Projective manifolds with small pluridegrees*

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Abstract

Let $\mathcal{L}$ be a very ample line bundle on a connected complex projective manifold $\mathcal{M}$ of dimension $n \geq 3$. Except for a short list of degenerate pairs $(\mathcal{M}, \mathcal{L})$, it follows that $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$ and that there exists a morphism $\pi : \mathcal{M} \to M$ expressing $\mathcal{M}$ as the blowup of a projective manifold M at a finite set B, with $\mathcal{K}_M := K_M + (n-2)L$ nef and big for the ample line bundle $L := (\pi_* \mathcal{L})^{**}$. The projective geometry of $(\mathcal{M}, \mathcal{L})$ is largely controlled by the pluridegrees $d_j := L^{n-j} \cdot (K_M + (n-2)L)^j$ for $j = 0, \ldots, n$, of $(\mathcal{M}, \mathcal{L})$. For example $d_0 + d_1$ is equal to the the genus of a curve section of $(\mathcal{M}, \mathcal{L})$, and d_2 is equal to the self-intersection of the canonical divisor of the minimal model of a surface section of $(\mathcal{M}, \mathcal{L})$. In this article a detailed analysis is made of the pluridegrees of $(\mathcal{M}, \mathcal{L})$. The restrictions found are used to give a new lower bound for the dimension of the space of sections of $\mathcal{K}_M$. The inequalities for the pluridegrees, that are presented in this article, will be used in a sequel to study the sheet number of the morphism associated to $|2(K_{\mathcal{M}} + (n-2)\mathcal{L})|$.

Introduction

Let $\mathcal{L}$ be a very ample line bundle on a connected complex projective manifold $\mathcal{M}$ of dimension $n \geq 3$. Except for a short list of degenerate pairs $(\mathcal{M}, \mathcal{L})$, it follows that $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$ and that there exists a morphism $\pi : \mathcal{M} \to M$ expressing $\mathcal{M}$ as the blowup of a projective manifold M at a finite set B, with $\mathcal{K}_M := K_M + (n-2)L$ nef and big for the ample line bundle $L := (\pi_* \mathcal{L})^{**}$. The projective geometry of $(\mathcal{M}, \mathcal{L})$ is largely controlled by the pluridegrees $d_j := L^{n-j} \cdot (K_M + (n-2)L)^j$ for $j = 0, \ldots, n$, of $(\mathcal{M}, \mathcal{L})$. For example $d_0 + d_1$ is equal to the the genus of a curve section of $(\mathcal{M}, \mathcal{L})$, and d_2 is equal to the self-intersection of the canonical divisor of the minimal model of a surface section of $(\mathcal{M}, \mathcal{L})$. In this article a detailed analysis is made of the pluridegrees of $(\mathcal{M}, \mathcal{L})$.

Here is a detailed summary of the results of this article. Theorems (2.1) details the exceptions to inequalities of the form $d_j \geq d_{j-1} + 2$ in dimensions ≥ 5, and Theorem (2.2) details the exceptions to inequalities of the form $2d_j \geq d_{j-1}$ in dimension 4.

Except for a short list of exceptions, $K_M + (n-3)\mathcal{K}$ is nef and big if $\dim M \geq 6$. Theorem (3.1) shows that under the assumption $\dim \mathcal{M} \geq 6$ we have with only a few exceptions that $d_j \geq 2\left(\frac{n-2}{n+1}\right) d_{j-1}$ for $j = 1, \ldots, n$.

Results that hold in the more difficult dimensions 3, 4, and 5 are presented in Theorems (4.1), (5.1), (6.2), and (7.1). As a first application of our results we give in Proposition (8.1) a new lower bound for the dimension of the space of sections of $\mathcal{K}_M$ when $n \geq 5$.

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In a sequel, we will use the results of this article to give sharp bounds for the degree of the morphism associated to $|2(K_{\mathcal{M}} + (n-2)\mathcal{L})|$.

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1 Background material

We work over the complex field $\mathbb{C}$. Throughout the paper we deal with projective varieties V. We denote by $\mathcal{O}_V$ the structure sheaf of V and by K_V the canonical bundle. For any coherent sheaf $\mathcal{F}$ on V, $h^i(\mathcal{F})$ denotes the complex dimension of $H^i(V, \mathcal{F})$.

Let L be a line bundle on V. The line bundle L is said to be *numerically effective* (*nef*, for short) if $L \cdot C \geq 0$ for all effective curves C on V. L is said to be *big* if $\kappa(L) = \dim V$, where $\kappa(L)$ denotes the Kodaira dimension of L. If L is nef then this is equivalent to $c_1(L)^n > 0$, where $c_1(L)$ is the first Chern class of L and $n = \dim V$.

1.1 Notation. The notation used in this paper are standard from algebraic geometry. Let us only fix the following.

$\approx$ (respectively $\sim$), the linear (respectively numerical) equivalence of line bundles;

$\chi(L) = \sum_i (-1)^i h^i(L)$, the Euler characteristic of a line bundle L;

$|L|$, the complete linear system associated to a line bundle L;

$\Gamma(L)$, the space of the global sections of a line bundle, L, on a variety V; we say that L is *spanned* if it is spanned at all points of V by $\Gamma(L)$;

$p_g(V) = h^0(K_V)$, the geometric genus, for V smooth;

$q(V) = h^1(\mathcal{O}_V)$, the irregularity, for V smooth;

$c_n(V)$, the n-th Chern class of the tangent bundle of V, for V smooth;

$\kappa(V) := \kappa(K_V)$, the Kodaira dimension, for V smooth.

Line bundles and divisors are used with little (or no) distinction. We almost always use the additive notation.

1.2 Genus formula. For a line bundle L on a variety V of dimension n the sectional genus, $g(L) = g(V, L)$, of (V, L) is defined by $2g(L) - 2 = (K_V + (n-1)L) \cdot L^{n-1}$. Note that if $|L|$ contains $n-1$ elements $H_1, \ldots, H_{n-1}$ meeting in a reduced irreducible curve C, then $g(L) = g(C) = 1 - \chi(\mathcal{O}_C)$, the arithmetic genus of C.

1.3 Castelnuovo's bound. Let C be a reduced irreducible projective curve. Assume that $\psi : C \to \mathbb{P}^N$ is a generically one-to-one morphism, and that $\psi(C)$ does not lie in any hyperplane. Let d denote the degree of $\psi(C)$ in $\mathbb{P}^N$. Let $g(C)$ be the arithmetic genus of C. Then Castelnuovo's bound (see, e.g., [8, Theorem 3.7]) reads

$$g(C) \leq \mathrm{Castel}(d, N) := \left[\frac{d-2}{N-1}\right] \left(d - N - \left(\left[\frac{d-2}{N-1}\right] - 1\right) \frac{N-1}{2}\right), \tag{1}$$

where $[x]$ means the greatest integer $\leq x$.

1.4 Reductions. (See e.g., [6, Chapters 7, 12]). Let $(\mathcal{M}, \mathcal{L})$ be a smooth variety of dimension $n \geq 2$ polarized with a very ample line bundle $\mathcal{L}$. A smooth polarized variety (M, L) is called a *reduction* of $(\mathcal{M}, \mathcal{L})$ if there is a morphism $\pi : \mathcal{M} \to M$ expressing $\mathcal{M}$ as the blowing p of X at a finite set of points, B, such that $L := (\pi_* \mathcal{L})^{**}$ is ample and $\mathcal{L} \approx \pi^* L - [\pi^{-1}(B)]$ or, equivalently, $K_{\mathcal{M}} + (n-1)\mathcal{L} \approx \pi^*(K_M + (n-1)L)$.

Note that there is a one-to-one correspondence between smooth divisors of $|L|$ which contain the set B and smooth divisors of $|\mathcal{L}|$.

Except for an explicit list of well understood pairs $(\mathcal{M}, \mathcal{L})$ (see in particular [6, §§7.2, 7.3, 7.4]) we can assume:

a) $K_{\mathcal{M}} + (n-1)\mathcal{L}$ is spanned and big, and $K_M + (n-1)L$ is very ample. Note that this reduction, (M, L), is unique up to isomorphism. We will refer to it as *the first reduction* of $(\mathcal{M}, \mathcal{L})$;

b) $K_M + (n-2)L$ is nef and big, for $n \geq 3$.

Since by the above we can assume that $K_M + (n-2)L$ is nef and big, from the Kawamata-Shokurov base point free theorem (see [10, §3]) we know that $|m(K_M + (n-2)L)|$, for $m \gg 0$, gives rise to a morphism $\varphi : M \to X$ with connected fibers and normal image. Thus there is an ample line bundle $\mathcal{K}$ on X such that $K_M + (n-2)L \approx \varphi^* \mathcal{K}$. The pair $(X, \mathcal{K})$ is known as the *second reduction* of $(\mathcal{M}, \mathcal{L})$. The morphism φ is very well behaved (see e.g., [6, §§7.5, 7.6, 7.7 and Chapter 12]). Furthermore X has terminal, 2-Gorenstein (i.e., $2K_X$ is a line bundle) isolated singularities and $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, where $\mathcal{D} := (\varphi_* L)^{**}$ is a 2-Cartier divisor such that $2L \approx \varphi^*(2\mathcal{D}) - \Delta$ for some effective divisor Δ on X which is φ-exceptional and $\dim \varphi(\Delta) \leq 1$ (see [6, (7.5.7)]). Notice that from $K_M + (n-2)L \approx \varphi^*(K_X + (n-2)\mathcal{D})$ and $2L \approx \varphi^*(2\mathcal{D}) - \Delta$ it follows that $2K_M \approx \varphi^*(2K_X) + 3\Delta$. For definition and properties of terminal singularities we also refer to [10].

If Y is an irreducible subvariety of X we will denote by $\mathcal{D}_Y$ the dual sheaf $(\mathcal{D}_{|Y})^{**}$, where $\mathcal{D}_{|Y}$ is the restriction of $\mathcal{D}$ to Y as a sheaf. Moreover, for any k-dimensional irreducible subvariety $Y \subset X$, we set

$$\mathcal{D}^k \cdot Y := \frac{(2\mathcal{D})^k \cdot Y}{2^k}.$$

We say that $(\mathcal{M}, \mathcal{L})$ is of *log-general type* if $\kappa(K_{\mathcal{M}} + (n-2)\mathcal{L}) = n$. Notice that this is equivalent to say that $K_M + (n-2)L$ is nef and big (see [6, (7.6.9)]). Let $\widehat{S}$ be the smooth surface obtained as transversal intersection of $n-2$ general members of $|\mathcal{L}|$ and let $S := \pi(\widehat{S})$ be the corresponding smooth surface in M. Since $K_M + (n-2)L$ is nef and big the canonical bundle K_S of S is nef and big, so that S is a minimal surface of general type (see also [6, (7.6.10)]). We have

$$K_S \cdot K_S < 9\chi(\mathcal{O}_S). \tag{2}$$

Indeed, Miyaoka's inequality yields $K_S \cdot K_S \leq 9\chi(\mathcal{O}_S)$. Note that the equality cannot happen. Otherwise S is ball quotient and hence a $K(\pi, 1)$, which contradicts [6, (1.3)].

For further properties of log-general type polarized pairs see e.g., [6, §13.2] and [7, (0.10)].

We recall the following special useful fact from [3, (4.41)].

