BETTI NUMBERS OF HYPERSURFACES AND DEFECTS OF LINEAR SYSTEMS

by

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Let $\underline{w} = (w_0, \dots, w_n)$ be a set of integer positive weights and denote by S the polynomial ring $\mathbb{C}[x_0, \dots, x_n]$ graded by the conditions $\deg(x_i) = w_i$ for $i = 0, \dots, n$. For any graded object M, let M_k denote the homogeneous component of degree k. Let $f \in S_N$ be a weighted homogeneous polynomial of degree N with respect to $\underline{w}$.

Let V be the hypersurface defined by $f = 0$ in the weighted projective space

$$\mathbb{P}(\underline{w}) = \text{Proj}\, S = \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^*$$

where the $\mathbb{C}^*$–action on $\mathbb{C}^{n+1}$ is defined by $t \cdot x = (t^{w_0} x_0, \dots, t^{w_n} x_n)$ for $t \in \mathbb{C}^*$, $x \in \mathbb{C}^{n+1}$. Assume that the singular locus $\Sigma(f)$ of f is 1–dimensional, namely

$$\Sigma(f) = \{x \in \mathbb{C}^{n+1} ; df(x) = 0\} = \{0\} \cup \bigcup_{i=1,s} \mathbb{C}^* a_i$$

for some points $a_i \in \mathbb{C}^{n+1}$, one in each irreducible component of $\Sigma(f)$.

Let G_i be the isotropy group of a_i with respect to the $\mathbb{C}^*$–action and let H_i be a small G_i–invariant transversal to the orbit $\mathbb{C}^* a_i$ at the point a_i. The isolated hypersurface singularity $(Y_i,a_i) = (H_i \cap f^{-1}(0),a_i)$ is called the transversal singularity of f along the branch $\overline{\mathbb{C}^* a_i}$ of the singular locus $\Sigma(f)$. Note that (Y_i,a_i) is in fact a G_i–invariant singularity.

The hypersurface V is a V–manifold (i.e. has only quotient singularities [8]) at all points, except at the points a_i where V has a hyperquotient singularity $(Y_i/G_i,a_i)$ in the sense of M. Reid [15].

In this paper we discuss an effective procedure to compute the Betti numbers $b_j(V) = \dim H^j(V)$ ($\mathbb{C}$ coefficients are used throughout) for such a weighted projective hypersurface V. It is known that only $b_{n-1}(V)$ and $b_n(V)$ are difficult to compute and that the Euler characteristic $\chi(V)$ can be computed (conjecturally in all, but surely in most of the interesting cases!) by a formula involving only the weights $\underline{w}$, the degree N and some local invariants of the G_i–singularities (Y_i,a_i), see [6], Prop. 3.19. Hence it is enough to determinee $b_n(V)$.

On the other hand, it was known since the striking example of Zariski involving sextic curves in $\mathbb{P}^2$ having six cusps situated (or not) on a conic [25], that $b_n(V)$ is a very subtle invariant depending not only on the data listed above for $\chi(V)$ but also on the position of the singularities of V in $\mathbb{P}(\underline{w})$.

In the next three special cases the determination of $b_n(V)$ has led to beautiful and mysterious (see H. Clemens remark in the middle of p. 141 in [2]) relations with the dimension of certain linear systems $\mathscr{A}$ of homogeneous polynomials vanishing at the singular set $\Sigma = \{a_1, \dots ,a_s\}$ of V :

(i) Some cyclic coverings of $\mathbb{P}^2$ ramified over a curve $B : b = 0$ (H. Esnault [12]). In fact the object of study in [12] are the Betti numbers of the associated Milnor fiber $F : b - 1 = 0$ in $\mathbb{C}^3$, but it is easy to see that they are completely determined by the Betti numbers of

$\overline{F}$, the closure of F in $\mathbb{P}^3$. And the closure $\overline{F}$ is a cyclic covering of $\mathbb{P}^2$ of degree $\deg B$ ramified over B . Beside several implicit results, one finds in [12] an explicit treatment of the Zariski example mentioned above.

(ii) Double coverings of $\mathbb{P}^3$ ramified over a surface $B : b = 0$ having only nodes as singularities (H. Clemens [2]). By a node we mean an A_1–singularity of arbitrary dimension. Note that such a covering is defined by the equation $b - t^2 = 0$ in the weighted projective space $\mathbb{P}(1, \dots ,1,e)$ with $2e = \deg B$ [7].

(iii) Odd dimensional hypersurfaces $X \subset \mathbb{P}^{2m}$ having only nodes as singularities (T. Schoen [17], J. Werner [24]).

In our paper we show that such relations exist without any restriction on the transversal singularities (Y_i,a_i) . The general answer is however not an obvious extension of the above special cases, i.e. the linear systems which occur are not defined by some (higher order) vanishing conditions on Σ , but by some subtle conditions depending on fine invariants of the singularities, i.e. the MHS (mixed Hodge structure) on the local cohomology groups $H^n_{a_i}(Y_i)$ [20]. Unlike the authors mentioned above, we do not use here the resolution of singularities (which is quite difficult to control in dimension ≥ 3) , but we essentially work on the complement $U = \mathbb{P}(\underline{w})\setminus V$, which is an affine V–variety and compute everything in terms of differential forms on U in the spirit of [13].

In this way we get in fact more than $b_n(V)$, namely we obtain a procedure to compute all the mixed Hodge numbers $h^{p,q}(H^n(V))$. See also Remark (2.7).

Let $F : f-1=0$ be the Milnor fiber of f in $\mathbb{C}^{n+1}$. Then F is a smooth affine hypersurface and $\tilde{H}^k(F) = 0$ except for $k = n-1$, n.

