ON THE STRUCTURE OF 4 FOLDS WITH A HYPERPLANE SECTION WHICH IS A $\mathbb{P}^1$ BUNDLE OVER A SURFACE THAT FIBRES OVER A CURVE

by

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In this article we want to analyze the structure of a 4 dimensional projective variety X which has a smooth ample divisor A that is a $\mathbb{P}^1$ bundle $\pi : A \longrightarrow S$ over a smooth surface S.

In [(Fa + So)2] , as a consequence of a more general result, the first and third authors determined the structure of X in the case the base S of the $\mathbb{P}^1$ bundle A has a cover $\tilde{S}$ with $h^{2,0}(\tilde{S}) \neq 0$. Here we look at the remaining cases except for those surfaces which are the projectivization of a stable rank two vector bundle over a curve (the result is obviously true for S rational).

The key point is to extend the morphism $\pi : A \longrightarrow S$ to a morphism $\bar{\pi} : X \longrightarrow S$. If the surface S has a morphism $\Psi : S \longrightarrow C$ onto a smooth curve C, then the morphism $\Psi \circ \pi : A \longrightarrow S$ extends to a morphism $\varphi : X \longrightarrow C$ (see [So1], Prop. V). Moreover the general fibre X_c of φ turns out to be a $\mathbb{P}^2$ bundle over a curve contained in S. We now construct $\bar{\pi} : X \longrightarrow S$ geometrically. The idea is to take a general fibre P of the general $\mathbb{P}^2$ bundle X_c and look at all the deformations of P in X. Using the "universal" family of such deformations we will get our desired map.

The main result is the following

<u>Theorem</u>: <u>Let</u> X <u>be a 4-dimensional projective variety which is a local complete intersection. Let</u> A <u>be an ample divisor on</u> X <u>which is a $\mathbb{P}^1$ bundle</u>, $\pi : A \longrightarrow S$ <u>over a smooth</u>

surface S. Assume that there is a surjective holomorphic map $\Psi : S \longrightarrow C$ with connected fibres, where C is a smooth curve. Then π can be extended to a holomorphic map $\bar{\pi} : X \longrightarrow S$ unless $S = \mathbb{P}_C(V)$ with V a stable rank two vector bundle on C. Moreover $\bar{\pi} : X \longrightarrow S$ is a $\mathbb{P}^2$ bundle.

The paper is organized as follows.

In §0 we recall some background material.

In §1 we study the structure of X in the case the surface S, base of the $\mathbb{P}^1$ bundle A has a surjective morphism $\Psi : S \longrightarrow C$ onto a curve.

In §2 we completely determine the structure of X in the case $S = \mathbb{P}^2$. Also, for completeness, we determine the structure of those X with an ample divisor A which is a $\mathbb{P}^1$ bundle over $\mathbb{P}^n$, with $n \geq 3$.

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§ 0 Background material

(0.1) Throughout this article the varieties considered will be projective and defined over $\mathbb{C}$. Given a variety X we denote its structure sheaf by $\mathcal{O}_X$. We do not distinguish between a holomorphic vector bundle E on a variety X and its sheaf of germs of holomorphic sections. We denote the tautological line bundle of E by ζ_E or $\mathcal{O}_{\mathbb{P}(E)}(1)$, where $\mathbb{P}(E) = E^V$ − {zero section} / $\mathbb{C}^*$ and E^V is the dual bundle of E. If Y is a subvariety of X we denote by $E_{|Y}$ the restriction of E to Y. For more details on vector bundles see [Ok+Sc+Sp].

(0.2) Let $p : X \longrightarrow Y$ be a map of projective varieties. We will use interchangeably the word morphism and holomorphic map, as well as rational map and meromorphic map.

(0.3) Let X be a projective variety. Let D be an effective Cartier divisor on X. We denote by $[D]$ or $\mathcal{O}_X(D)$ the line bundle defined by D. If L is a line bundle on X, let $|L|$ denote the linear system of all Cartier divisors associated to L.

(0.4) By F_r with $r \geq 0$ we denote the rth Hirzebruch surface. F_r is the unique $\mathbb{P}^1$ bundle $\pi : F_r \longrightarrow \mathbb{P}^1$ over $\mathbb{P}^1$ with a section E satisfying $E \cdot E = -r$. By $\tilde{F}_r$ with $r \geq 1$ we denote the surface obtained from F_r by blowing down E.

The next result will be used often. We will state it for the convenience of the reader and refer to [So2], (0.6.1) for a proof.

(0.5) Lemma Let X be a normal irreducible compact surface. Let L be an ample line bundle on X, with a smooth $C \in |L|$ being a rational curve and $C \subseteq X_{reg}$. Then L is very ample and either

a) X is F_r and $L = [E] \otimes [f]^k$ with $k \geq r+1$, or

b) X is $\widetilde{F}_r$ and $p^*L = [E] \otimes [f]^r$ where $p : F_r \longrightarrow \widetilde{F}_r$ is the map that blows down E. (Here f denotes a fibre of $\pi : F_r \longrightarrow \mathbb{P}^1$) .

