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A LIFT INTO SIEGEL MODULAR FORMS OVER THE THETA GROUP IN DEGREE TWO AND THE CHIRAL SUPERSTRING MEASURE

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Abstract. We prove in degree two, that the Siegel modular form of D'Hoker and Phong that gives the chiral superstring measure is a lift. This gives a fast algorithm for computing its Fourier coefficients. We prove a general lifting from Jacobi cusp forms of half integral index $t/2$ over the theta group $\Gamma_1(1,2)$ to Siegel modular cusp forms over certain subgroups $\Gamma^{\mathrm{para}}(t;1,2)$ of paramodular groups. The theta group lift given here is a modification of the Gritsenko lift.

1. Introduction

We construct a lifting L from Jacobi cusp forms of index $t/2$ for the theta group $\Gamma_1(1,2)$ to Siegel modular forms on subgroups $\Gamma^{\mathrm{para}}(t;1,2)$ of the paramodular groups $\Gamma^{\mathrm{para}}(t)$:

$$L : J_{k,t/2}^{\mathrm{cusp}}(\Gamma_1(1,2)^J) \to M_k\left(\Gamma^{\mathrm{para}}(t;1,2)\right).$$

Our construction imitates the construction of the lift due to Gritsenko, $\mathrm{Grit} : J_{k,m}(\mathrm{SL}_2(\mathbb{Z})^J) \to M_k\left(\Gamma^{\mathrm{para}}(m)\right)$, which sends Jacobi forms of index m on $\mathrm{SL}_2(\mathbb{Z})$ to Siegel modular forms on the paramodular group $\Gamma^{\mathrm{para}}(m)$, see [8]. Although we proceed in greater generality, our main interest is the case where t is odd. In order to properly call L a lift, we should really discuss the L-series of the lifted Siegel modular form but here we content ourselves with giving the Fourier coefficients.

Theorem 1. *Let $t \in \mathbb{N}$ and $k \in \mathbb{Z}$. There is a monomorphism*

$$L : J_{k,t/2}^{\mathrm{cusp}}(\Gamma_1(1,2)^J) \to M_k\left(\Gamma^{\mathrm{para}}(t;1,2)\right)$$

such that if $\phi \in J_{k,t/2}^{\mathrm{cusp}}(\Gamma_1(1,2)^J)$ has the Fourier expansion

$$\phi(\tau,z) = \sum_{n,r\in\mathbb{Z}:\ tn-r^2>0,\, n>0} c(n,r)e(\tfrac{1}{2}n\tau + rz),$$

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for $\tau \in \mathcal{H}_1$ *and* $z \in \mathbb{C}$*, then* $L(\phi) \in M_k\left(\Gamma^{\mathrm{para}}(t;1,2)\right)$ *has the Fourier expansion*

$$L(\phi)(\Omega) = \sum_{\substack{T=\left(\begin{smallmatrix} n & r \\ r & m \end{smallmatrix}\right):\, t|m, \\ mn-r^2>0,\, n>0, m>0.}} a(T)e(\tfrac{1}{2}\mathrm{tr}(T\Omega)),$$

for $\Omega \in \mathcal{H}_2$*, where*

$$a(\begin{pmatrix} n & r \\ r & m \end{pmatrix}) = (-1)^{(m/t+1)(n+1)} \sum_{\substack{a|(n,r,m/t) \\ a \text{ odd}}} a^{k-1}\, c(\frac{mn}{ta^2}, \frac{r}{a}).$$

If $t \not\equiv 0 \mod 4$ *then* $L(\phi)$ *is a cusp form.*

Although the lifting L is adequately described as an imitation of the Gritsenko lift, the choice of Hecke operators used to construct L was not obvious to us. The special case $t = 1$ is a lifting from Jacobi cusp forms of index $1/2$ for the Jacobi theta group $\Gamma_1(1,2)^J$ to Siegel modular cusp forms for the theta group $\Gamma_2(1,2)$,

$$L : J^{\mathrm{cusp}}_{k,1/2}(\Gamma_1(1,2)^J) \to S_k\left(\Gamma_2(1,2)\right).$$