Moreover, one has again a "simple" formula computing the Euler characteristic $\chi(F)$ in terms of $\underline{w}$, N and the singularities (Y_i,a_i), [6], Prop. 3.19. Hence it is enough to compute $b_{n-1}(F)$. And the results described in this paper combined with some results in [6] allow one to compute not only $b_{n-1}(F)$, but also all the Hodge numbers $h^{p,q}(H^{n-1}(F))$, as explained in Corollary (3.6) below in the special case when all the transversal singularities are of type A_1. For related computations of Betti numbers of Milnor fibers of non isolated singularities see Siersma [18] and van Straten [22].

It will turn out that in order to get very explicit results the assumption that the transversal singularities (Y_i,a_i) are weighted homogeneous is quite helpful. In particular, we establish several explicit formulas as in the special cases (i)–(iii) above in the last section of our paper.

During this paper we recall and use some of our results in [6]. But all the results in this area should perhaps be regarded as attempts to understand and to generalize Griffiths fundamental work in [13].

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§ 1. A global and a local spectral sequence

Since $U = \mathbb{P}(\underline{w}) \setminus V$ is an affine V–variety, it follows by (a slightly more general version of) Grothendieck Theorem [14], [21] that the cohomology of U can be computed using the deRham complex $A^{\cdot} = H^{0}(U, \Omega^{\cdot}_{U})$, where $\Omega^{\cdot}_{U}$ denotes the sheaves complex of algebraic differential forms on U.

The complex $A^{\cdot}$ has a polar filtration defined as follows

(1.1) $$F^{s}A^{j} = \{\omega \in A^{j} ;\ \omega \text{ has a pole along } V \text{ of order at most } j-s\}$$

for $j-s \geq 0$ and $F^{s}A^{j} = 0$ for $j-s < 0$.

By the general theory of spectral sequences, the filtration F^{s} gives rise to an E_1–spectral sequence $(E_r(U), d_r)$ converging to $H^{\cdot}(U)$. For more details see [6] and also H. Terao [23].

Let $F^{s}H^{\cdot}(U) = \operatorname{im}\{H^{\cdot}(F^{s}A^{\cdot}) \longrightarrow H^{\cdot}(A) = H^{\cdot}(U)\}$ be the filtration induced on $H^{\cdot}(U)$ by the polar filtration on $A^{\cdot}$. Note that on the cohomology algebra $H^{\cdot}(U)$ one has also the canonical (mixed) Hodge filtration F^{s}_{H} constructed by Deligne [3]. It is not difficult to prove the next result, see [6], Theorem (2.2).

(1.2) Proposition

One has $F^{s}H^{\cdot}(U) \supset F^{s+1}_{H}H^{\cdot}(U)$ for any s and $F^{0}H^{\cdot}(U) = F^{1}_{H}H^{\cdot}(U) = H^{\cdot}(U)$.

For an example where the above inclusion is strict we refer to [6], (2.6).

Since we shall be concerned especially with $H^{n}(U)$, we recall the explicit description of A^{n}, given by Griffiths in the homogeneous case [13] and by Dolgachev in the weighted homogeneous case [8]. Let Ω^k denote the S–module of algebraic differential k–forms on $\mathbb{C}^{n+1}$, graded by the condition $\deg(x_i) = \deg(dx_i) = w_i$ for $i = 0, \dots, n$. Consider the differential n–form $\Omega \in \Omega^n_w$ with $w = w_0 + \dots + w_n$ given by

$$(1.3) \qquad \Omega = \sum_{i=0,n} (-1)^i w_i x_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n .$$

Then any element $\omega \in A^n$ may be written in the form

$$(1.4) \qquad \omega = \frac{h\,\Omega}{f^t} \text{ for some } h \in S_{tN-w}$$

and, if h is not divisible by f, then t is precisely the order of the pole of ω along V.

Next we consider a similar spectral sequence, but associated this time to a (local) hypersurface singularity. Let $g : (\mathbb{C}^n,0) \longrightarrow (\mathbb{C},0)$ be an analytic function germ and let $(Y,0) = (g^{-1}(0),0)$ be the associated hypersurface singularity. Let $\Omega^{\cdot}_g$ denote the localization of the stalk at the origin of the analytic de Rham complex for $\mathbb{C}^n$ with respect to the multiplicative system $\{g^s, s \geq 0\}$.

Choose $\varepsilon > 0$ small enough such that Y has a conic structure in the closed ball $B_\varepsilon = \{y \in \mathbb{C}^n; |y| \leq \varepsilon\}$ [1]. Since $B_\varepsilon \setminus Y$ is a Stein manifold, Theorem 2 in [14] implies the next result

(1.5) Proposition

$$H^{\cdot}(B_\varepsilon \setminus Y) = H^{\cdot}(\Omega^{\cdot}_g) .$$

One may define a polar filtration F^s on $\Omega^{\cdot}_g$ exactly as in (1.1) and get an E_1–spectral sequence $(E_r(Y),d_r)$ converging to $H^\cdot(B_\varepsilon \setminus Y)$. Assume from now on that $(Y,0)$ is an isolated singularity. Even then the spectral sequence $(E_r(Y),d_r)$ is quite complicated, e.g. one has the next result [6], Cor. (3.10').

(1.6) Proposition

The spectral sequence $(E_r(Y),d_r)$ degenerates at E_2 if and only if the singularity $(Y,0)$ is weighted homogeneous (i.e. there exist suitable coordinates $y_1, \dots, y_n$ on $\mathbb{C}^n$ around the origin and suitable weights $v_i = wt(y_i)$ such that $(Y,0)$ can be defined by a weighted homogeneous polynomial g, of degree M say, with respect to the weights $\underline{v} = (v_1, \dots, v_n))$.

