Topogonov's Theorem for Metric Spaces II

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In this note we correct some errors in "Toponogov's Theorem for metric spaces" (henceforth referred to as [P3]), prove a "rigidity" theorem, and generalize Toponogov's Maximal Diameter Theorem.

We use the same notation and references as in [P3], and refer to results in [P3] by number only (e.g., "Lemma 2" refers to Lemma 2, [P3]). The most important error in [P3] is the omission of geodesic completeness from the hypothesis of the main theorem, which should should have been stated:

Theorem A. *If X is geodesically complete of curvature $\geq k$, then every proper triangle in X is A1 and every proper wedge in X is A2.*

Definition. We say that a wedge $(\gamma_{ab}, \beta_{ac})$ is *A2 with equality* if there is a representative wedge $(\overline{\gamma}_{AB}, \overline{\beta}_{AC})$ in S_k (i.e., whose sides are minimal with $L(\overline{\gamma}_{AB}) = L(\gamma_{ab})$, $L(\overline{\beta}_{AC}) = L(\beta_{ac})$, $\alpha(\overline{\gamma}_{AB}, \overline{\beta}_{AC}) = \alpha(\gamma_{ab}, \beta_{ac})$) and $d(B, C) = d(b, c)$.

Using results of [P2] one can formulate rigidity theorems analogous to the rigidity part of Toponogov's theorem in the

Riemannian case, but the statements are long, and at present we have no applications for a theorem stronger than what we give below.

Theorem R. *Suppose X is geodesically complete of curvature* $\geq k$ *and* (γ_1, γ_2) *is proper and A2 with equality, with representative* $(\bar{\gamma}_1, \bar{\gamma}_2)$ *in* S_k. *Let* $L_i = L(\gamma_i)$, $i = 1, 2$. *Then for all* $0 \leq t \leq L_2$, $d(\gamma_1(L_1), \gamma_2(t)) = d(\bar{\gamma}_1(L_1), \bar{\gamma}_2(t))$. *In addition, if* γ_2 *is minimal,* $d(\gamma_1(s), \gamma_2(t)) = d(\bar{\gamma}_1(s), \bar{\gamma}_2(t))$ *for all* $0 \leq s \leq L_1$.

Theorem D. *If* $k > 0$, *X is geodesically complete of curvature* $\geq k$ *and dia* $(X) = \pi/\sqrt{k}$, *then X is isometric to* S^n_k *for some n.*

Before proving Theorems R and D, we give corrections to [P3]. In the statement of Lemma 5, "there exists a $\chi > 0$" should be replaced by "for all sufficiently small $\chi > 0$." Rather than giving a list of corrections for the proof of Theorem A we will simply give below a simplified and corrected proof in its entirety. What follows should replace the arguments in [P3] beginning with the last paragraph on page 6 to the beginning of the proof of Theorem C on page 11. We assume throughout this proof that X is geodesically complete (although this is only directly used in Step 2).

For $0 < D < \pi/\sqrt{k}$, fix a closed ball $B = \overline{B}(p, D) \subset X$ and a cover U of $\overline{B}(p, 2D)$ by regions of curvature $\geq k$, and let $\chi(U) < D$ be as in Lemma 5 and also less than 1/12 of a Lebesque number of U. Let $\tau(U)$ small enough that if $c\overline{\alpha}$, $\overline{\gamma}$ are unit geodesics in S_k with $\alpha(\overline{\alpha}, \overline{\gamma}) \leq \tau(U)$, then for all $0 \leq t \leq D$, $d(\overline{\alpha}(t), \overline{\gamma}(t)) \leq \chi(U)$. If $\alpha, \beta : [0, 1] \to B$ are minimal curves starting at p, we call a proper triangle (α, γ, β) *p-based*. A p-based triangle (α, γ, β) is U-*thin* if $\alpha(\alpha, \beta) \leq \tau(U)$ and γ is minimal. At present we do not require that γ lie in B in either definition, but $\chi(U) < D$ implies γ lies in B(p, 2D). Consider the following statements:

S1(n,m). *If (α, γ, β) is* U-*thin such that $(n-1)\cdot\chi(U) \leq L(\alpha) \leq n\cdot\chi(U)$ and $(m-1)\cdot\chi(U) \leq L(\beta) \leq m\cdot\chi(U)$, then (α, γ, β) is A1.*

S2(n,m). *If (α, γ, β) is* U-*thin such that $(n-1)\cdot\chi(U) \leq L(\alpha) \leq n\cdot\chi(U)$ and $(m-1)\cdot\chi(U) \leq L(\beta) \leq m\cdot\chi(U)$, then (α, β) is A2.*

S3(n). *If (α, γ, β) is p-based and lies in $\overline{B}(p, n\cdot\chi(U)$, then (α, γ, β) is A1.*

Note that by monotonicity S1(n,m) and S3(n) state equivalently that (α, γ) and (β, γ) are A2. S1(6,6), S2(6,6), and S3(6) are true by the way $\chi(U)$ was chosen. We will prove by induction that S3(n) holds for $n \leq (D-3\chi) / \chi$.

