 3$, we set $\mu_y^+(A) = [|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|]_y^+$. Note that, due to the balancing condition, this definition does not depend on the choice of the order in the sequence A. If $r \ge 4$, then, for each pair $1 \le i < j \le m$, we form the two balanced sequences

- A'_{ij} consisting of the vectors $\boldsymbol{a}_k$, $1 \le k \le r$, $k \ne i, j$, and one more vector $\boldsymbol{a}_{ij} := \boldsymbol{a}_i + \boldsymbol{a}_j$,
- $A''_{ij} = (\boldsymbol{a}_i, \boldsymbol{a}_i, -\boldsymbol{a}_{ij})$.

Then we set

$$\mu_y^+(A) = \sum_{1 \le i < j \le m} \mu_y^+(A'_{ij}) \cdot \mu_y^+(A''_{ij}) . \tag{9}$$

It is easy to see that $\mu_y^+(A)$ does not depend on the choice of the order in A.

(3) The refined cuspidal multiplicity of a marked vertex. Given a marked vertex $V \in \Gamma^0 \cap \boldsymbol{p}$ and the directing vectors $\boldsymbol{a}_1, ..., \boldsymbol{a}_r$ of all the edges incident to, we set (cf. [1, Formula (9)])

$$RCM_y(\Gamma, h, \boldsymbol{p}, V) = \mu_y^+(A_V), \quad A_v = (\boldsymbol{a}_1, ..., \boldsymbol{a}_r) . \tag{10}$$

(4) The refined cuspidal multiplicity of an unmarked vertex of valency ≥ 4. Given an unmarked vertex $V \in \Gamma^0 \setminus \boldsymbol{p}$ of valency $r \ge 4$ and the directing vectors $\boldsymbol{a}_1, ..., \boldsymbol{a}_r$ of all its incident edges of Γ, ordered so that $\boldsymbol{a}_1, \boldsymbol{a}_2$ direct the edges regularly oriented towards V, we set

$$RCM_y(\Gamma, h, \boldsymbol{p}, V) = [|\boldsymbol{a}_1 \wedge \boldsymbol{a}_2|]_y^- \cdot \mu_y^+(A'_V), \quad A'_V = (\boldsymbol{a}_1 + \boldsymbol{a}_2, \boldsymbol{a}_3, ..., \boldsymbol{a}_r) . \tag{11}$$

(5) The refined cuspidal multiplicity of a regular plane rational marked tropical curve. Given $[(\Gamma, h, \boldsymbol{p})] \in \mathcal{M}^e_{0,\overline{n},\overline{m}}(\mathbb{R}^2, \Delta)$, define

$$RCM_y(\Gamma^{\rm lab}, h, \boldsymbol{p}) = \prod_{V \in \Gamma^0} RCM_y(\Gamma, h, \boldsymbol{p}, V), \quad RCM_y(\Gamma, h, \boldsymbol{p}) = \frac{RCM_y(\Gamma^{\rm lab}, h, \boldsymbol{p})}{|\mathrm{Aut}(\Gamma, h, \boldsymbol{p})|} . \tag{12}$$

2.3 The invariance statement

Theorem 2.3 *Let $\Delta \subset \mathbb{Z}^2 \setminus \{0\}$ be a balanced, nondegenerate multiset, $\overline{m} \in \mathbb{Z}_+^{\infty,*}$, $\overline{n} \in \mathbb{Z}_+^{\infty}$, and let (2), (3) and the restriction (R) hold. Then the expression*

$$RC_y(\Delta, \overline{m}, \boldsymbol{x}) := \sum_{[(\Gamma, h, \boldsymbol{p})] \in (\mathrm{Ev}^e)^{-1}(\boldsymbol{x})} RCM_y(\Gamma, h, \boldsymbol{p}) \tag{13}$$

does not depend on the choice of $\boldsymbol{x} \in \mathbb{R}^{2n} \setminus Y^{2n-1}$ with Y^{2n-1} defined by (4).