If this is the case, then the limit term $E_\infty = E_2$ can be described quite explicitly as follows [6], Example (3.6). In fact we restrict our attention only to the terms $E_\infty^{n-t,t}$ for $t \geq 0$, since this is all we need in the sequal.

Let $M(g) = \mathcal{O}_n / J_g$ be the Milnor algebra of g, where $J_g = \left[\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n} \right]$ is the Jacobian ideal of g [5]. Note that in our case $M(g)$ has a grading induced by the weights $\underline{v}$. Then one has a $\mathbb{C}$–linear identification

$$(1.7) \qquad E_\infty^{n-t,t}(Y) = M(g)_{tM-v}$$

with $v = v_1 + \dots + v_n$, by associating to the class of a monomial y^α in $M(g)_{tM-v}$ the class of the differential form $y^\alpha \cdot g^{-t} \cdot \omega_n$, where $\omega_n = dy_1 \wedge \dots \wedge dy_n$. Since $Y \setminus \{0\}$ is a smooth divisor in $B_\varepsilon \setminus \{0\}$, the Poincaré residue map

$$H^n(B_\varepsilon \setminus Y) \xrightarrow{R} H^{n-1}(Y \setminus \{0\})$$

in the associated Gysin sequence [21] is an isomorphism (assume $n \geq 3$ from now on). Moreover, the exact sequence of the pair $(Y, Y\setminus\{0\})$ gives an isomorphism

$$H^{n-1}(Y\setminus\{0\}) \xrightarrow{\delta} H^n(Y, Y\setminus\{0\}) = H^n_0(Y)$$

where $H^{\cdot}_0(Y)$ denote the local cohomology groups of Y at the origin. Note that this cohomology $H^{\cdot}_0(Y)$ carries a natural MHS according to Steenbrink [20] and Durfee [10]. Finally we get an isomorphism

$$(1.8) \qquad H^n(B_\varepsilon\setminus Y) = H^n_0(Y)$$

and in this way the filtration F^s on $\Omega^{\cdot}_g$ induces a filtration F^s on $H^n_0(Y)$. It is easy to check, using (1.7) and Steenbrink description of the MHS on $H^n_0(Y)$ when $(Y,0)$ is weighted homogeneous [19], that in this case F^s coincide with the Hodge filtration F^s_H for all s and that $H^n_0(Y)$ has a pure Hodge structure of weight n.

Consider next a semi weighted homogeneous singularity $Y_1 : g_1 = g + g'$, where g is as above and all the monomials in g' have degrees $> M$ with respect to the weights $\underline{v}$ [5]. Inspite of the fact that the corresponding spectral sequence $(E_r(Y_1), d_r)$ is much more complicated, we can obtain directly (by some obvious μ–constant arguments) the next simple description of the cohomology group $H^n(B_\varepsilon\setminus Y_1)$. Let $\{y^\alpha g^{-t_\alpha}\omega_n\ ;\ \alpha \in A\}$ be a basis for $H^n(B_\varepsilon\setminus Y)$ obtained as above. Then the forms $\{y^\alpha g_1^{-t_\alpha}\omega_n\ ;\ \alpha \in A\}$ give a basis for $H^n(B_\varepsilon\setminus Y_1)$. Here of course $t_\alpha = (\deg(y^\alpha)+v)\cdot M^{-1}$. Moreover, using the fact that in a μ–constant deformation the dimensions of the Hodge filtration subspaces remain constant, it follows that on $H^n_0(Y_1)$ the polar filtration coincides with the Hodge filtration, exactly as in the weighted homogeneous case.

In general, one may compute the MHS on $H_0^n(Y)$ if one knows the MHS on the cohomology $H^{n-1}(Y_\infty)$ of the Milnor fiber Y_∞ of the singularity $(Y,0)$, since $H^{n-1}(Y\setminus\{0\})$ is just the fixed part in $H^{n-1}(Y_\infty)$ under the monodromy action and δ is an isomorphism of MHS.

We say that the singularity $(Y,0)$ is nondegenerate if $H_0^n(Y) = 0$. The name comes from the fact that this condition is equivalent to the Milnor lattice of $(Y,0)$ being nondegenerate [4]. Otherwise the singularity $(Y,0)$ is called degenerate. We make next a list of the simplest nondegenerate and degenerate singularities, using terminology which is standard in Singularity Theory [5], [9].

(1.9) Examples (nondegenerate singularities)

(i) If $n = \dim Y + 1$ is odd, then the singularities A_k, D_k, E_6, E_7 and E_8 are nondegenerate

(ii) If $n = \dim Y + 1$ is even, then the singularities A_{2k}, E_6 and E_8 are nondegenerate.

For more examples we refer to Ebeling [11].

(1.10) Examples (degenerate singularities)

(i) Assume that $n = 2t$ is even and that we consider an A_{2k-1} singularity, i.e.

$$g = y_1^{2k} + y_2^2 + \dots + y_n^2, \quad v_1 = 1, \quad v_j = k \text{ for } j > 1$$

$v = 1 + (2t - 1)k$, $M = 2k$. The graded pieces $M(g)_j$ of the Milnor algebra are nontrivial only for $j \in \{0,1, \dots ,2k-2\}$. Hence the equality $sM - v = j$ has a unique solution in this range, namely $s = t$, $j = k-1$.

