ON SOME DIFFERENTIALS IN THE MOTIVIC COHOMOLOGY SPECTRAL SEQUENCE

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Abstract. The main purpose of this paper is to compute the first non-zero differential in the motivic cohomology spectral sequence with p-local coefficients in terms of the motivic Steenrod operations.

The motivic cohomology spectral sequence is an algebraic-geometrical analogue of the Atiyah–Hirzebruch spectral sequence in topology. For smooth varieties, it has the second term consisting of motivic cohomology groups and converges to algebraic K-theory.

The spectral sequence was initially constructed for fields by Bloch and Lichtenbaum in their unpublished preprint [BL]. Further, two constructions for varieties were given in papers of Friedlander and Suslin [FS] and Grayson [Gr]. The equivalence of their approaches was established in [Su3].

The behavior of differentials in the motivic cohomology spectral sequence is quite similar to the topological case. Being taken with rational coefficients the sequence collapses at its E_2-term (see Levine [Le]).

The next natural question in the row is to describe possible non-trivial differentials of order p in the spectral sequence with p-local coefficients.

In topology the goal was achieved by Buchstaber [Bu]. In the current paper we establish the parallel result for the motivic cohomology spectral sequence. Philosophically, presented proof is similar to Buchstaber's one, but the technique is certainly rather different.

The strategy of the proof is the following: First, I show, using Adams operations, that the first non-trivial differential may appear only in E_p-term (Proposition 1.2). Then, computing the motivic Steenrod algebra in the corresponding degree, it is possible to show that the differential in question is proportional to the Steenrod algebra generator (Proposition 2.2). Finally, to check that the proportionality coefficient is not 0, I construct an example of a variety with non-trivial differentials (Theorem 4.1).

The computation of the p-local Steenrod algebra is based on Voevodsky's result on the structure of the motivic Steenrod algebra with finite coefficients. Since the latter result is proven only for fields of characteristic 0, we should restrict ourself to this case. However, all other arguments work perfectly for arbitrary fields. Unfortunately, currently I also have no satisfactory proof of Proposition 2.1 that makes the main result a bit hypothetical. In topology, similar result easily follows from the fact that n-th Eilenberg–Mac Lane space in n-connected.

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1. Main Result

Let k be a field. Let us fix some prime $p > 2$ different from the field characteristic and denote by $\mathbb{Z}_{(p)}$ the localization of the group of integers $\mathbb{Z}$ outside the prime ideal (p).

Consider the Motivic Cohomology spectral sequence (see [FS]) with coefficients in $\mathbb{Z}_{(p)}$:

$$E_2^{i,j} = H^{i-j}(X, \mathbb{Z}_{(p)}(-j)) \Rightarrow K_{-i-j}(X, \mathbb{Z}_{(p)}). \tag{1.1}$$

(Everywhere in this paper H denotes the Motivic cohomology groups.) It is shown by Voevodsky [Vo2], that the Motivic cohomology groups coincide with the corresponding higher Chow groups. Namely,

$$H^{i-j}(X, \mathbb{Z}_{(p)}(-j)) = CH^{-j}(X, -i-j; \mathbb{Z}_{(p)}). \tag{1.2}$$

Differentials in this spectral sequence act as: $d_n \colon E_n^{i,j} \to E_n^{i+n,j-n+1}$.

The purpose of the current paper is to compute the first non-trivial differential in the spectral sequence above. Namely, we are going to prove the following Theorem:

Theorem 1.1.

$$d_i = \begin{cases} 0 \text{ for } 1 < i < p \\ a(\mathfrak{B} \circ P^1 \circ \mathrm{red}) \text{ for } i = p. \end{cases} \tag{1.3}$$

Here $a \neq 0$ denotes some constant lying in the group $\mathbb{Z}/p$, the operations red, $\mathfrak{B}$, *and P^1 are coefficient reduction, Bockstein homomorphism, and first $\mathbb{Z}/p$-Steenrod power operation, correspondingly.*

In this section we prove the first statement of the theorem. Then, the differential in question will be described as an element of the corresponding Steenrod algebra, which will be computed in Sections 2 and 3. Finally, in Section 4 we construct an example of a variety for which the corresponding differential in the spectral sequence is non-trivial. This completes the proof of the Theorem.

Proposition 1.2. $d_n = 0$ *for* $p - 1 \nmid n - 1$.