Step 1. *S1(n,n) and S2(n,n) imply S2(n, n+1).*

Proof. Fix a U-thin triangle (α, γ, β) such that $n \cdot \chi(U) \leq L(\alpha) \leq (n+1) \cdot \chi(U)$ and $(n-1) \cdot \chi(U) \leq L(\beta) \leq n \cdot \chi(U)$. Let q lie on α such that $d(p, q) = L(\beta)$, let $x = \alpha(1)$, $y = \beta(1)$ and η be minimal from y to q. If ν is the segment of α from p to q, we obtain from S2(n,n) that (β, ν) is A2 and from S1(n,n) that (ν, η) is A2. S2(n,n) implies $\text{dia}\{x, y, q\} \leq 3\chi(U)$; if ζ is the segment of α from q to x we have that both (η, ζ) and (ζ, γ) are A2, and that (α, β) is A2 follows from Lemma 1. □

Step 2. *S3(n) implies that if ω is minimal from p to a point $a \in B(p, (n-1) \cdot \chi(U))$ and ξ is minimal starting at a with $L(\xi) \leq 4\chi(U)$, then (ω, ξ) is A2.*

Proof. Let $R' = L(\omega)$, assume both ω and ξ are unit, and let $x = \xi(L(\xi))$. Choose a representative $(\bar{\omega}, \bar{\xi})$ in S_k, denoting the corresponding points with capitals. Let $\bar{\mu}$ be unit minimal from P to X, $R = \min\{R', L(\bar{\mu})\}$, and $\bar{\kappa}$ be minimal from A to $\bar{\mu}(R)$. Since $n \leq (D-3\chi) / \chi$, $L(\bar{\omega}) + L(\bar{\xi}) \leq D$, and by Lemma 5, for all s, $d(P, \bar{\kappa}(s)) < R + \chi(U) \leq n \cdot \chi(U)$. For any sufficiently small $\delta > 0$, by Lemma 2 and geodesic completeness there exists a geodesic $\kappa : [0, 1] \to X$ starting at a of length $L = L(\bar{\kappa})$ with $|\alpha(\kappa, \omega) - \alpha(\bar{\kappa}, \bar{\omega})| < \delta$ and $|\alpha(\kappa, \xi) - \alpha(\bar{\kappa}, \bar{\xi})| < \delta$. For small enough δ, S3(n) implies that $d(p, \kappa(s)) < n \cdot \chi(U)$ for all s and (κ, ω) is A2. On the other hand, by the triangle inequality $L(\bar{\kappa}) \le 8\chi(U)$ and dia $\{\kappa(1), a, x\} < 12\chi(U)$; thus (κ, ξ) is A2. Lemma 4 now implies (ω, ξ) is A2. □

Step 3. *S1(m,m), S2(m,m), for all m ≤ n, and S3(n) imply S1(n,n+1).*

Proof. Let (α, γ, β) be as above. The proof that (α, γ) is A2 is similar to the argument in Step 1. Let a be the point on β such that $d(a, y) = \chi(U)$, $R = d(p, a)$, ω denote the segment of β from p to a and ξ be minimal from a to x. By the triangle inequality (and the fact that $\alpha(\alpha, \beta) \le \tau(U)$) $L(\xi) \le 4\chi(U)$ and Step 2 implies (ω, ξ) is A2. By a proof similar to that of Step 1, S1(n,n) and S2(n,n) imply (α, ω) is A2. If λ denotes the segment of β from a to y, (ξ, λ, γ) is also A1, and the proof is complete by Lemma 1. □

Step 4. *S1(n,n+1) and S2(n,n+1) imply S1(n+1,n+1) and S2(n+1,n+1).*

Proof. This is a straightforward application of Lemma 1. □

Step 5. *S1(m,m), S2(m,m), for all m ≤ n+1, and S3(n) imply S3(n+1,n+1) (and the induction is complete).*