It follows by (1.7) that $\dim H^n(B_\varepsilon \setminus Y) = 1$ and that a generator of $H^n(B_\varepsilon \setminus Y)$ is provided in this case by the form $\beta = y^{k-1} g^{-t} \omega_n$.

Note moreover that the class of a form $\gamma = h \cdot g^{-t} \omega_n$ (with $h \in \mathcal{O}_n$) in $H^n(B_\varepsilon \setminus Y)$ is precisely

$$[\gamma] = \frac{1}{(k-1)!} \frac{\partial^{k-1} h}{\partial y_1^{k-1}} (0) \cdot [\beta]$$

It follows from [19] that β is a class of type (t,t) with respect to the MHS on $H_0^{2t}(Y)$.

(ii) Assume that $n = 2t + 1$ is odd and let $g = 0$ be the usual weighted homogeneous equation for a unimodal singularity of type $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$. Then it is known that the weights $\underline{v}$ and the degree M of g satisfy the next equality

$$\deg(\mathrm{hess}(g)) = nM - 2v = M = \deg(g)$$

where $\mathrm{hess}(g) = \det \left[\frac{\partial^2 g}{\partial y_i \partial y_j} \right]$ is the <u>hessian</u> of g and also $M(g)_j = 0$ for $j > M$, see [5], [16]. Hence the equality $sM - v = j$

has just two solutions with $j \leq M$, namely $j = 0$, $s = t$ and $j = M$, $s = t + 1$. The differential forms

$$\beta_1 = g^{-t}\omega_n \text{ and } \beta_2 = \mathrm{hess}(g) \cdot g^{-t-1}\omega_n$$

form a basis of $H^n(B_\varepsilon \setminus Y)$ in this case and it follows from [19] that β_1 has type $(t + 1,t)$ and β_2 has type $(t,t + 1)$ with respect to the MHS on $H^n_0(Y)$.

Note that the class of a differential form $\gamma = h \cdot g^{-t}\omega_n$ with $h \in \mathcal{O}_n$ is just

$$[\gamma] = h(0)\,[\beta_1]\ .$$

In what follows we are particularly interested by the local cohomology groups $H^n_{a_i}(V)$ corresponding to the hyperquotient singularities of V.

The obvious isomorphisms

$$\text{(1.11)} \qquad H^n_{a_i}(V) = H^n_{a_i}(Y_i/G_i) = H^n_{a_i}(Y_i)^{G_i}$$

shows that $H^n_{a_i}(V)$ can be computed (together with its MHS) as the fixed part of the natural action of G_i on $H^n_{a_i}(Y)$. This description is quite effective as soon as we have explicit forms giving a basis for $H^n_{a_i}(Y)$. Note also that it may happen that $H^n_{a_i}(V) = 0$ even if $H^n_{a_i}(Y_i) \neq 0$.

(1.12) Example

Let $(Y,0)$ be the A_{2k-1} singularity considered in (1.10.i) and let $G = \{\pm 1\}$ act on $(Y,0)$ by the rule $(-1) \cdot y = (y_1, -y_2, y_3, \dots, y_n)$. Then

$$(-1) \cdot [\beta] = -[\beta]$$

and hence $H_0^n(Y)^G = 0$.

§ 2. A basic MHS exact sequence

Let $\mathbb{P}^* = \mathbb{P}(\underline{w}) \setminus \Sigma$, $V^* = V \setminus \Sigma$ and consider the exact cohomology sequence of the pair $(\mathbb{P}^*, \mathbb{P}^* \setminus V^*)$:

(2.1)

$$\longrightarrow H^k(\mathbb{P}^*, \mathbb{P}^* \setminus V^*) \xrightarrow{j^*} H^k(\mathbb{P}^*) \xrightarrow{i^*} H^k(\mathbb{P}^* \setminus V^*) \xrightarrow{\delta} H^{k+1}(\mathbb{P}^*, \mathbb{P}^* \setminus V^*) \longrightarrow .$$

Note that there is a Thom isomorphism

$$T : H^{k-1}(V^*) \longrightarrow H^{k+1}(\mathbb{P}^*, \mathbb{P}^* \setminus V^*)$$

obtained as follows. Let $X = \mathbb{C}^{n+1} \setminus \Sigma(f)$ and $D = f^{-1}(0) \setminus \Sigma(f)$. Then D is a smooth divisor in X and hence there is an usual Thom isomorphism $T : H^{k-1}(D) \longrightarrow H^{k+1}(X, X \setminus D)$. Since the normal bundle of D in X may be chosen $\mathbb{C}^*$–invariant, it follows that T is compatible with the $\mathbb{C}^*$–actions which exist on both sides. Hence T induces an isomorphism between the fixed parts

$$H^{k-1}(D)^{\mathbb{C}^*} = H^{k-1}(V^*) \xrightarrow{T} H^{k+1}(\mathbb{P}^*, \mathbb{P}^* \setminus V^*) = H^{k+1}(X, X\setminus D)^{\mathbb{C}^*} .$$

In the same way, the Poincaré residue

$$R : H^k(X\setminus D) \longrightarrow H^{k-1}(D)$$

induces a map

$$R : H^k(\mathbb{P}^* \setminus V^*) \longrightarrow H^{k-1}(V^*)$$

such that $T \cdot R = \delta$.

It is easy to show that in the middle dimensions $j^* = 0$ and that if we define the primitive cohomology of V^* by $H_0(V^*) = \ker(j^* \circ T)$, then this has the expected properties. For instance one may define in the same way the primitive cohomology of V , denoted $H_0^{\cdot}(V)$ and the inclusion $\iota : V^* \longrightarrow V$ induces a morphism $\iota_0^* : H_0^{\cdot}(V) \longrightarrow H_0^{\cdot}(V^*)$ and carries isomorphically the nonprimitive part in $H^{\cdot}(V)$ onto the nonprimitive part in $H^{\cdot}(V^*)$ (except of course the top dimension).

