L-functions for Jacobi forms and the basis problem

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Abstract

This paper deals with Jacobi forms Φ on $\mathbb{H}\times\mathbb{C}$. The Rankin-Selberg doubling method is employed to study properties of the standard L-function of Hecke-Jacobi eigenforms. It is shown that every analytic Klingen-Jacobi Eisenstein series attached to Φ has a meromorphic continuation on the whole complex plane. Hecke-Jacobi cusp eigenforms of weight $k > 4$ and $k \equiv 0$ mod 4 can written explicitly as a linear combination of theta series. Finally the basis problem of Jacobi forms of square-free index is solved.[1]

Introduction

Jacobi forms are a mixture of modular forms and elliptic functions, examples are Jacobi theta functions and Fourier coefficients of Siegel modular forms.

In the early eighties Eichler and Zagier, stimulated by the proof of the Saito-Kurokawa conjecture, developed a systematic theory of Jacobi forms on $\mathbb{H}\times\mathbb{C}$ [E-Z85]. One knows that Jacobi forms play a role in several contexts:

- the theory of modular forms of half-integral weight
- the theory of Maass lift
- the theory of elliptic cusp forms
- the theory of Heegner points
- the theory of Borcherds products.

Recently Jacobi forms turned out to be very useful in studying properties of Siegel modular forms. Kohnen and Böcherer [B-K93], (see also [Br98]) showed that one can obtain a very good estimate for the Fourier coefficients of a Siegel modular form by analyzing the Fourier coefficients of Jacobi-Poincaré series. In [He96], we established a functional equation of the twisted spinor L-function, i.e. the L-function attached to $\mathrm{GSp}_2 \times \mathrm{Gl}_2$ and proved *special value* results, by using the Saito-Kurokawa conjecture and the Maass lift.

This paper deals with another aspect and application of the theory of Jacobi forms. We study the standard L-function of a Hecke-Jacobi cusp eigenform of even weight $k > 4$ on $\mathbb{H} \times \mathbb{C}$ by the Rankin-Selberg doubling method. We prove that every such

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form can be written as a linear combination of theta series. This generalizes works of Murase [Mu89], [Mu91], Arakawa [Ar94], Sugano [Su95] and Dulinski [Du97] on this problem.

Parts of this work is based on ideas of Paul Garrett [Ga84], [Ga89] [Ga92] and Siegfried Böcherer [Boe83], [Boe85]. They calculated the Rankin-Selberg convolution of a diagonal-restricted Siegel Eisenstein series and a Siegel cusp form. They obtained an integral expression of the standard L-function and the Klingen Eisenstein series associated to a Siegel cusp eigenform. Then Böcherer used this formula to solve the basis problem for Siegel modular forms of weight k and degree n, where $k \equiv 0 \bmod 4$ and $k > 2n$. He proved that every modular form with this data can be written as a linear combination of theta series (Siegel's class-invariants).

Murase [Mu89],[Mu91] showed that the ideas of Böcherer and Garrett can also be used to study standard L-functions attached to Jacobi forms Φ on $\mathbb{H}_n \times \mathbb{C}^{r,n}$, where $\mathbb{H}_n$ is Siegel's upper half-space of degree n. More precisely, he proved under certain assumptions on the index of the Jacobi form that the standard L-function has a functional equation (for example in the case $n = r = 1$, only the index 1 is allowed). Then Arakawa [Ar94] used a variation of the formula of Murase to solve the basis problem for Jacobi forms on $\mathbb{H}_n \times \mathbb{C}^{1,n}$ of index 1. Supplementary to this, Dulinski was able to generalize the approach of Murase in the case of Jacobi forms on $\mathbb{H} \times \mathbb{C}$ to square free index.

In this paper we prove the following. Let Φ be a Jacobi cusp form on $\mathbb{H} \times \mathbb{C}$, which is an eigenform with respect to the (full) Hecke-Jacobi algebra. First of all we show that every analytic Klingen Eisenstein series attached to Φ has a meromorphic continuation on the whole complex plane. This generalizes our previous work [He97]. We prove that Φ can be explicitly written as a linear combination of theta series. Moreover we study standard L-functions $L(\Phi, s)$ attached to Φ, when Φ is a newform (in the sense of [S-Z88]) or the index of Φ is square free. One ingredient of our proof is a certain rationality theorem in the Hecke-Jacobi algebra due to Gritsenko [Gr95]. In contrast to the previous works on the standard L-function of a Jacobi forms, our proof does not depend on the nature of the index, because we allow ourselves to work in a Hecke-Jacobi algebra, which is non commutative and has zero divisors. We believe that our method can be used to study standard L-functions of Jacobi forms on $\mathbb{H}_n \times \mathbb{C}^{1,n}$. But much work remains to be done to get a better understanding of the underlying Hecke-Jacobi algebra.

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1 On a Dirichlet series

This section deals with the properties of a certain kind of Dirichlet series, attached to the Jacobi symmetric-square operator and a Hecke-Jacobi eigenform Φ. We start with a short description of the underlying Hecke algebra.

1.1 Hecke-Jacobi theory

In this section we present a selfcontained introduction of a rationality theorem in the Hecke-Jacobi algebra attached to the Jacobi group. The theorem was used to study Kohnen and Skoruppa's [K-S89] approach towards the spinor zeta function of a Siegel modular form of degree 2. We show that such a theorem can also be used to study standard L-functions of Jacobi forms. One of the ingredients is the doubling method in the Rankin-Selberg method [Ar94]. This has the advantage that we do not have to care about the index of the underlying Jacobi forms, because we can mainly work in the Hecke-Jacobi algebra itself. This leads to an uniform approach to standard L-functions of Jacobi forms.