Remark 2.4 *In case of Δ primitive, $n_i = 0$ for all $i \ge 1$, and $m_i = 0$ for all $i \ge 2$ (i.e., $(\Gamma, h, \boldsymbol{p})$ trivalent without marked vertices), the cuspidal invariant $RC_y(\Delta, \overline{n}, \overline{m})$ coincides with the Block-Göttsche refined invariant $N^{\Delta,\delta}_{trop}(y)$ for δ chosen so that the counted tropical curves are rational [2].*

In case of Δ primitive, $n_i = m_i = 0$ for all $i \ge 2$ (i.e., $(\Gamma, h, \boldsymbol{p})$ trivalent but with some vertices marked), the invariant $RC_y(\Delta, \overline{n}, \overline{m})$ coincides with refined broccoli invariant as defined by Göttache and Schroeter [6].

At last, in case of $m_i = 0$ for all $i \ge 2$ (i.e., all unmarked vertices trivalent), the invariant $RC_y(\Delta, \overline{n}, \overline{m})$ coincides with the refined descendant invariant defined in [1]. The only novelty of the present note is that we allow on unmarked vertex of arbitrary valency.

Similarly to [1, Proposition 2.4], our invariant $RC_y(\Delta, \overline{m})$ is often a rational function of y:

Proposition 2.5 *If under hypotheses of Theorem 2.3, in addition, $\Delta \subset \mathbb{Z}^2 \setminus 2\mathbb{Z}^2$ (i.e., does not contain even vectors), then we have*

$$RC_y(\Delta, \overline{m}) = \frac{F(y + y^{-1})}{(y + 2 + y^{-1})^k}\ , \tag{14}$$

where $k \ge 0$ and F is a nonzero polynomial of degree

$$\deg F = |\mathrm{Int}P(\Delta) \cap \mathbb{Z}^2| + \frac{|\partial P(\Delta) \cap \mathbb{Z}^2| - |\Delta|}{2} + k\ ,$$

where $P(\Delta)$ is the Newton polygon constructed in the proof of Lemma 1.3. Furthermore,

$$k \le \sum_{i \ge 2} i(n_{2i} + n_{2i+1}) + \frac{1}{2} \sum_{j \ge 4} (j - 3) m_j\ . \tag{15}$$

Proof. The argument used in the proof of [1, Proposition 2.4] word-for-word applies in the considered situation. We only make a couple of comments. The computation of $\deg F$ uses the construction of a regular tropical curve in the proof of Lemma 1.3 and also Remark 1.4. The last summand in the right-hand side of (15) (as compared with [1, Inequality (14)]) comes from the fact that an unmarked vertex of valency $j > 3$ contributes to the denominator at most $j - 3$ factors $y^{1/2} + y^{-1/2}$. $\square$

In general, the denominator in formula (14) is unavoidable as noticed in [1, Corollary 3.3].

2.4 Proof of the invariance

It will be convenient to consider labeled tropical curves. In view of formulas (6) and (12), the invariance of $RC_y(\Delta, \overline{n}, \overline{m}, \boldsymbol{x})$ is equivalent to the invariance of $RC^{\mathrm{lab}}_y(\Delta, \overline{n}, \overline{m}, \boldsymbol{x})$. [I]

[I] The latter expression is defined by formula (13), where we sum up over all labeled corves.

So, we choose two generic configurations $\boldsymbol{x}(0), \boldsymbol{x}(1) \in \mathbb{R}^{2n} \setminus Y^{2n-1}$. There exists a continuous path $\boldsymbol{x}(t) \in \mathbb{R}^{2n}$, $0 \le t \le 1$, connecting the chosen configurations, that avoids X^{2n-2}, but may finitely many times hit cells of X^{2n-1}, which may cause changes in the structure of $(\mathrm{Ev}^e)^{-1}(\boldsymbol{x}(t))$. We shall consider all possible wall-crossing phenomena and verify the constancy of $RC_y^{\mathrm{lab}}(\Delta, \overline{m}, \overline{m}, \boldsymbol{x}(t))$ (as a function of t) in the in these events.