As a result of this definition and since $\mathbb{P}^* \setminus V^* = U$, we get the next

(2.2) Lemma

The Poincaré residue $R : H^k(U) \longrightarrow H_0^{k-1}(V^*)$ is a type $(-1,-1)$ isomorphism of MHS .

Consider now the long exact sequence of MHS [20]:

$$\longrightarrow H^k_\Sigma(V) \longrightarrow H^k(V) \longrightarrow H^k(V^*) \xrightarrow{\delta} H^{k+1}_\Sigma(V) \longrightarrow$$

and note that excision gives us the next isomorphism of MHS .

$$H^k_\Sigma(V) = \bigoplus_{i=1,s} H^k_{a_i}(V) = \bigoplus_{i=1,s} H^k_{a_i}(Y_i)^{G_i} .$$

Hence $H^k_\Sigma(V)$ is a computable object as soon as we know enough about the transversal singularities (Y_i, a_i) .

The final part of the above long exact sequence, Lemma (2.2) and our remark on ι_0^* give us the next exact sequence of MHS

(2.3) $$H^n(U) \xrightarrow{\theta} H^n_\Sigma(V) \longrightarrow H^n_0(V) \longrightarrow 0$$

with $\theta = \delta R$ a morphism of type $(-1,-1)$. (There is no danger to confuse the primitive cohomology $H^\cdot_0(V)$ with some local cohomology of V , since $0 \notin \mathbb{P}(\underline{w})$). Let t be the maximal positive integer such that $F^t_H H^n_\Sigma(V) = H^n_\Sigma(V)$. Then using the strict compatibility of MHS morphisms with the Hodge filtrations F_H [3] we get a finer version of (2.3), namely

$$F^{t+1}_H H^n(U) \xrightarrow{\theta} H^n_\Sigma(V) \longrightarrow H^n_0(V) \longrightarrow 0 .$$

Using now Proposition (1.2) it follows that the composition

$$F^t H^n(U) \hookrightarrow H^n(U) \xrightarrow{\theta} H^n_\Sigma(V)$$

has exactly the same image as θ .

Let T^t be the linear map given by the obvious composition

$$S_{(n-t)N-w} \xrightarrow{\sim} F^tA^n \longrightarrow F^tH^n(U) \longrightarrow H^n_\Sigma(V) .$$

We may summarize our result as follows

(2.4) Theorem

The image of the linear map T^t is a MH substructure in $H^n_\Sigma(V)$ and $H^n_0(V)$ with its canonical MHS is isomorphic to the quotient $H^n_\Sigma(V)/\mathrm{im}\,(T^t)$.

Note that the proof in [20], Theorem (1.13) adapts to our more general situation and shows that $H^n_0(V)$ has a pure Hodge structure of weight n . Consider now a subset $\Sigma' \subset \Sigma$ defined as follows:

$$\Sigma' = \{a_i \in \Sigma \; ; \; H^n_{a_i}(V) \neq 0\} .$$

We may call Σ' the set of essential singularities of V . It is clear that we may replace $H^n_\Sigma(V)$ with $H^n_{\Sigma'}(V)$ everywhere. More important, note that $T^t(h) = 0$ means that h satisfies certain (linear) conditions $\mathscr{C}$ at the points $a_i \in \Sigma'$. Indeed, it is easy to check that θ corresponds to the composition of the morphism

$$H^n(U) \xrightarrow{\rho} \bigoplus_{a_i \in \Sigma'} H^n(D_i \setminus V)$$

induced by the restriction of n–forms (with D_i being an open neighbourhood of a_i in $\mathbb{P}(\underline{w})$ of the form $D_i = B_i/G_i$, for B_i a small ball in H_i centered at a_i and G_i–invariant) with the isomorphism induced essentially by local Poincaré residue isomorphisms

$$\bigoplus_i H^n(D_i \backslash V) \xrightarrow[\sim]{R} \bigoplus_i H^{n-1}(V \cap D_i \backslash \{a_i\}) \xrightarrow[\sim]{} \bigoplus_i H^n_{a_i}(V) = H^n_{\Sigma'}(V) .$$

Let $\mathscr{A} = \ker T^t$ be the linear system in $S_{(n-t)N-w}$ defined by the conditions $\mathscr{C}$. We define the <u>defect</u> of the linear system $\mathscr{A}$ by the formula

$$\mathrm{def}(\mathscr{A}) = \dim H^n_{\Sigma'}(V) - \mathrm{codim}\ \mathscr{A}$$

i.e. the difference between the number of linear conditions in $\mathscr{C}$ and the codimension of $\mathscr{A}$ in $S_{(n-t)N-w}$. It is clear that $\mathrm{def}(\mathscr{A})$ depends not only on $\mathscr{A}$ but also on the set of conditions $\mathscr{C}$ used to define it and that $\mathrm{def}(\mathscr{A}) = 0$ says that the conditions in $\mathscr{C}$ are independent. With this definition, we may state the next.