Let G^+Sp_n denote the group of symplectic similitudes of degree n with positive similitude factor. Let $\mathcal{R} = \mathbb{Z}, \mathbb{Z}_{[\frac{1}{p}]}$ or $\mathbb{Q}$. We define $G^J_{\mathcal{R}} = \{\left(\begin{smallmatrix} \beta & \\ 0_3 & * \end{smallmatrix}\right) \in G^+Sp_2(\mathcal{R}) \,|\, \beta > 0\}$ and put $\Gamma^J = G^J_{\mathbb{Z}} \cap Sp_2(\mathbb{Z})$. Then we associate to the Hecke pairs $(\Gamma^J\,,\, G^J_{\mathbb{Q}})$ and $(\Gamma^J\,,\, G^J_{\mathbb{Z}_{[\frac{1}{p}]}})$ the Hecke algebras $\mathcal{H}^J$ and $\mathcal{H}^J_p$, respectively. Next we distinguish several elements in $\mathcal{H}^J$. Let $[a_1, a_2, a_3, a_4]$ denote a diagonal matrix in $\mathbb{C}^{4,4}$. We define $T^{J,r}(n) = \Gamma^J[1, n^{-1}, 1, n]\Gamma^J$, $\Delta_n = \Gamma^J[n, n, n, n]\Gamma^J$, $\Lambda^r_-(p) = [p^{-1}, 1, p, 1]$ and $\Lambda^r_+(p) = \Gamma^J[p, 1, p^{-1}, 1]\Gamma^J$. Further let

$$\nabla^r_n = \sum_{b \bmod n} \Gamma^J \begin{pmatrix} 1 & 0 & \frac{b}{n} & 0 \\ 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 1 \end{pmatrix} \Gamma^J \quad \text{and} \quad \Xi^r_n = \sum_{\lambda,\mu,\kappa \bmod n} \Gamma^J \begin{pmatrix} 1 & \frac{\lambda}{n} & \frac{\kappa}{n} & \frac{\mu}{n} \\ 0 & 1 & \frac{\mu}{n} & 0 \\ & 0 & 1 & -\frac{\lambda}{n} \\ & & 0 & 1 \end{pmatrix}.$$

Let $\tilde{\mathcal{H}}^J_0$ be a subalgebra of $\mathcal{H}^J$ generated by $T^{J,r}, \Delta, \nabla^r, \Xi^r$ and $\tilde{\mathcal{H}}^J_{0,p} = \tilde{\mathcal{H}}^J_0 \cap \mathcal{H}^J_p$. Then $\tilde{\mathcal{H}}^J_0 = \otimes_p \tilde{\mathcal{H}}^J_{0,p}$. Now everything is prepared to state the rationality theorem. It seems to have first appeared in a paper of Gritsenko ([Gr95] section 5, Theorem 5.8).

Theorem 1.1 (**Gritsenko**) *The formal power series* $\sum_{n=o}^{\infty} T^{J,r}(p^n)(\frac{X}{p})^n$ *is a rational function with left denominator*

$$Q^J_p(X) = 1 - \Big(p^{-1}T^{J,r}(p) - (p-1)p^{-1}\nabla^r_p\Big)X + p\big(p^{-2}\Xi^r_p - p^{-1}\nabla^r_p + 1\big)X^2.$$

More precisely we have

$$(1) \qquad Q^J_p(X) \sum_{n=0}^{\infty} T^{J,r}(p^n)\left(\tfrac{X}{p}\right)^n = P^J_p(X),$$

where $P_p^J(X) = 1 + (p-1)p^{-1}\nabla_p^r X - \big((p-1)p^{-1}\nabla_p^r + 1\big)X^2$.

In this paper we would like to apply this result to study Arakawa's [Ar94] approach to the standard L-function of a Jacobi form in more detail. As already practiced in [He96], we simulate the behavior of certain Hecke-Jacobi operators in the underlying local Hecke-Jacobi algebra $\tilde{\mathcal{H}}_{0,p}^J$ to show the various specializations, depending on the index t of the underlying representation space of Jacobi forms. We have to distinguish between three cases: $p \nmid t$, $p|t$ with $p^2 \nmid t$ and $p^2|t$. For further simplification we put $n \parallel m$ if $n \mid m$ and $(n, \frac{m}{n}) = 1$.

$\boxed{\text{CASE I}}$ Let $\nabla_p^r = \Xi_p^r = 0$ and $T^{J,r}(p) = \lambda$. Then

$$\begin{aligned} Q_p^J(X) &= 1 - p^{-1}\lambda X + pX^2 \\ P_p^J(X) &= 1 - X^2. \end{aligned}$$

$\boxed{\text{CASE II}}$ Let $\nabla_p^r = p$, $p^{-2}\Xi_p^r = \varepsilon$, where $\varepsilon = \pm 1$ and $T^{J,r}(p) = \lambda$.

$$\begin{aligned} Q_p^J(X) &= 1 - (p^{-1}\lambda - p + 1)X + p\,\varepsilon\, X^2 \\ P_p^J(X) &= (1 + pX)\,(1 - X). \end{aligned}$$

(a) Let $\lambda = -p(\varepsilon + 1)$.

$$Q_p^J(X) = (1 + pX)\,(1 + \varepsilon\, X)$$

(b) Let $\varepsilon = 1$ and $\lambda = \lambda' - p + p^2$.

$$Q_p^J(X) = 1 - p^{-1}\lambda' X + pX^2$$

$\boxed{\text{CASE III}}$ Let $\nabla_p^r = p$, $p^{-2}\Xi_p^r = 0$ and $T^{J,r}(p) = -p$.

$$\begin{aligned} Q_p^J(X) &= (1 + pX) \\ P_p^J(X) &= (1 + pX)\,(1 - X). \end{aligned}$$

In section 3 we will give an explanation for the made assumptions and its connection to L-functions of Jacobi forms.

1.2 Zeta functions of Hecke-Jacobi eigenforms

We denote by $\mathcal{J}_{k,t}^{\text{cusp}}$ the space of Jacobi cusp forms on $\mathcal{D} = \mathbb{H} \times \mathbb{C}$ of weight k and index t with respect to the Jacobi group Γ^J. Moreover let $\langle\, ,\, \rangle_J$ be the Petersson scalar product. By $\hat{\mathcal{J}}_{k,t}^{\text{cusp}}$ we denote the space of completed cusp forms on $\mathbb{H}_2$. Here $\mathbb{H}_n$ denotes Siegel's upper half space of degree n. We frequently switch between the notation Φ and $\hat{\Phi}$ for the same Jacobi form. For more details the reader is referred to the standard reference [E-Z85] and [A-H98], section 1.