The groups that arise in this construction have natural geometric interpretations. The moduli space $\Gamma^{\mathrm{para}}(t)\backslash\mathcal{H}_2$ is the equivalence classes of polarized abelian surfaces whose polarization has type $E_t = \left(\begin{smallmatrix} 0 & T \\ -T & 0 \end{smallmatrix}\right)$ with $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right)$. This implies that there exists a divisor on the abelian surface with Chern class E_t. To each equivalence class of type E_t polarized abelian surface in $\Gamma^{\mathrm{para}}(t;1,2)\backslash\mathcal{H}_2$, one may associate a distinguished rank t vector space of sections of a divisor of Chern class E_t, compare the transformation of theta functions under the paramodular group in [12], page 175. For many purposes, the theta group $\Gamma_g(1,2)$ is just as natural, or even more natural, than the full modular group Γ_g. For example, the theta series of an integral unimodular lattice of even rank is always automorphic with respect to the theta group for a character, whereas the theta series is only automorphic with respect to the full modular group when the lattice happens to be even. We can also connect the lift L with elliptic modular forms on the theta group $\Gamma_1(1,2)$ if we make use of multiplication by the theta function $\theta[0] \in J_{1/2,1/2}(\Gamma_1(1,2)^J, v_\theta)$. Here, v_θ is the multiplier of the theta function and takes values in the eighth roots of unity.

Corollary 2. *For* $k \in \mathbb{N}$*, with* $4|k$*, there are monomorphisms*

$$S_{k-\frac{1}{2}}\left(\Gamma_1(1,2), v_\theta^{2k-1}\right) \overset{\cdot\theta[0](z,\tau)}{\longrightarrow} J^{\mathrm{cusp}}_{k,1/2}(\Gamma_1(1,2)^J) \overset{L}{\longrightarrow} S_k\left(\Gamma_2(1,2)\right).$$

The point is that L, which is defined with respect to the theta group, is just as fundamental as any member of the Saito-Kurokawa family of lifts, to which L belongs. For a general context and for an extended family of lifts, see the thesis of F. Clery [3].

An Application.

D'Hoker and Phong [5] have computed the chiral superstring measure $d\nu^{(g)}[e] = \Xi_g[e]\, d\mu^{(g)}$ in g=2 and it is determined by $\Xi_2[0] \in S_8\,(\Gamma_2(1,2))$, which can be defined, for example, as a polynomial of degree 16 in the thetanullwerte, see [10]:

$$\Xi_2[0] = \frac{1}{1024}\left(2\,\theta\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)^{16} - \theta\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)^{8}\sum_{\zeta \text{ even}}^{10}\theta[\zeta]^8 + 2\,\theta\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)^{4} F\right), \tag{1}$$

with $F =$

$$\theta\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)^4 + \theta\left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)^4 + \theta\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)^4$$
$$+\theta\left(\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)^4 + \theta\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right)^4 + \theta\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)^4\theta\left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)^4.$$

The solution $\Xi_2[0]$ may be variously viewed as a Siegel modular form, a Teichmuller modular form or as a binary invariant depending upon whether it is viewed as a section over the moduli space of abelian varieties, curves or hyperelliptic curves. In the first setting, the Ansatz of D'Hoker and Phong [5][2][10][11] asks for a family of Siegel modular forms satisfying: 1) $\Xi_{g_1+g_2}[0]\left(\left(\begin{smallmatrix}\Omega_1&0\\0&\Omega_2\end{smallmatrix}\right)\right) = \Xi_{g_1}[0](\Omega_1)\Xi_{g_2}[0](\Omega_2)$ for Ω_1, Ω_2 in the Jacobian loci. 2) $\operatorname{tr}(\Xi_g[0])$ vanishes on the Jacobian locus. 3) The family $\{\Xi_g[0]\}$ is uniquely determined on the Jacobian loci by the genus one solution $\Xi_1[0] = \theta_{00}^4\eta^{12}$. This Ansatz can be satisfied through $g \leq 5$ but is thought unlikely to extend further [14]. Over the hyperelliptic locus, however, the corresponding conditions are solved for all g by a family of binary invariants, see [15]. As of this writing it remains an open question whether the corresponding conditions can be satisfied by a Teichmuller modular form beyond $g = 5$. See [13] for an entry to the physics literature. We write $T = \left(\begin{smallmatrix}n&r\\r&m\end{smallmatrix}\right) = [n, r, m]$ and in Table 1 give some Fourier coefficients for $a\,(T; \Xi_2[0])$ using the above polynomial in the thetanullwerte (1).

Table 1. Fourier coefficients for $\Xi_2[0]$.