§1. Proof of the main theorem

(1.0) Theorem Let X be a four dimensional projective variety which is a local complete intersection. Let A be an ample divisor on X which is a $\mathbb{P}^1$ bundle, $\pi : A \longrightarrow S$ over a smooth surface S. Assume that there is a surjective holomorphic map $\Psi : S \longrightarrow C$ with connected fibres, where C is a smooth curve. Then π can be extended to a holomorphic map $\bar{\pi} : X \longrightarrow S$ unless $S = \mathbb{P}_C(V)$ with V a stable rank two vector bundle on C (see Remark (1.0.1)). Moreover $\bar{\pi} : X \longrightarrow S$ is a $\mathbb{P}^2$ bundle.

(1.0.1) Remark We do not need to assume that $\Psi : S \longrightarrow C$ has connected fibres and that C is smooth. In fact if otherwise we can Remmert-Stein factorize $\Psi = s \circ r$ where $r : X \longrightarrow C'$ is a holomorphic map onto a smooth curve C' and $s : C' \longrightarrow C$ is a finite to one holomorphic map. Then the theorem is true unless $S = \mathbb{P}_{C'}(V)$ where V is a stable rank two vector bundle on C'.

Proof of the theorem. We notice that $\dim \mathrm{Sing}\,(X) \leq 0$ since the ample divisor A on X is smooth. The holomorphic map $\Psi \circ \pi$ extends to a holomorphic map $\varphi : X \longrightarrow C$, see [So1] Prop V or [Fu]. Let X_c and A_c denote the general fibre of φ and $\Psi \circ \pi$ respectively. Note that A_c is a geometrically ruled surface over $\Psi^{-1}(c)$ and moreover A_c is an ample divisor on X_c. We claim that either α) X_c is a $\mathbb{P}^2$ bundle over $\Psi^{-1}(c)$ and A_c is the tautological line bundle on X_c, or

β) $(\Psi \circ \pi)^{-1}(c) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and X_c is a $\mathbb{P}^2$ bundle over $\mathbb{P}^1$ where the canonical projection $X_c \to \mathbb{P}^1$ is not an extension of $\pi : A_c \to \Psi^{-1}(c)$ $(\simeq \mathbb{P}^1)$.

Proof of the claim The general fibre of Ψ is a smooth curve of genus $g \geq 0$. If $g > 0$ or if $g = 0$ and $A_c \simeq F_r$ with $r > 0$, where F_r is as in (0.4), then using [Ba2] we conclude that X_c is a $\mathbb{P}^2$ bundle over $\Psi^{-1}(c)$ and A_c is the tautological line bundle on X_c. If $g = 0$ and $A_c \simeq F_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ then we will show that

(*) $\operatorname{Pic}(X_c) \simeq \operatorname{Pic}(A_c) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Therefore the result will follow from [Ba1] once we know (*).

Proof of (*) From the following diagram

$$\begin{array}{ccc} H_2(A,\mathbb{Q}) & \longrightarrow & H_2(X,\mathbb{Q}) \\ \uparrow & & \uparrow \\ H_2(A_c,\mathbb{Q}) & \longrightarrow & H_2(X_c,\mathbb{Q}) \end{array}$$

we see that $\dim H_2(X_c,\mathbb{Q}) = 1$ is possible only if the two rulings of A_c $(\simeq F_0)$ get identified in X. But the two rulings were in different homology classes in A therefore they cannot go in the same homology class in X. Using Kronecker duality and the first Lefschetz theorem we conclude that $\text{Pic}(X_c) \simeq \text{Pic}(A_c)$. □

The proof of the theorem will be split up in two parts. We will treat case α) first and then the case β).

<u>Case α)</u> Fix a general $\mathbb{P}^2$ which is a fibre of $X_c \to \Psi^{-1}(c)$ and denote it by P. Using the fact that $P \subseteq X_c \subseteq X$ and the exact sequence of normal bundles

$$0 \to N_{P/X_c} \to N_{P/X} \to N_{X_c/X|P} \to 0$$

it is straightforward to see that $N_{P/X} = \mathcal{O}_P \oplus \mathcal{O}_P$, where $N_{P/X}$ is the normal bundle of P in X, and that $H^1(P, N_{P/X}) = 0$. Under the above assumption, using a basic result on Hilbert schemes, it follows that there exist irreducible projective varieties $\mathcal{W}$ and $\mathcal{Z}$ with the following properties:

1) $\mathcal{W} \subseteq \mathcal{Z} \times X$ and the map $p : \mathcal{W} \to \mathcal{Z}$ induced by the product projection is a flat surjection,

2) there is a smooth point $a \in \mathcal{Z}$ with p of maximal rank in a neighborhood of $p^{-1}(a)$ and $p^{-1}(a)$ is identified with $P \simeq \mathbb{P}^2$ via q, where $q : \mathcal{W} \to X$ is the map induced by the product projection.