Proposition 1.5 *Let $\mathcal{M}$ be a smooth n-fold and let $\mathcal{L}$ be a very ample line bundle on $\mathcal{M}$. Assume that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^N$ with $2n > N$. Assume that the first reduction, (M, L), of $(\mathcal{M}, \mathcal{L})$ exists. If $\widehat{d} := \mathcal{L}^n \neq 2^n - 1$, then $(\mathcal{M}, \mathcal{L}) \cong (M, L)$.*

1.6 Pluridegrees. Let $(\mathcal{M}, \mathcal{L})$, (M, L) be as in (1.4). Define the *pluridegrees*, for $j = 0, \ldots, n = \dim \mathcal{M}$, by

$$\widehat{d_j} := (K_{\mathcal{M}} + (n-2)\mathcal{L})^j \cdot \mathcal{L}^{n-j} \ \text{ and } \ d_j := (K_M + (n-2)L)^j \cdot L^{n-j}.$$

If γ denotes the number of points blown up under $\pi : \mathcal{M} \to M$, then because $K_{\mathcal{M}} + (n-2)\mathcal{L} \approx K_M + (n-2)L + \sum_i E_i$, E_i the exceptional divisors, the invariants $\widehat{d_j}$, d_j are related by

$$\widehat{d_j} = d_j - (-1)^j \gamma.$$

We put $\widehat{d} := \widehat{d_0}$, $d := d_0$. If $K_M + (n-2)L$ is nef, by the generalized Hodge index theorem (see e.g., [6, (2.5.1), (13.1)]) one has

$$d_j^2 \geq d_{j-1} d_{j+1}, \ \ j = 1, \ldots, n-1, \tag{3}$$

and the parity Lemma (13.1.1) of [6] says that

$$d_j \equiv d_{j+1} \bmod(2) \ \text{ for } \ j = 0, \ldots, n-1 \ \text{ and } \ j \text{ even}. \tag{4}$$

Moreover if $K_M + (n-2)L$ is nef and big, i.e., if $(\mathcal{M}, \mathcal{L})$ is of log-general type, the d_j's are positive.

If the second reduction, $(X, \mathcal{D})$, $\varphi : M \to X$, with $\mathcal{D} = (\varphi_* L)^{**}$, $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, of $(\mathcal{M}, \mathcal{L})$ exists, then we define

$$d'_j := \mathcal{K}^j \cdot \mathcal{D}^{n-j}, \ \ j = 0, \ldots, n, \ d' := d'_0.$$

Note that

$$d_j = d'_j \ \text{ for } \ j \geq 2. \tag{5}$$

To see this recall that $2L \approx \varphi^*(2\mathcal{D}) - \Delta$ for some effective Cartier divisor Δ which is φ-exceptional (see (1.4)) and compute, for $j \geq 2$,

$$\begin{aligned} 2^{n-j} d_j &= (K_M + (n-2)L)^j \cdot (2L)^{n-j} \\ &= (\varphi^*\mathcal{K})^j \cdot (\varphi^*(2\mathcal{D}) - \Delta)^{n-j} = 2^{n-j} \mathcal{K}^{n-j} \cdot \mathcal{D}^j = 2^{n-j} d'_j, \end{aligned}$$

where the last but one equality follows from the fact that $\dim \varphi(\Delta) \leq 1$.

The first assertion of the following proposition is a special case of the general formula of Kleiman [11, (I, 37), Section D, p. 313] (see also [6, (8.2.1), (13.1.5)]). The second assertion follows from the proof of Lemma (2.3) in [7] (see also [6, (13.10)]).

Proposition 1.7 (Double point formula for surfaces) *Let $\mathcal{M}$ be a smooth n-fold polarized by a very ample line bundle $\mathcal{L}$, $n \geq 2$. Assume that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^N$ with $N \geq n+2$. Let $\widehat{S}$ be the smooth surface obtained as transversal intersection of $n-2$ general members of $|\mathcal{L}|$. Let (M, L), $\pi : \mathcal{M} \to M$, be the first reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.4). Let $S := \pi(\widehat{S})$. Let $\widehat{d_j}$, d_j, $j = 0, \ldots, n$, $\widehat{d_0} = \widehat{d}$, be the pluridegrees of $(\mathcal{M}, \mathcal{L})$, (M, L) respectively. Then we have the following inequalities (note that $g(\mathcal{L}) = g(L)$, $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_{\widehat{S}})$.*

1. $\widehat{d}(\widehat{d} - 5) - 10(g(\mathcal{L}) - 1) + 12\chi(\mathcal{O}_{\widehat{S}}) \geq 2\widehat{d}_2$, *with equality if $N = n+2$.*

2. *Assume $d + \widehat{d} \geq 7$, e.g., $\widehat{d} \geq 4$. Then*

$$d(d-5) - 10(g(L) - 1) + 12\chi(\mathcal{O}_S) \geq 2d_2,$$

or, by using the genus formula (1.2),

$$d(d-10) + 12\chi(\mathcal{O}_S) \geq 2d_2 + 5d_1.$$

The following is a consequence of the double point formula above (see also [6, (13.1.6)]).

Proposition 1.8 (Double point formula) *Let $\mathcal{M}$ be a smooth 3-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that the first reduction, (M, L), of $(\mathcal{M}, \mathcal{L})$ exists. Let $\widehat{d_j}$, d_j, $0 \leq j \leq 3$, be the pluridegrees of $(\mathcal{M}, \mathcal{L})$, (M, L) respectively. Let γ be the number of points blown up under the first reduction map. Let S be a smooth element in $|L|$. Then*

$$44h^0(K_{\mathcal{M}} + \mathcal{L}) + 58\chi(\mathcal{O}_S) + 2h^0(K_{\mathcal{M}}) + 4 \geq 12d_2 + 17d_1 + d_3 + (20 - \widehat{d})\widehat{d} + 5\gamma.$$

The following is a consequence of the Tsuji inequality (see [14, (5.2)]), the log version of the usual Yau inequality (see also [6, (13.1.7), (13.1.8)]).

Proposition 1.9 (Tsuji inequality) *Let $\mathcal{M}$ be a smooth 3-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that the first reduction, (M, L), of $(\mathcal{M}, \mathcal{L})$ exists. Let d_j, $0 \leq j \leq 3$, be the pluridegrees of (M, L). Assume that $K_M + L$ is nef. Then we have*

$$(K_M + L)^3 + \frac{8}{3}K_S \cdot L_S \leq 32(2h^0(K_M + L) - \chi(\mathcal{O}_S)),$$

or

$$h^0(K_M + L) \geq \frac{\chi(\mathcal{O}_S)}{2} + \frac{d_1}{24} + \frac{d_3}{64}.$$

The following lower bound for the degree is not optimal, but it is sufficient for our purposes.

Lemma 1.10 *Let $\mathcal{M}$ be a smooth n-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let (M, L) be the first reduction of $(\mathcal{M}, \mathcal{L})$. Then either $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^{n+1}$ or $d \geq \widehat{d} := \mathcal{L}^n \geq 8$.*

Proof. We can assume that $\Gamma(\mathcal{L})$ embeds $\mathcal{M}$ in $\mathbb{P}^N$ with $N \geq n+2$. Let $\widehat{S}$ be the smooth surface obtained as transversal intersection of $n-2$ general members of $|\mathcal{L}|$. Since $\widehat{S}$ is of general type we have by a result of Castelnuovo (see [6, (8.1)], [13, (0.6)]) that $\widehat{d} > 2(N-n)+2 \geq 6$. Thus $d \geq \widehat{d} \geq 7$. Assume $\widehat{d} = 7$. Then $N = n+2$ since otherwise $\widehat{d} > 2(N-n)+2 \geq 8$. Thus by Castelnuovo's bound we conclude that $g(C) \leq 6$ for any smooth curve section C of $\mathcal{M} \subset \mathbb{P}^{n+2}$. Therefore, by the genus formula, $7+d_1 \leq 7+\widehat{d}_1 = 2g(C)-2 \leq 10$, or $d_1 \leq 3$. But then $d_2 \leq 1$ by the Hodge index relation $d_1^2 \geq dd_2$. Since $(\mathcal{M},\mathcal{L})$ is of log-general type we know that $d_j \geq 1$ for each $j = 1,\ldots,n$. Therefore we conclude that $d_2 = 1$ and $d_2^2 \geq d_1 d_3$ implies $d_1 = 1$. This gives the absurdity that $d = 1$. Q.E.D.

We need the following useful fact.

Lemma 1.11 *Let L be a line bundle on an irreducible projective variety V and let $p : V \to Y$ be a birational surjective morphism with Y normal. Then we have an inclusion*

$$H^0(V,L) \subset H^0(Y,(p_*L)^{**}).$$

Proof. Let Z be the union of the singular locus of Y and the exceptional set of the morphism p. Then Z is a closed subset of codimension ≥ 2 such that $Y \setminus Z$ is smooth, and p induces an isomorphism $V \setminus p^{-1}(Z) \cong Y \setminus Z$. A section $s \in H^0(V,L)$ gives a section $s' \in H^0(V \setminus p^{-1}(Z), L) \cong H^0(Y, (p_*L)^{**}_{Y\setminus Z}) \cong H^0(Y,(p_*L)^{**})$, where the last isomorphism follows from the facts that Z is of codimension ≥ 2 in Y and Y is normal. Q.E.D.

2 Exceptions to nefness and bigness

Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M},\mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M},\mathcal{L}) \to (M,L)$, $\varphi : (M,L) \to (X,\mathcal{D})$, $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, be the first and the second reduction of $(\mathcal{M},\mathcal{L})$ as in (1.4). In this section we study the exceptions to nefness and bigness of $K_X + (n-3)\mathcal{K}$ when $n \geq 5$, and the exceptions to nefness and bigness of $K_X + 3\mathcal{K}$ for $n = 4$.

First assume $n \geq 5$. From [6, (7.7.2), (7.7.3), (7.7.5), (7.7.6)] we know that $K_X + (n-3)\mathcal{K}$ is nef and big unless either

i) $n = 6$, $(X,\mathcal{K}) \cong (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))$; or

ii) $n = 5$, $(X,\mathcal{K}) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}})$, $\mathcal{Q}$ a quadric in $\mathbb{P}^6$; or

iii) $n = 5$, X is a $\mathbb{P}^4$-bundle over a smooth curve, $\mathcal{K}_F \approx \mathcal{O}_{\mathbb{P}^4}(1)$ for any fiber F; or

iv) $n = 5$, X is a singular 2-Gorenstein Fano 5-fold described in [6, (7.7.5)]; or

v) $(X,\mathcal{K})$ is a scroll over a normal 4-fold Y, $\psi : X \to Y$, $\mathcal{K}_{|F} \approx \mathcal{O}_{\mathbb{P}^4}(1)$ for a general fiber F; or

vi) $(X,\mathcal{K})$ is a quadric fibration, $\psi : X \to Y$, over a normal 3-fold Y; or

vii) $(X,\mathcal{K})$ is a Del Pezzo fibration, $\psi : X \to Y$, over a normal surface Y; or

viii) $(X,\mathcal{K})$ is a Mukai fibration, $\psi : X \to Y$, over a smooth curve Y; or

ix) X is a Fano n-fold with $K_X \approx -(n-3)\mathcal{K}$ if n is even and $2K_X \approx -2(n-3)\mathcal{K}$ if n is odd.

In cases i), ii) one has $d_n = d_6 = 1$, $d_n = d_5 = 2$ respectively.

In cases v), vi), vii), viii) we have $K_X + (n-3)\mathcal{K} \approx \psi^*\mathcal{H}$ if n is even and $2(K_X + (n-3)\mathcal{K}) \approx \psi^*\mathcal{L}$ if n is odd, for some ample line bundle $\mathcal{H}$ on Y. In both cases, since $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, we conclude that $\mathcal{K} - \mathcal{D}$ is numerically effective. In this case, dotting with $\mathcal{K}^{n-1}$, we get the relation

$$d'_n > d'_{n-1} \text{ on } X,$$

or, recalling (5),

$$d_n > d_{n-1} \text{ on } M. \tag{6}$$

By using the Hodge index relation (3), we find

$$d_n > d_{n-1} > d_{n-2} > \cdots > d_1 > d. \tag{7}$$

According to whether n is odd or even we have the following bounds:

$$d_k \geq d_{k-1} + 2 \text{ for } k = 1, \ldots, n, \text{ and hence } d_n \geq d + 2n, \text{ if } n \text{ is odd,}$$

and

$$d_n \geq d_{n-1} + 1,\ d_k \geq d_{k-1} + 2 \text{ for } k = 1, \ldots, n-1, \text{ and hence } d_n \geq d + 2n - 1, \text{ if } n \text{ is even.}$$

Assume n is odd. By parity (4) we have

$$d_n \geq d_{n-1} + 2.$$

Let t be the biggest integer, $t < n$, such that $d_t < d_{t-1} + 2$. Since $d_t > d_{t-1}$, thus we get $d_t = d_{t-1} + 1$. Therefore $d_{t-1}d_{t+1} \leq d_t^2 = d_t d_{t-1} + d_t$, or

$$d_{t-1}(d_{t+1} - d_t) \leq d_t.$$

By the assumption on t one has $d_{t+1} - d_t \geq 2$. Then we find $2d_{t-1} \leq d_t$, whence $2d_{t-1} \leq d_t = d_{t-1} + 1$. This gives $d_{t-1} = 1$ and hence $d = 1$ by the Hodge index inequalities. Therefore $\widehat{d} = 1$ and $(\mathcal{M}, \mathcal{L}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, contradicting the log-general type assumption.