Proof. As it was shown by Mark Levine [Le], the Adams operations ψ_k in $K_*(X)$ can be extended to operations acting on the whole Motivic Cohomology spectral sequence. Moreover, their action on the E_2-term is given by the relation: $\psi_k(\alpha) = k^q \alpha$ for $\alpha \in H^*(X, \mathbb{Z}(q))$. Therefore, all topological arguments proposed by Buchstaber [Bu] work in this case as well. Since Adams operations commute with differentials, we have:

$$d_i \psi_k = \psi_k d_i \colon H^*(X, \mathbb{Z}(q)) \to H^{*+2i-1}(X, \mathbb{Z}(q+i-1)).$$

Therefore, one has: $k^q(k^{i-1} - 1)d_i = 0$ and the differential d_i annihilates as multiplied by the

$$\underset{\substack{k=2\ldots\infty \\ p\nmid k}}{\text{g.c.d.}} \{k^q(k^{i-1} - 1)\} \overset{\text{def}}{=} M(q, i). \tag{1.4}$$

(These integers are sometimes called Kervaire–Milnor constants[1].) Since for $p-1 \nmid i-1$ we have: $p \nmid M(*,i)$, the differentials of these degrees vanish. □

Corollary 1.3. *We have: $pd_p = 0$.*

Proof. Since one has: $M(*,p) = p$, the corollary follows. □

Lemma 1.4. *Differential d_n is a bistable cohomological operation of degree $(2n-1, n-1)$ on the category of Voevodsky's motivic spaces* [Vo4].

Proof. Since every space can be obtained as a colimit of smooth schemes and the Motivic spectral sequence is functorial, one can extend it term-wise to the whole category of spaces.

To prove stability, it is sufficient to show that the following diagram commutes:

$$\begin{array}{ccc} H^i(X,\mathbb{Z}(j)) & \xrightarrow{d_n} & H^{i+2n-1}(X,\mathbb{Z}(j+n-1)) \\ \Sigma\downarrow & & \downarrow\Sigma \\ H^{i+2}(T\wedge X,\mathbb{Z}(j+1)) & \xrightarrow{d_n} & H^{i+2n+1}(T\wedge X,\mathbb{Z}(j+n)), \end{array} \tag{1.5}$$

where Σ denotes the T-suspension morphism and T is the Tate object. The space $T \wedge X$ is T-homotopically equivalent to $(\mathbb{P}^1,\infty)\wedge X$. Motivic cohomology groups of the latter space appear as shifted cohomology groups of X itself and the isomorphism may be delivered by multiplication with the Tate element σ. Since the spectral sequence differentials satisfy the Leibnitz rule, it is suffices to check that $d_n(\sigma) = 0$ that follows by the dimension reasons. □

2. Computation of $\mathbb{Z}_{(p)}$-Steenrod algebra torsion

Denote the set of bistable cohomological operations of degree $\{i,j\}$, sending motivic cohomology with coefficients in a group S to one with coefficients in some group T by $\mathcal{OP}^{i,j}(S,T)$. In particular, set $\mathcal{A}^{i,j}(S) = \mathcal{OP}^{i,j}(S,S)$. For an Eilenberg–Mac Lane space $K(S,n)$ one can choose a universal element ε in the group $H^n(K(S,n))$, corresponding to the identical morphism of the space $K(S,n)$. Applying the T-suspension map to the element ε, one obtains the element $\Sigma\varepsilon$, i.e. the homotopy class of morphisms: $\Sigma K(S,n) \xrightarrow{f_n} K(S,n+1)$. Finally, using this morphism, one can construct an inverse system of the groups $H^{i+2n,i+n}(K(S,n),T)$ as shown in the diagram below.

$$\begin{tikzcd} H^{i+2n+2,i+n+1}(K(S,n+1),T) \arrow[r, "f_n^*"] \arrow[dr] & H^{i+2n+2,i+n+1}(\Sigma K(S,n),T) \arrow[d, "\simeq\ \Sigma"] \\ & H^{i+2n,i+n}(K(S,n),T) \end{tikzcd} \tag{2.1}$$

One sets:

$$\mathcal{OP}^{i,j}(S,T) = \lim_{\substack{\longleftarrow \\ n}} H^{i+2n,j+n}(K(S,n),T). \tag{2.2}$$

For motivic cohomology all the construction above can be made explicit. Following Voevodsky [Vo3, Vo4], for a smooth scheme X over k one considers $\mathbb{Z}_{tr}(X)$

[1] Probably, after their paper [KM]

the presheaf of abelian groups on the category Sm/k of smooth algebraic varieties of finite type, which takes U to the abelian group generated by all cycles on $X \times U$, which are finite and equidimensional over U. For an abelian group A set $A_{tr} := A \otimes \mathbb{Z}_{tr}$ and one defines $K(A,n)$ as the pointed sheaf of sets associated with the presheaf

$$K^{pre}_{A,n} : U \mapsto A_{tr}(\mathbb{A}^n)(U)(U)/A_{tr}(\mathbb{A}^n - \{0\})(U). \tag{2.3}$$