Trace: 2
[1, 0, 1] 1
Trace: 3
[1, 0, 2] 6
Trace: 4
[1, 0, 3] 0, [2, 0, 2] 64, [2, 1, 2] 0

Trace: 5
[1, 0, 4] −64, [2, 0, 3] 252, [2, 1, 3] −84
Trace: 6
[1, 0, 5] −84, [2, 0, 4] 384, [2, 1, 4] −512, [3, 0, 3] 1080, [3, 1, 3] −384
Trace: 7
[1, 0, 6] 252, [2, 0, 5] 28, [2, 1, 5] −1107, [3, 0, 4] 0, [3, 1, 4] 0
Trace: 8
[1, 0, 7] 512, [2, 0, 6] 0, [2, 1, 6] 0, [3, 0, 5] −4608, [3, 1, 5] 792,
[4, 0, 4] −4096, [4, 1, 4] 4608, [4, 2, 4] 0.

A rapid method exists for computing these Fourier coefficients because $\Xi_2[0]$ is a lift. Consider $\Phi = \theta^{11} F_2 - 16\,\theta^7 F_2^2 \in S_{15/2}\,(\Gamma_0(4)^*, \tilde{v}_\theta^{15})$, where $\tilde{v}_\theta : \Gamma_0(4)^* \to \mathbb{C}^*$ is conjugate to v_θ; note $\Gamma_1(1,2)$ is conjugate to $\Gamma_0(4)^*$ via $\Gamma_1(1,2) = \left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)^{-1}\Gamma_0(4)^*\left(\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right)$. We also note that Φ is not in the Kohnen plus space and that its Fourier expansion begins:

$$\begin{aligned}\Phi(\tau) =& q + 6q^2 - 64q^4 - 84q^5 + 252q^6 + 512q^7 - 384q^8 - 1107q^9\\ &+28q^{10} + 3724q^{13} + 792q^{14} - 4608q^{15} + 4096q^{16} - 168q^{17}\\ &-15390q^{18} + 5376q^{20} + 1944q^{21} + 27676q^{22} + 10752q^{23} - 16128q^{24}\\ &-11635q^{25} - 20748q^{26} - 32768q^{28} - 31836q^{29} + 79704q^{30}\\ &+21504q^{31} + 24576q^{32} + 60984q^{33} - 107464q^{34} + 70848q^{36}\\ &-41492q^{37} - 20748q^{38} - 124416q^{39} - 1792q^{40} + 63504q^{41}\\ &-68616q^{42} + 215460q^{45} + 175640q^{46} + 64512q^{47} - 315783q^{49}\\ &+O(q^{50}).\end{aligned}$$

Use Corollary 2 to define a Jacobi form $\phi(\tau,z) = \theta_{00}(z,\tau)\,\Phi(\tau/2) \in J^{\mathrm{cusp}}_{8,1/2}(\Gamma_1(1,2)^J)$. The lift $L(\phi)$ is then in the one dimensional space $S_8\,(\Gamma_2(1,2))$. By checking agreement on one Fourier coefficient we conclude $\Xi_2[0] = L(\phi)$ and obtain the formula

$$a\left(\begin{pmatrix} n & r \\ r & m\end{pmatrix}; \Xi_2[0]\right) = (-1)^{(m+1)(n+1)} \sum_{\substack{a|(n,r,m)\\ a \text{ odd}}} a^7 c\left(\frac{mn-r^2}{a^2}; \Phi\right).$$

Thus, the entries in Table 1 can be easily verified from the q-expansion of the elliptic modular form Φ.

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2. Groups

The justifications for working out a variant of the Saito-Kurokawa lift are the precise specification of the group of automorphy and the cuspidality of the lift. For index $1/2$, the lift $L(\phi)$ is automorphic with respect to the theta group $\Gamma_2(1,2)$. For index $t/2$, this role is played by $\Gamma^{\mathrm{para}}(t;1,2)$, a subgroup of the paramodular group $\Gamma^{\mathrm{para}}(t)$. In order to determine the group of automorphy for the lift we will need to know generators of $\Gamma^{\mathrm{para}}(t;1,2)$. The thesis of Delzeith [4] shows that $\Gamma^{\mathrm{para}}(t)$ is generated by its translations and by $J(t) = \left(\begin{smallmatrix} 0 & T^{-1} \\ -T & 0 \end{smallmatrix}\right)$ for $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right)$. In order to show the cuspidality of the lift for $t \not\equiv 0 \mod 4$, we require coset decompositions of $\mathrm{Sp}_2(\mathbb{Q})$ with respect to $\Gamma^{\mathrm{para}}(t)$ and $\Gamma^{\mathrm{para}}(t;1,2)$. Let $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right) \in \mathrm{GL}_{2g}(\mathbb{Z})$.