Assume n is even. By parity (4) we have

$$d_{n-1} \geq d_{n-2} + 2.$$

Let t be the biggest integer, $t \leq n$, such that $d_t < d_{t-1} + 2$. Since $d_t > d_{t-1}$, then we have $d_t = d_{t-1} + 1$. It may happen that $d_n = d_{n-1} + 1$. If this is not the case we may assume $t < n$ and exactly the same argument as above leads to a contradiction.

The arguments above show the following result.

Theorem 2.1 *Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$, with $n \geq 5$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let (M, L), $(X, \mathcal{D})$, $\mathcal{K} \approx (n-2)\mathcal{D}$ be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.4). Let d_j, $j = 0, \ldots, n$, $d_0 = d$, be the pluridegrees of (M, L). Assume that $K_X + (n-3)\mathcal{K}$ is not nef and big. Then either*

1. $d_k \geq d_{k-1} + 2$ *for* $k = 1, \ldots, n$, *and hence* $d_n \geq d + 2n$, *if* n *is odd and*
   $d_n \geq d_{n-1} + 1$, $d_k \geq d_{k-1} + 2$ *for* $k = 1, \ldots, n-1$, *and hence* $d_n \geq d + 2n - 1$, *if* n *is even; or*

2. $n = 6$, $(X, \mathcal{K}) \cong (\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(1))$*; or*

3. $n = 5$, $(X, \mathcal{K}) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}})$, $\mathcal{Q}$ *a quadric in* $\mathbb{P}^6$*; or*

4. $n = 5$, X *is a* $\mathbb{P}^4$*-bundle over a smooth curve,* $\mathcal{K}_{|F} \approx \mathcal{O}_{\mathbb{P}^4}(1)$ *for any fiber* F*; or*

5. $n = 5$, X *is a singular* 2*-Gorenstein Fano* 5*-fold described in* [6, (7.7.5)]*; or*

6. X *is a Fano* n*-fold with* $K_X \approx -(n-3)\mathcal{K}$ *if* n *is even and* $2K_X \approx -2(n-3)\mathcal{K}$ *if* n *is odd.*

We consider now the case $n = 4$. From [4, (2.1)] we know that $K_X + 3\mathcal{K}$ is nef and big unless either

i) $(X, \mathcal{K}) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$; or

ii) X is a Gorenstein Fano 4-fold with $-K_X \approx 3\mathcal{K}$ and $-4\mathcal{K} \approx 6\mathcal{D}$; or

iii) there exists a holomorphic map $\psi : X \to C$, where C is a smooth curve, $4K_X + 6\mathcal{D} \approx \psi^* H$ for some ample line bundle H on C. In this case $K_X + 3\mathcal{K} \approx \psi^* H$, i.e., $(X, \mathcal{K})$ is a quadric fibration over C; or

iv) there exists a holomorphic map $\psi : X \to S$, where S is a smooth surface, $4K_X + 6\mathcal{D} \approx \psi^* H$ for some ample line bundle H on S. In this case $K_X + 3\mathcal{K} \approx \psi^* H$, i.e., $(X, \mathcal{K})$ is a scroll over S.

In case i), $d_4 = 2$. In cases iii) and iv), $4K_X + 6\mathcal{D}$ is effective. From [6, (7.6.1)] we know that

$$H^0(X, 4K_X + 6\mathcal{D}) \cong H^0(M, 4K_M + 6L),$$

so that $4K_M + 6L = 4(K_M + 2L) - 2L$ is effective on M. Therefore, dotting with $(K_M + 2L)^j \cdot L^{3-j}$, $j = 0, 1, 2, 3$, we obtain

$$2d_j > d_{j-1}, \quad j = 1, 2, 3, 4.$$

The arguments above show the following result.

Theorem 2.2 *Let* $\mathcal{M}$ *be a smooth connected* 4*-fold polarized by a very ample line bundle* $\mathcal{L}$*. Assume that* $(\mathcal{M}, \mathcal{L})$ *is of log-general type. Let* (M, L), $(X, \mathcal{D})$, $\mathcal{K} \approx 2\mathcal{D}$ *be the first and the second reduction of* $(\mathcal{M}, \mathcal{L})$ *as in* (1.4)*. Let* d_j, $j = 0, 1, 2, 3, 4$, $d_0 = d$, *be the pluridegrees of* (M, L)*. Assume that* $K_X + 3\mathcal{K}$ *is not nef and big. Then either*

1. $2d_j > d_{j-1}$, $j = 1, 2, 3, 4$*; or*

2. $(X, \mathcal{K}) \cong (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$, $d_4 = 1$*; or*

3. X *is a Gorenstein Fano* 4*-fold with* $-K_X \approx 3\mathcal{K}$ *and* $-4K_X \approx 6\mathcal{D}$*.*

3 The case $n \geq 6$

Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.4). In this section we consider the general case when $n \geq 6$. From the results of §2 we can assume that

- $K_X + (n-3)\mathcal{K}$ is nef and big.

It should be noted that $K_X + (n-3)\mathcal{K}$ is always nef under the log-general type assumption if $n \geq 7$ (see [2, (3.1)]). We can prove the following general result.

Theorem 3.1 *Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$, with $n \geq 6$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$. Assume that $K_X + (n-3)\mathcal{K}$ is nef and big. Let d_j be the pluridegrees of (M, L), $j = 0, \ldots, n$, $d_0 = d$. Then either*

1. $d_j \geq 2\left(\frac{n-1}{n+1}\right) d_{j-1}$, $j = 1, \ldots, n$, *and hence* $d_n \geq \left(\frac{2n-2}{n+1}\right)^n d$*; or*
2. $n = 8$ *and there is a birational morphism* $\psi : X \to \mathbb{P}^8$ *such that* $K_X + (n-3)\mathcal{D} \approx \psi^* \mathcal{O}_{\mathbb{P}^8}(1)$*; or*
3. $n = 6$, $\widehat{d} = \mathcal{L}^6 = 364$ *and* $d_j \geq \frac{13}{16} d_{j-1}$, $j = 1, \ldots, 6$.

Proof. First assume $n \geq 7$. Thus Theorem (3.1) of [2] applies to say that $(n+1)K_{\mathcal{M}} + n(n-3)\mathcal{L}$ is effective unless $n = 8$ and we are as in case 2). From [6, (7.6.1)] we know that

$$H^0(\mathcal{M}, (n+1)K_{\mathcal{M}} + n(n-3)\mathcal{L}) \cong H^0(M, (n+1)K_M + n(n-3)L).$$

Therefore, up to the special case with $n = 8$,

$$(n+1)K_M + n(n-3)L = (n+1)(K_M + (n-2)L) - 2(n-1)L$$

is effective. Thus, dotting with $(K_M + (n-2)L)^{j-1} \cdot L^{n-j}$, we get

$$d_j \geq 2\left(\frac{n-1}{n+1}\right) d_{j-1}, \quad j = 1, \ldots, n.$$

Hence in particular

$$d_n \geq \left(\frac{2n-2}{n+1}\right)^n d.$$

Assume now $n = 6$. Thus from [2, (4.1)] we know that $13K_{\mathcal{M}} + 36\mathcal{L}$ is effective. If moreover $\widehat{d} = \mathcal{L}^6 \neq 364$, then $7K_{\mathcal{M}} + 18\mathcal{L}$ is effective.

Let us consider first the general case $\widehat{d} \neq 364$. From [6, (7.6.1)] we know that

$$H^0(\mathcal{M}, 7K_{\mathcal{M}} + 18\mathcal{L}) \cong H^0(M, 7K_M + 18L).$$

Therefore $7K_M + 18L = 7(K_M + 4L) - 10L$ is effective. Thus dotting with $7(K_M + (n - 2)L)^{j-1} \cdot L^{6-j}$, we get

$$d_j \geq \frac{10}{7} d_{j-1} = \left(\frac{2n-2}{n+1}\right) d_{j-1}, \quad j = 1, \dots, 6, \tag{8}$$

as in 1). In the special case $\widehat{d} = 364$, the same argument gives

$$13 d_j \geq 16 d_{j-1}, \quad j = 1, \dots, 6, \tag{9}$$

as in case 3). Q.E.D.

We have stronger bounds in special cases.

Remark 3.2 Notation and assumptions as in (3.1). Assume that either X is not Gorenstein or $\mathcal{D}$ is not Cartier. Then we know from [2, (3.1)] that $K_{\mathcal{M}} + (n-4)\mathcal{L}$ is effective. By using again [6, (7.6.1)] we thus conclude that $K_M + (n-3)L = K_M + (n-2)L - 2L$ is effective. Therefore, dotting with $(K_M + (n-2)L)^{j-1} \cdot L^{n-j}$, we get the stronger bound $d_j > 2d_{j-1}$, $j = 1, \dots, n$, and hence $d_n > 2^n d$.

We have the following numerical conditions, expressed in terms of pluridegrees, for a given variety $\mathcal{V}$ to be a k-fold section of the polarized pair $(\mathcal{M}, \mathcal{L})$.

Remark 3.3 Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$ with either $n \geq 7$ or $n = 6$ and $\mathcal{L}^6 \neq 364$, as in (3.1). Let $\mathcal{V}$ be the k-fold section of $\mathcal{M}$ obtained as transversal intersection of $n - k$ general members of $|\mathcal{L}|$, $k = 2, \dots, n-1$. Let $\mathcal{L}_{\mathcal{V}}$ be the restriction of $\mathcal{L}$ to $\mathcal{V}$. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$ be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$. Let $V := \pi(\mathcal{V})$, $Y := \varphi(V)$. Let L_V and $\mathcal{D}_Y$ be the restrictions of L and $\mathcal{D}$ to V and Y respectively. Thus intersections are compatible with reductions, i.e., (V, L_V), $(Y, \mathcal{D}_Y)$ are the first and the second reduction of $(\mathcal{V}, \mathcal{L}_{\mathcal{V}})$.

Let $d_j' s$ be the pluridegrees of (M, L) and let

$$d_j(V) := (K_V + (k-2)L_V)^j \cdot L_V^{k-j} (= (K_M + (n-2)L^j) \cdot L^{n-j})$$

be the pluridegrees of (V, L_V). We have

$$d_k = d_k(V), \ k \geq 1, \ \text{ and } \ d = L^n = L_V^k.$$

Assuming $n \neq 8$ and $\mathcal{L}^n \neq 364$, we have from (3.1)

$$d_k(V) = d_k \geq \left(\frac{2n-2}{n+1}\right)^k d, \tag{10}$$

i.e., $\mathcal{V}$ cannot be a k-fold section of $(\mathcal{M}, \mathcal{L})$ as soon as $d_k(V) = d_k < \left(\frac{2n-2}{n+1}\right)^k d$.

In particular, for $k = 2$, inequality (10) gives a lower bound for the Euler characteristic of the surface section $V = S$. Indeed Miyaoka's inequality $d_2 < 9\chi(\mathcal{O}_S)$ combined with (10) gives

$$\chi(\mathcal{O}_S) > \frac{4}{9} \left(\frac{n-1}{n+1}\right)^2 d. \tag{11}$$

4 The case $n = 5$

Let $\mathcal{M}$ be a smooth connected 5-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + 3\mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.4). From the results of §2 we can assume that

- $K_X + 2\mathcal{K}$ is nef and big.