In [Ar94], Theorem 2.9, Arakawa proved the following relation between analytic Siegel Jacobi Eisenstein series and analytic Klingen-Jacobi Eisenstein series. Roughly speaking, he transfered a well-known theorem of Garrett and Böcherer [He96], Theorem 1.11 into the setting of Jacobi forms, which was a non trivial task. Let $k, t, n \in \mathbb{N}$ and k even. We denote by $E_n^{k,t}((\tau, z); s)$ the analytic Siegel-Jacobi Eisenstein series on $\mathcal{D}_{n,1} = \mathbb{H}_n \times \mathbb{C}^{1,n}$ of weight k and index t. Let further $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$. Then we denote by $E_{n,1}^{k,t}((\tau, z), \Phi, s)$ the analytic Klingen-Jacobi Eisenstein series attached to Φ on $\mathcal{D}_{n,1}$ of weight k and index t. Here $s \in \mathbb{C}$ with $s \gg 0$ (see [Ar94], section 2.4 for a explicit definition and more details).

Let $F : \mathbb{H}_n \longrightarrow \mathbb{C}$, $\quad k \in \mathbb{Z}$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G^+Sp_n(\mathbb{R})$. Then

$$(F|_k M)(z) := \mu(M)^{nk - \frac{n(n+1)}{2}} j(M, z)^{-k} F(M(z)),$$

where $j(M, z) = \det(c\, z + d)$ and $\mu(M)$ is the factor of similitudes of M. Let $X \in \mathcal{H}^J$. Then we define the action of $X = \sum a_j \; (\Gamma^J \, g_j)$ on the space of Jacobi forms by

$$\hat{\Phi}|_k \mathcal{D}(X) = \sum_j \; a_j \; \hat{\Phi}|_k \; g_j. \tag{2}$$

Definition 1.2 *Let $\Phi \in \mathcal{J}_{k,t}^{cusp}$. Then Φ is denoted by Hecke-Jacobi eigenform if*

$$\hat{\Phi}|_k \mathcal{D}(T^{J,r}(n)) = \lambda(\Phi, n) \; \hat{\Phi},$$

i.e. Φ is an eigenform for all operators $T^{J,r}(n)$ with eigenvalues $\lambda(\Phi, n)$.

Remark 1.3 *Let $\Phi \in \mathcal{J}_{k,t}^{cusp}$ be a Hecke-Jacobi eigenform. Then Φ is an eigenform with respect to all elements of $\tilde{\mathcal{H}}_0^J$. This follows from the formula*

$$T^{J,r}(p^2) = \big(T^{J,r}(p)\big)^2 - (p-1)\nabla_p^r T^{J,r}(p) - p\big(\Xi_p^r - \nabla_p^r + p^2 + p\big). \tag{3}$$

Let $s \in \mathbb{C}$. The Jacobi symmetric-square operator is formally defined by

$$\begin{aligned} S^J(s) &= \sum_{n=1}^{\infty} T^{J,r}(n)\, n^{-s} \\ &= \prod_p S_p^J(s), \end{aligned} \tag{4}$$

where $S_p^J(s) = \sum_{j=0}^{\infty} T^{J,r}(p^j)\, p^{-js}$.

Lemma 1.4 *Let $\Phi \in \mathcal{J}_{k,t}^{cusp}$ be a Hecke-Jacobi eigenform. Let $s \in \mathbb{C}$ with Re$(s) > 4$. Then the following identity is well-defined*

$$\hat{\Phi}|_k \mathcal{D}(S^J(s)) = Z^J(s,\Phi)\ \hat{\Phi}, \tag{5}$$

where $Z^J(s,\Phi) = \sum_{n=1}^{\infty} \lambda(\Phi,n)\, n^{-s}$. Moreover we have the Euler product

$$Z^J(s,\Phi) = \prod Z_p^J(s,\Phi) \text{ with } Z_p^J(s,\Phi) = \sum_{j=0}^{\infty} \lambda(p^j,\Phi)\, p^{-js}.$$

Here the series $Z^J(s,\Phi)$ is absolutely convergent.

Proof:
This has been proven in [Ar94], section 2.

$\square$

Theorem 1.5 *(**Arakawa**) Let $m \geq 1$ and k an even natural number. Let $\Phi \in \mathcal{J}_{k,t}^{cusp}$ be a Hecke-Jacobi eigenform and $k + 2Re(s) > m + 3$. Let $(\tau, z) \in \mathcal{D}_{m,1}$. Then we have*

$$\left\langle \Phi(\tau', z')\, ;\, E_{m+1}^{k,t}\Big(\begin{pmatrix} -\overline{\tau} & 0 \\ 0 & \tau' \end{pmatrix}, (\overline{z},\, z'), s \Big) \right\rangle_J \tag{6}$$

$$= \mu(k,t,1,s)\ Z^J(2s+k,\Phi)\ E_{m,1}^{k,t}(\Phi,(\tau,z),s). \tag{7}$$

In particular if $m = 1$, then the right hand side of this equation is equal to

$$\mu(k,t,1,s) Z^J(2s+k,\Phi)\ \ \Phi^{\sharp}(\tau,z).$$

Here $\mu(\ )$ is a certain constant depending only on k, t and s ([Ar94] (2.4)).

We have proven in [He97] that analytic Siegel-Jacobi Eisenstein series have a meromorphic continuation on the whole complex plane. Thus we have

Proposition 1.6 *Let $\Phi \in \mathcal{J}_{k,t}^{cusp}$ be a Hecke-Jacobi eigenform. Then $Z^J(s,\Phi)$ has a meromorphic continuation to the whole complex plane.*

At this point we do not know if it can happen that $Z^J(s,\Phi)$ is identically zero.

2 Non-vanishing of the local L-factors

Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ be a Hecke-Jacobi eigenform. Let $s \gg 0$. We are interested in the question if the local L-factor $Z_p^J(s,\Phi)$ can vanish. We have

$$\hat{\Phi}|_k \mathcal{D}(S_p^J(2s+k)) = Z_p^J(2s+k)\hat{\Phi}. \tag{8}$$

On the other hand, we can also calculate the left side of this equation by performing Theorem 1.1. This will show that $Z_p^J(s)$ is a rational function in p^{-s}, if we assume that Φ is a Hecke-Jacobi eigenform. More precisely: Let $X = p^{-(2s+k)}$. Then

$$\hat{\Phi}|_k \mathcal{D}(Q_p^J(pX) S_p^J(X)) = \hat{\Phi}|_k \mathcal{D}(P_p^J(pX)), \tag{9}$$

by abuse of notation. Here we have interpreted $f(X) \in \mathcal{H}_{0,p}^J[X]$ as an operator in the obvious way.