Definition 3. *For $\mathbb{F} = \mathbb{R}, \mathbb{Q}$ or $\mathbb{Z}$, define groups of matrices:*

$$\mathrm{Sp}_g(\mathbb{F}) = \{\gamma \in M_{2g\times 2g}(\mathbb{F}) : \gamma J \gamma' = J\},$$
$$\mathrm{GSp}_g^+(\mathbb{F}) = \{\gamma \in M_{2g\times 2g}(\mathbb{F}) : \exists \mu(\gamma) \in \mathbb{F}^+ : \gamma J \gamma' = \mu(\gamma) J\}.$$

The theta group of genus g is

$$\Gamma_g(1,2) = \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_g(\mathbb{Z}) : A'C, B'D \text{ have even diagonal entries}\}.$$

The real symplectic group $\mathrm{Sp}_g(\mathbb{R})$ has a natural action on the Siegel upper half space $\mathcal{H}_g$. For a domain $\mathbb{D} \subseteq \mathbb{C}$, let $V_g(\mathbb{D})$ be the g-by-g symmetric matrices with coefficients in $\mathbb{D}$. For $\mathbb{D} \subseteq \mathbb{R}$, let $\mathcal{P}_g(\mathbb{D})^{\mathrm{semi}} \subseteq V_g(\mathbb{D})$ be the semidefinite elements and let $\mathcal{P}_g(\mathbb{D})$ be the definite elements. Let $\mathcal{H}_g$ be the Siegel upper half space of degree g, the subset of $V_g(\mathbb{C})$ with positive definite imaginary part. The symplectic group $\mathrm{Sp}_g(\mathbb{R})$ acts on $\Omega \in \mathcal{H}_g$ via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \circ \Omega := (A\Omega + B)(C\Omega + D)^{-1}.$$

Here we think of elements of $\mathrm{Sp}_g(\mathbb{R})$ as consisting of four $g \times g$ blocks. The group of symplectic similitudes $\mathrm{GSp}_g^+(\mathbb{Q})$ is useful in the construction of Hecke algebras. The theta function $\theta[0]^8$ is automorphic with respect to theta group $\Gamma_g(1,2)$. Because $\Gamma_g(1,2)$ is closed under transposition, we may also use the conditions that AB' and CD' are even matrices.

Definition 4. *The parabolic subgroup of the symplectic group is*

$$\Gamma_\infty(\mathbb{F}) = \{\begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{F})\}.$$

Also define

$$G\Gamma_\infty(\mathbb{F}) = \{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in \mathrm{GSp}_2^+(\mathbb{F}) \}.$$

Denote $\Gamma_2(1,2)_\infty = \Gamma_2(1,2) \cap \Gamma_\infty(\mathbb{Z})$.

For an element $\gamma \in \mathrm{GSp}_2^+(\mathbb{F})$ to be in $G\Gamma_\infty(\mathbb{F})$, it suffices that the second column be of the correct form. Introduce the notation $(\gamma)_2$ to mean the second column of γ written as a row 4-tuple for typesetting convenience.

Lemma 5. $G\Gamma_\infty(\mathbb{F}) = \{\gamma \in \mathrm{GSp}_2^+(\mathbb{F}) : (\gamma)_2 = (0, *, 0, 0)$ *for* $* \in \mathbb{F}\}$.

Proof. Writing $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, we assume that $A_{12} = C_{12} = C_{22} = 0$ and need to show that $C_{21} = D_{21} = 0$. The defining conditions for $\mathrm{GSp}_2^+(\mathbb{F})$ are AB', CD' symmetric and $AD' - BC' = \mu I$ and we deduce:

$$C_{21}D_{11} = C_{11}D_{21}, \quad A_{11}D_{11} - B_{11}C_{11} = \mu, \quad A_{11}D_{21} = B_{11}C_{21}.$$

The conclusion then follows since $\mu > 0$ and

$$\begin{aligned} \mu C_{21} &= A_{11}D_{11}C_{21} - B_{11}C_{11}C_{21} \\ &= A_{11}C_{11}D_{21} - B_{11}C_{11}C_{21} = C_{11}\left(A_{11}D_{21} - B_{11}C_{21}\right) = 0; \\ \mu D_{21} &= A_{11}D_{11}D_{21} - B_{11}C_{11}D_{21} \\ &= D_{11}B_{11}C_{21} - B_{11}C_{11}D_{21} = B_{11}\left(D_{11}C_{21} - C_{11}D_{21}\right) = 0. \end{aligned}$$