We can prove the following result.

Theorem 4.1 *Let $\mathcal{M}$ be a smooth connected 5-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + 3\mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$. Assume that $K_X + 2\mathcal{K}$ is nef and big. Let d_j's, $d_0 = d$, $j = 0, \ldots, 5$, be the pluridegrees of (M, L). Then either*

1. *$d \geq 14$, $d_1 \geq 19$, $d_2 \geq 24$, $d_3 \geq 30$, $d_4 \geq 37$, $d_5 \geq 44$; or*
2. *$\mathcal{M}$ is the complete intersection of a quadric and a quintic in $\mathbb{P}^7$, $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $(\mathcal{M}, \mathcal{L}) \cong (M, L)$, $d = 10$, $d_j = 2^j d$, $j = 1, 2, 3, 4, 5$; or*
3. *$\mathcal{M}$ is the complete intersection of a quadric and a sextic in $\mathbb{P}^7$, $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $(\mathcal{M}, \mathcal{L}) \cong (M, L)$, $d = 12$, $d_j = 3^j d$, $j = 1, 2, 3, 4, 5$; or*
4. *$\mathcal{M}$ is the complete intersection of a cubic and a quartic in $\mathbb{P}^7$, $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $(\mathcal{M}, \mathcal{L}) \cong (M, L)$, $d = 12$, $d_j = 2^j d$, $j = 1, 2, 3, 4, 5$.*

Proof. Consider the line bundle $\mathcal{F} := 2(K_X + 2\mathcal{D})$ on X. Since $K_X + 2\mathcal{K} \approx 3(K_X + 2\mathcal{D})$ is nef and big we conclude that

$$\mathcal{F} \text{ is nef and big.} \tag{12}$$

Moreover

$$2\mathcal{F} - K_X = 3K_X + 8\mathcal{D} = K_X + 2\mathcal{D} + 2(K_X + 3\mathcal{D})$$

is nef and big since by the above $K_X + 2\mathcal{D}$ is nef and big and $\mathcal{K} \approx K_X + 3\mathcal{D}$ is ample. Thus the Kawamata-Shokurov basepoint free theorem (see e.g., [6, (1.5.1)]) applies to say that

$$t\mathcal{F} \text{ is spanned by global sections for sufficiently large integers } t. \tag{13}$$

From [6, (7.2.9)] (see also [2, (0.2.8)]) we know that $h^i(t(K_X + 2\mathcal{D}) = 0$ for all positive integers t and for $i > 0$. Then in particular

$$h^i(t\mathcal{F}) = 0 \text{ for } i > 0,\ t > 0. \tag{14}$$

Thus in view of conditions (12), (13), (14), Proposition (0.5) of [2] applies to say that either

$$h^0(K_X + 5\mathcal{F}) = h^0(11K_X + 20\mathcal{D}) > 0,$$

or there exists a birational morphism $\psi : X \to \mathbb{P}^5$ such that $\mathcal{F} = 2(K_X + 2\mathcal{D}) \approx \psi^*\mathcal{O}_{\mathbb{P}^5}(1)$. The latter case is easily ruled out. Indeed we have $(\psi_* K_X)^{**} \approx \mathcal{O}_{\mathbb{P}^5}(-6)$ and $(\psi_*\mathcal{D})^{**} \approx \mathcal{O}_{\mathbb{P}^5}(a)$ for some integer a. Hence we get the numerical contradiction $-12 + 4a = 1$.

Thus we conclude that $11K_X + 20\mathcal{D}$ is effective. We claim that $2(11K_M + 20L)$ is effective. To see this recall that $2K_M \approx \varphi^*(2K_X) + 3\Delta$ and $2L \approx \varphi^*(2\mathcal{D}) - \Delta$, for some effective Cartier divisor Δ (see (1.4)). Thus

$$22K_M + 40L \approx \varphi^*(22K_X + 40\mathcal{D}) + (33 - 20)\Delta.$$

Hence $2(11K_M + 20L) \approx 22\mathcal{K}_M - 26L$ is effective, where $\mathcal{K}_M := K_M + 3L$. Thus dotting with $\mathcal{K}_M^{j-1} \cdot L^{5-j}$, we get

$$11d_j \geq 13d_{j-1}, \quad j = 1, 2, 3, 4, 5. \tag{15}$$

From Lemma (1.10) we know that $d \geq \widehat{d} \geq 8$. Then inequalities (15) give $d_1 \geq 10$.

Assume that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^N$, $N \geq 9$. Suppose $\widehat{d} := \mathcal{L}^5 = 14$. Then $d \geq 14$ and $d_1 \geq 17$ by (15). Castelnuovo's bound (1) gives

$$g := g(\mathcal{L}) \leq \mathrm{Castel}(14, 5) \leq 15.$$

Then we find the numerical contradiction $31 \leq d_1 + d = \widehat{d} + \widehat{d_1} = 2g - 2 \leq 28$.

Exactly the same argument, by using (15) and (1), rules out the low degree cases $8 \leq \widehat{d} \leq 13$.

Thus $d \geq \widehat{d} \geq 15$. From (15) we obtain

$$d \geq 15, \ d_1 \geq 18, \ d_2 \geq 22, \ d_3 \geq 26, \ d_4 \geq 31, \ d_5 \geq 37.$$

If $d_1 = 18$, then $d \geq 16$ by parity, contradicting $d_1^2 = 324 \geq dd_2 \geq 330$. Therefore $d_1 \geq 19$. Then $d_2^2 \geq d_1 d_3 \geq 494$ yields $d_2 \geq 23$. If $d_3 = 26$, then $d_2 \geq 24$ by parity, contradicting $d_3^2 = 676 \geq d_2 d_4 \geq 744$. Therefore $d_3 \geq 27$. If $d_3 = 27$, then $d_2 \geq 25$ by parity, which contradicts $d_3^2 = 729 \geq d_2 d_4 \geq 775$. Thus $d_3 \geq 28$, so that (15) gives $d_4 \geq 34$ and $d_5 \geq 41$. Summarizing, we have

$$d \geq 15, \ d_1 \geq 19, \ d_2 \geq 23, \ d_3 \geq 28, \ d_4 \geq 34, \ d_5 \geq 41.$$

From $d_2^2 \geq d_1 d_3 \geq 532$ we get $d_2 \geq 24$, so that (15) gives $d_3 \geq 29$, $d_4 \geq 35$, $d_5 \geq 42$. If $d_3 = 29$, then $d_2 \geq 25$ by parity, contradicting $d_3^2 = 841 \geq d_2 d_4 \geq 875$. Thus $d_3 \geq 30$, so that (15) yields $d_4 \geq 36$, $d_5 \geq 43$. Then we have

$$d \geq 15, \ d_1 \geq 19, \ d_2 \geq 24, \ d_3 \geq 30, \ d_4 \geq 36, \ d_5 \geq 43.$$

Finally, if $d_4 = 36$, then $d_5 \geq 44$ by parity, which contradicts $d_4^2 = 1296 \geq d_3 d_5 \geq 1320$. Therefore $d_4 \geq 37$ and (15) gives $d_5 \geq 44$, as in case 1).

Thus we can assume that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^N$, $N \leq 8$. Then by Barth-Lefschetz theorem we know that $\mathrm{Pic}(\mathcal{M}) \cong \mathbb{Z}[\mathcal{L}]$. Note that this condition implies $\mathcal{M} \cong M \cong X$. Moreover $\mathcal{L} \approx \mathcal{O}_{\mathcal{M}}(1)$, $K_{\mathcal{M}} \approx \mathcal{O}_{\mathcal{M}}(r)$ for some $r \in \mathbb{Z}$ and hence

$$d_j = (r+3)d_{j-1} = (r+3)^j d, \quad j = 1, 2, 3, 4, 5. \tag{16}$$

From (15) one has $r + 3 \geq 2$. Note that for the sectional genus g we have the upper bound (see e.g., [1, p. 118])

$$2g - 2 \leq \frac{d(d-4)}{2}.$$

Therefore, recalling that $r+3 \geq 2$, we have $2g-2 \geq 3d$ and hence the inequality

$$3d \leq \frac{d(d-4)}{2},$$

which gives $d \geq 10$. Hence $d_1 \geq 12$ by (15). Let $\mathcal{V}$ be the smooth 3-fold section of $\mathcal{M}$ obtained as transversal intersection of two general members of $|\mathcal{L}|$. Let $\mathcal{L}_{\mathcal{V}}$ be the restriction of $\mathcal{L}$ to $\mathcal{V}$. Then $\mathcal{V}$ is embedded by $|\mathcal{L}_{\mathcal{V}}|$ in $\mathbb{P}^{N-2}$ as 3-fold of degree $d = \widehat{d} = \mathcal{L}_{\mathcal{V}}^3$. Moreover $(\mathcal{V}, \mathcal{L}_{\mathcal{V}})$ is of log-general type since $(\mathcal{M}, \mathcal{L})$ is of log-general type and $(\mathcal{V}, \mathcal{L}_{\mathcal{V}})$ coincides with its own first reduction since $(\mathcal{M}, \mathcal{L}) \cong (M, L)$. In what follows we use the lists from [5, Chapter 6] of 3-folds of degree ≤ 12 in $\mathbb{P}^5$.

Let $d = 10$. Then $\mathcal{M}$ lies in $\mathbb{P}^7$ since otherwise $g \leq \text{Castel}(10,4) \leq 9$, which leads to the contradiction $22 \leq d + d_1 = 2g-2 \leq 16$. Since by (16) all the pluridegrees are multiple of $d = 10$, we see that the only possible case is when $\mathcal{V}$ is the complete intersection of a quadric and a quintic in $\mathbb{P}^5$. Then $\mathcal{M}$ is the complete intersection of a quadric and a quintic in $\mathbb{P}^7$, and $d_j = 2^j d$, $j = 1,2,3,4,5$ as in case 2).

Thus we can assume $d \geq 11$. Note that the case $d = 11$ doesn't happen. Indeed in this case d_1 must be odd by parity and hence by (16) we have $d_1 = (2k+1)d$ for some positive integer k. Then in particular $d_1 \geq 3d$. Since d is odd we have the upper bound (see e.g., [1, p. 118])

$$2g-2 \leq \frac{(d-1)(d-3)}{2} - 2.$$

This gives the numerical contradiction $38 = 2g-2 = d_1 + d \geq 4d \geq 44$.

Thus $d \geq 12$. Assume that $d = 12$ and that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^7$. Then $|\mathcal{L}_{\mathcal{V}}|$ embeds $\mathcal{V}$ in $\mathbb{P}^5$ and, recalling (16), we see that either $\mathcal{V}$ is the complete intersection of a quadric and a sextic in $\mathbb{P}^5$ or $\mathcal{V}$ is the complete intersection of a cubic and a quartic in $\mathbb{P}^5$. Accordingly, either $\mathcal{M}$ is the complete intersection of a quadric and a sextic in $\mathbb{P}^7$, with $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $d_j = 3^j d$, $j = 1,2,3,4,5$, as in case 3), or $\mathcal{M}$ is the complete intersection of a cubic and a quartic in $\mathbb{P}^7$, with $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $d_j = 2^j d$, $j = 1,2,3,4,5$, as in case 4).

Thus we can assume that $d = 12$ and that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^8$. Then Castelnuovo's bound gives $g \leq \text{Castel}(12,4) \leq 15$. Since, by (16), $d_1 = (r+3)d$ and $r+3 \geq 2$ we find the contradiction $36 \leq 3d \leq d + d_1 = 2g-2 \leq 30$.