If A is a commutative ring, there exist multiplication morphisms

$$K(A,n) \wedge K(A,m) \to K(A,n+m), \tag{2.4}$$

induced by the morphism sending a pair of cycles to their external product. The presheaf $A_{tr}(\mathbb{P}^n,\infty)(U)$ gives another equivalent model of the motivic Eilenberg–Mac Lane spaces. This fact, together with existence of the canonical morphism $(\mathbb{P}^n,\infty) \to A_{tr}(\mathbb{P}^n,\infty)$ enables us to endow the sequence of the spaces $K(A,n)$ with bonding maps

$$(\mathbb{P}^1,\infty) \wedge K(A,n) \to K(A,n+1) \tag{2.5}$$

and enrich this sequence with the structure of a $(\mathbb{P}^1,\infty)$-spectrum.[2]

Later we will need the following statement:

Proposition 2.1. $\varprojlim_n{}^1 H^{i+2n,j+n}(K(S,n),T) = 0$.

Proposition 2.2. *The p-torsion in the $\mathbb{Z}_{(p)}$-Steenrod algebra is equal to $\mathbb{Z}/p$, i.e. ${}_p\mathcal{A}^{2p-1,p-1}(\mathbb{Z}_{(p)}) = \mathbb{Z}/p$.*

Proof. Consider a universal coefficient sequence corresponding to the coefficient sequence

$$0 \longrightarrow \mathbb{Z}_{(p)} \xrightarrow{\ p\ } \mathbb{Z}_{(p)} \xrightarrow{\ \mathrm{red}\ } \mathbb{Z}/p \longrightarrow 0. \tag{2.6}$$

Since the higher inverse limits vanish, it looks as

$$0 \to \mathcal{A}^{i,j}(\mathbb{Z}_{(p)})/p \to \mathcal{OP}^{i,j}(\mathbb{Z}_{(p)},\mathbb{Z}/p) \to {}_p\mathcal{A}^{i+1,j}(\mathbb{Z}_{(p)}) \to 0. \tag{2.7}$$

In Section 4 we construct a nontrivial operation d_p of degree $(2p-1,p-1)$ with $\mathbb{Z}_{(p)}$-coefficients. Our investigation of Adams operations implies that $pd_p = 0$. Therefore, the group ${}_p\mathcal{A}^{2p-1,p-1}(\mathbb{Z}_{(p)})$ is non-trivial. Below, in Proposition 3.1 we show that $\mathcal{OP}^{2p-2,p-1}(\mathbb{Z}_{(p)},\mathbb{Z}/p) = \mathbb{Z}/p$. Hence, there is an epimorphism $\mathbb{Z}/p \to {}_p\mathcal{A}^{2p-1,p-1}(\mathbb{Z}_{(p)})$. This completes the proof. □

3. Computation of the groups $\mathcal{OP}^{*,p-1}(\mathbb{Z}_{(p)}, Z/p)$

Everywhere in the current section P^1 denotes the first Steenrod operation in the mod-p Steenrod algebra and red denotes the coefficient reduction operation from $\mathbb{Z}_{(p)}$ to $\mathbb{Z}/p$.

The purpose of the current section is to prove the following proposition:

Proposition 3.1.

$$\mathcal{OP}^{m,p-1}(\mathbb{Z}_{(p)}, Z/p) = \begin{cases} \mathbb{Z}/p \text{ for } m = 2p-2 \\ \mathbb{Z}/p \text{ for } m = 2p-1 \\ 0 \quad \text{otherwise.} \end{cases}$$

[2] It's even become a symmetric motivic spectrum (see [Ja]).

The following operations can be taken as generators in the non-trivial degrees: $P^1 \circ \mathrm{red}$ *for* $m = 2p-2$ *and* $\beta \circ P^1 \circ \mathrm{red}$ *for* $m = 2p-1$*, correspondingly. Here* β *is the Bockstein homomorphism, corresponding to the short exact sequence*

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0.$$

Remark 3.2. Since the second grading of all groups considered in this section is equal to $p-1$, we will omit it for shortness.

Before we prove the Proposition, let us recall the following fact due to Voevodsky [Vo, Vo3]:

Proposition 3.3.

$$\mathcal{A}^m(Z/p) = \begin{cases} \mathbb{Z}/p & \text{for } m = 2p-2 \\ \mathbb{Z}/p \oplus \mathbb{Z}/p & \text{for } m = 2p-1 \\ \mathbb{Z}/p & \text{for } m = 2p \\ 0 & \text{otherwise.} \end{cases}$$

and the following operations can be taken as generators in the corresponding degrees: P^1*,* $\{\beta \circ P^1, P^1 \circ \beta\}$*, and* $\beta \circ P^1 \circ \beta$*.*

We will also need the following lemma.