Proof. Let (α, γ, β) be p-based, with

$\gamma : [0, 1] \to \overline{B}(p, (n+1)\cdot\chi(U))$. We first claim the following: If ζ is minimal from p to $q = \gamma(t)$, for some t, $t_i \to t$ and η_i is minimal from p to $\gamma(t_i)$, then for all sufficiently large i, $(\zeta_i, \gamma_i, \eta_i)$ is A1, where γ_i is γ restricted to the interval between t_i and t. By using two subsequences, if necessary, we can assume $\lim_{i\to\infty} \alpha(\eta_i, \zeta)$ is either 0 or $2\epsilon > 0$. In the first case the proof is complete by S1(m,m) for $m \le n+1$. In the second case $\alpha(\eta_i, \zeta) > \epsilon$ for all large i. Choosing a subsequence if necessary we can find a minimal η from p to q such that $\alpha(\eta_i, \eta) \to 0$; in particular, (η_i, γ_i) is A2 for all sufficiently large i by S1(m,m) for $m \le n+1$. On the other hand, let a be the point on ζ such that $d(a, q) = 2\cdot\chi(U)$, ω denote the segment of ζ from p to a, ν that from a to q, and μ_i be minimal from a to $\gamma(t_i)$. Since $L(\omega) + L(\mu_i) \to L(\eta_i)$, if $(\overline{\zeta}, \overline{\eta}_i)$ represents (ζ, η_i) in S_k then $\alpha(\overline{\zeta}, \overline{\eta}_i) \to 0$. Now $\alpha(\eta_i, \zeta) > \epsilon$ implies (ζ, η_i) is A2 for large i. By Step 2, (ω, μ_i) is A2. Since γ_i is minimal for large enough i and (μ_i, ν), (ν, γ_i) are A2, the proof of the claim is complete by Lemma 1.

For $s > 0$, let γ_s denote $\gamma|_{[0,s]}$, and denote by A1(s) the statement: *for every minimal β_s from p to $\gamma(s)$, $(\alpha, \gamma_s, \beta_s)$ is A1*. The above claim implies that A1(δ) is true for sufficiently small $\delta > 0$, and the claim and Lemma 1 prove that if A1(T) is true for some T, then A1(T+δ) is true. Likewise, if A1(s) is true for all $s < T$ then A1(T) is true; it follows that A1(T) holds for all T. □

Proof of Theorem A. Step 5 implies that every p-based triangle in $\overline{B}(p, D-3\chi(U))$ is A1. Letting $\chi(U) \to 0$ we conclude that every proper triangle (α, γ, β) in X such that $d(\alpha(0), \gamma) < \pi/\sqrt{k}$ is A1. The proof is now complete for $k \le 0$, and is easily completed for $k > 0$ using a limit argument and Lemma 1. □

Before proving Theorem R we reconcile the conclusion of Theorem A with our original definition of curvature bounded below (in the sense of Rinow, cf. [P2], [R]).

Proposition 1. *If X is geodesically complete of curvature $\ge k$ then all of X is a region of curvature $\ge k$.*

Proof. By definition, we need to show that if $(\gamma_1, \gamma_2, \gamma_3)$ is a triangle of minimal curves in X represented by $(\overline{\gamma}_1, \overline{\gamma}_2, \overline{\gamma}_3)$ in S_k, then $d(\gamma_1(s), \gamma_3(t)) \ge d(\overline{\gamma}_1(s), \overline{\gamma}_3(t))$ for all s and t. We assume all curves are unit parameterized and s, t > 0. By monotonicity we may show equivalently that if γ is minimal from $\gamma_1(s)$ to $\gamma_3(t)$, $\alpha = \gamma_1|_{[0,s]}$, $\beta = \gamma_3|_{[0,t]}$, $(\overline{\alpha}, \overline{\gamma}, \overline{\beta})$ represents (α, β, γ) in S_k, and $\overline{\mu}$ and $\overline{\nu}$ are extensions of $\overline{\alpha}$ and $\overline{\beta}$ of length $a = L(\gamma_1)$ and $b = L(\gamma_3)$, respectively, then $d(\gamma_1(a), \gamma_3(b)) \le d(\overline{\mu}(a), \overline{\nu}(b))$. Suppose first that $t = b$ and let $\zeta = \gamma_1|_{[s,a]}$ and $\overline{\zeta} = \overline{\mu}|_{[s,a]}$. By A1, $\alpha(\overline{\alpha}, \overline{\gamma}) \le \alpha(\alpha, \gamma)$, so $\alpha(\overline{\gamma}, \overline{\zeta}) \ge \alpha(\gamma, \zeta)$ and by A2 $d(\gamma_1(a), \gamma_3(b)) \le d(\overline{\mu}(a), \overline{\nu}(b))$. Now suppose $t < b$. Let η