To relax notations we simply denote labeled tropical curves by $(\Gamma, h, \boldsymbol{p})$ or and write $RC_y^{\mathrm{lab}}(t)$ for $RC_y^{\mathrm{lab}}(\Delta, \overline{m}, \boldsymbol{x}(t))$.

Let $\boldsymbol{x}(t^*)$ be generic in an $(2n-1)$-dimensional cell of X^{2n-1}. Denote by H_0 the germ of this cell at $\boldsymbol{x}(t^*)$ and by $H_+, H_- \subset \mathbb{R}^{2n}$ the germs of the halfspaces with common boundary H_0. Let $T^* = (\Gamma, h, \boldsymbol{p}) \in (\mathrm{Ev}^e)^{-1}(\boldsymbol{x}(t^*))$ be as described in Lemma 1.5(a,b), and let $F_0 \subset \widehat{\mathcal{M}}^{e,\mathrm{lab}}_{0,\overline{m}}(\mathbb{R}^2, \Delta)$ be the germ at T^* of the $(2n-1)$-cell projected by Ev^e onto H_0. We shall analyze the $2n$-cells of $\widehat{\mathcal{M}}^{e,\mathrm{lab}}_{0,\overline{m}}(\mathbb{R}^2, \Delta)$ attached to F_0, their projections onto H_+, H_-, and prove the constancy of $RC_y^{\mathrm{lab}}(t)$, $t \in (t^* - \eta, t^* + \eta)$, $0 < \eta \ll 1$.

(1) Suppose that T^* is as in Lemma 1.5(a), i.e., it has a marked point p_1 at a vertex $V \in \Gamma^0$ of valency $i_1 + i_2 + 2$, with incident edges $E_0, ..., E_{i_1+i_2+1}$ directed by the vectors $\boldsymbol{a}_j := \boldsymbol{a}_V(E_j)$, $0 \le j \le i_1 + i_2 + 1$, and we assume that the limit of the regular orientation is such that E_0 is incoming and all other edges incident to V are outgoing. Without loss of generality we can suppose that the path $\boldsymbol{x}(t)$, in a neighborhood of t^*, is as follows: $x_1 = h(p_1) \in \mathbb{R}^2$ moves along a smooth germ transversal to the fixed line L through the segment $h(E_0)$, while $\boldsymbol{x} \setminus \{x_1\}$ remains fixed.

Assume that $i_2 = 0$. Then, in the deformation, the marked point p_1 moves from V to one of the edges $E_1, ..., E_{i_1+1}$. Note that the sign of $\boldsymbol{a}_0 \wedge \boldsymbol{a}_j$ determines whether the tropical curve with a marked point on E_j, $1 \le j \le i_1 + 1$, is mapped to H_+ or H_-. Hence, in view of the former formula in (12) and formula (11), the constancy of $RC_y^{\mathrm{lab}}(t)$ is equivalent to the relation

$$\sum_{j=1}^{i_1+1} [\boldsymbol{a}_0 \wedge \boldsymbol{a}_j]_y^- \cdot \mu_y^+(A_j) = 0, \quad \text{where } A_j = (\boldsymbol{a}_0 + \boldsymbol{a}_j, (\boldsymbol{a}_k)_{k \ne 0,j}) \,. \tag{16}$$

If $i_1 = 1$, the balancing condition, which reads $\boldsymbol{a}_0 + \boldsymbol{a}_1 + \boldsymbol{a}_2 = 0$, and the definition $\mu_y^+(A_1) = \mu_y^+(A_2) = 1$ imply (16). If $i_1 \ge 2$, then (16) is equivalent to [1, Formula (18)].