Lemma 3.4. *The groups* $\mathcal{OP}(\mathbb{Z}/p, -)$ *and* $\mathcal{OP}(-, \mathbb{Z}/p)$ *are* p*-torsion groups.*

Proof. All elements of these groups are stable cohomological operations from (or to) cohomology groups with $\mathbb{Z}/p$ coefficients. Since all such operations are additive and the addition in the groups corresponds to the addition of the operations, one has: $p\alpha = 0$ for any operation α of the considered type. □

Proposition 3.5.

$$\mathcal{OP}^m(\mathbb{Z}/p, \mathbb{Z}_{(p)}) = \begin{cases} \mathbb{Z}/p & \text{for } m = 2p-1 \\ \mathbb{Z}/p & \text{for } m = 2p \\ 0 & \text{otherwise.} \end{cases}$$

The following operations can be taken as generators in the non-trivial degrees: $\mathfrak{B} \circ P^1$ *for* $m = 2p-1$ *and* $\mathfrak{B} \circ P^1 \circ \beta$ *for* $m = 2p$*, respectively. Here* β *is as above and* $\mathfrak{B}$ *is the Bockstein homomorphism, corresponding to the coefficient short exact sequence*

$$0 \longrightarrow \mathbb{Z}_{(p)} \xrightarrow{p} \mathbb{Z}_{(p)} \longrightarrow \mathbb{Z}/p \longrightarrow 0.$$

Remark 3.6. One can easily verify that $\beta = \mathrm{red} \circ \mathfrak{B}$.

Proof. Consider a cohomology long exact sequence corresponding to the coefficient short sequence

$$0 \longrightarrow \mathbb{Z}_{(p)} \xrightarrow{p} \mathbb{Z}_{(p)} \xrightarrow{\mathrm{red}} \mathbb{Z}/p \longrightarrow 0. \tag{3.1}$$

It looks as

$$0 \longrightarrow \mathcal{OP}^i(\mathbb{Z}/p, \mathbb{Z}_{(p)}) \xrightarrow{\mathrm{red}} \mathcal{A}^i(\mathbb{Z}/p) \xrightarrow{\mathfrak{B}} \mathcal{OP}^{i+1}(\mathbb{Z}/p, \mathbb{Z}_{(p)}) \longrightarrow 0. \tag{3.2}$$

(This long exact sequence splits into short fragments since the multiplication by p map is here zero.) Setting $i > 2p$, one has, due to 3.3: $\mathcal{OP}^i(\mathbb{Z}/p, \mathbb{Z}_{(p)}) = 0$. Similarly, for $i < 2p-2$, one has: $\mathcal{OP}^i(\mathbb{Z}/p, \mathbb{Z}_{(p)}) = 0$. Letting $i = 2p$, one obtains

an isomorphism: $\mathcal{OP}^{2p}(\mathbb{Z}/p, \mathbb{Z}_{(p)}) \overset{\text{red}}{\simeq} \mathcal{A}^{2p}(\mathbb{Z}/p) = \mathbb{Z}/p$ and one can see that the preimage of the generator $\beta \circ P^1 \circ \beta$ in $\mathcal{A}^{2p}(\mathbb{Z}/p) = \mathbb{Z}/p$ is the operation $\mathfrak{B} \circ P^1 \circ \beta$. Finally, taking $i = 2p-1$, one gets the short-exact sequence:
(3.3)

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{OP}^{2p-1}(\mathbb{Z}/p, \mathbb{Z}_{(p)}) \arrow[r, "\text{red}"] & \mathbb{Z}/p \oplus \mathbb{Z}/p \arrow[r, "\mathfrak{B}"] & \mathbb{Z}/p \arrow[r] \arrow[d, equal] & 0 . \\ & & & \mathcal{OP}^{2p}(\mathbb{Z}/p, \mathbb{Z}_{(p)}) & \end{tikzcd}$$

Obviously, the generator $\mathfrak{B} \circ P^1$ of $\mathcal{OP}^{2p-1}(\mathbb{Z}/p, \mathbb{Z}_{(p)})$ passes to the generator $\beta \circ P^1$ of $\mathcal{A}^{2p-1}(\mathbb{Z}/p)$ and $P^1 \circ \beta$ passes to $\mathfrak{B} \circ P^1 \circ \beta \in \mathcal{OP}^{2p}(\mathbb{Z}/p, \mathbb{Z}_{(p)})$. □

Proposition 3.7. *There exists the following natural isomorphism:* $\mathcal{OP}^i(\mathbb{Z}/p, \mathbb{Z}_{(p)}) \simeq \mathcal{OP}^{i-1}(\mathbb{Z}_{(p)}, \mathbb{Z}/p)$. *The groups on the right-hand-side are generated by the operations* $P^1 \circ \text{red}$ *and* $\beta \circ P^1 \circ \text{red}$ *in the degrees* $i = 2p-2,\ 2p-1$, *correspondingly.*

Before we start to prove this proposition, let us make a remark about Eilenberg–Mac Lane spaces. Using the functoriality of the given above construction, one can build two morphisms of spaces $\times p$ and red (multiplication by p and coefficient reduction), corresponding to the maps $\mathbb{Z}_{(p)} \overset{\times p}{\to} \mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(p)} \overset{\text{red}}{\to} \mathbb{Z}/p$, respectively.