Let us denote the eigenvalues of $Q_p^J(pX)$ and $P_p^J(pX))$ by $Q_p^{J,\Phi}(pX)$ and $P_p^{J,\Phi}(pX)$. Since these eigenvalues can be viewed as non-trivial polynomials with respect to X, we can see that the quotient of these eigenvalues is holomorphic and has no zeros for $\text{Re}(s) \gg 0$.

Equipped with the absolute convergence of $Z^J(s,\Phi)$ for $\text{Re}(s) > 4$ we only have to analyze the numerator on $Z^J(s,\Phi)$ in more detail. We distinguish between two cases: $p \nmid t$ and $p \mid t$. This gives either $p^{-1}\nabla_p^r = 0$ or 1. The value of $P_p^{J,\Phi}(pX))$ is therefore $(1 - p^2X^2)$ or $(1 - pX)(1 - X)$. This shows that this is holomorphic in s and $P_p^{J,\Phi}(p^{-2s-k+1}) \neq 0$ for $k + 2\text{Re}(s) > 4$. We have

$$\hat{\Phi}|_k \mathcal{D}(S_p^J(X)) = \frac{P_p^{J,\Phi}(pX))}{Q_p^{J,\Phi}(pX)}\ \hat{\Phi}. \tag{10}$$

Now we are in a good situation. We have proven the following global non-vanishing result.

Theorem 2.1 *Let k be an even natural number larger than 4. Let $\Phi \in \mathcal{J}_{k,t}^{cusp}$ be a Hecke-Jacobi eigenform. Let $s \in \mathbb{C}$ with Re$(s) > 4$. Then $Z_p(s,\Phi)$ never vanishes and*

$$Z^J(s,\Phi) = \prod_p Z_p^J(s,\Phi) \neq 0. \tag{11}$$

Corollary 2.2 *Let $\Phi \in \mathcal{J}_{k,t}^{cusp}$ be a Hecke-Jacobi eigenform and $m \in \mathbb{N}$. Then the analytic Klingen-Jacobi Eisenstein series*

$$E_{m,1}^{k,t}(\Phi,(\tau,z),s)$$

associated to Φ on $\mathcal{D}_{n,1}$ of weight k and index t has a meromorphic continuation on the whole complex plane.

Corollary 2.3 *Let t be square-free and $\Phi \in \mathcal{J}_{k,t}^{cusp}$. Then $E_{m,1}^{k,t}((\tau,z),\Phi,s)$ has a meromorphic continuation on the whole complex plane.*

Proof:

It is well-known that for t square-free the space of Jacobi cusp forms has always a basis of Hecke-Jacobi eigenforms ([Gr95], section 5 & 6).

$\square$

Corollary 2.4 *Let k be an even natural number larger than 4 and $\Phi \in \mathcal{J}_{k,t}^{cusp}$ be a Hecke-Jacobi eigenform. Then*

$$Z^J(k,\Phi) \neq 0.$$

3 Local computations

Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ be a Hecke-Jacobi eigenform. We would like to calculate $Z_p^J(2s+k,\Phi)$ explicitly for every prime number p. It is convenient to distinguish between the three cases: $p \nmid t,\ p \parallel t,$ and $p^2|t$ (see section 1.1).

In the last section we have proven for the local factors of the Dirichlet series $Z^J(s,\Phi)$:

$$Z_p^J(2s+k,\Phi) = \frac{P_p^{J,\Phi}(pX)}{Q_p^{J,\Phi}(pX)}, \tag{12}$$

where $X = p^{-(2s+k)}$.

Already in the beginning of this paper, we have simulated the different cases in the Hecke-Jacobi algebra $\tilde{\mathcal{H}}_{0,p}^J$. Before we refine these formulas, we give some convenient definitions. Let $\zeta_p(s) = 1 - p^{-s}$.

3.1 Newforms

We start by introducing some virtual definitions for local L-factors associated to a Hecke-Jacobi form at the unramified primes.

Definition 3.1 *Let $p \nmid t$ and let $\Phi \in \mathcal{J}_{k,t}^{cusp}$ be a Hecke-Jacobi eigenform. Let $\hat{\Phi}|_k T^{J,r}(p) = \lambda\hat{\Phi}$ and $\lambda^{\mathsf{EZ}} := p^{k-3}\lambda$. We define*

$$\begin{aligned}
\tilde{L}_p^J(s,\Phi) &= 1 - \lambda\, p^{-s} + p^3\, p^{-2s} \\
\tilde{L}_p^{\mathsf{EZ}}(s,\Phi) &= 1 - \lambda^{\mathsf{EZ}} p^{-s} + p^{2k-3}\, p^{-2s}.
\end{aligned}$$

The definition of $\tilde{L}_p^J(s,\Phi)$ seems to be natural in the context of L-functions for Jacobi forms, but $\tilde{L}_p^{\mathsf{EZ}}(s,\Phi) = \tilde{L}_p^J(s-k+3,\Phi)$ stresses a hidden correspondance between elliptic cusp forms φ of weight $2k-2$ and level t and Jacobi cusp forms Φ of weight k and index t. For example, let $t=1$ and let $L_p(s,\varphi)$ be the Hecke polynomial attached to the eigenform φ. Then $\tilde{L}_p^{\mathsf{EZ}}(s,\Phi) = L_p(s,\varphi)$.