(1.0.2) <u>Lemma</u> <u>There exists a Zariski open neighborhood</u> U <u>of</u> a, <u>where</u> a <u>is as in 2), such that for every</u> $z \in U$

i) $p^{-1}(z) = W_z$ <u>is isomorphic to</u> $\mathbb{P}^2$ <u>and it is a</u> <u>fibre of</u> $X_c \to \Psi^{-1}(c)$ <u>for some</u> $c \in C$,

ii) $W_z \cap A = f(\simeq \mathbb{P}^1)$, <u>where</u> f <u>is a fibre of</u> π .

<u>Proof</u> From 2) above there exists a smooth neighborhood U of a in Z such that $p^{-1}(U) \to U$ and $q^{-1}(A) \cap p^{-1}(U) \to U$ are smooth morphisms. Note that $A \cap W_a = \mathbb{P}^1$. Moreover using the fact that small deformations of $\mathbb{P}^2$ and $\mathbb{P}^1$ are $\mathbb{P}^2$ and $\mathbb{P}^1$ respectively we conclude that the fibres of the maps $p_{|p^{-1}(U)}$ and $q_{|q^{-1}(A) \cap p^{-1}(U)}$ are $\mathbb{P}^2$ and $\mathbb{P}^1$ respectively. On the other hand a morphism φ from $\mathbb{P}^2 \subseteq X$ to C is constant. Hence any fibre of $p_{|p^{-1}(U)}$ is contained in a fibre of φ. Therefore the rest of (1.0.2) is obvious

□

(1.0.3) <u>Lemma</u> <u>The intersection number</u> $A \cdot A \cdot W_z = 1$ <u>for</u> <u>every</u> $z \in Z$. <u>And if</u> $W_z = \overline{W}_z \cup \{\text{embedded part}\}$ <u>then</u> $\overline{W}_z$ <u>is reduced and irreducible.</u>

<u>Proof</u> By α) we have that $\mathcal{O}_X(A)_{|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$. Hence $(A \cdot A \cdot \mathbb{P}^2)_X = (\mathcal{O}_X(A)_{|\mathbb{P}^2} \cdot \mathcal{O}_X(A)_{|\mathbb{P}^2})_{\mathbb{P}^2} = 1$, which implies that $A \cdot A \cdot W_z = 1$ since the intersection number is preserved by flat maps. Clearly $\overline{W}_z$ is reduced and irreducible (since $A \cdot A \cdot W_z = 1$).

□

Note that the general fibre of the morphism $\Psi : S \to C$ is either isomorphic to $\mathbb{P}^1$ or to a curve of positive genus.

(1.0.4) Lemma For every $z \in Z, W_z \not\subseteq A$.

Proof Let $z \in Z$ and let $\{z_n\}$ be a sequence of points in Z such that $\lim_{n\to\infty} z_n = z$ and $W_{z_n} \simeq \mathbb{P}^2$ for every n. The above is possible by (1.0.2). Now use the fact that $\varphi(W_{z_n})$ is one point for every n, to conclude that $\varphi(W_z)$ is also one point. Assume that $W_z \subseteq A$.

Since $\pi : A \to S$ is a $\mathbb{P}^1$ bundle and since $(\Psi \circ \pi)(W_z) = c$, with c a point in C, we get that $\Phi = \pi_{|W_z} : W_z \to \pi(W_z)$ is a $\mathbb{P}^1$ bundle. Note that $\pi(W_z) \subseteq \Psi^{-1}(c)$. To continue the proof of the lemma we distinguish two cases:

Case 1 The general fibre of Ψ is isomorphic to $\mathbb{P}^1$.
If $\Psi^{-1}(c)$ with c as above is isomorphic to $\mathbb{P}^1$ then W_z is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$. Moreover there exists an ample line bundle $([A]_{|W_z})$ on W_z whose selfintersection is 1. This last fact is impossible.

If $\Psi^{-1}(c)$ is singular then $\Psi^{-1}(c) = \Sigma n_i C_i$ with $C_i \simeq \mathbb{P}^1$. Also $\pi(W_z) = C_i$ for some i otherwise we would get a contradiction with the fact that W_z is irreducible. Hence

W_z is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ which is impossible as noticed earlier.

<u>Case 2</u> The general fibre of Ψ is isomorphic to a curve of positive genus.
Take a general fibre of $W \longrightarrow Z$ and consider all the lines on such fibre. Let T denote the irreducible component of the Hilbert scheme of X parametrizing such lines. Denote by M the universal family. Thus $M \subseteq T \times X$. Note that every fibre of M is irreducible and reduced (since $L \cdot M_t = L \cdot \mathbb{P}^1 = 1$, where M_t is a fibre of M over T).

<u>Claim</u> Every fibre of $M \rightarrow T$ has $\mathbb{P}^1$ as normalization.