(2.5) <u>Corollary</u>

$$\dim H^n_0(V) = \mathrm{def}(\mathscr{A}) .$$

The next section contains several examples where it is possible to work out explicitely the conditions $\mathscr{C}$ and hence to state several special cases of Corollary (2.5) in more down–to–earth terms. When on $H^n_\Sigma(V)$ the polar filtration F^s coincides with the Hodge filtration F^s_H (this is the case for instance when all the singularities (Y_i, a_i) are weighted homogeneous), one may increase the number t (and hence <u>decrease the degree</u>

of the elements in $S_{(n-t)N-w}$) by the following simple observation. We present only the case $n = 2m + 1$ is odd since we shall apply this in the next section and leave the analogue statement in the case n even to the reader. As remarked above, $H_0^n(V)$ has a pure Hodge structure of weight n and it is clear that

$$\dim H_0^n(V) = 2 \sum_{i>m} h^{i,n-i}(H_0^n(V)) .$$

Let $\tilde{T}^{m+1}$ be the composition

$$S_{(n-m-1)N-w} \xrightarrow{\sim} F^{m+1} A \longrightarrow F^{m+1} H^n(U) \longrightarrow F^{m+1} H^n_\Sigma(V)$$

and let $\tilde{\mathcal{S}}$ be the linear system $\ker \tilde{T}^{m+1}$.

If we set as above

$$\operatorname{def}(\tilde{\mathcal{S}}) = \dim F^{m+1} H^n_\Sigma(V) - \operatorname{codim} \tilde{\mathcal{S}}$$

then we get the next result.

(2.6) Corollary

$$\dim H_0^{2m+1}(V) = 2 \operatorname{def}(\tilde{\mathcal{S}}) .$$

(2.7) Remark

Unlike $H_0^n(V)$ which has a pure Hodge structure of weight n , the middle cohomology group $H^{n-1}(V)$ has in general a nonpure Hodge structure, whose associated MHS numbers can be computed as follows (at least in the homogeneous case).

In the MHS sequence

$$H_0^{n-1}(V) \longrightarrow H_0^{n-1}(V^*) \longrightarrow H_\Sigma^n(V) \xrightarrow{\;j\;} H_0^n(V) \longrightarrow 0$$

used above, one has

(i) $H_\Sigma^n(V)$ has weights $\geq n$, i.e. $W_{n-1}H_\Sigma^n(V) = 0$ by Durfee [10].

(ii) $H_0^{n-1}(V)$ has weights $\leq n-1$, i.e. $W_{n-1}H_0^{n-1}(V) = H_0^{n-1}(V)$ since V is proper [3].

It follows that one can determine $h^{p,q}(H_0^{n-1}(V^*))$ for $p+q=m \geq n$ from short exact sequences

$$0 \longrightarrow Gr_m^W H_0^{n-1}(V^*) \longrightarrow Gr_m^W H_\Sigma^n(V) \xrightarrow{\;j\;} Gr_m^W H_0^n(V) \longrightarrow 0$$

(using of course computations with linear systems to determine the kernel of j). Using duality results for the MHS on $H_0^{\cdot}(V)$ and on $H^{\cdot}(U)$ explained in [6] and Lemma (2.2) we get

$$h^{p,q}(H_0^{2n-s-1}(V)) = h^{n-p,n-q}(H^s(U)) = h^{n-p-1,n-q-1}(H_0^{s-1}(V^*))$$

for any p,q and s .

Hence the above short exact sequences give all the numbers $h^{p,q}(H_0^{n-1}(V))$ for $p+q < n-1$.

To determine the remaining MHS numbers, it is enough to recall that the coefficient of $(n-p)$ in the spectrum $Sp(f)$ of f is precisely

$$\sum_s h^{p,s}(H^n(U)) - \sum_t h^{p,t}(H^{n-1}(U)) \ .$$

This formula contains exactly one unknown number, namely

$$h^{p,n+1-p}(H^n(U)) = h^{n-p,p-1}(H_0^{n-1}(V)) \ .$$

On the other hand, the spectrum $Sp(f)$ is computed (at least in the case of a homogeneous polynomial f) explicitly in terms of the spectra of the transversal singularities (Y_i,a_i) by J. Steenbrink in his recent (unpublished) manuscript: "The spectrum of hypersurface singularities".

As a result, in this way one is able to determine all the MHS numbers for V, V^* and U, provided one knows enough about the transversal singularities (Y_i,a_i).

In particular, one gets the next obvious consequences of this discussion.

(2.8) Corollary

(i) $H^{n-1}(V)$ has a pure Hodge structure of weight $(n-1)$ if and only if the morphism j above is an isomorphism. This can be rephrased by saying that $\mathrm{codim}(\mathscr{A}) = 0$, i.e. the conditions $\mathscr{C}$ in (2.5) are automatically satisfied by all the polynomials in $S_{(n-t)N-w}$.

(ii) The subspace $W_{n-3}H^{n-1}(V)$ depends on the transversal singularities (Y_i,a_i), but not on their position.

By general properties of Hodge structures it follows that the subspace $W_{n-2}H^{n-1}(V)$ is precisely the kernel of the cup–product pairing

$$H^{n-1}(V) \times H^{n-1}(V) \longrightarrow H^{2n-2}(V) = \mathbb{C} \ .$$

Moreover, when $\dim(V)$ is even, one can use in the usual way the numbers $h^{p,q}(H^{\cdot}(V))$ to compute the signature (μ_+,μ_0,μ_-) of the cup–product pairing over $\mathbb{R}$ [19].

(2.9) <u>Corollary</u>

V is a $\mathbb{C}$–homology manifold (i.e. there are no essential singularities for V) if and only if the cohomology algebra $H^{\cdot}(V)$ is a Poincaré algebra (i.e. for any k the cup–product pairing

$$H^k(V) \times H^{2n-2-k}(V) \longrightarrow H^{2n-2}(V) = \mathbb{C}$$

is non degenerate).