We will need the following statement:

Lemma 3.8. *For every n the sequence* $K(\mathbb{Z}_{(p)}, n) \overset{\times p}{\to} K(\mathbb{Z}_{(p)}, n) \overset{\text{red}}{\to} K(\mathbb{Z}/p, n)$ *yields the long exact sequence:*

$$\cdots \to H^{*-1}(K(\mathbb{Z}_{(p)}, n)) \to H^*(K(\mathbb{Z}/p, n)) \overset{\text{red}^*}{\to} H^*(K(\mathbb{Z}_{(p)}, n)) \to \cdots$$

Proof. One should check that the space $K(\mathbb{Z}/p, n)$ is homotopically equivalent to the cone of the morphism $\times p$. Since the composite red $\circ \times p$ is trivial, by the universality of the cone, one has the morphism $Cone(\times p) \to K(\mathbb{Z}/p, n)$. Let us also note that both the spaces $K(\mathbb{Z}/p, n)$ and $Cone(\times p)$ represent the same cohomology group $H^*(-, \mathbb{Z}/p)$. The homotopy equivalence follows by the abstract nonsense. □

Proof of Proposition 3.7. Applying the previous lemma and taking into account vanishing of the higher inverse limits, one gets the following commutative diagram:
(3.4)

$$\begin{tikzcd} & & 0 & & \\ & & \mathcal{OP}^{i+1}(\mathbb{Z}/p, \mathbb{Z}_{(p)}) \arrow[u] \arrow[dr, dotted] & & \\ 0 \arrow[r] & \mathcal{OP}^{i-1}(\mathbb{Z}_{(p)}, \mathbb{Z}/p) \arrow[r] & \mathcal{A}^i(\mathbb{Z}/p) \arrow[r] \arrow[u] & \mathcal{OP}^i(\mathbb{Z}_{(p)}, \mathbb{Z}/p) \arrow[r] & 0 \\ & & \mathcal{OP}^i(\mathbb{Z}/p, \mathbb{Z}_{(p)}) \arrow[u] \arrow[ul] & & \\ & & 0 \arrow[u] & & \end{tikzcd}$$

Assuming that the bottom diagonal arrow is an isomorphism, one can construct the dotted arrow and show, using simple diagram hunting that it is an isomorphism as well. It's also obvious that the bottom diagonal arrow exists and becomes an isomorphism for $i < 2p - 2$. The induction runs. □

4. Non-triviality of the differential

The purpose of the current section is to construct a variety that has a non-trivial differential in its p-local Motivic spectral sequence. Let k be the field as above. Let $\mathcal{D}$ be a central simple algebra over k of prime degree p. Denote by G the norm variety $SL_{1,\mathcal{D}}$. Since G is a twisted form of SL_p, its dimension is $p^2 - 1$.

In the current section we prove the following statement:

Theorem 4.1. *Consider the Motivic spectral sequence corresponding to the variety G. Its second term is*

$$E_2^{i,j} = H^{i-j}(G, \mathbb{Z}_{(p)}(-j)) \Rightarrow K_{-i-j}(G, \mathbb{Z}_{(p)}).$$

Then the differential $d_p \colon E_2^{1,-2} \to E_2^{p+1,-p-1}$ in this spectral sequence is non-trivial.

First of all, consider the Brown-Gersten-Quillen spectral sequence (BGQ), converging to the K-groups of G and having the second term:

$$E^2_{st}(G) = H^s(G, \mathcal{K}_{-t}) \ \Rightarrow \ K_{-s-t}(G),$$

where $H^s(X, \mathcal{K}_{-t})$ are $\mathcal{K}$-cohomology groups. From now on we will omit mentioning the coefficient ring in the notation whenever it is possible.

The following statement implies Theorem 4.1.

Theorem 4.2. *The differential $d_p \colon E_2^{1,-2} \to E_2^{p+1,-p-1}$ in the considered BGQ spectral sequence is non-trivial.*

In order to show that 4.2 implies 4.1, let's, first, mention that the differentials in question act from the diagonal $s + t = -1$ to $s + t = 0$. Now the desired statement easily follows from the proposition below.