<u>Proof of the claim</u> Consider a curve B in T through a point t'. Also choose B of positive genus. Let M_B denote the inverse image of B under the natural projection $M \longrightarrow T$. Note that most fibres of $M_B \longrightarrow B$ are linear $\mathbb{P}^1$'s since B is chosen of positive genus. If we take a minimal model of a desingularization of $\tilde{M}_B$, where $\tilde{M}_B$ denotes the normalization of M_B, we get a ruled surface over the normalization of B. This last conclusion follows from the fact that M_B has infinitely many $\mathbb{P}^1$'s and from the fact that the genus of B is positive. Thus since going from $M_B \longrightarrow$ normalization $\longrightarrow$ desingularization $\longrightarrow$ minimal model does not destroy a positive genus curve and the normalization of $M_{t'}$ goes in a fibre of a $\mathbb{P}^1$ bundle we conclude that every

fibre of $M \longrightarrow T$ has $\mathbb{P}^1$ as a normalization.

□

Now choose 2 points $(a,b) \subseteq W_z$ with $\Phi(a) \neq \Phi(b)$. Let $(x_n, y_n) \subseteq W_{z_n}$ be a sequence of pairs of points such that $\lim_{n\to\infty} x_n = a$ and $\lim_{n\to\infty} y_n = b$.

Let M_{t_n} be a sequence of lines containing (x_n, y_n). The limit of M_{t_n} is (maybe after passing to a subsequence) a line M_t containing the point (a,b). As shown in our previous claim, M_t is birational to $\mathbb{P}^1$ so $\Phi(M_t) = \mathbb{P}^1$. Therefore $W_z \longrightarrow R_g$ is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$. Thus our lemma is proved.

□

(1.0.5) <u>Lemma</u> $W_z \cap A = f$, <u>where</u> f <u>is a fibre of</u> π. <u>(The equality here is only setwise)</u>.

<u>Proof</u> By (1.0.2) we can take a sequence of points $\{z_n\}$ in Z with $\lim_{n\to\infty} z_n = z$, such that $\lim_{n\to\infty} W_{z_n} = W_z$, $W_{z_n} \simeq \mathbb{P}^2$ for all n and $W_{z_n} \cap A =$ fibre of π. Hence $W_z \cap A = f + C$, where f is a fibre of π and C is a possibly empty effective 1-cycle. From (1.0.3) and the fact that A is ample it follows that $C = \emptyset$.

□

(1.0.6) <u>Lemma</u> <u>For every fibre</u> f <u>of</u> $\pi, \{z \in Z \mid W_z \supset f\}$ is finite.

<u>Proof</u> Assume otherwise. Then there is a curve $Y \subseteq Z$ such that for every $y \in Y$, $W_y \supseteq f$. Note that

$(\bigcup_{y\in Y} W_y) \cap A = f$ by (1.0.5). On the other hand $\bigcup_{y\in Y} W_y$ is a divisor on X. Thus $\dim((\bigcup_{y\in Y} W_y) \cap A) \geq 2$. This contradiction proves our lemma.

□

Notice that (1.0.5) and (1.0.6) yield the following important lemma.

(1.0.7) Lemma Assume the same hypotheses as in (1.0). Then there exist Zariski open sets $W_1 \subseteq W$ and $X_1 \subseteq X$ such that $A \subseteq X_1$ and $q_{|W_1}: W_1 \to X_1$ is an isomorphism. Moreover Z is isomorphic to S.

Proof: The proof of the first conclusion is obvious from Zariski's Main Theorem. To get the second conclusion we use the universality of the Hilbert schemes.

□

From (1.0.7) it follows that there exists a meromorphic map $\bar{\pi}: X \dashrightarrow S$, where $\bar{\pi} = p \circ q^{-1}$, such that $\bar{\pi}_{|A}$ is holomorphic. We claim that $\bar{\pi}$ is holomorphic. In fact the set on which $\bar{\pi}$ is not holomorphic in finite since $\bar{\pi}_{|A}$ is holomorphic and A is an ample divisor on X. Therefore by Riemann extension theorem $\bar{\pi}$ is holomorphic.

Before passing to the case β) we will show that the above $\bar{\pi}$ gives to X the structure of a $\mathbb{P}^2$ bundle over S.

By construction the general fibre of $\bar{\pi}$ is $\mathbb{P}^2$. Also

$\mathcal{O}_X(A)_{|\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(1)$. As for the possible singular fibre F of $\bar{\pi}$, we notice that F is reduced and irreducible since $L\cdot L\cdot F = 1$. Since $\mathbb{P}^1$ is an hyperplane section of F it is well known, see (0.5) that F is either F_r with $r \geq 0$ or $\tilde{F}_r$ with $r \geq 1$, where F_r and $\tilde{F}_r$ are as in (0.4). There are no F_r with an ample line bundle of degree 1. Among the $\tilde{F}_r$ the only one with an ample line bundle of degree 1 is $\tilde{F}_1 \simeq \mathbb{P}^2$. Now we use a theorem of Hironaka ([Hi],Th. 1.8) to conclude that $\bar{\pi}: X \to S$ is a $\mathbb{P}^2$ bundle.

Let us now consider the case β).