Next we would like to recall the concept of newforms in the context of Jacobi forms introduced by Skoruppa and Zagier [S-Z88]. The space of Jacobi newforms is defined by

$$\hat{\mathcal{J}}_{k,t}^{\text{cusp,new}} = \left(\sum_{\substack{l,d>0,\, ld^2>1 \\ ld^2|t}} \hat{\mathcal{J}}_{k,t/(ld^2)}^{\text{cusp}}|_k \mathcal{D}\Big(\Lambda_-^r(d)\, T_-^r(l)\Big) \right)^{\perp}, \tag{13}$$

here $\perp$ denotes the orthogonal complement with respect to the Petersson scalar product of Jacobi forms. It remains to define the operators $T^r_-(p)$ and $T^r_+(n)$. We have $T^r_-(p) = \Delta_p^{-1}\Gamma^J[1,1,p,p]\Gamma^J$ and $T^r_+(p) = \Delta_p^{-1}\Gamma^J[p,1,1,p]\Gamma^J$. In [Gr95], page 80 Gritsenko has mentioned that this can be simplified to

(14)
$$\hat{\mathcal{J}}_{k,t}^{\text{cusp,new}} = \bigcap_{p|t} \text{Ker}\left(\mathcal{J}_{k,t}^{\text{cusp}}|_k\mathcal{D}\Big(T^r_+(p)\Big)\right) \cap \text{Ker}\left(\mathcal{J}_{k,t}^{\text{cusp}}|_k\mathcal{D}\Big(\Lambda^r_+(p)\Big)\right).$$

Facts:

- The space $\mathcal{J}_{k,t}^{\text{cusp,new}}$ has a basis of Hecke Jacobi eigenforms.
-

$$\mathcal{J}_{k,t}^{\text{cusp}} = \mathcal{J}_{k,t}^{\text{cusp,new}} \oplus \bigoplus_{\substack{l,d>0\\ ld^2|t\\ ld^2>1}} \mathcal{J}_{k,t/(ld^2)}^{\text{cusp, new}}|_k\mathcal{D}\Big(\Lambda^r_-(d)T^r_-(l)\Big). \tag{15}$$

Gritsenko has introduced another interesting subspace of $\mathcal{J}_{k,t}^{\text{cusp}}$, the space

(16)
$$\mathcal{J}_{k,t}^{\text{cusp, G}} := \bigcap_{p|t} \text{Ker}\left(\mathcal{J}_{k,t}^{\text{cusp}}|_k\mathcal{D}\Big((T^r_+(p))^2\Big)\right) \cap \text{Ker}\left(\mathcal{J}_{k,t}^{\text{cusp}}|_k\mathcal{D}\Big(\Lambda^r_+(p)\Big)\right).$$

This space has some remarkable properties: It has a basis of Hecke-Jacobi eigenforms and in the case of t square-free, it exhausts the whole space,

$$\mathcal{J}_{k,t}^{\text{cusp, G}} = \mathcal{J}_{k,t}^{\text{cusp}}. \tag{17}$$

For arbitrary t we have

$$\mathcal{J}_{k,t}^{\text{cusp,new}} \subset \mathcal{J}_{k,t}^{\text{cusp, G}} \subset \mathcal{J}_{k,t}^{\text{cusp}}. \tag{18}$$

3.2 Explicit form of $Z_p^J(s,\Phi)$

Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ be a Hecke-Jacobi newform. We will determine in this section the explicit nature of the p^{th} - local factor of $Z^J(s,\Phi)$. In addition we will also show in this paper, that if the index t is square free, the assumption on Φ to be a Hecke eigenform is sufficient, to compute $Z_p^J(s,\Phi)$. This improves the results of Murase [Mu89], Arakawa [Ar94] and Dulinski [Du97] for Jacobi forms on $\mathbb{H}\times\mathbb{C}$. We would like to mention, that our approach and proof is new, even if we restrict ourself to the known cases of the papers stated above. We believe that this method should also work for Jacobi forms on $\mathbb{H}_n \times \mathbb{C}^{1,n}$. But this would involve much more complicated calculations and much work on the underlying Hecke-Jacobi theory. We will come back to this task in another paper.

As usual, in this section λ and ε will be the eigenvalues of $\Phi \in \mathcal{J}_{k,t}^{\text{cusp,new}}$ with respect to $T^{J,r}(p)$ and $p^{-2}\Xi_p^r$. Let $X = p^{-(2s+k)}$. In the following we distinguish between three cases:

CASE I: $p \nmid t$, CASE II: $p \parallel t$ and CASE III: $p^2 \mid t$.

The arguments below will justify the assumptions and computations already done in the Hecke-Jacobi algebra (see section 1.1). The starting point of everything which follows is the formula given in (12). Let us start with the *generic*

$\boxed{\text{CASE I}}$ It is obvious that if $p \nmid t$, then $\hat{\Phi}|_k\mathcal{D}(\nabla_p^r) = \hat{\Phi}|_k\mathcal{D}(\Xi_p^r) = 0$. This shows that the assumptions in section 1.1 in CASE I are satisfied. Hence we have

$$\begin{aligned} Z_p^J(2s+k,\Phi) &= \frac{\zeta_p(4s+2k-2)}{\tilde{L}_p^J(2s+k,\Phi)} \\ &= \frac{\zeta_p(4s+2k-2)}{\tilde{L}_p^{\text{EZ}}(2s+2k-3,\Phi)}. \end{aligned}$$

Next we devote our interest to the situation $p \mid t$. Since $\hat{\Phi}|_k\mathcal{D}\big(p^{-1}\nabla_p^r\big) = \hat{\Phi}$, we have

$$P_p^J(X) = (1+pX)(1-X), \tag{19}$$

$$Q_p^J(X) = 1 - \big(p^{-1}\lambda - p + 1\big)X + p\,\varepsilon\, X^2. \tag{20}$$

$\boxed{\text{CASE II}}$ Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$ be a Hecke-Jacobi newform and $p \parallel t$. The fact that $p \parallel t$ links our operator Ξ_p^r with the Atkin-Lehner operator W_p introduced in [E-Z85], section 5.

Remark 3.2 *([Gr95], Theorem 5.10) Let $t = m\,p$ and $(m,p) = 1$ and $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$, then*

$$\hat{\Phi}|_k\mathcal{D}(p^{-2}\Xi_p^r) = \hat{\Phi}|_k W_p, \tag{21}$$

where W_p denotes the involution given in [E-Z85], page 60.