Proposition 4.3 (Suslin [Su2])**.** *For a norm-variety there exists a canonical morphism between corresponding motivic spectral sequence and BGQ spectral sequence inducing isomorphisms on the diagonals $s + t = -1$ and $s + t = 0$.*

Remark 4.4. Let's mention that both spectral sequences under consideration are concentrated in the region: $s \geq 0$, $t \leq 0$, and $s + t \leq 0$.

Let's now look at the BGQ spectral sequence above. Its first term can be rewritten as a direct sums of K-groups:

$$E_1^{st} = \coprod_{x \in X^{(s)}} K_{-s-t} F(x). \tag{4.1}$$

Here $X^{(s)}$ denotes the set of all generic points of codimension s in X and $F(x)$ are residue field of the corresponding point. In particular, one has:

$$E_2^{s,-s} = H^s(G, \mathcal{K}_s) = \mathrm{Coker}\left(\coprod_{x \in X^{(s-1)}} K_1 F(x) \to \coprod_{x \in X^{(s)}} K_0 F(x) \right) \tag{4.2}$$

$$= \mathrm{Coker}\left(\coprod_{x \in X^{(s-1)}} F^*(G) \to \coprod_{x \in X^{(s)}} \mathbb{Z}_{(p)} \right) = CH^s(G, \mathbb{Z}_{(p)}).$$

Therefore, the 0-th diagonal of the E^2-term of BGQ spectral sequence consists of the Chow groups. So that, the spectral sequence looks like this:

$$\begin{tikzcd}
CH^0(G) & 0 & \cdots & & \\
H^0(G,\mathcal{K}_1) & CH^1(G) & 0 & \cdots & \\
H^0(G,\mathcal{K}_2) & H^1(G,\mathcal{K}_2) \arrow[ddrrr, "d_p"] & CH^2(G) & 0 & \vdots \\
\vdots & \vdots & \vdots & \ddots & 0 \\
H^0(G,\mathcal{K}_{p+1}) & H^1(G,\mathcal{K}_{p+1}) & \cdots & & CH^{p+1}(G)
\end{tikzcd}$$

It's shown, for example, in [Le2] that $K_0(G) = \mathbb{Z} \cdot 1$, where 1 lies in codimension 0. Therefore, since the spectral sequence converges to the adjoint filtration on the K-groups and one has $K^{(i/i+1)} = 0$ for $i > 0$, all the positively indexed Chow groups should be annihilated by some differentials.

On the other hand, one can conclude from the previous section (or show it applying the Chern character), that $d_i = 0$ for $i \leq p-1$. So, one has $CH^i(G) = 0$ for $i \leq p$ and in order to show the non-triviality of the differential $d_p \colon H^1(G, \mathcal{K}_2) \to CH^{p+1}(G)$ (i.e. prove Theorem 4.2), one just should check the following claim:

Proposition 4.5. *For a non-split algebra $\mathcal{D}$, one has: $CH^{p+1}(SL_{1,\mathcal{D}}) \neq 0$.*

We try to compute the group $CH^{p+1}(SL_{1,\mathcal{D}})$ in a different way. Denote by $X = SB(\mathcal{D})$ the Severi-Brauer variety of dimension $p-1$ corresponding to the algebra $\mathcal{D}$. For the projection map $G \times X \to G$ consider a filtration of the base by codimension of points and write down the corresponding spectral sequence (see Rost[Ro]):

$$E_1^{st}(n) = \coprod_{g \in G^{(s)}} H^t(X_{F(g)}, \mathcal{K}_{n-s}) \;\Rightarrow\; H^{s+t}(G \times X, \mathcal{K}_n), \tag{4.3}$$

where $X_{F(g)} = X \times \mathrm{Spec}\, F(g)$ is a fiber over the generic point g. The E_1-term of this spectral sequence is concentrated in the strip given by the conditions: $0 \leq t \leq p-1$ and $s+t \leq n$. These dimension conditions easily imply the following statement.

Lemma 4.6. *For $n > p$ one has: $E_2^{n-p-1,p-1}(n) = 0$.*

Proof. Observe, first, that due to a result of Suslin [Su], $H^*(G\times X, \mathcal{K}_*)$ is a module over $H^*(X, \mathcal{K}_*)$ generated by Chern classes $c_j \in H^j(G \times X, \mathcal{K}_{j+1})$. This implies that $CH^i(G \times X) = 0$ for $i > p-1$. So, the spectral sequence converges to zero in the degree $s+t = n$. Since there is no differentials affecting the term $E_2^{n-p-1,p-1}(n)$ of the spectral sequence, one has: $E_2^{n-p-1,p-1} = E_\infty^{n-p-1,p-1} = 0$. □

Below we will also need one statement due to Merkurjev and Suslin, which we reproduce here.