<u>Case β)</u> Let $c \in C$ be a general point. We take a general rational curve ℓ in $A_c = (\Psi \circ \pi)^{-1}(c) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ such that $\ell \cdot \ell = 0$ and ℓ is not a fibre of π. From now on we denote by ℓ the ruling of $\mathbb{P}^1 \times \mathbb{P}^1$ which is not a fibre of π. It is straightforward to see that

$$N_{\ell/A} = \mathcal{O}_\ell \oplus \mathcal{O}_\ell \quad \text{and} \quad H^1(\ell, N_{\ell/A}) = 0 .$$

Denote by S' the irreducible component of the Hilbert scheme of A parametrizing flat deformations of ℓ in A and by $\mathcal{Y}$ the universal family. Thus $\mathcal{Y} \subseteq S' \times A$. Denote by $p: \mathcal{Y} \to S'$ and $q: \mathcal{Y} \to A$ the maps induced by the product projections. Note that such deformations fill up the whole space A, i.e., $q(\mathcal{Y}) = A$.

<u>Claim 1</u> $\Psi: S \to C$ <u>is a geometrically ruled surface.</u>

<u>Proof of claim 1.</u> Assume that there exists a point $c_0 \in C$ such that $\Psi^{-1}(c_0)$ is a singular fibre. Then the number of irreducible components of $\Psi^{-1}(c_0)$ is at least 2. Let $\{c_n\}$ be a sequence of points in C approaching the point c_0. Let $\{\ell_n\}$ be the corresponding sequence of lines in $\mathcal{Y}$. Thus $\lim_{n\to\infty} \pi(\ell_n) = \Psi^{-1}(c_0)$, where the equality is only setwise (Here we have identified ℓ_n with $q(\ell_n)$). But the above equality is impossible since $A\cdot\ell_n = 1$ for all n, while the number of irreducible components of $\Psi^{-1}(c)$ is at least 2 and A is an ample divisor.

□

We note that for every $c \in C$, $(\Psi\circ\pi)^{-1}(c) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. In fact since S is geometrically ruled it follows that for every $c \in C$, $(\Psi\circ\pi)^{-1}(c) \simeq F_r$ with $r \geqq 0$. Assume that there exists a $c_0 \in C$ such that $(\Psi\circ\pi)^{-1}(c_0) \simeq F_r$ with $r > 0$. Then $A\cdot f = 1$, where f is a fibre of π. Note that $X_{c_0} = \varphi^{-1}(c_0)$ is a local complete intersection containing F_r, with $r > 0$ as an ample divisor. Therefore X_{c_0} is a $\mathbb{P}^2$ bundle over $\Psi^{-1}(c_0)$ (see [(Fa+So)1] for a short proof and cf. [Ba2] for the case X_{c_0} smooth). Thus we would be in the case α).

Let S' and $\mathcal{Y}$ be as before. We denote by ℓ_s the fibre of $\mathcal{Y}$ over $s \in S'$. Clearly the smooth fibres of the flat family $\mathcal{Y}$ are isomorphic to $\mathbb{P}^1$. Recall that $A\cdot\ell_s = 1$. Hence the Hilbert polynomial $\chi(\mathcal{O}_{\ell_s}(A|_{\ell_s})^{\otimes n})$ of ℓ_s is equal to $n+1$.

Let $s \in S'$ be such that ℓ_s is singular. Denote by $\overline{\ell}_s$ the one dimensional closed subscheme of ℓ_s defined by removing the embedded points of ℓ_s.

Claim 2 $\ell_s = \overline{\ell}_s$ and S' is smooth.

Proof of Claim 2 Note that since $\overline{\ell}_s$ is contained in a fibre of $\Psi \circ \pi$ which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and since $A \cdot \overline{\ell}_s = 1$ it follows that $\overline{\ell}_s$ is a fibre of $\mathbb{P}^1 \times \mathbb{P}^1$, so $\overline{\ell}_s = \mathbb{P}^1$. In order to see that $\ell_s = \overline{\ell}_s$ we consider the following exact sequence

$$(1.0.8) \qquad 0 \longrightarrow T \longrightarrow \mathcal{O}_{\ell_s} \longrightarrow \mathcal{O}_{\overline{\ell}_s} \longrightarrow 0$$

where the sheaf T is the torsion part of $\mathcal{O}_{\ell_s}$. Tensoring (1.0.8) with $(A_{|\ell_s})^{\otimes n}$ and using the fact that the Euler characteristic is additive on a short exact sequence it follows that

$$\chi(\mathcal{O}_{|\ell_s}(A_{|\ell_s})^{\otimes n}) = \chi(T \otimes \mathcal{O}_{\ell_s}(A_{|\ell_s})^{\otimes n}) + \chi(\mathcal{O}_{\overline{\ell}_s}(A_{|\overline{\ell}_s})^{\otimes n}).$$

Note that the Hilbert polynomial of ℓ_s and of $\overline{\ell}_s$ are equal. Thus T is the 0-sheaf. To see that S' is smooth note that $N_{\ell_s/A} = \mathcal{O}_\ell \oplus \mathcal{O}_\ell$. Therefore it follows that S' is smooth at s.

(1.0.9) Remark i) $\mathcal{Y}$ is isomorphic to A

ii) A is a $\mathbb{P}^1$ bundle $\sigma : A \longrightarrow S'$ over S'.