So, assume that $i_2 \ge 1$. Then, in the deformation, V splits into a marked (i_2+2)-valent vertex p_1 and an unmarked (i_1+2)-valent vertex V' mapped to the line L. Denote by E_0' the edge connecting the vertices V, V' of the deformed curve. The sign of $\boldsymbol{a}_0 \wedge \boldsymbol{a}_0'$, where $\boldsymbol{a}_0' = \boldsymbol{a}_V(E_0')$, determines whether the Ev^e-image of the deformed curve belongs to H_+ or to H_-. Then the sought constancy will follow from the relation (see Figure 1(a))

$$\sum_{\substack{I \cup J = \{1, ..., i_1+i_2+1\} \\ |I| = i_1, |J| = i_2+1}} [\boldsymbol{a}_0 \wedge \boldsymbol{a}_0']_y^- \cdot \mu_y^+(A_I) \cdot \mu_y^+(B_J) = 0 \,, \tag{17}$$

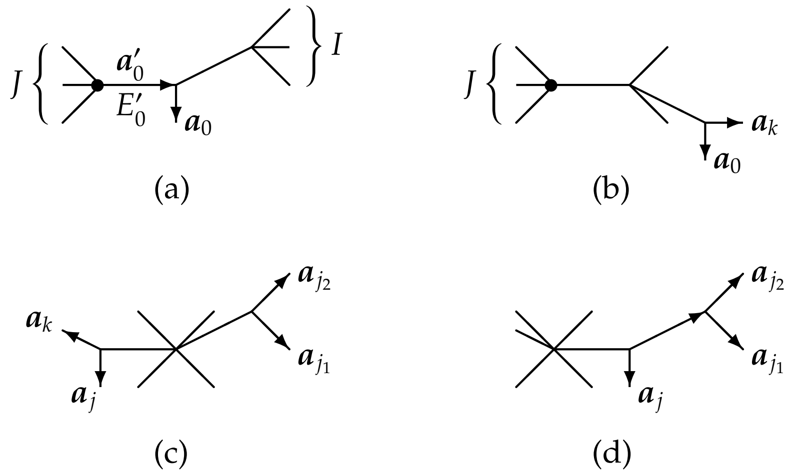

Figure 1: Geometric illustration to the invariance statement

where

$$A_I = (\boldsymbol{a}_0, -\boldsymbol{a}'_0, (\boldsymbol{a}_s\ :\ s \in I)), \quad B_J = (\boldsymbol{a}'_0, (\boldsymbol{a}_s\ :\ s \in J)), \quad \boldsymbol{a}'_0 = -\sum_{s\in J} \boldsymbol{a}_s\ .$$

Using [1, Formula (18)], we rewrite (17) in the form (see Figure 1(b))

$$\sum_{\substack{I\cup J=\{1,\dots,i_1+i_2+1\}\\ |I|=i_1,|J|=i_2+1}} \sum_{k\in I} [\boldsymbol{a}_0 \wedge \boldsymbol{a}_s]^-_y \cdot \mu^+(A_{I,k}) \cdot \mu^+_y(B_J) = 0, \quad A_{I,k} = (\boldsymbol{a}_0 + \boldsymbol{a}_k, -\overline{a}'_0, (\boldsymbol{a}_s\ :\ s \in I\backslash\{k\}))\,,$$

or, equivalently, as

$$\sum_{k=1}^{i_1+i_2+1} \left([\boldsymbol{a}_0 \wedge \boldsymbol{a}_k]^-_y \cdot \sum_{\substack{I\cup J=\{1,\dots,i_1+i_2+1\}\\ k\in I,|I|=i_1,|J|=i_2+1}} \mu^+_y(A_{I,k})\mu^+_y(B_J) \right) = 0\,. \tag{18}$$