□

The parabolic group $\Gamma_\infty(\mathbb{R})$ is used in the construction of Fourier Jacobi expansions. The intersection of this parabolic subgroup with the theta group may be constructed in terms of more elementary groups as follows: Consider the Heisenberg group $H(\mathbb{F}) = \mathbb{F}^3 = \{(\lambda, v, k) \in \mathbb{F}^3\}$ with the multiplication $(\lambda_1, v_1, k_1)(\lambda_2, v_2, k_2) = (\lambda_1 + \lambda_2, v_1 + v_2, k_1 + k_2 + \begin{vmatrix} \lambda_1 & v_1 \\ \lambda_2 & v_2 \end{vmatrix})$. The theta group $\Gamma_1(1,2)$ produces two orbits in $H(\mathbb{Z})$ under the action that sends (λ, v, k) to $(\lambda, v, k)(\sigma \oplus 1)$ for $\sigma \in \Gamma_1(1,2)$:

$$\begin{aligned} H_e(\mathbb{Z}) &= \{(\lambda, v, k) \in \mathbb{Z}^3 : \lambda v + k \text{ is even } \}, \\ H_o(\mathbb{Z}) &= \{(\lambda, v, k) \in \mathbb{Z}^3 : \lambda v + k \text{ is odd } \}. \end{aligned}$$

We denote by $\Gamma_1(1,2)^J$ the semidirect product $\Gamma_1(1,2) \ltimes H_e(\mathbb{Z})$. By choosing the orbit which is a subgroup, this notation is consistent with that for the level one Jacobi group $\mathrm{SL}_2(\mathbb{Z})^J = \mathrm{SL}_2(\mathbb{Z}) \ltimes H(\mathbb{Z})$.

Lemma 6. *The following multiplicative homomorphisms are injective:* $i : \mathrm{GL}_2(\mathbb{R}) \to G\Gamma_\infty(\mathbb{R})$*, given by*

$$i(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & 0 & b & 0 \\ 0 & ad-bc & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $w : H(\mathbb{R}) \to \Gamma_\infty(\mathbb{R})$ *given by*

$$w(\lambda, v, k) = \begin{pmatrix} 1 & 0 & 0 & v \\ \lambda & 1 & v & k \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have a group isomorphism $\Gamma_1(1,2) \ltimes H_e(\mathbb{Z}) \to \Gamma_2(1,2)_\infty/\{\pm I\}$*, sending* $(g, h) \mapsto \pm\, i(g)w(h)$*. Let* $\epsilon = \mathrm{diag}(1, -1, 1, -1)$*. We have an exact sequence*

$$\{I\} \to \langle w\left(H_e(\mathbb{Z})\right), \epsilon\rangle \to \Gamma_2(1,2)_\infty \to \Gamma_1(1,2) \to \{I\}$$

given by sending $\gamma \in \Gamma_2(1,2)_\infty$ *to* $\left(\begin{smallmatrix} \gamma_{11} & \gamma_{13} \\ \gamma_{31} & \gamma_{33} \end{smallmatrix}\right) \in \Gamma_1(1,2)$*.*

Proof. Left to reader. □

Definition 7. *For* $t \in \mathbb{N}$*, define the paramodular group to be*

$$\Gamma^{\mathrm{para}}(t) = \left\{ \begin{pmatrix} * & t* & * & * \\ * & * & * & */t \\ * & t* & * & * \\ t* & t* & t* & * \end{pmatrix} : * \in \mathbb{Z} \right\} \cap \mathrm{Sp}_2(\mathbb{Q}).$$

Define the paramodular theta group, $\Gamma^{\mathrm{para}}(t; 1, 2) =$

$$\{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma^{\mathrm{para}}(t) : A'C = \left(\begin{smallmatrix} a & * \\ * & bt \end{smallmatrix}\right), B'D = \left(\begin{smallmatrix} c & * \\ * & d/t \end{smallmatrix}\right), a, b, c, d \in 2\mathbb{Z}\}.$$