To see i) note that $\mathcal{Y}$ is birational to A. Moreover $\mathcal{Y}$ is in one to one correspondence with A, since for every $a \in A$ there exists a unique $\ell \subseteq A_c$ containing a, where $c = (\Psi \circ \pi)(a)$. Hence $\mathcal{Y}$ is isomorphic to A. From i) it follows that there is a morphism $\sigma = p \circ q^{-1}$ from A onto S' whose fibres are isomorphic to $\mathbb{P}^1$. Moreover $\mathcal{O}_A(A)|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1)$. Thus ii) is clear.

Claim 3 S' is geometrically ruled over C.

Proof of Claim 3 Let $c \in C$ and let f_c be the ruling of $A_c (\simeq \mathbb{P}^1 \times \mathbb{P}^1)$ corresponding to the map π. By the universality of the Hilbert schemes, the above f_c induces a closed subscheme f'_c in S', with f'_c a rational curve. To show that there exists a morphism from S' onto C we will distinguish the case $g(C) > 0$ and $g(C) = 0$ where $g(C)$ denotes the genus of C.

In the case $g(C) > 0$ it follows that $H^1(S', \mathcal{O}_{S'}) \neq 0$. We get the following diagram

$$
*) \qquad
\begin{array}{ccccc}
A & \xrightarrow{\Psi \circ \pi} & C & \longrightarrow & J(C) \simeq \mathrm{Alb}(A) \\
\sigma \downarrow & & \downarrow j & & \wr\wr \\
S' & \xrightarrow[\alpha]{} & \alpha(S') & \hookrightarrow & \mathrm{Alb}(S')
\end{array}
$$

where α is the Albanese map. In the above diagram we have used the fact that $Alb(A) \simeq Alb(S) \simeq J(C)$. Note that $\dim \alpha(S') = 1$. We claim that $j: C \longrightarrow \alpha(S')$ is an isomorphism. Using the Riemann-Hurwitz formula the above claim is clear for $g(C) > 1$. For $g(C) = 1$ we get that the morphism j is a covering map. But this is impossible by the commutativity of the first square diagram in *). Therefore we get a morphism $\tau: S' \longrightarrow C$, with $\tau = j^{-1} \circ \alpha$. Also f'_c (the closed subscheme induced in S' by f_c) are fibres of τ. Therefore S' is generically ruled over C. To see that S' is geometrically ruled we assume otherwise. Then there exists a fibre $F = \sum_i n_i C_i$. Let $c = \tau(F)$. Note that $\sigma^{-1}(F) = \Sigma n_i F_i$, where each F_i is a $\mathbb{P}^1$ bundle over C_i. By the commutativity of the first square diagram in *) we see that $\Sigma n_i F_i = \sigma^{-1}(F) = \varphi^{-1}(c) = \mathbb{P}^1 \times \mathbb{P}^1$ which is impossible.

If $g(C) = 0$ then $H^1(S', \mathcal{O}_{S'}) = 0$. Thus there exists a line bundle L on S' such that the linear system $|L|$ contains infinitely many f'_c where f'_c is the closed subscheme induced in S' by f_c. It follows immediately form

$$0 \longrightarrow \mathcal{O}_{S'} \longrightarrow L \longrightarrow L|_{f'_c} \longrightarrow 0$$

that $\dim|L| = 1$. Also it can be easily seen that the linear system $|L|$ is base point free. Hence it defines a morphism onto $\mathbb{P}^1$. The general fibre of such morphism is isomorphic to $\mathbb{P}^1$. Therefore by Noether's lemma S' is rational. The same argument as in the case $g(C) > 0$, shows that S' is geometrically ruled.

From the above proof it also follows that the elements of $|L|$ are exactly $\{f'_c\}_{c\in C}$.

□

Thus we have the following commutative diagram

$$\begin{array}{ccccc} & & A & & \\ & \sigma \swarrow & & \searrow \pi & \\ S' & & & & S \\ & \tau \searrow & & \swarrow \psi & \\ & & C & & \end{array}$$

We will now show that the case β) cannot occur unless $S = \mathbb{P}_C(V)$ where V is a stable rank two vector bundle on C. (Obviously does not occur if S is rational ruled).

By the universality of the fibre product of S and S' over C we get a morphism $A \longrightarrow S \times_C S'$ which is an isomorphism by Zariski's Main Theorem. The surfaces S and S' are geometrically ruled over C therefore there exist rank two vector bundles V and V' over C such that $S = \mathbb{P}(V)$ and $S' = \mathbb{P}(V')$. For the triple X, A and S' the hypotheses of (1.0) are satisfied. Moreover we are in the case α). Hence we conclude that the morphism $\sigma : A \longrightarrow S'$ extends to a morphism $\tilde{\sigma} : X \longrightarrow S'$ and that $\tilde{\sigma} : X \longrightarrow S'$ is a $\mathbb{P}^2$ bundle. Therefore we have the following exact sequence of vector bundles on S'