Proposition 4.7 ([MS, Corollary 8.7.2])**.** *Let $\bar{k}$ be the algebraic closure of k. For a Severi–Brauer variety X of dimension $p-1$, set $\bar{X} = X \times \operatorname{Spec} \bar{k}$. Then*

$$H^i(X, \mathcal{K}_i) = CH^i(X) = p\mathbb{Z}_{(p)} \subset \mathbb{Z}_{(p)} = CH^i(\bar{X}) \tag{4.4}$$

and

$$H^i(X, \mathcal{K}_{i+1}) = \operatorname{Nrd} \mathcal{D}^* \subset \bar{k}^* = H^i(\bar{X}, \mathcal{K}_{i+1}), \tag{4.5}$$

provided that $1 \le i \le p-1$. (Here Nrd *denotes the group of the reduced norms.)*

Proposition 4.8. *The following equalities hold:*

$$E_2^{p+1-t,t}(p+1) = \begin{cases} 0, & \text{for } t \neq 0 \\ CH^{p+1}(G), & \text{for } t = 0. \end{cases}$$

Proof. The case $t = p-1$ is exactly the conclusion of Lemma 4.6. Applying Proposition 4.7, one gets the isomorphisms

$$\begin{aligned} E_1^{p+1-t-m,t}(p+1) &= \coprod_{g \in G^{(p+1-t-m)}} H^t(X_{F(g)}, \mathcal{K}_{t+m}) \\ \simeq \coprod_{g \in G^{(p+1-t-m)}} H^{p-1}(X_{F(g)}, \mathcal{K}_{p-1+m}) &= E_1^{p+1-t-m,p-1}(2p-t) \end{aligned} \tag{4.6}$$

for $1 \le t \le p-1$ and $m = 0, 1$. Taking into account the functoriality of BGQ spectral sequence and Lemma 4.6, one obtains the commutative diagram:

$$\begin{tikzcd} E_1^{p+1-t-1,t}(p+1) \arrow[r] \arrow[d, equal] & E_1^{p+1-t,t}(p+1) \arrow[d, equal] \\ E_1^{p-t,p-1}(2p-t) \arrow[r, two heads] & E_1^{p+1-t,p-1}(2p-t) \end{tikzcd} \tag{4.7}$$

that yields the equality: $E_2^{p+1-t,t}(p+1) = E_2^{p+1-t,p-1}(2p-t) = 0$ by Lemma 4.6. This proves the first case of the proposition. The group $E_2^{p+1,0}(p+1)$ is a cokernel of the map

$$\coprod_{g \in G^{(p)}} H^0(X_{F(g)}, \mathcal{K}_1) \to \coprod_{g \in G^{(p+1)}} H^0(X_{F(g)}, \mathcal{K}_0).$$

Therefore, it equals to

$$\operatorname{Coker} \left(\coprod_{g \in G^{(p)}} F(g)^* \to \coprod_{g \in G^{(p+1)}} \mathbb{Z} \right) = CH^{p+1}(G). \tag{4.8}$$

□

From now on we are going to consider only the case $n = p + 1$. So, we omit for good mentioning n in the notation. Consider the term $E_2^{1,p-1}$ that equals to the middle-term homology of the complex:

(4.9)
$$\coprod_{g\in G^{(0)}} H^{p-1}(X_{F(g)}, \mathcal{K}_{p+1}) \to \coprod_{g\in G^{(1)}} H^{p-1}(X_{F(g)}, \mathcal{K}_{p}) \to \coprod_{g\in G^{(2)}} H^{p-1}(X_{F(g)}, \mathcal{K}_{p-1})$$

and denote it by V. There are no differentials coming to this term and the only potentially non-trivial differential leaving V is $d_p\colon V \to E_p^{p+1,0} = CH^{p+1}(G)$. To complete the proof of Proposition 4.5, it suffices to check the following:

Proposition 4.9. *The map $d_p\colon V \to CH^{p+1}(G)$ is non-zero.*

Proof. The spectral sequence yields the following boundary short exact sequence: $H^p(G\times X, \mathcal{K}_{p+1}) \xrightarrow{\varphi} V \xrightarrow{d_p} CH^{p+1}(G)$. So that, it is sufficient to establish that φ is not an epimorphism. Consider the base-change commutative diagram corresponding to the morphism $\operatorname{Spec}\bar{k} \to \operatorname{Spec} k$, where $\bar{k}$ is the algebraic closure of k.

$$\begin{array}{ccc} H^p(G\times X, \mathcal{K}_{p+1}) & \xrightarrow{\varphi} & V \\ \chi\downarrow & & \psi\downarrow \\ H^l(\bar{G}\times \bar{X}, \mathcal{K}_{p+1}) & \xrightarrow{\bar{\varphi}} & \bar{V} \end{array} \tag{4.10}$$