Thus $d \geq 13$. Assume that $d = 13$ and that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^7$. Since d_1 must be odd by parity, we have $d_1 \geq (2k+1)d$ for some positive integer k from (16). Castelnuovo's bound gives $g \leq \text{Castel}(13,3) \leq 30$. If $k \geq 2$, i.e., $d_1 \geq 5d$, we find the contradiction $78 \leq 6d = d + d_1 = 2g-2 \leq 58$. Therefore $k = 1$, i.e., $d_1 = 3d$, so that $g = 2d+1 = 27$ doesn't reach the maximum according to Castelnuovo's bound. Thus, since the hyperplane curve section of $\mathcal{M}$ lies in $\mathbb{P}^3$, we can use Gruson-Peskine bound which gives (see e.g., [6, (1.4.9)])

$$g \leq \frac{d(d-3)}{6} + 1 = \frac{68}{3},$$

or $g \leq 22$. This contradicts the above equality $g = 27$.

Therefore we have to consider the case when $d = 13$ and $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^8$. Again, as above, $d_1 \geq 3d$. Castelnuovo's bound yields $g \leq \text{Castel}(13,4) \leq 18$ and then we get the contradiction $52 \leq 4d \leq d_1 + d = 2g-2 \leq 34$.

Thus we can assume $d \geq 14$, so that (16) gives, since $r+3 \geq 2$,

$$d \geq 14,\ d_1 \geq 28,\ d_2 \geq 56,\ d_3 \geq 112,\ d_4 \geq 224,\ d_5 \geq 448$$

and we fall again in case 1). Q.E.D.

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5 The case $n = 4$

Let $\mathcal{M}$ be a smooth connected 4-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + 2\mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.4). From the results of §2 we can assume that

- $K_X + 3\mathcal{K}$ is nef and big.

We can prove the following result.

Theorem 5.1 *Let $\mathcal{M}$ be a smooth connected 4-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + 2\mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$. Assume that $K_X + 3\mathcal{K}$ is nef and big. Let d_j's, $d_0 = d$, $j = 0, 1, 2, 3, 4$, be the pluridegrees of (M, L). Then either*

1. *$d \geq 11$, $d_1 \geq 9$, $d_2 \geq 7$, $d_3 \geq 5$, $d_4 \geq 3$; or*
2. *$\mathcal{M}$ is the complete intersection of a quadric and a quartic in $\mathbb{P}^6$, $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $(\mathcal{M}, \mathcal{L}) \cong (M, L)$, and $d_j = 8$, $j = 0, 1, 2, 3, 4$; or*
3. *$\mathcal{M}$ is the complete intersection of two cubics in $\mathbb{P}^6$, $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $(\mathcal{M}, \mathcal{L}) \cong (M, L)$, and $d_j = 9$, $j = 0, 1, 2, 3, 4$; or*
4. *$\mathcal{M}$ is the complete intersection of of a quadric and a quintic in $\mathbb{P}^6$, $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $(\mathcal{M}, \mathcal{L}) \cong (M, L)$, and $d = 10$, $d_j = 2d_{j-1}$, $j = 1, 2, 3, 4$.*

Proof. Since $K_X + 3\mathcal{K} \approx 4\mathcal{K} - 2\mathcal{D}$ is nef and big, dotting with $\mathcal{K}^j \cdot \mathcal{D}^{3-j}$, $j = 2, 3$, we obtain $d'_4 > d'_3/2$, $d'_3 > d'_2/2$ on X. Therefore, by using (5), we have $d_4 > d_3/2$, $d_3 > d_2/2$ on M. Thus Hodge index inequalities (3) give $d_2 > d_1/2$, $d_1 > d/2$. Then we get

$$d_4 > \frac{d_3}{2} > \frac{d_2}{4} > \frac{d_1}{8} > \frac{d}{16}. \tag{17}$$

In particular $d_4 \geq 2$ since otherwise $d = \widehat{d} = 1$ and hence $(\mathcal{M}, \mathcal{L}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, contradicting the log-general type assumption. From Lemma (1.10) we know that $d \geq \widehat{d} := \mathcal{L}^4 \geq 8$. Then inequalities (17) give

$$d \geq 8,\ \ d_1 \geq 5,\ \ d_2 \geq 3,\ \ d_3 \geq 2,\ \ d_4 \geq 2.$$

By using repeatedly the Hodge index inequalities we obtain $d_1 \geq 7$, $d_2 \geq 6$, $d_3 \geq 4$. Then $d_4 > d_3/2$ from (17) gives $d_4 \geq 3$, and hence $d_3^2 \geq d_2 d_4 = 18$ yields $d_3 \geq 5$. If $d_1 = 7$, then $d \geq 9$ by parity. This contradicts $d \leq d_1^2/d_2 \leq 49/6 < 9$. Therefore $d_1 \geq 8$, so that $d_2^2 \geq d_1 d_3 \geq 40$ gives $d_2 \geq 7$. Thus we have

$$d \geq 8,\ \ d_1 \geq 8,\ \ d_2 \geq 7,\ \ d_3 \geq 5,\ \ d_4 \geq 3. \tag{18}$$

Assume now that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^N$, $N \geq 7$. If $\widehat{d} = \mathcal{L}^4 = 8, 9, 10$, Castelnuovo's bound $g := g(\mathcal{L}) \leq \mathrm{Castel}(\widehat{d}, 4)$ gives respectively $g \leq 5$, $g \leq 7$, $g \leq 9$. In each case $18 \leq d_1 + d = \widehat{d_1} + \widehat{d} = 2g - 2 \leq 16$ leads to a numerical contradiction. Thus $d \geq \widehat{d} \geq 11$ and $d_1^2 \geq dd_2 \geq 77$ gives $d_1 \geq 9$. Then, recalling (18), we are in case 1).

Thus we can assume that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^6$. Therefore by Barth-Lefschetz theorem we know that $\mathrm{Pic}(\mathcal{M}) \cong \mathbb{Z}[\mathcal{L}]$. Note that this condition implies $\mathcal{M} \cong M \cong X$. Moreover $\mathcal{L} \cong \mathcal{O}_{\mathcal{M}}(1)$, $K_{\mathcal{M}} \cong \mathcal{O}_{\mathcal{M}}(r)$ for some $r \in \mathbb{Z}$ and hence

$$d_j = (r+2)d_{j-1} = (r+2)^j d, \quad j = 1, 2, 3, 4. \tag{19}$$

Let $\mathcal{V}$ be a general smooth member of $|\mathcal{L}|$. Let $\mathcal{L}_{\mathcal{V}}$ be the restriction of $\mathcal{L}$ to $\mathcal{V}$. Then $\mathcal{V}$ is a smooth 3-fold of degree $d = \widehat{d} = \mathcal{L}_{\mathcal{V}}^3$ embedded by $|\mathcal{L}_{\mathcal{V}}|$ in $\mathbb{P}^5$. Moreover $(\mathcal{V}, \mathcal{L}_{\mathcal{V}})$ is of log-general type since $(\mathcal{M}, \mathcal{L})$ is of log-general type and $(\mathcal{V}, \mathcal{L}_{\mathcal{V}})$ coincides with its own first reduction since $(\mathcal{M}, \mathcal{L}) \cong (M, L)$. In what follows we use the lists from [5, Chapter 6] of 3-folds of degree ≤ 12 in $\mathbb{P}^5$.

Assume $d (= \widehat{d}) = 8$. There are only two possible cases. In the first case $K_{\mathcal{V}} + \mathcal{L}_{\mathcal{V}}$ is not nef and big, contradicting the log-general type assumption. In the second case $\mathcal{V}$ is the complete intersection of a quadric and a quartic in $\mathbb{P}^5$, so that $\mathcal{M}$ is a complete intersection of type $(2, 4)$ in $\mathbb{P}^6$ and $d_j = 8$, $j = 0, 1, 2, 3, 4$, as in 2).

Let $d (= \widehat{d}) = 9$. Then from the lists we see that $\widehat{d}_1 = d_1 \leq 13$ and hence (19) gives $r + 2 = 1$, i.e., $d_j = d_{j-1}$, $j = 1, 2, 3, 4$. This is not possible unless $\mathcal{V}$ is the complete intersection of two cubic hypersurfaces in $\mathbb{P}^5$. In this case $\mathcal{M}$ is the complete intersection of two cubics in $\mathbb{P}^6$ and $d_j = 9$, $j = 0, 1, 2, 3, 4$, as in 3).

Assume $d (= \widehat{d}) = 10$. Then either $\mathcal{V}$ is the complete intersection of a quadric and a quintic in $\mathbb{P}^5$, so that $\mathcal{M}$ is a complete intersection of type $(2, 5)$ in $\mathbb{P}^6$, $K_{\mathcal{M}} \approx \mathcal{O}_{\mathcal{M}}$, $r = 0$, $d_j = 2d_{j-1}$, $j = 1, 2, 3, 4$, and we are as in case 4), or $d_3 = 1$, which contradicts (18), or $d_1 = d_2 = d_3 = d_4 = 12$. From (19) we have $d_j = (r+2)^j d = (r+2)^j 10$, $j = 1, 2, 3, 4$, so we find again a contradiction.

Thus we conclude that $d (= \widehat{d}) \geq 11$ and $d_1^2 \geq dd_2 \geq 77$ gives $d_1 \geq 9$. Then, recalling (18), we fall again in case 1). Q.E.D.

6 The case $n = 3$, I

Let $\mathcal{M}$ be a smooth connected 3-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + \mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.4). Let d_j, $j = 0, 1, 2, 3$, $d_0 = d$, be the pluridegrees of (M, L).

First, let us state some general relations. Let S be a smooth general member of $|L|$. Set $\mathcal{K}_M := K_M + L$ and $h := h^0(K_M + L)$. Recall that $h \geq 2$ from [7, (1.2)]. Since $\mathcal{K}_M$ is nef and big, the Riemann-Roch formula yields

$$\begin{aligned} h^0(K_M + t\mathcal{K}_M) &= -\chi(t\mathcal{K}_M) \\ &= \frac{t^3 d_3}{6} + \frac{t^2(d_3 - d_2)}{4} + \frac{t\mathcal{K}_M \cdot ((\mathcal{K}_M - L)^2 + c_2(M))}{12} - \chi(\mathcal{O}_M). \end{aligned}$$

Since $\mathcal{K}_M \cdot (\mathcal{K}_M - L)^2 = d_3 - 2d_2 + d_1$,

$$\mathcal{K}_M \cdot c_2(M) = K_M \cdot c_2(M) + L \cdot c_2(M) = -24\chi(\mathcal{O}_M) + 12\chi(\mathcal{O}_S) - d_2 - d_1,$$

and $\chi(\mathcal{O}_M) = \chi(\mathcal{O}_S) - h$, we find

$$h^0(K_M + t\mathcal{K}_M) = \frac{t^3 d_3}{6} + \frac{t^2 d_3}{4} - \frac{t^2 d_2}{4} + \frac{t d_3}{12} - \frac{t d_2}{4} - (t+1)\chi(\mathcal{O}_S) + h(2t+1).$$

For $t = 1$ we have

$$2h^0(K_M + \mathcal{K}_M) + 4\chi(\mathcal{O}_S) + d_2 = d_3 + 6h. \tag{20}$$

For $t = 2$ we have

$$2h^0(K_M + 2\mathcal{K}_M) + 6\chi(\mathcal{O}_S) + 3d_2 = 5d_3 + 10h. \tag{21}$$

For $t = 3$ we have

$$h^0(K_M + 3\mathcal{K}_M) + 4\chi(\mathcal{O}_S) + 3d_2 = 7d_3 + 7h. \tag{22}$$

We have the following four exceptions to $K_X + 2\mathcal{K}$ being nef from [2, (2.2)].

i) $d_3 = 1$, $(X, \mathcal{D}) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))$.

ii) $d_3 = 2$, $(X, \mathcal{D}) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(1))$, where $\mathcal{Q}$ is a quadric in $\mathbb{P}^4$ and $\mathcal{O}_{\mathcal{Q}}(1)$ is the restriction of $\mathcal{O}_{\mathbb{P}^4}(1)$ to $\mathcal{Q}$, under the usual embedding $\mathcal{Q} \subset \mathbb{P}^4$.

iii) $d_3 = 4$, and X is the cone over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ with $2\mathcal{D} \approx 7\xi$, where ξ is the tautological line bundle of the cone.

iv) X is a $\mathbb{P}^2$-bundle, $p : X \to C$, over a smooth curve C with $K_X + 3\mathcal{K} \approx p^*H$ for some ample line bundle H on C.