$$(1.0.9) \qquad 0 \longrightarrow \mathcal{O}_{S'} \longrightarrow E \xrightarrow{\gamma} F \longrightarrow 0$$

with $X = \mathbb{P}(E)$ and $A = \mathbb{P}(F)$ is embedded in X via the map γ. Since for every $c \in C$ $(\tau \circ \sigma)^{-1}(c) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ we have that $F|_{\tau^{-1}(c)} = \mathcal{O}_{\mathbb{P}^1}(a_c) \oplus \mathcal{O}_{\mathbb{P}^1}(a_c)$. It is an easy check to see that a_c is independent of c in C. Thus we can omit the subscript c. Consider the vector bundle $F \otimes \xi^{-a}$ where ξ is the tautological line bundle of V'. By the base change theorem $\tau_*(F \otimes \xi^{-a}) = \tilde{V}$ is a vector bundle on C of rank two. Thus (1.0.9) becomes

$$(1.0.10) \qquad 0 \longrightarrow \mathcal{O}_{S'} \longrightarrow E \longrightarrow \tau^*\tilde{V} \otimes \xi^{a} \longrightarrow 0$$

(1.0.11) Lemma $S = \mathbb{P}(\tilde{V})$

Proof. Note that $A = \mathbb{P}(F) = \mathbb{P}(\tau^*V \otimes \xi^a) = \mathbb{P}(\tau^*V)$. Also $A = S \times_C S' = \mathbb{P}(V) \times_C S' = \mathbb{P}(\tau^*\tilde{V})$. Therefore there exists a line bundle L on S' such that $\tau^*\tilde{V} = \tau^*V \otimes L$. Taking the 0-th direct image via τ on both sides of the equality we get that $\tilde{V} = V \otimes \tau_*L$. Also τ_*L is a line bundle since $L|_{\tau^{-1}(c)}$ is trivial. Hence $\mathbb{P}(\tilde{V}) = \mathbb{P}(V \otimes \tau_*L) = \mathbb{P}(V) = S$.

(1.0.12) Lemma *If* $\tilde{V}$ *is not a stable vector bundle on* C *then* A *is not an ample divisor on* X.

Proof It is enough to show that the sequence (1.0.10) splits. Since $\tilde{V}$ is a vector bundle of rank 2 on the curve C which is not stable, there exists an exact sequence

$$0 \longrightarrow M \longrightarrow \tilde{V} \longrightarrow N \longrightarrow 0$$

such that $\deg M \geq \deg N$. If we pull back the above exact sequence via τ and we tensor it with ξ^a we get

$$(1.0.13) \qquad 0 \longrightarrow \tau^*M \otimes \xi^a \longrightarrow \tau^*\tilde{V} \otimes \xi^a \longrightarrow \tau^*N \otimes \xi^a \longrightarrow 0.$$

Note that $\tau^*N \otimes \xi^a$ is ample. Hence $\tau^*M \otimes \xi^a$ is ample since $\deg M \geq \deg N$. Therefore using the cohomology sequence associated to the dual sequence of (1.0.13), the ampleness of $\tau^*N \otimes \xi^a$ and of $\tau^*M \otimes \xi^a$ and the fact that $a > 1$, we conclude that $H^1(S', (\tau^*\tilde{V} \otimes \xi^a)^V) = 0$.
(Note that $a = 1$ would imply that (1.0.10) splits).

□

Thus we have shown that the case β) does not occur unless $S = \mathbb{P}_C(V)$ with V a stable rank 2 vector bundle on C.

§2. $\mathbb{P}^1$ bundles over $\mathbb{P}^n$ with $n \geq 2$ as ample divisors

(2.0) Theorem *Let* X *be a projective local complete intersection. Let* A *be an ample divisor on* X *which is a* $\mathbb{P}^1$ *bundle* $p : A \longrightarrow \mathbb{P}^2$ *over* $\mathbb{P}^2$. *Then* X *is a* $\mathbb{P}^2$ *bundle over* $\mathbb{P}^2$ *unless* $A \simeq \mathbb{P}^1 \times \mathbb{P}^2$.

Proof. We claim that the map $p : A \longrightarrow \mathbb{P}^2$ extends to a map $\tilde{p} : X \longrightarrow \mathbb{P}^2$ unless $A \simeq \mathbb{P}^1 \times \mathbb{P}^2$. Think of p as the map associated to the linear system $|p^* \mathcal{O}_{\mathbb{P}^2}(1)|$. To show that the map p extends it is enough to check that the sections of $\Gamma(A, p^* \mathcal{O}_{\mathbb{P}^2}(1)$ can be extended to X as sections of L where L is the unique extension of $p^* \mathcal{O}_{\mathbb{P}^2}(1)$ to X. Now to show that the sections extend it is sufficient to prove that $H^1(X, L \otimes [-A]) = 0$. This is implied by $H^1(A, (L \otimes [-A]^t)_{|A}) = 0$ for all $t > 0$, see [So 1] or [(Fa+So) 2]. Let $F \in \left|p^* \mathcal{O}_{\mathbb{P}^2}(1)\right|$, i.e., $F = p^{-1}(\ell)$ where ℓ is a linear hyperplane of $\mathbb{P}^2$. Using the long cohomology sequence associated to the following exact sequence

$$0 \longrightarrow K_A \otimes [A]^t \otimes [F]^{-1} \longrightarrow K_A \otimes [A]^t \longrightarrow (K_A \otimes [A]^t)_{|F} \longrightarrow 0,$$

the Kodaira vanishing theorem and the fact that F is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$, we get that

$H^1(A, L_A \otimes [-A]^t{}_{|A}) = 0$ for all $t > 0$ unless $F = F_0$, with F_0 as in (0.4).