<u>Proof</u>

If $H^{\cdot}(V)$ is a Poincaré algebra, it follows that $H^n_0(V) = 0$. Then using (2.8 i) and the above description of the kernel of the cup–product on $H^{n-1}(V)$ it follows that $H^n_\Sigma(V) = 0$, i.e. there are no essential singularities for V .
The other implication is standard.

Similar consideration lead to the computation of the MHS numbers of $H^n(F)$, but we leave the details for the reader (use the same method as in the proof of (3.6) below).

§ 3. Some examples

Let us discuss first the case when dim V is even. Then the simplest singularities which are degenerate in this case are $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$.

(3.1) Proposition

Let $V \subset \mathbb{P}(\underline{w})$ be a hypersurface with $\deg V = N$ and $\dim V = 2m$. Assume that the set Σ' of essential singularities for V consists only of singularities a_i whose associated transversal singularities are of type $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$. Then the only (possibly) nonzero Hodge numbers of $H_0^{2m+1}(V)$ are given by the next formula

$$h^{m,m+1}(H_0^{2m+1}(V)) = h^{m+1,m}(H_0^{2m+1}(V)) = \operatorname{def}(\tilde{\mathscr{S}})$$

where the linear system $\tilde{\mathscr{S}}$ is defined by

$$\tilde{\mathscr{S}} = \{h \in S_{mN-w} \, ; \, h|\Sigma' = 0\} .$$

Proof Use (1.10. ii) and (2.6).

(3.2) Corollary (including Zariski example [25], [12])

Let $B \subset \mathbb{P}^{2m}$ be a hypersurface of degree N having only isolated singularities and let $V \longrightarrow \mathbb{P}^{2m}$ be a cyclic covering of order 6 ramified over B. Assume that all the points $a_i \in \Sigma'$ correspond to points $\bar{a}_i \in B$ such that B has an A_2 singularity at $\bar{a}_i$. Let $\bar{\Sigma}$ denote the set of all these points $\bar{a}_i$.

Then the only (possibly) nonzero Hodge numbers of $H_0^{2m+1}(V)$ are given by the next formula $h^{m,m+1}(H_0^{2m+1}(V)) = h^{m+1,m}(H_0^{2m+1}(V)) = \mathrm{def}(\bar{\mathscr{A}})$ where the linear system $\bar{\mathscr{A}}$ is defined by $\bar{\mathscr{A}} = \{h \in H^0(\mathbb{P}^{2m}, \mathcal{O}(mN - 2m - 1 - N/6))$; $h \mid \bar{\Sigma} = 0\}$.

Proof

Let $b = 0$ be an equation for B. Then V is a hypersurface defined by the equation $b - t^6 = 0$ in the weighted projective space $\mathbb{P}(1, \ldots, 1, N/6)$ and all the singularities $a_i \in \Sigma'$ have associated transversal singularities (Y_i, a_i) of type $\tilde{E}_8$. Hence we can apply (3.1) and note that an element $h \in S_{mN-w}$ with $w = 2m + 1 + N/6$ can be written as a sum $h = \Sigma h_j t^j$ where h_j is a homogeneous polynomial in $x_0, x_1, \ldots, x_{2m}$ of degree $\deg(h_j) = mN - w - jN/6$.

Moreover the condition $h \mid \Sigma' = 0$ is clearly equivalent to $h_0 \mid \bar{\Sigma} = 0$.

Assume from now on that $\dim V = 2m - 1$ is odd. Then the simplest degenerate singularities are A_{2k-1} for $k \geq 1$.

(3.3) Proposition

Let V be a hypersurface in $\mathbb{P}(\underline{w})$ with $\dim V = 2m - 1$, $\deg V = N$ and such that any essential singularity $a_i \in \Sigma'$ corresponds to a transversal singularity of type A_1. Then the only (possibly) nonzero Hodge number of $H_0^{2m}(V)$ is given by the formula

$h^{m,m}(H_0^{2m}(V)) = \text{def}(\mathcal{A})$ where

$$\mathcal{A} = \{h \in S_{mN-w} , h | \Sigma' = 0\} .$$

Proof Use (1.10 i) with $k = 1$ aand (2.5) with $t = m$.

Note that (3.3) extends the computations of Betti numbers in Clemens [2], Schoen [17] and Werner [24].

A more complicated example involving several types of A_{2k-1}–singularities is the next.

(3.4) Proposition

Let $V \subset \mathbb{P}(w_0, \dots, w_{2m})$ be a hypersurface of degree N such that the set Σ' of essential singularities satisfies the next two conditions:

(i) Σ' is contained in the hyperplane $x_0 = 0$

(ii) any transversal singularity (Y_i, a_i) corresponding to a point $a_i \in \Sigma'$ is of type A_{2k+1} for some k and $(Y_i \cap H_0, a_i)$ is an A_1–singularity in (H_0, a_i) , where H_0 denotes the affine hyperplane $x_0 = 0$. Let $\Sigma_k = \{a_i \in \Sigma' ; (Y_i, a_i)$ is of type $A_{2k+1}\}$ and for any k with $\Sigma_k \neq \phi$ consider the linear system

$$\mathcal{A}_k = \{h \in \bar{S}_{mN-w-kw_0} ; h | \Sigma_k = 0\} .$$

Then the only possible nonzero Hodge number of $H_0^{2m}(V)$ is given by the formula

$$h^{m,m}(H_0^{2m}(V)) = \sum_{k,\Sigma_k \neq \phi} \operatorname{def}(\mathscr{A}_k) .$$

Here $\bar{S}$ denotes the polynomial ring $\mathbb{C}[x_1, \dots, x_{2m}]$ graded by the conditions $\deg(x_i) = w_i$ for $i \geq 1$.