The desired statement easily follows from the following two claims:

Lemma 4.10. $\operatorname{Im}\chi$ *is divisible by p.*

Proof. This follows from the above mentioned (see the proof of Lemma 4.6) decomposition

$$H^p(G\times X, \mathcal{K}_{p+1}) = \coprod_{i>0} c_i CH^{p-i}(X) \tag{4.11}$$

and the fact that the map $CH^i(X) \to CH^i(\bar{X})$ is a multiplication by p due to Proposition 4.7. $\square$

Proposition 4.11. *The map $\psi\colon V \to \bar{V}$ is an epimorphism.*

Proof. First, consider the BGQ spectral sequence converging to the K-groups of the Severi–Brauer variety X. This spectral sequence has no non-trivial differentials affecting the two highest diagonals. Moreover, if the base field is algebraically closed, all the differentials in the spectral sequence vanish (see [MS, 8.6.2]).

The infinity term of BGQ consists of consequent factor-filtration groups of $K(X)$. Taking into account the triviality of differentials mentioned in the previous paragraph, there exist the boundary maps:

$$H^{p-1}(X, \mathcal{K}_{p-1+m}) \to K_m(X)^{(p-1)}, \tag{4.12}$$

where $m = 0, 1, 2$. These maps are isomorphisms for $m = 0, 1$ and for $m = 2$, provided that the base field is algebraically closed. Since $(p-1)!$ is invertible in the coefficient ring, the topological filtration coincide with the γ-filtration. The desired filtration is generated by the image of the corresponding γ-operation. Using Quillen's computation [Qu] one has isomorphisms $K_m(X)^{(p-1)} \simeq K_m(\mathcal{D}^{\otimes(p-1)})$. Denoting $\mathcal{D}^{\otimes(p-1)}$ by $\mathcal{F}$, we obtain the maps: $H^{p-1}(X_g, \mathcal{K}_{p-1+m}) \xrightarrow{\rho_m} K_m(F(g); \mathcal{F})$

for $m = 0, 1, 2$, which are isomorphisms for $m = 0, 1$ and isomorphism for $m = 2$ provided that the base-field is algebraically closed. As a result, one gets the map ρ_* from the complex (4.9) to the complex:

$$K_2(F(G); \mathcal{F}) \to \coprod_{g \in G^{(1)}} K_1(F(g); \mathcal{F}) \to \coprod_{g \in G^{(2)}} K_0(F(g); \mathcal{F}), \tag{4.13}$$

inducing the epimorphism map $\tilde{\rho}$ on the middle-term homology groups. The latter map becomes an isomorphism after passing back to the algebraic closure and yields the following commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\psi"] \arrow[d, "\tilde{\rho}"'] & \bar{V} \arrow[d, equal] \\ H^1(G, \mathcal{K}_2; \mathcal{F}) \arrow[r, "\omega"] & H^1(\bar{G}, \mathcal{K}_2; \bar{\mathcal{F}}). \end{tikzcd} \tag{4.14}$$

Let us, finally, show that the bottom arrow is an epimorphism. First, observe that $\bar{G} = SL_n(\bar{k})$ and $H^1(\bar{G}, \mathcal{K}_2; \bar{\mathcal{F}}) = H^1(SL_n, \mathcal{K}_2) = \mathbb{Z}_{(p)}$ with a natural choice of a generator, given by the first Chern class (see [Su, Theorem 2.7]). Consider the base-change diagram:

$$\begin{tikzcd} K_1(G; \mathcal{F}) \arrow[r, "c_1"] \arrow[d, "f"'] & H^1(G, \mathcal{K}_2; \mathcal{F}) \arrow[d, "\omega"] \\ K_1(SL_n) \arrow[r, "\bar{c}_1"] & H^1(SL_n, \mathcal{K}_2) \end{tikzcd} \tag{4.15}$$

Consider the universal element $\alpha \in K_1(G; \mathcal{F})$ defined as in [Su, Section 4]. It is constructed in such a way that its image $f(\alpha)$ in $K_1(SL_n)$ is the universal matrix element. Then, due to [Su, Theorem 2.7], $\bar{c}_1 f(\alpha) = 1$. Hence, the map ω is an epimorphism and so is ψ. □

Proof of Proposition 4.9 Assume, that the map φ is an epimorphism. From the proof of the previous Lemma we already know that $\bar{V} = \mathbb{Z}_{(p)}$. Since ψ is also an epimorphism, we can chose an element $x \in H^p(G \times X, \mathcal{K}_{p+1})$ such that $\psi\varphi(x) = 1$. Then, by Lemma 4.10, one has that $1 = \bar{\varphi} i(x)$ is p-divisible. Contradiction. □

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