Note that since A is a $\mathbb{P}^1$ bundle over $\mathbb{P}^2$ we have that $A = \mathbb{P}(V)$, where V is a rank 2 vector bundle on $\mathbb{P}^2$. In the case $F = F_0$ we have that for every line ℓ in $\mathbb{P}^2$, $V_{|\ell} = \mathcal{O}_\ell(a_\ell) \oplus \mathcal{O}_\ell(a_\ell)$. Also it is easy to see that a_ℓ is independent of ℓ. Therefore the vector bundle V is uniform and so $V = \mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(a)$. Therefore $A = \mathbb{P}(V) \simeq \mathbb{P}^1 \times \mathbb{P}^2$. Thus the map p extends to a holomorphic map $\tilde{p} : X \longrightarrow \mathbb{P}^2$ unless $A \simeq \mathbb{P}^1 \times \mathbb{P}^2$. Now the same argument as in [(Fa + So) 2], (3.0) shows that X is a $\mathbb{P}^2$ bundle over $\mathbb{P}^2$.

□

(2.1) Theorem Let X be a projective local complete intersection. Let A be an ample divisor on X which is a $\mathbb{P}^1$ bundle $p : A \longrightarrow \mathbb{P}^n$ over $\mathbb{P}^n$. If $n \geq 3$ then $A \simeq \mathbb{P}^1 \times \mathbb{P}^n$ and hence X is a $\mathbb{P}^{n+1}$ bundle over $\mathbb{P}^1$.

Proof. Note that $A = \mathbb{P}(V)$ for some rank 2 vector bundle V on $\mathbb{P}^n$. We can assume, without loss of generality that V is normalized.

We will prove the theorem for $n = 3$. The same proof yields the general case also. Let $F = p^{-1}(\mathbb{P}^2)$, where $\mathbb{P}^2$ is a hyperplane of $\mathbb{P}^3$. Let $L \in \mathrm{Pic}(X)$ be such that $L_A = [F]$. If $\Gamma(X,L) \longrightarrow \Gamma(A,L_A) \longrightarrow 0$ then the map p extends to X. And we will have the contradiction that $n \leq 2$, see [So 1], Prop. V. Thus we can assume that $H^1(X, L \otimes [A]^{-1}) \neq 0$. This implies that $H^1(A, L_A \otimes [A]^{-t}_{|A}) \neq 0$ for some $t > 0$. For such t we consider the following exact sequence

$$0 \longrightarrow K_A \otimes [A]^t \otimes [F]^{-1} \longrightarrow K_A \otimes [A]^t \longrightarrow K_F \otimes [A]^t_F \otimes [F]^{-1}_F \longrightarrow 0 .$$

From the long exact cohomology sequence associated to the above sequence, Kodaira vanishing theorem and the fact that $H^3(A, K_A \otimes [A]^t \otimes [F]^{-1}) \neq 0$ by hypothesis, it follows that $H^2(F, K_F \otimes [A]^t_F \otimes [F]^{-1}_F) \neq 0$.

Note that F is a $\mathbb{P}^1$ bundle $p_F : F \longrightarrow \mathbb{P}^2$ over $\mathbb{P}^2$. Let $\tilde{F} = p_F^{-1}(\mathbb{P}^1)$, where $\mathbb{P}^1$ is a hyperplane of $\mathbb{P}^2$. We consider the sequence

$$0 \longrightarrow K_F \otimes [A]^t_F \otimes [\tilde{F}]^{-1} \longrightarrow K_F \otimes [A]^t_F \longrightarrow K_{\tilde{F}} \otimes [A]^t_{\tilde{F}} \otimes [F]^{-1}_{\tilde{F}} \longrightarrow 0 .$$

And now, as above, we conclude that $H^1(\tilde{F}, K_{\tilde{F}} \otimes [A]^t_{\tilde{F}} \otimes [\tilde{F}]^{-1}_{\tilde{F}}) \neq 0$. This together with the fact $\tilde{F}$ is a $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ implies that $\tilde{F} = F_0$, where F_0 is as in (0.4). Therefore we conclude that $V_{|\ell}$ is trivial for all lines $\ell \subseteq \mathbb{P}^3$, which implies that V is trivial. Thus $A \simeq \mathbb{P}^1 \times \mathbb{P}^3$. But $A (\simeq \mathbb{P}^1 \times \mathbb{P}^3)$ is ample on X. Hence X is a $\mathbb{P}^{3+1}$ bundle, see [So1].

□

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