Proof

According to Theorem (2.4) we have to analyse the kernel of T^m on S_{mN-w} .

Write an element $h \in S_{mN-w}$ as a sum $h = \Sigma\, h_j x_0^j$ with $h_j \in \bar{S}_{mN-w-jw_0}$. If $a_i \in \Sigma_k$, then the component of $T^m(h)$ corresponding to $H^n_{a_i}(V)$ is zero if and only if $h_k(a_i) = 0$, i.e. if $h_k \in \mathscr{A}_k$, use (1.10 i) and the second part of the condition (ii) above.

It follows from (3.4) that the singularities situated in one Σ_k do not interact at all with the singularities situated in a different Σ_ℓ (with $\ell \neq k$) and this fact is not at all obvious from purely topological considerations.

A special case of (3.4) is the next

(3.5) Corollary

Let $B \subset \mathbb{P}^{2m-1}$ be a hypersurface of degree N having only isolated singularities. Let e be a divisor of N and let $V \longrightarrow \mathbb{P}^{2m-1}$ be a cyclic covering of order e ramified over B . Assume that all the essential singularities of V $a_i \in \Sigma'$ correspond to points $\bar{a}_i$ which are nodes on B . Let $\bar{\Sigma}$ denote the set of all these nodes $\bar{a}_i$.

Then either

(i) e is odd, $\Sigma' = \phi$ and $H_0^{2m}(V) = 0$, or

(ii) e is even, N is even and the only possibly nonzero Hodge number of $H_0^{2m}(V)$ is given by $h^{m,m}(H_0^{2m}(V)) = \mathrm{def}(\mathscr{A})$ where

$$\mathscr{A} = \{h \in H^0(\mathbb{P}^{2m-1}, \mathcal{O}(mN - 2m - N/2), h \,|\, \bar{\Sigma} = 0\} .$$

Proof Apply (3.4) with $\Sigma' = \Sigma_k$ for $2k + 2 = e$, $w_0 = N|e$, $w_1 = \ldots = w_{2m} = 1$.

Note that the answer in case (ii) does not depend on the degree e of the covering $V \longrightarrow \mathbb{P}^{2m-1}$!

(3.6) Corollary

Let $F : f - 1 = 0$ be the Milnor fiber of the weighted homogeneous polynomial f. Assume that all the transversal singularities of f are nodes. Then:

(i) $b_{n-1}(F) = 0$ if n and N are both odd;

(ii) If $n = 2m$ is even, then the only possibly nonzero Hodge number of $H^{n-1}(F)$ is given by $h^{m,m}(H^{n-1}(F)) = \mathrm{def}(\mathscr{A})$ where

$$\mathscr{A} = \{h \in S_{mN-w} \,;\, h \,|\, \Sigma' = 0\}$$

with Σ' the set of essential singularities for $V : f = 0$. Moreover in this case $H^{n-1}(F) = H^{n-1}(F)_0$, i.e. all the elements in $H^{n-1}(F)$ are fixed under the monodromy operator h^* .

(iii) If $n = 2m - 1$ is odd and N is even, then the only possibly nonzero Hodge number of $H^{n-1}(F)$ is given by $h^{m-1,m-1}(H^{n-1}(F)) = \mathrm{def}(\mathscr{A}')$, where $\mathscr{A}' = \{h \in S_{mN-w-N/2}$; $h \mid \tilde{\Sigma} = 0\}$ with $\tilde{\Sigma}$ the set of essential singularities for $\tilde{V} : f - t^N = 0$ in $\mathbb{P}(\underline{w},1)$. Moreover in this case $H^{n-1}(F) = H^{n-1}(F)_{\neq 0}$, i.e. there is no nonzero element fixed under the monodromy operator h^* .

Proof

For $a \in \mathbb{Z}/N\mathbb{Z}$, let $H^{\cdot}(F)_a$ denote the eigenspace of h^* corresponding to the eigenvalue t^a . If we set $H^{\cdot}(F)_{\neq 0} = \bigoplus_{a \neq 0} H^{\cdot}(F)_a$, then one clearly has the decomposition $H^{\cdot}(F) = H^{\cdot}(F)_0 \oplus H^{\cdot}(F)_{\neq 0}$. It follows from [6], (1.19) and (2.5) that one has isomorphisms $H^{n-1}(F)_0 = H^n_0(V)$ and $H^{n-1}(F)_{\neq 0} = H^{n+1}_0(\tilde{V})$ which are (in some precise way) compatible with the MHS. See the remarks after (2.5) in [6].

Assume first that $n = 2m$ is even. Then all the singularities of $\tilde{V}$ are nondegenerate and hence $H^{n+1}_0(\tilde{V}) = 0$. The result follows using (3.3). Assume next that $n = 2m - 1$ is odd. Then all the singularities of V are nondegenerate and hence $H^n_0(V) = 0$. If N is also odd, the same is true for $\tilde{V}$ and we get the case (i) above. If N is even, then the singularities in $\tilde{\Sigma}$ are of type A_{N-1} and we can apply (3.4). Note that since $\tilde{\Sigma}$ is contained in the hyperplane $t = 0$, we regard $\tilde{\Sigma}$ as a subset in $\mathbb{P}(\underline{w})$. Recall that the monodromy operator $h^* : H^{\cdot}(F) \longrightarrow H^{\cdot}(F)$ is induced by the mapping

$$h : F \longrightarrow F \, , \; h(x) = (t^{w_0} x_0, \ldots , t^{w_n} x_n) \text{ for } t = \exp(2\pi i/N) .$$

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