For a given k, the term $\sum_{I,J} \mu^+_y(A_{I,k})\mu^+_y(B_J)$ in the left-hand side of (18) can be written (cf. [1, Section 2.5, proof of Lemma 2.5]) as the sum of the expressions $\mu^+_{y,\alpha}(C_k, \boldsymbol{a}_0 + \boldsymbol{a}_k, E)$, where $C_k = (\boldsymbol{a}_0 + \boldsymbol{a}_k, (\boldsymbol{a}_s\ :\ 1 \le s \le i_1 + i_2 + 1,\ s \ne k))$, α runs over all combinatorial types of trivalent trees having $i_1 + i_2 + 1$ leaves and containing a point, whose complement consists of two trees with $i_1 + 1$ and $i_2 + 2$ leaves, respectively, which E runs over the leaves of the former subtree. It follows from [1, Formula (24)] that

$$\sum_{\substack{I\cup J=\{1,\dots,i_1+i_2+1\}\\ k\in I,|I|=i_1,|J|=i_2+1}} \mu^+_y(A_{I,k})\mu^+_y(B_J) = \Phi_1(z) \sum_{\tau\in S_{i_1+i_2}} z^{\tau\Lambda(C_k)}\,, \tag{19}$$

where $C_k = \{\boldsymbol{a}_s\ :\ 1 \le s \le i_1 + i_2 + 1,\ s \ne k\}$, $z^2 = y$, S_k is the permutation group of k elements, $C = (\boldsymbol{b}_s\}_{1\le s\le |C|}$, and

$$\tau\Lambda(C) = \sum_{1\le s<t\le |C|} \boldsymbol{b}_{\tau(s)} \wedge \boldsymbol{b}_{\tau(t)}\ .$$

Plugging (19) to (18) and using relations

$$[\boldsymbol{a}_0\wedge\boldsymbol{a}_k]_y^-=\frac{z^{\boldsymbol{a}_0\wedge\boldsymbol{a}_k}-z^{\boldsymbol{a}_k\wedge\boldsymbol{a}_0}}{z-z^{-1}},\quad \boldsymbol{a}_0=-\boldsymbol{a}_1-\ldots-\boldsymbol{a}_{i_1+i_2+1}\ ,$$

we obtain in the left-hand side of (18)

$$\Phi_2(z)\left(\sum_{k=1}^{i_1+i_2+1}\sum_{\tau\in S_{i_1+i_2}}z^{\tau\Lambda(C_k)+\sum_{s\ne0}\boldsymbol{a}_k\wedge\boldsymbol{a}_s}-\sum_{k=1}^{i_1+i_2+1}\sum_{\tau\in S_{i_1+i_2}}z^{\tau\Lambda(C_k)+\sum_{s\ne0}\boldsymbol{a}_s\wedge\boldsymbol{a}_k}\right)$$

$$=\Phi_2(z)\left(\sum_{\sigma\in S_{i_1+i_2+1}}z^{\sigma\Lambda(C)}-\sum_{\sigma\in S_{i_1+i_2+1}}z^{\sigma\Lambda(C)}\right)=0$$

(where $C=\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{i_1+i_2+1}\}$).

(2) Suppose that $T^*=(\Gamma,h,\boldsymbol{p})$ is as in Lemma 1.5(b), i.e., it results from a collision of two unmarked vertices of valency 3 and $r\ge3$. Then Γ has an unmarked vertex V of valency $r+1$. Let $E_j\in\Gamma^1$, $j=1,\ldots,r+1$, be the edges incident to V, and the limit of the regular orientation is such that E_1,E_2,E_3 are incoming, while the other edges are outgoing. Denote $\boldsymbol{a}_j=\boldsymbol{a}_V(E_j)$, $j=1,\ldots,r+1$. We use the same symbols E_j for the corresponding edges of curves $T(t)\in\mathcal{M}^{e,\mathrm{lab}}_{0,\overline{n},\overline{m}}(\mathbb{R}^2,\Delta)$ obtained in a deformation of T^* along the path $\boldsymbol{x}(t)$, $t\in(t^*-\eta,t^*+\eta)$, no confusion will arise.