Newforms have the property, that they lie in the kernel of $T_+^r(p)$. The list given in [He96], (section 3.3, number 31) gives the identity

$$\Delta_p T_+^r(p) T_-^r(p) = T^{J,r}(p)\nabla_p^r + p\nabla_p^r + \Xi_p^r.$$

Putting these two informations together shows that $pT^{J,r}(p) + p^2 + \Xi_p^r$ is trivial on the space of newforms. Thus we have (by abuse of notation)

$$T^{J,r}(p) = -p\big(1+\varepsilon\big) \text{ and} \tag{22}$$

$$Z_p^J(2s+k,\Phi) = \frac{\zeta_p(4s+2k-2)}{1+\varepsilon\,p^{-(2s+k-1)}\big)\big(1+p^{-(2s+k-1)}\big)},$$

where $\varepsilon = \pm 1$.

$\boxed{\text{CASE III}}$ Let $\Phi \in \mathcal{J}_{k,t}^{\text{cusp,new}}$ be a Hecke-Jacobi newform and $p^2 \nmid t$. Then formula (42) given in the list in [He96], section 3.3 for $\Lambda_+^r(p)\Lambda_-^r(p)$ leads finally to

$$\hat{\Phi}|_k \mathcal{D}(\Xi_p^r) = 0.$$

The formulae given in (20) and (22) lead us to $T^{J,r}(p) = -p$. Hence we have $Q_p^J(X) = (1 + pX)$ and

$$Z_p^J(2s+k,\Phi) = \frac{\zeta_p(4s+2k-2)}{1+p^{-(2s+k-1)})}.$$

Finally we obtain the desired theorem on the local nature of $Z^J(s,\Phi)$.

Theorem 3.3 *Let k be an even natural number. Let $\Phi \in \mathcal{J}_{k,t}^{cusp}$ be a Hecke-Jacobi newform and $k + 2Re(s) > 4$. Then we have*

$$\frac{Z_p^J(2s+k,\Phi)}{\zeta(4s+2k-2)} = \begin{cases} \tilde{L}_p^{EZ}(2s+2k-3,\Phi)^{-1} & \text{if } p \nmid t \\ (1+\varepsilon\, p^{-(2s+k-1)})^{-1}(1+p^{-(2s+k-1)})^{-1} & \text{if } p \parallel t \\ (1+p^{-(2s+k-1)})^{-1} & \text{if } p^2 \mid t \end{cases}.$$

Let $S^k(\Gamma_0(N))$ denote the space of cusp forms of weight k with respect to $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod N\}$. The results of Skoruppa and Zagier [S-Z88] on a Hecke-equivariant map between $\mathcal{J}_{k,t}^{\text{cusp}}$ and a certain subspace $\mathcal{M}_{2k-2}^{\text{cusp},-}(N)$ of $S^{2k-2}(\Gamma_0(N))$ suggests the following definition of a L-function attached to a Hecke-Jacobi eigenform in $\mathcal{J}_{k,t}^{\text{cusp,new}}$.

Definition 3.4 *Let k be an even natural number and $\Phi \in \mathcal{J}_{k,t}^{cusp}$ be a Hecke-Jacobi newform. Let $Re(s) \gg 0$ and ε_p^{Φ} be the eigenvalue with respect to $p^{-2}\Xi_p^r$, then the (standard) L-function of Φ is defined by*

$$L(s,\Phi) = \prod_{p \| t}(1+\varepsilon_p^{\Phi}\, p^{k-2}p^{-s})^{-1} \prod_{p \nmid t} \tilde{L}_p^{EZ}(s,\Phi)^{-1}. \tag{23}$$

Remark 3.5 *Let $\varphi \in S^{2k-2}(\Gamma_0(1))$ correspond to the Hecke-Jacobi eigenform $\Phi \in \mathcal{J}_{k,1}^{cusp}$ by the Saito-Kurokawa lift and let $L(s,\varphi)$ be the Hecke L-function of φ. Then*

$$L(s,\Phi) = L(s,\varphi). \tag{24}$$

Summarizing everything gives the following main result of this section.

Theorem 3.6 *Let k be an even natural number and $Re(s) > 4$. Let $\Phi \in \mathcal{J}_{k,t}^{cusp}$ be a Hecke-Jacobi newform. Then we have*

$$Z^J(s,\Phi) = \zeta(2s-2)^{-1}\Big(\prod_{p|t}(1+p^{-(s-1)})\Big)^{-1} L(s+k-3,\Phi). \tag{25}$$

Moreover let $k > 4$. Then $L(2k-3,\Phi) \neq 0$.

Let us specialize the formula given in Theorem 3.6 to $t = 1$. If we combine this with the identity (24) given in the remark we get

$$Z^J(s, \Phi) = \frac{L(s+k-3, \varphi)}{\zeta(2s-2)}. \tag{26}$$

This result was also obtained by Arakawa [Ar94] (section 6.1) with a totally different proof.

3.3 Beyond newforms

As already mentioned, there exists a subspace $\mathcal{J}_{k,t}^{\text{cusp,G}}$ of $\mathcal{J}_{k,t}^{\text{cusp}}$ which contains the space of newforms and possesses a basis of Hecke-Jacobi eigenforms. We would like to study the Dirichlet series $Z^J(s, \Phi)$ attached to $\Phi \in \mathcal{J}_{k,t}^{\text{cusp}}$, where t is square free in more detail. We remind the reader, that in this case $\mathcal{J}_{k,t}^{\text{cusp}} = \mathcal{J}_{k,t}^{\text{cusp,G}}$ and

$$\mathcal{J}_{k,t}^{\text{cusp}} = \mathcal{J}_{k,t}^{\text{cusp,new}} \oplus \bigoplus_{l|t,\, l>1} \mathcal{J}_{k,t/l}^{\text{cusp, new}}|_k \mathcal{D}\Big(T_-^r(l)\Big). \tag{27}$$

The decomposition suggest how one can construct a *good basis* of the space of Jacobi cusp forms.

Lemma 3.7 *Let t be a square free natural number and $\Psi \in \mathcal{J}_{k,t/l}^{cusp}$, where $l \mid t$, be a Hecke-Jacobi eigenform. Then*

$$\hat{\Psi}|_k \mathcal{D}\big(T_-^r(l)\big)$$

is also a Hecke-Jacobi eigenform.