Let's now analyze the special case iv) of a variety $\mathcal{M}$ with second reduction a $\mathbb{P}^2$-bundle X over a curve C of genus $q = h^1(\mathcal{O}_X)$ and $\mathcal{K}_{|F} \approx \mathcal{O}_{\mathbb{P}^2}(1)$ for any fiber $F \cong \mathbb{P}^2$.

Note that, since X has rational singularities, $h^i(\mathcal{O}_M) = h^i(\mathcal{O}_X) = 0$ for $i = 2, 3$. Hence

$$\chi(\mathcal{O}_S) - h = \chi(\mathcal{O}_M) = 1 - q, \tag{23}$$

i.e.,

$$h = p_g(S). \tag{24}$$

Note also that, since $K_X + t\mathcal{K} \approx (\varphi_*(K_M + t\mathcal{K}_M))^{**}$ for $t \geq 1$, from Lemma (1.11) one has an inclusion

$$H^0(M, K_M + t\mathcal{K}_M) \subset H^0(X, K_X + t\mathcal{K}), \quad t \geq 1. \tag{25}$$

Then, since $K_X + t\mathcal{K}$ is not effective for $t = 1, 2$ in case iv), we conclude that $h^0(K_M + \mathcal{K}_M) = h^0(K_M + 2\mathcal{K}_M) = 0$. Thus (20) and (21) give respectively

$$4\chi(\mathcal{O}_S) + d_2 = d_3 + 6h$$

and

$$6\chi(\mathcal{O}_S) + 3d_2 = 5d_3 + 10h.$$

By subtracting we obtain

$$\chi(\mathcal{O}_S) + d_2 = 2d_3 + 2h.$$

The last two relations give

$$3\chi(\mathcal{O}_S) + d_3 = 4h \tag{26}$$

as well as

$$10h = d_2 + 7\chi. \tag{27}$$

Lemma 6.1 *Notation as at the beginning of this section. Assume that X is a $\mathbb{P}^2$-bundle, $p : X \to C$, over a smooth curve C of genus q with $K_X + 3\mathcal{K} \approx p^*H$ for some ample line bundle H on C as in case iv) above. Let $h := h^0(K_M + L)$ and let S be a smooth member of $|L|$. Then we have $d_3 \geq 2$ with equality implying $q = 1$, $\chi(\mathcal{O}_S) = 2$, $h = 2$, $d_2 = 6$, $d \geq 24$, $d_1 \geq 13$.*

Proof. Assume $d_3 = 1$. Then from (24) and (26) we get $h = 4 - 3q$. Thus

$$p_g(S) = 4 - 3q. \tag{28}$$

This implies that either $q = 0$ or $q = 1$. If $q = 0$ then $h = 4$. Since $\mathcal{K}$ is nef (indeed ample) on a projective bundle, then it is spanned. Hence $|\mathcal{K}|$ defines a morphism $\sigma : X \to \mathbb{P}^3$. Since $d_3 = d_3' = (\mathcal{K}')^3 = 1$ and $\mathcal{K}$ is ample we conclude that σ is generically one-to-one and finite, and therefore σ is an isomorphism. This contradicts the assumption that X is a $\mathbb{P}^2$-bundle.

If $q = 1$ then from (28) we get $p_g(S) = \chi(\mathcal{O}_S) = 1$. Thus (26) gives the absurdity that $h = 1$ (see [7, (1.2)]). If $d_3 = 2$, then we have from (24) and (26),

$$p_g(S) = 5 - 3q. \tag{29}$$

This implies that either $q = 0$ or $q = 1$. If $q = 0$, then $h = p_g(S) = 5$. As above, since $\mathcal{K}$ is nef on a projective bundle, then it is spanned. Hence $|\mathcal{K}|$ defines a morphism $\sigma : X \to \mathbb{P}^4$. Since $d_3 = d_3' = (\mathcal{K}')^3 = 2$, the image of σ is a quadric in $\mathbb{P}^4$ and σ is generically one-to-one. Since $\mathcal{K}$ is ample we conclude that σ is finite and hence X is isomorphic to a quadric in $\mathbb{P}^4$. Again this contradicts the fact that X is a $\mathbb{P}^2$-bundle.

If $q = 1$, then by (29), $p_g(S) = \chi(\mathcal{O}_S) = 2$. Thus (24) and (27) lead to $h = 2$ and $d_2 = 6$. Since $(\mathcal{M}, \mathcal{L})$ is of log-general type we have $d \geq 8$ by Lemma (1.10). Then $d_1^2 \geq dd_2 \geq 48$ gives $d_1 \geq 7$. If $d_1 = 7$ then $d = 8$ by the above inequality, contradicting the parity condition. By combining the double point formula (1.7) with Hodge index inequalities (3) we obtain $d \geq 24$, $d_1 \geq 13$. Q.E.D.

As a consequence of the results above, we can describe the case $d_3 = 1$.

Theorem 6.2 *Let $\mathcal{M}$ be a smooth connected 3-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + \mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$. Let $h := h^0(K_M + L)$ and let S be a smooth member of $|L|$. If $d_3 = 1$ then either $(X, \mathcal{D}) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))$ or X is a Gorenstein Del Pezzo threefold with $K_X \approx -2\mathcal{K}$, $h = 3$, $\chi(\mathcal{O}_S) = 4$, $d \geq 9$, $d_1 \geq 6$, $d_2 = 3$.*

Proof. If $K_X + 2\mathcal{K}$ is not nef we are in one of the special cases i)—iv). In case i) we are done, cases ii), iii) are not possible since we are assuming $d_3 = 1$, and case iv) as well is not possible by Lemma (6.1).

Thus we can assume that $K_X + 2\mathcal{K}$ is nef, so that $K_X + 3\mathcal{K} = 4\mathcal{K} - \mathcal{D}$ is ample. Therefore $(4\mathcal{K} - \mathcal{D}) \cdot \mathcal{K} \cdot \mathcal{K} > 0$, i.e., $4d_3' - d_2' > 0$ and hence $d_2' = d_2 \leq 3$.

If $d_2 = 1$ we have, by the Hodge index relations, $d_1 = d = 1$ and hence $(\mathcal{M}, \mathcal{L}) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$, contradicting the log-general type assumption. Since $d_2 \neq 2$ by parity, we conclude that $d_2 = 3$. Also, the log-general type assumption gives $d \geq 8$ (see Lemma

(1.10)) and therefore $d_1^2 \geq dd_2 \geq 24$ yields $d_1 \geq 5$. But $d_1 = 5$ implies $d = 8$, which contradicts again parity. Then $d_1 \geq 6$.

Assume $K_X + \mathcal{K} \approx 2\mathcal{K} - \mathcal{D}$ effective. Then $(2\mathcal{K} - \mathcal{D}) \cdot \mathcal{K} \cdot \mathcal{K} \geq 0$ gives $2d_3' > d_2'$, or $2d_3 > d_2$. This contradicts $d_3 = 1$, $d_2 = 3$. Thus we can assume $h^0(K_X + \mathcal{K}) = 0$ and therefore (20) and (25) yield $4\chi(\mathcal{O}_S) + d_2 = d_3 + 6h$, or $2\chi(\mathcal{O}_S) + 1 = 3h$. Since $h = p_g(S)$ by (24), this gives $3 = h + 2q$. Since $h \geq 2$ we conclude that

$$q = 0, \quad h = 3, \quad \text{and} \quad \chi(\mathcal{O}_S) = 4.$$

Then using the double point formula (1.7) we have

$$d(d-10) \geq 2d_2 + 5d_1 - 12\chi(\mathcal{O}_S) \geq -12,$$

and hence $d \geq 9$. From $d_1^2 \geq dd_2 \geq 27$ we obtain $d_1 \geq 6$.

Assume now $h^0(K_X + 2\mathcal{K}) \geq 1$. Note that $K_X + 2\mathcal{K}$ is a rank 1 reflexive sheaf (since K_X is a reflexive sheaf and $\mathcal{K}$ is a line bundle). Then a section $s \in H^0(X, K_X + 2\mathcal{K})$ vanishes on a Cartier divisor. But the ampleness of $\mathcal{K}$ and the equality

$$(K_X + 2\mathcal{K}) \cdot \mathcal{K} \cdot \mathcal{K} = (3\mathcal{K} - \mathcal{D}) \cdot \mathcal{K} \cdot \mathcal{K} = 3d_3 - d_2 = 0$$

imply that s vanishes on a subvariety $Y \subset X$ of codimension ≥ 2. This gives a contradiction, unless $K_X + 2\mathcal{K} \approx \mathcal{O}_X$ (and $h^0(K_X + 2\mathcal{K}) = 1$). Thus X is a Gorenstein Del Pezzo threefold with $K_X \approx -2\mathcal{K}$.

To conclude the proof it remains to show that $h^0(K_X + 2\mathcal{K}) \geq 1$. In view of (25) it suffices to show that $h^0(K_M + 2\mathcal{K}_M) \geq 1$. If $\chi(-2\mathcal{K}_M) \neq 0$ we have, since $\mathcal{K}_M$ is nef and big,

$$\chi(-2\mathcal{K}_M) = -\chi(K_M + 2\mathcal{K}_M) = -h^0(K_M + 2\mathcal{K}_M) \neq 0,$$

so we are done. Thus we can assume $\chi(-2\mathcal{K}_M) = 0$, i.e., $h^0(K_M + 2\mathcal{K}_M) = 0$ and hence also

$$h^0(K_M + \mathcal{K}_M) = \chi(K_M + \mathcal{K}_M) = -\chi(-\mathcal{K}_M) = 0,$$

i.e., $\chi(-\mathcal{K}_M) = 0$. Moreover $\chi(\mathcal{K}_M) = \chi(K_M + L) = h^0(K_M + L) = h = 3$. Therefore from the exact sequence

$$0 \to -L \to \mathcal{O}_M \to \mathcal{O}_S \to 0$$

we obtain

$$\chi(\mathcal{O}_M) = \chi(\mathcal{O}_S) + \chi(-L) = 4 - \chi(K_M + L) = 1.$$

Now a standard polynomial argument, by using $\chi(\mathcal{K}_M) = 3$, $\chi(\mathcal{O}_M) = 1$, $\chi(-\mathcal{K}_M) = 0$ and $\chi(-2\mathcal{K}_M) = 0$ and the bigness of $\mathcal{K}_M$ leads to a numerical contradiction. Q.E.D.

Remark 6.3 Notation as in (6.2). Note that the case of X being a Gorenstein Del Pezzo threefold with $K_X \approx -2\mathcal{K}$, of invariants $h = 3$, $\chi(\mathcal{O}_S) = 4$, $d \geq 9$, $d_1 \geq 6$, $d_2 = 3$ is possible. The example in [12] has $\mathcal{L} = 3H$ very ample with $K_{\mathcal{M}} \approx -2H$ and degree $\mathcal{L}^3 = 27$. Thus all reductions are isomorphic with $\mathcal{K} = H$ having $h^0(\mathcal{K}) = 3$, $d_3 = \mathcal{K}^3 = 1$.

7 The case $n = 3$, II

Let $\mathcal{M}$ be a smooth connected 3-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + \mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$ as in (1.4). Let d_j, $j = 0, 1, 2, 3$, $d_0 = d$, be the pluridegrees of (M, L). Let $h := h^0(K_M + L)$ and let S be a general member of $|L|$. Set also $\mathcal{K}_M := K_M + L$.

In this section we study the case $d_3 = 2$. We will work under the assumption that $K_X + 2\mathcal{K}$ is nef. If this is not the case, in view of the exceptions listed in §6, we know that either $(X, \mathcal{D}) \cong (\mathcal{Q}, \mathcal{O}_{\mathcal{Q}}(1))$, $\mathcal{Q}$ smooth quadric in $\mathbb{P}^4$, or X is as in Lemma (6.1) (see also Remark (6.3)).