The list of possible curves $T(t)$ is as follows:

- either, for some $1\le j\le3$, a curve $T(t)$ has a trivalent vertex V_1 incident to the edges E_j, E_k, $k=4,\ldots,r+1$, and the edge E_0 that joins V_1 with the vertex V_2; in turn, V_2 is incident to E_0, E_{j_1}, E_{j_2}, where $\{1,2,3\}\setminus\{j\}=\{j_1,j_2\}$, and E_s, $s=4,\ldots,r+1$, $s\ne k$;
- or, for some $1\le j\le3$, a curve $T(t)$ has a trivalent vertex V_1 incident to the edges E_{j_1}, E_{j_2}, and the edge E_0 that joins V_1 with the vertex V_2; in turn, V_2 is incident to E_0, E_s, $s=4,\ldots,r+1$.

The regular orientation of E_0 is given in the former and in the latter case by the vectors

$$\boldsymbol{a}_0=\boldsymbol{a}_{V_2}(E_0)=\boldsymbol{a}_j+\boldsymbol{a}_k\quad\text{and}\quad \boldsymbol{a}_0=\boldsymbol{a}_{V_1}(E_0)=\boldsymbol{a}_{j_1}+\boldsymbol{a}_{j_2}\ ,$$

respectively. In both the cases, the sign of $\boldsymbol{a}_j\wedge\boldsymbol{a}_0$ determines whether $\mathrm{Ev}^e(T(t))$ belongs to H_+ or H_-. Introduce $\varepsilon_j=\pm1$, $j=1,2,3$, so that $\varepsilon_j\cdot\mathrm{sign}(\boldsymbol{a}_j\wedge\boldsymbol{a}_0)=1$ points to H_+ for all $j=1,2,3$. Then the required constancy relation reads

$$\sum_{j=1}^3\varepsilon_j\cdot[|\boldsymbol{a}_{j_1}\wedge\boldsymbol{a}_{j_2}|]_y^-\cdot\left(\sum_{k=4}^{r+1}[\boldsymbol{a}_j\wedge\boldsymbol{a}_k]_y^-\cdot\mu_y^+(A_k)+[\boldsymbol{a}_j\wedge(-\boldsymbol{a}_{j_1}-\boldsymbol{a}_{j_2})]_y^-\cdot\mu_y^+(A)\right),\qquad(20)$$

where

$$A_k=(\boldsymbol{a}_{j_1}+\boldsymbol{a}_{j_2},\boldsymbol{a}_j+\boldsymbol{a}_k,(\boldsymbol{a}_l\ :\ l\in K\setminus\{k\})),\quad A=(\boldsymbol{a}_1+\boldsymbol{a}_2+\boldsymbol{a}_3,(\boldsymbol{a}_l)_{l\in K})$$

(see Figure 1(c,d)).

If $r = 3$, then (20) turns into

$$\sum_{j=1}^{3} \varepsilon_j \cdot [\overline{a}_j \wedge (-\overline{a}_{j_1} - \overline{a}_{j_2})]_y^- \cdot [|\overline{a}_{j_1} \wedge \overline{a}_{j_2}|]_y^- = 0 \,, \tag{21}$$

which reflects a collision of two trivalent vertices with Block-Göttsche refined multiplicities, and in which case (21) appears to be a particular case of the invariance statement in [7, Theorem 1] (see a detailed treatment in [7, Pages 5313-5316]).

If $r \geq 4$, relation [1, Formula (18)] (cf. also (16)) yields that

$$\sum_{k=4}^{r+1} \left([\boldsymbol{a}_j \wedge \boldsymbol{a}_k]_y^- \cdot \mu_y^+(A_k) \right) = -[\boldsymbol{a}_j \wedge (\boldsymbol{a}_{j_1} + \boldsymbol{a}_{j_2})]_y^- \cdot \mu_y^+(A) \,,$$

we obtain in the left-hand side of (20)

$$2 \sum_{j=1}^{3} \left(\varepsilon_j \cdot [|\boldsymbol{a}_{j_1} \wedge \boldsymbol{a}_{j_2}|]_y^- \cdot [\boldsymbol{a}_j \wedge (-\boldsymbol{a}_{j_1} - \boldsymbol{a}_{j_2})]_y^- \right) \,,$$

which vanishes in view of (21).

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