Proof:
Let $p|l$. Then is suffices to show that $\hat{\Psi}|_k \mathcal{D}\big(T_-^r(p)\big)$ is an eigenform with respect to $T^{J,r}(p)$ and $p^{-2}\Xi_p^r$. First of all we would like to remark that since t is square free, $(t/l, p) = 1$ implies $\hat{\Psi}|_k \mathcal{D}(T_+^r(p)) = \hat{\Psi}|_k \mathcal{D}(\nabla_p^r) = 0$.

We have (again we use results of the list given in [He96], section 3.3):

$$\begin{aligned} \hat{\Psi}|_k \mathcal{D}(T_-^r(p)) \mid_k \mathcal{D}(T^{J,r}(p)) &= p(p-1)\hat{\Psi}|_k \mathcal{D}(T_-^r(p)) + \hat{\Psi}|_k \mathcal{D}(T^{J,r}(p) T_-^r(p)), \\ \hat{\Psi}|_k \mathcal{D}(T_-^r(p)) \mid_k \mathcal{D}(p^{-2}\Xi_p^r) &= \hat{\Psi}|_k \mathcal{D}(T_-^r(p)), \end{aligned}$$

which proves the lemma.

$\square$

It is obvious, that the arguments employed in the above proof of the lemma, do not work for arbitrary index. With the results of the lemma in mind we can construct a certain distinguished basis of $\mathcal{J}_{k,t}^{\text{cusp}}$ for t square free. Let us start with the decomposition in

(27). For every $l \mid t$, we fix a normalized basis $(\Phi_j^l)_{j=1}^{d_l}$ of Hecke-Jacobi eigenforms of the underlying space of newforms $\mathcal{J}_{k,t/l}^{\text{cusp,new}}$ (of course this will be only relevant if $\mathcal{J}_{k,t/l}^{\text{cusp,new}}$ is non-trivial). Then

$$\mathbf{B} = \bigcup_{l \mid t} \left(\hat{\Phi}_j^l \mid_k \mathcal{D}(T_-^r(l)) \right)_{j=1}^{d_l} \tag{28}$$

is a basis of Hecke-Jacobi eigenforms of $\mathcal{J}_{k,t}^{\text{cusp}}$ for t square free. Let $\Psi \in B$, i.e there exists a $l \in \mathbb{N}$ and $1 \leq j \leq d_l$ such that $\hat{\Psi} = \hat{\Phi}_j^l \mid_k \mathcal{D}(T_-^r(l))$. Let $l > 1$ and $p_1 \cdot \ldots \cdot p_{l_r} = l$ be the prime number decomposition of l. Let λ and λ' be the eigenvalues of Ψ and Φ_j^l with respect to $T^{J,r}(p_i)$. Then

$$\lambda = p_i(p_i - 1) + \lambda',$$

and $\hat{\Psi} \mid_k \mathcal{D}(p_i^{-2}\Xi_{p_i}^r) = \hat{\Psi}$.

Definition 3.8 *Let k, t be natural numbers. Let k be even and t square free. We call the basis B of $\mathcal{J}_{k,t}^{cusp}$ defined in (28), the generalized newform basis of $\mathcal{J}_{k,t}^{cusp}$. Elements are called generalized newforms.*

Proposition 3.9 *Let t be square free and k even. Let $\Phi \in \mathcal{J}_{k,t}^{cusp}$ be a generalized newform, i.e*

$$\hat{\Psi} = \hat{\Phi}_j^l \mid_k \mathcal{D}(T_-^r(l))$$

for suitable Hecke-Jacobi eigenforms $\Phi_j^l \in \mathcal{J}_{k,t/l}^{cusp,new}$. Then

$$\frac{Z_p^J(2s+k,\Phi)}{\zeta(4s+2k-2)} = \begin{cases} \tilde{L}_p^{EZ}(2s+2k-3,\Phi_j^l)^{-1} & \text{if } p \nmid t \\ (1+\varepsilon\, p^{-(2s+k-1)})^{-1}(1+p^{-(2s+k-1)})^{-1} & \text{if } p \mid \frac{t}{l} \\ \frac{(1+p^{-(2s+k-2)})}{(1+p^{-(2s+k-1)})}\tilde{L}_p^{EZ}(2s+2k-3,\Phi_j^l)^{-1} & \text{if } p \mid l \end{cases}.$$

Let $p \nmid t$. Then $\tilde{L}_p^{EZ}(s,\Phi) = \tilde{L}_p^{EZ}(s,\Phi_j^l)$. Let $p \mid \frac{t}{l}$, then the eigenvalue of Φ and Φ_j^l with respect to $p^{-2}\Xi_p^r$ is ε.

Let $\hat{\Psi} = \hat{\Phi}_j^l \mid_k \mathcal{D}(T_-^r(l))$ be a generalized newform. Then we associate to Φ the (standard) L-function

$$\begin{aligned} L(s,\Phi) \;=\; & \prod_{p \mid l}(1+p^{k-1}p^{-s}) \prod_{p \mid (t/l)} (1+\varepsilon_p^{\Phi}\, p^{k-2}p^{-s})^{-1} \\ & \prod_{p \mid l} \tilde{L}_p^{\text{EZ}}(s,\Phi_j^l)^{-1} . \prod_{p \nmid t} \tilde{L}_p^{\text{EZ}}(s,\Phi)^{-1}. \end{aligned}$$

This is compatible with the Definition 3.4 for the L-function attached to $\Phi \in \mathcal{J}_{k,t}^{\text{cusp,new}}$.

4 Global results and the basis problem

Let us first summarize what we have proven in the last sections.