Since $K_X + 2\mathcal{K}$ is nef we have $(K_X + 2\mathcal{K}) \cdot \mathcal{K} \cdot \mathcal{K} = (3\mathcal{K} - \mathcal{D}) \cdot \mathcal{K} \cdot \mathcal{K} \geq 0$ and hence

$$3d_3 \geq d_2.$$

Therefore, by parity, $d_2 = 2, 4, 6$. If $d_2 = 2$, then $d_2^2 \geq d_1 d_3 = 2d_1$ gives $d_1 = 2$ and hence $d = 2$. Then we would be in the special case of the quadric in $\mathbb{P}^4$, contradicting the nefness assumption.

Thus either $d_2 = 6$ or $d_2 = 4$. We have the following result, whose proof is given in the two paragraph below, where the cases $d_2 = 6$ and $d_2 = 4$ are treated separately. Recall that $d \geq 8$ by Lemma (1.10).

Theorem 7.1 *Let $\mathcal{M}$ be a smooth connected 3-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M}, \mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M}, \mathcal{L}) \to (M, L)$, $\varphi : (M, L) \to (X, \mathcal{D})$, $\mathcal{K} \approx K_X + \mathcal{D}$, be the first and the second reduction of $(\mathcal{M}, \mathcal{L})$. Let d_j, $j = 0, 1, 2, 3$, $d_0 = d$, be the pluridegrees of (M, L). Let $h := h^0(K_M + L)$ and let S be a general member of $|L|$. Assume that $K_X + 2\mathcal{K}$ is nef and big. Further assume that $d_3 = 2$. Then either*

1. $\chi(\mathcal{O}_S) = 5$, $h = 4$, $d_2 = 6$, $d_1 \geq 9$, $d \geq 11$, *or*
2. $\chi(\mathcal{O}_S) = 4$, $h = 3$, $d_2 = 4$ *and* $(d, d_1) = (11, 7), (12, 8), (14, 8), (16, 8)$.

7.2 The case $d_2 = 6$. The nefness of $K_X + 2\mathcal{K}$ and the ampleness of $\mathcal{K}$ imply nefness and bigness of $2(K_X + 2\mathcal{K}) - K_X$. Thus, by the Kawamata-Shokurov basepoint free theorem, we conclude that $N(2(K_X + 2\mathcal{K}))$ has a section for $N \gg 0$, and hence $N(K_X + 2\mathcal{K})$ has a section for $N \gg 0$, i.e., $K_X + 2\mathcal{K}$ is $\mathbb{Q}$-effective. From

$$(K_X + 2\mathcal{K}) \cdot \mathcal{K}^2 = (3\mathcal{K} - \mathcal{D}) \cdot \mathcal{K}^2 = 3d_3' - d_2' = 3d_3 - d_2 = 0$$

it thus follows that $N(K_X + 2\mathcal{K}) \approx \mathcal{O}_X$ and therefore $-K_X$ is ample. This implies

$$q := q(S)(= q(X)) = 0. \tag{30}$$

From $d_1^2 \geq dd_2 \geq 48$ we get $d_1 \geq 7$. If $d_1 = 7$, then by parity $d \geq 9$, so that $d_1^2 \geq dd_2$ yields to a contradiction. Therefore

$$d_1 \geq 8. \tag{31}$$

From

$$(K_X + \mathcal{K}) \cdot \mathcal{K}^2 = (2\mathcal{K} - \mathcal{D}) \cdot \mathcal{K}^2 = 2d_3' - d_2' = 2d_3 - d_2 < 0$$

we conclude that $h^0(K_X + \mathcal{K}) = 0$. Recalling (25) we have $h^0(K_M + \mathcal{K}_M) = 0$. Then relation (20) gives $4\chi(\mathcal{O}_S) + d_2 = d_3 + 6h$, or

$$2\chi(\mathcal{O}_S) + 2 = 3h. \tag{32}$$

Noether inequality gives $d_2 \geq \chi(\mathcal{O}_S) - 6$, so that $\chi(\mathcal{O}_S) \leq 6$ and hence (32) yields either $\chi(\mathcal{O}_S) = h = 2$, or $\chi(\mathcal{O}_S) = 5$, $h = 4$.

Note that the case $\chi(\mathcal{O}_S) = h = 2$ is not possible. Indeed from $\chi(\mathcal{O}_S) = 2$ and $q = 0$ we have $p_g(S) = 1$. Then we see that $h = h^0(K_M + L) = 2$ is not possible by considering the cohomology sequence associated to the exact sequence

$$0 \to K_M \to K_M + L \to K_S \to 0.$$

Therefore $\chi(\mathcal{O}_S) = 5$, $h = 4$. Since $q = 0$ one has $p_g(S) = 4$. From the double point formula (1.7), 2) we have $d(d-10) + 48 \geq 5d_1$. Recalling (31) we obtain $d \geq 10$.

Assume that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^N$ with $N \geq 6$. Then $d \geq 11$. Indeed, if $d = 10$, Castelnuovo's bound gives $g(\mathcal{L}) \leq \mathrm{Castel}(10, 4) = 9$, and hence we find the contradiction

$$18 \leq d + d_1 = \widehat{d} + \widehat{d}_1 = 2g(\mathcal{L}) - 2 \leq 16.$$

Assume now that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^5$. Since $d \geq 10$, [3, (4.4.1)] applies to say that $\mathcal{M} \cong M$. Thus the double point formula (1.7), 2) for $d = 10$ gives $12\chi(\mathcal{O}_S) = 2d_2 + 5d_1$, whence the numerical contradiction $60 = 12 + 5d_1$. Therefore we conclude that $d \geq 11$.

From $d_1^2 \geq dd_2 \geq 66$ we also get $d_1 \geq 9$. Summarizing, we have the following numerical bounds:

$$d_3 = 2, \;\; d_2 = 6, \;\; d_1 \geq 9, \;\; d \geq 11.$$

Example 7.3 Let us construct an explicit example satisfying the numerical bounds above. Let $p : \mathcal{M} \to \mathbb{P}^3$ be a finite double cover of $\mathbb{P}^3$ branched along a smooth quartic surface B. Then $K_{\mathcal{M}} \approx p^*\mathcal{O}_{\mathbb{P}^3}(-2)$. Let $\mathcal{L} := p^*\mathcal{O}_{\mathbb{P}^3}(3)$, so that $K_{\mathcal{M}} + \mathcal{L} \approx p^*\mathcal{O}_{\mathbb{P}^3}(1)$. Note that $\mathcal{L}$ and $K_{\mathcal{M}} + 2\mathcal{L} \approx p^*\mathcal{O}_{\mathbb{P}^3}(4)$ are very ample (see e.g., [12]). Hence in particular $(\mathcal{M}, \mathcal{L})$ is isomorphic to its own first reduction (M, L). The invariants are

$$d = 2\mathcal{O}_{\mathbb{P}^3}(3)^3 = 54, d_1 = 2\mathcal{O}_{\mathbb{P}^3}(1) \cdot \mathcal{O}_{\mathbb{P}^3}(3)^2 = 18, d_2 = 2\mathcal{O}_{\mathbb{P}^3}(1)^2 \cdot \mathcal{O}_{\mathbb{P}^3}(3) = 6, d_3 = 2\mathcal{O}_{\mathbb{P}^3}(1)^3 = 2.$$

Moreover $h = h^0(K_{\mathcal{M}} + \mathcal{L}) = h^0(p^*\mathcal{O}_{\mathbb{P}^3}(1)) = 4$. To see this, consider the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(1) \to \mathcal{O}_{\mathbb{P}^3}(1)_{|B} \to 0.$$

Then $h^0(\mathcal{O}_{\mathbb{P}^3}(1)_{|B}) = h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$. On $\mathcal{M}$ we have the exact sequence (identifying B with $p^{-1}(B)$)

$$0 \to p^*\mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_B \to 0.$$

Tensoring with $K_{\mathcal{M}} + \mathcal{L}$ and computing the cohomology we obtain

$$h = h^0(K_{\mathcal{M}} + \mathcal{L}) = h^0((K_{\mathcal{M}} + \mathcal{L})_{|B}) = h^0(\mathcal{O}_{\mathbb{P}^3}(1)_{|B}) = 4.$$

Note also that $K_{\mathcal{M}} + \mathcal{L}$ is ample and spanned and gives a two-to-one map (see [12]).

7.4 The case $d_2 = 4$. Let us first prove the following

Claim 7.5 $h^0(K_X + \mathcal{K}) = 0$.

Proof. Assume otherwise. Then from

$$(K_X + \mathcal{K}) \cdot \mathcal{K}^2 = (2\mathcal{K} - \mathcal{D}) \cdot \mathcal{K}^2 = 2d_3' - d_2' = 2d_3 - d_2 = 0$$

we see that $2(K_X + \mathcal{K}) \approx \mathcal{O}_X$. Hence we have a section of $2(K_X + \mathcal{K})$ and hence a section of $K_X + \mathcal{K}$ which doesn't vanish at regular points of X. This implies that $K_X + \mathcal{K} \approx \mathcal{O}_X$.

Thus X is Gorenstein, $q(X) = 0$, $h^0(K_X + \mathcal{K}) = 1$, $h^2(\mathcal{O}_X) = h^3(\mathcal{O}_X) = 0$ by Kodaira vanishing and $\chi(\mathcal{O}_X) = 1$. Since X is Gorenstein we have

$$K_M + \mathcal{K}_M \approx \varphi^*(K_X + \mathcal{K}) + J,$$

where J is an integral divisor. Therefore J is a line bundle since M is smooth. Recalling that $h^0(K_M + \mathcal{K}_M) \leq h^0(K_X + \mathcal{K}) = h^0(\varphi_*(K_M + \mathcal{K}_M)^{**})$ by Lemma (1.11), we conclude from $h^0(K_X + \mathcal{K}) = 1$ that $h^0(K_M + \mathcal{K}_M) = 1$. Thus relation (20) yields

$$2 = 2h^0(K_M + \mathcal{K}_M) = d_3 + 6h - d_2 - 4\chi(\mathcal{O}_S).$$

Since $\chi(\mathcal{O}_S) = h + 1$ from (23), and $d_3 = 2$, $d_2 = 4$, we find

$$3h = 2(\chi(\mathcal{O}_S) + 1).$$

Noether formula gives $4 = d_2 \geq 2\chi(\mathcal{O}_S) - 6$, i.e., $\chi(\mathcal{O}_S) \leq 5$. Recalling that $h \geq 2$ we infer that either $h = 4$, $\chi(\mathcal{O}_S) = 5$, or $h = \chi(\mathcal{O}_S) = 2$. Exactly the same argument as in paragraph (7.2) rules out the case $h = \chi(\mathcal{O}_S) = 2$. Then $d_2 = 2\chi(\mathcal{O}_S) - 6$, so that S is a Horikawa surface. Then from [9, Lemma (1.1)] we know that K_S is spanned since $d_2 = K_S^2$ is even. Thus from the exact sequence

$$0 \to K_M \to K_M + L \to K_S \to 0,$$

by using $h^2(\mathcal{O}_M) = h^1(K_M) = 0$, we infer that $\mathcal{K}_M = K_M + L$ is spanned. Let $\psi : M \to \mathbb{P}^{h-1} = \mathbb{P}^3$ be the surjective map associated to $|\mathcal{K}_M|$. Since $(K_M + L)^3 = d_3 = 2$, the morphism ψ is generically one-to-one. Since $\mathcal{K}_M \approx \varphi^*\mathcal{K}$, ψ factors through the second reduction map φ, $\psi = f \circ \varphi$, $f : X \to \mathbb{P}^3$, and f is finite since $\mathcal{K}$ is ample. We have $\mathcal{K} \approx f^*\mathcal{O}_{\mathbb{P}^3}(1)$ and $h = h^0(\mathcal{K}_M) = h^0(\mathcal{K}) = 4$. Let B be the branch locus of ψ. Then $\mathcal{O}_{\mathbb{P}^3}(B) \approx \mathcal{O}_{\mathbb{P}^3}(2b)$, $b := \deg B$. One has $L \approx \psi^*\mathcal{O}_{\mathbb{P}^3}(a)$ for some integer a, $K_M \approx \psi^*\mathcal{O}_{\mathbb{P}^3}(b-4)$ and hence

$$\psi^*\mathcal{O}_{\mathbb{P}^3}(a + b - 4) \approx \mathcal{K}_M \approx \psi^*\mathcal{O}_{\mathbb{P}^3}(1).$$

Therefore $a + b = 5$. If $b = 1$, then $a = 4$, which gives the contradiction $d_2 = \mathcal{K}_M^2 \cdot L = 2\mathcal{O}_{\mathbb{P}^3}(1)^2 \cdot \mathcal{O}_{\mathbb{P}^3}(3) = 6$.