be minimal from $\gamma_3(b)$ to $\alpha(s)$ and let $(\bar{\alpha}, \bar{\eta}, \bar{\gamma}_3)$ be a representative in S_k. Then if $\bar{\kappa}$ is the extension of $\bar{\alpha}$ of length a, by the above argument and monotonicity, $d(\bar{\kappa}(a), \bar{\eta}(b)) \geq d(\gamma_1(a), \gamma_3(b))$ and $d(\bar{\alpha}(s), \bar{\gamma}_3(t)) \leq d(\alpha(s), \gamma_3(t))$. The proposition now follows from monotonicity. □

Proof of Theorem R. The proof when $\alpha(\gamma_1, \gamma_2) = 0$ or 1 is trivial; we assume otherwise below. The second statement of Theorem R follows immediately from Proposition 1 and A2. If γ_2 is not minimal, partition the domain of γ_2 into finitely many intervals $[t_i, t_{i+1}]$ such that the restriction α_i of γ_2 to $[t_i, t_{i+1}]$ is minimal. Let β_i be minimal from $\gamma_1(L_1)$ to $\alpha_i(t_i)$ (e.g. $\beta_1 = \gamma_1$). Then by an argument similar to the proof of Lemma 1 we see that (β_i, α_i) is A2 with equality for all i, and that if $\bar{\beta}_i$ is minimal in S_k from $\bar{\gamma}_1(L_1)$ to $\bar{\gamma}_2(t_i)$, then $L(\bar{\beta}_i) = L(\beta_i)$. The proof is now finished by the special case proved above. □

Proof of Theorem D. By Corollary B, we can find points p, $q \in X$ such that $d(p, q) = \pi/\sqrt{k}$. Choosing a minimal curve from p to q we can apply A2 (via Theorem A) to conclude that every geodesic of length $\pi/\sqrt{k}$ starting at p is minimal from p to q, and geodesics starting at q behave likewise. Therefore the exponential map (cf. [P2]) is a homeomorphism on $B(0, \pi/\sqrt{k}) \subset T_p = R^n$ (and X is homeomorphic to a sphere). We identify T_p with the tangent space at a point on the sphere, and lift the metric of the sphere to $B(0, \pi/\sqrt{k})$. It now suffices to prove that the exponential map is an isometry, i.e., by Theorem R, if α, β are minimal from p to q then $(\alpha|_{[0,t]}, \beta|_{[0,t]})$ is A2 with equality for all large enough $t < \pi/\sqrt{k}$. Using geodesic completeness we extend α to a geodesic γ passing through q and returning to p. Then γ is minimal on any interval [a, b], where $a = c - \epsilon$, $b = c + \epsilon$, $c = \pi/\sqrt{k}$, and small enough $\epsilon > 0$. If $\eta = \gamma|_{[0,a]}$ and $\nu = -\gamma|_{[b,2c]}$ (i.e. with parameterization reversed), then by A1, $\alpha(\eta, \nu) = \pi$ (i.e., γ is a closed geodesic). Thus (η, ν) is A2 with equality. Since $\alpha(\alpha, \beta) + \alpha(\beta, -\nu) = \pi$, from the triangle inequality and A2 we obtain the desired conclusion. □

We do not know of a counterexample to Theorem A with geodesic completeness removed from the hypothesis; however, the diameter theorem obviously does not hold in this case--e. g. a hemisphere. For a more interesting example, one can "suspend" RP^n (with the metric of constant curvature 1) by attaching two "endpoints" to the warped product, using the sine function, of RP^n and $[0, \pi]$. A simple argument due to K. Grove shows that the resulting space X satisfies the conclusion of Theorem A with $k = 1$. On the other hand, dia $X = \pi$, but X is not a manifold, let alone a sphere. Of course, X is not geodesically complete at the "endpoints." In fact, from Theorem A, [P1] (since the endpoints are codimension 2 they cannot form a boundary), and Theorem D we obtain the following theorem, where $S_{sine}X$ denotes the suspension described above:

Theorem S. *If X is a complete Riemannian manifold of sectional curvature ≥ 1 then the following are equivalent:*

a) $S_{sine}X$ has curvature $\leq K$ for some K,

b) $S_{sine}X$ is geodesically complete, and

c) X is isometric to a standard sphere.