Theorem 4.1 *Let k be an even natural number and Re$(s) > 4$. Let $\Phi \in \mathcal{J}_{k,t}^{cusp}$ be a Hecke-Jacobi eigenform. Then the Dirichlet series $Z^J(s,\Phi)$, defined in Lemma 1.4, has a meromorphic continuation on the whole complex space and $Z^J(k,\Phi) \neq 0$ for $k > 4$.*

If we assume Φ to be a (generalized) newform, generalized if t is square free. Then we have

$$Z^J(s,\Phi) = \zeta(2s-2)^{-1}\Big(\prod_{p|t}(1+p^{-(s-1)})\Big)^{-1} L(s+k-3,\Phi). \tag{29}$$

Let $k > 4$ then

$$L(2k-3,\Phi) \neq 0.$$

4.1 Siegel's formula for Jacobi forms

Let $m, n, t \in \mathbb{N}$. By A_n^+ we denote the set of semi-integral positive definite matrices of size n and $A_n^+(t) = \{\left(\begin{smallmatrix} * & * \\ * & t\end{smallmatrix}\right) \in A_n^+\}$. Two elements $T, T' \in A_{m+1}^+(t)$ are said to be in the same $\mathbf{t-class}$, if there exists $\gamma = \{\left(\begin{smallmatrix} u & 0 \\ y & 1\end{smallmatrix}\right)$ with $u \in \mathsf{SL}_m(\mathbb{Z})$, $y \in \mathbb{Z}^{1,m}$ such that $T' = T[\gamma]$. Moreover T, T' are in the same $\mathbf{t-genus}$, if for each prime integer p, $\gamma = \left(\begin{smallmatrix} u & 0 \\ y & 1\end{smallmatrix}\right)$ can be choosen in such a way that $u \in \mathsf{SL}_m(\mathbb{Z}_p)$ and $y \in \mathbb{Z}_p^{1,m}$. Each $t-$genus consists of finitely many $t-$classes (cf. [Ar93], Proposition 2.1). We denote by $H(t)$ the number of $t-$classes of the $t-$genus for given t. This number is called the $t-$class number of t.

Definition 4.2 *Let $Q = \begin{pmatrix} M & (q/2)^t \\ q/2 & t \end{pmatrix} \in A_{m+1}^+(t)$. We define the theta series $\Theta_Q^{(n)}(\tau,z)$, where $(\tau,z) \in \mathcal{D}_{n,1}$ by*

$$\Theta_Q^{(n)}(\tau,z) = \sum_{G \in \mathbb{Z}^{m+1,n}} e\{Q[G]\,\tau + z^t(q,2t)G\}. \tag{30}$$

It is known that $\Theta_Q^{(n)}(\tau,z)$ is a Jacobi form on $\mathbb{H}_n \times \mathbb{C}^{1,n}$ of index t and weight $\frac{m+1}{2}$, if $2\,Q$ is unimodular. Let $E(Q)$ be the order of the finite group

$$\left\{P = \begin{pmatrix} u & 0 \\ y & 1\end{pmatrix} \in \mathbb{Z}^{m+1,m+1} \;\mid\; u \in \mathsf{SL}_m(\mathbb{Z}) \text{ and } Q[P] = Q\right\}.$$

Arakawa has proven [Ar93]

Theorem 4.3 *(**Analytic Siegel's formula**) Assume that $m > 2n+3$. Let $Q \in A^+_{m+1}(t)$ with $2Q$ unimodular. Let $Q_1, Q_2, \ldots, Q_H$ be a complete set of representatives of the $t-$classes in the given $t-$genus of Q. Then*

$$\left(\sum_{j=1}^{H} \frac{\Theta^{(n)}_{Q_j}(\tau,z)}{E(Q_j)}\right) \Big/ \left(\sum_{j=1}^{H} \frac{1}{E(Q_j)}\right) = E_n^{\frac{m+1}{2},t}(\tau,z). \tag{31}$$

It is useful to introduce the following map. Let

$$\Lambda : \begin{cases} \mathcal{J}^{\mathsf{cusp}}_{k,t} & \longrightarrow & \mathcal{J}^{\mathsf{cusp}}_{k,t} \\ \Phi & \mapsto & \left\langle \Phi \, , \, E_2^{k,t}\begin{pmatrix} \overline{\tau} & 0 \\ 0 & * \end{pmatrix}, (\overline{z}, *)) \right\rangle_J. \end{cases}$$

Then Λ is a linear map. Moreover, if Φ is a Hecke-Jacobi eigenform, then it is also an eigenfunction of Λ. Now, let the eigenvalue of Φ with respect to Λ be non zero, then Siegel's theorem can be used to give an explicit describtion of Φ as a linear combination of theta series [Boe83].

Theorem 4.4 *Let k, t be natural numbers and let $k \equiv 0 \bmod 4$ and $k > 4$. Let $\Phi \in \mathcal{J}^{cusp}_{k,t}$ be a Hecke-Jacobi eigenform. Then Φ can be explicitly written as a linear combination of theta series $\Theta^{(1)}_{Q_j}(\tau,z)$, where $Q_1, Q_2, \ldots, Q_H$ is a complete set of representatives of the $t-$classes in the given $t-$genus of suitable $Q \in A^+_{2k}(t)$ with $2Q$ unimodular.*

$$\Phi(\tau,z) = \left(\mu(k,t,1,0)\, Z^J(k,\Phi)\right)^{-1} \sum_{j=1}^{H} \frac{\langle \Phi, \Theta^{(1)}_{Q_j}\rangle_J}{m_Q E(Q_j)}\, \Theta^{(1)}_{Q_j}(\tau,z),$$

where $m_Q = \sum_j 1/E(Q_j)$.

Proof:
First of all we have to show hat the assumptions of Siegel's Theorem are satisfied. Let $2Q_8$ denote an even unimodular positive definite symmetric matrix of size 8 associated with the E_8 lattice. We may choose Q_8 such that $Q_8 \in A^+_8(t)$. Because the classical theta series on $\mathbb{H}$ attached to E_8 is proportional to the elliptic Eisenstein series of weight 4, and hence has positive Fourier coefficients. We have already proven in this paper that $Z^J(k,\Phi) \neq 0$ for Hecke-Jacobi eigenforms Φ. Finally Theorem 1.5 and Siegel's Theorem complete the proof of this theorem.

□

Corollary 4.5 *Let k, t be natural numbers. Let $k > 4$, $k \equiv 0 \bmod 4$ and t be square free. Then the space $\mathcal{J}_{k,t}$ is $\mathbb{C}-$ linearly spanned by the theta series $\Theta^{(1)}_{Q_j}(\tau,z)$, $(1 \leq j \leq H)$. Here $Q_1, Q_2, \ldots, Q_H$ are given as in the Theorem 4.4.*

Proof:
This follows from the results of Theorem 4.4 and a structure theorem [Ar94], section 4.2 on the space of Jacobi forms $\mathcal{J}_{k,t}$. Here a certain maximality condition on the index t is essential.

$\square$

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