To conclude the proof of the claim it remains to show that the case $b \geq 3$ does not occur. To see this consider the exact sequences (identifying B with $p^{-1}(B)$)

$$0 \to \mathcal{O}_{\mathbb{P}^3}(5 - 2b) \to \mathcal{O}_{\mathbb{P}^3}(5 - b) \to \mathcal{O}_{\mathbb{P}^3}(5 - 2b)_{|B} \to 0$$

and

$$0 \to L - B \approx \psi^*\mathcal{O}_{\mathbb{P}^3}(5 - 2b) \to L \to L_B \to 0.$$

If $b \geq 3$ then $5 - 2b < 0$, so that

$$h^0(\mathcal{O}_{\mathbb{P}^3}(5-b)) = h^0(\mathcal{O}_{\mathbb{P}^3}(5-2b)_{|B}) = h^0(L_B) \geq h^0(\mathcal{O}_{\mathbb{P}^3}(5-b)).$$

Thus $h^0(L) = h^0(\mathcal{O}_{\mathbb{P}^3}(5-b))$, i.e., all sections of L pullback from $\mathbb{P}^3$. Therefore $|L|$ doesn't give a birational map. This contradicts the fact that L is very ample outside of a finite set of points. $\square$

Recalling also Lemma (1.11), from now on we can thus assume that $h^0(K_X + \mathcal{K}) = h^0(K_M + \mathcal{K}_M) = 0$. Then since $d_3 = 2$, $d_2 = 4$, from relation (20) we find

$$2\chi(\mathcal{O}_S) + 1 = 3h. \tag{33}$$

Noether formula $d_2 \geq 2\chi(\mathcal{O}_S) - 6$ gives again $\chi(\mathcal{O}_S) \leq 5$. Thus, recalling that $h \geq 2$, (33) gives

$$\chi(\mathcal{O}_S) = 4, \quad h = 3.$$

From $d_3 d_1 = 2d_1 \leq d_2^2 = 16$ we have $d_1 \leq 8$. Then $dd_2 = 4d \leq d_1^2 \leq 64$ yields $d \leq 16$. From (1.7), 2) one has $d(d-10)+40 \geq 5d_1$. Since $d_1 \geq 6$, this implies $d \geq 9$. If $d = 9$, then $d_1 \geq 7$ by parity and hence the inequality above gives again a contradiction. Therefore $d \geq 10$. Then $d_1^2 \geq dd_2 \geq 40$ yields $d_1 \geq 7$. Thus we have

$$d_3 = 2, \quad d_2 = 4, \quad 8 \geq d_1 \geq 7, \quad 16 \geq d \geq 10. \tag{34}$$

Assume $d = 10$. Then $\widehat{d} \leq 10$ and $d_1 = 8$ by parity. Suppose that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^N$ with $N \geq 6$. Then $g(\mathcal{L}) \leq \mathrm{Castel}(10,4) = 9$. This gives the numerical contradiction

$$16 \geq 2g(\mathcal{L}) - 2 = \widehat{d} + \widehat{d}_1 = d + d_1 \geq 18.$$

Thus if $d = 10$ we see that $|\mathcal{L}|$ embeds $\mathcal{M}$ in $\mathbb{P}^5$. By combining Lemma (1.10) and Proposition (1.5) we conclude that $(\mathcal{M},\mathcal{L}) \cong (M,L)$, $d = \widehat{d} = 10$, $d_1 = \widehat{d}_1 = 8$. Since there are no 3-folds in $\mathbb{P}^5$ of degree 10 with $d_1 = 8$ (see [5, Chapter 6]), we conclude that $d \geq 11$.

Recalling (34) and by using parity, we have the following possibilities for d, d_1. Precisely, $(d, d_1) = (11,7), (12,8), (14,8), (16,8)$. Note that if $d = 13$, then $d_1 = 7$ by parity and $d_1^2 \geq dd_2 \geq 52$ gives a contradiction. Similarly, if $d = 15$, then $d_1 = 7$, and we contradict again $d_1^2 \geq dd_2$.

8 A lower bound for the number of sections of the second adjoint bundle

Let $\mathcal{M}$ be a smooth connected n-fold polarized by a very ample line bundle $\mathcal{L}$, $n \geq 3$. Assume that $(\mathcal{M},\mathcal{L})$ is of log-general type. Let $\pi : (\mathcal{M},\mathcal{L}) \to (M,L)$, $\varphi : (M,L) \to (X,\mathcal{D})$, $\mathcal{K} \approx K_X + (n-2)\mathcal{D}$, be the first and the second reduction of $(\mathcal{M},\mathcal{L})$ as in (1.4). Let $h := h^0(K_{\mathcal{M}} + (n-2)\mathcal{L})$. In [7, (1.2), (3.3)] we showed that $h \geq 2$ under the log-general type assumption and moreover, if $\kappa(\mathcal{M}) \geq 0$, that $h \geq 5$ with equality only if $n = 3$ and $(\mathcal{M},\mathcal{L})$ a degree 5 hypersurface in $\mathbb{P}^4$. In this section we prove that for $n \geq 5$, under the further assumption that $K_X + (n-3)\mathcal{K}$ is nef and big (see §2) and by using the pluridegrees bounds stated in the previous sections, one has the general lower bound

$$h \geq 5. \tag{35}$$

Let $\mathcal{V}$ be the smooth 5-fold section obtained as transversal intersection of $n-5$ general members of $|\mathcal{L}|$ and let $\mathcal{L}_{\mathcal{V}}$ be the restriction of $\mathcal{L}$ to $\mathcal{V}$. The Kodaira vanishing theorem yields

$$h^0(K_{\mathcal{M}}+(n-2)\mathcal{L}) \geq h^0(K_{\mathcal{V}}+3\mathcal{L}_{\mathcal{V}}).$$

Thus, since we have better bounds as soon as n increases, we can assume $n=5$ and (35) will be a consequence of the following result.

Proposition 8.1 *Let $\mathcal{M}$ be a smooth connected 5-fold polarized by a very ample line bundle $\mathcal{L}$. Assume that $(\mathcal{M},\mathcal{L})$ is of log-general type. Let $\pi:(\mathcal{M},\mathcal{L})\to(M,L)$, $\varphi:(M,L)\to(X,\mathcal{D})$, $\mathcal{K}\approx K_X+3\mathcal{D}$, be the first and the second reduction of $(\mathcal{M},\mathcal{L})$. Let d_j's, $j=0,1,2,3,4,5$, $d_0=d$, be the pluridegrees of (M,L). Assume that $K_X+2\mathcal{K}$ is nef and big. Then $h:=h^0(K_{\mathcal{M}}+3\mathcal{L})\geq 5$.*

Proof. Let S be the smooth surface obtained as transversal intersection of $n-2=3$ general members of $|L|$. Then S is a minimal surface of general type. Since $d_2\geq 24$ by (4.1), from Miyaoka's inequality (2) we infer that

$$\chi(\mathcal{O}_S)\geq 3.$$

Assume $\chi(\mathcal{O}_S)=3$. Then since $d_1\geq 19$, $d_2\geq 24$ from (4.1), the double point formula (1.7), 2) gives $d(d-10)\geq 107$, and hence $d\geq 17$. Therefore (15) gives

$$d\geq 17,\ \ d_1\geq 21,\ \ d_2\geq 25,\ \ d_3\geq 30,\ \ d_4\geq 36,\ \ d_5\geq 43.$$

If $d_2=25$, then $d_3\geq 31$ by parity, so that we find the contradiction $d_2^2=625\geq d_1d_3\geq 651$. Therefore $d_2\geq 26$ and hence $d_1^2\geq dd_2\geq 442$ yields $d_1\geq 22$. Thus (1.7), 2) gives $d(d-10)\geq 126$, or $d\geq 18$. If $d_2=26$, then $d_3\geq 31$ by (15) and hence $d_3\geq 32$ by parity, so that $d_2^2\geq d_1d_3\geq 704$ gives $d_2\geq 27$. Then $d_1^2\geq dd_2=486$ yields $d_1\geq 23$, as well as $d_3\geq 32$, $d_4\geq 37$, $d_5\geq 44$ by (15). Summarizing, we have

$$d\geq 18,\ \ d_1\geq 23,\ \ d_2\geq 27,\ \ d_3\geq 32,\ \ d_4\geq 37,\ \ d_5\geq 44.$$

From $d_2^2\geq d_1d_3\geq 736$ we get $d_2\geq 28$. Thus by (15) we obtain $d_3\geq 34$, $d_4\geq 41$, $d_5\geq 49$ and therefore $d_3^2\geq d_2d_4\geq 1184$ yields $d_3\geq 35$. Again, (15) gives $d_4\geq 42$, $d_5\geq 50$. From $d_2^2\geq d_1d_3\geq 805$ we find $d_2\geq 29$. If $d_1=23$ then $d\geq 19$ by parity, so that $d_2\leq d_1^2/d$ leads to a contradiction. Thus we have

$$d\geq 18,\ \ d_1\geq 24,\ \ d_2\geq 29,\ \ d_3\geq 35,\ \ d_4\geq 42,\ \ d_5\geq 50. \tag{36}$$

Note that Miyaoka inequality $\chi(\mathcal{O}_S)>d_2/9$ yields $\chi(\mathcal{O}_S)\geq 4$. Let $\mathcal{V}$ be the smooth 3-fold section of $\mathcal{M}$ obtained as transversal intersection of two general members of $|\mathcal{L}|$ and let $\mathcal{L}_{\mathcal{V}}$ be the restriction of $\mathcal{L}$ to $\mathcal{V}$. Let $h':=h^0(K_{\mathcal{V}}+\mathcal{L}_{\mathcal{V}})$. Then by Kodaira vanishing one has $h\geq h'$. Thus it suffices to show that $h'\geq 5$.

By using (36), and recalling that d_j equals the pluridegrees $d_j(\mathcal{V})$ of $(\mathcal{V},\mathcal{L}_{\mathcal{V}})$, $j=0,1,2,3$, Tsuji's inequality (1.9) reads $h'\geq 4$. Clearly we can assume $\chi(\mathcal{O}_S)=4$ since otherwise (1.9) gives

$$h'\geq\frac{5}{2}+1+\frac{35}{64},\ \text{ i.e., }\ h'\geq 5,$$

and we would be done.

Note also that we can assume $h^0(K_{\mathcal{V}}) = 0$, since otherwise a section of $\Gamma(K_{\mathcal{V}})$ gives an embedding $\Gamma(\mathcal{L}_{\mathcal{V}}) \hookrightarrow \Gamma(K_{\mathcal{V}} + \mathcal{L}_{\mathcal{V}})$ and hence $h' := h^0(K_{\mathcal{V}} + \mathcal{L}_{\mathcal{V}}) \geq 5$. Then double point formula (1.8) yields

$$44h' + 58\chi(\mathcal{O}_S) + 4 \geq 12d_2 + 17d_1 + d_3 + (20 - d)d,$$

or, with $h' = \chi(\mathcal{O}_S) = 4$, $(d - 20)d \geq 269$. This implies $d \geq 30$ and therefore $d_1 \geq 36$, $d_2 \geq 43$, $d_3 \geq 51$ by (15). Thus Tsuji's inequality (1.9) gives

$$h \geq 2 + \frac{36}{24} + \frac{51}{64}, \quad \text{or} \quad h \geq 5.$$

Q.E.D.

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