The moduli space interpretation of these groups was mentioned in the Introduction. The groups $\Gamma^{\mathrm{para}}(t)$ and $\Gamma^{\mathrm{para}}(t; 1, 2)$ have a common normalizer $\mu_t \in \mathrm{Sp}_2(\mathbb{R})$ with the property that $\mu_t^2 = -I$; we have

$$\mu_t = \begin{pmatrix} 0 & \sqrt{t} & 0 & 0 \\ \frac{-1}{\sqrt{t}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{t}} \\ 0 & 0 & -\sqrt{t} & 0 \end{pmatrix}.$$

We now determine the parabolic part of the paramodular groups. For $t \in \mathbb{N}$, define γ_t as below and set $\Gamma_2(1,2)_\infty[t] = \langle \Gamma_2(1,2)_\infty, \gamma_t \rangle$, the group generated by $\Gamma_2(1,2)_\infty$ and γ_t inside $\mathrm{Sp}_2(\mathbb{Q})$, where

$$\gamma_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2/t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lemma 8. *We have* $\Gamma_\infty(\mathbb{Q}) \cap \Gamma^{\mathrm{para}}(t;1,2) = \Gamma_2(1,2)_\infty[t]$.

Proof. The "$\supseteq$" inclusions is easy. Take $\delta \in \Gamma_\infty(\mathbb{Q}) \cap \Gamma^{\mathrm{para}}(t;1,2)$, and write

$$\delta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a & 0 & b & c \\ d & \epsilon_1 & e & f/t \\ g & 0 & h & i \\ 0 & 0 & 0 & \epsilon_2 \end{pmatrix}$$

for some $a,b,c,d,e,f,g,h,i \in \mathbb{Z}$ and $\epsilon_1, \epsilon_2 \in \{-1,+1\}$. The diagonal of $A'C$ and the upper left entry of $B'D$ are even integers because $\delta \in \Gamma^{\mathrm{para}}(t;1,2)$; the lower right entry of $B'D$ is an even multiple of $1/t$. So $ci + \epsilon_2 f/t = 2z/t$ for some $z \in \mathbb{Z}$. In particular, $tci + \epsilon_2 f$ is even. We proceed by cases.

If t is odd, then let $\beta = \gamma_t^{\epsilon_2 f(t-1)/2}\delta$ and we see that its $(2,4)$ entry is $\epsilon_2^2 f = f$ and so $\beta \in \Gamma^{\mathrm{para}}(t;1,2) \cap \Gamma_\infty(\mathbb{Z})$. We now show that the lower right entry of its "$B'D$" is even; it is $ci + \epsilon_2 f \equiv tci + \epsilon_2 f \equiv 0 \mod 2$ because t is odd. Thus $\beta \in \Gamma_2(1,2)_\infty$. Then $\delta = \gamma_t^{-\epsilon_2 f(t-1)/2}\beta \in \Gamma_2(1,2)_\infty[t]$.

If t is even, then the condition that $tci + \epsilon_2 f$ is an even integer forces f to be even. Then let $\beta = \gamma_t^{-\epsilon_2 f/2 - cit/2}\delta$ to see that its $(2,4)$ entry is $-\epsilon_2 ci$ and so $\beta \in \Gamma^{\mathrm{para}}(t;1,2) \cap \Gamma_\infty(\mathbb{Z})$. We now show that the lower right entry of its "$B'D$" is an even integer; it is $ci - \epsilon_2^2 ci = 0$. Thus $\beta \in \Gamma_2(1,2)_\infty$ and $\delta = \gamma_t^{\epsilon_2 f/2 + cit/2}\beta \in \Gamma_2(1,2)_\infty[t]$. $\square$

Proofs about generators are best done inside an integral version of the group. To this end, denote $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right)$ and $I_t = \left(\begin{smallmatrix} I & 0 \\ 0 & T \end{smallmatrix}\right)$, and $E_t = \left(\begin{smallmatrix} 0 & T \\ -T & 0 \end{smallmatrix}\right)$. For any group G, denote $G^I = I_t^{-1} G I_t$. Then

$$\begin{aligned} (2) \quad \Gamma^{\mathrm{para}}(t;1,2)^I = \{ & g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_4(\mathbb{Z}) : g' E_t g = E_t \\ & \text{and } \left(T^{-1}\alpha' T \gamma\right)_0 \equiv 0 \mod 2, \quad \left(T^{-1}\beta' T \delta\right)_0 \equiv 0 \mod 2 \}. \end{aligned}$$

The presentation (2) makes it clear that the integral version of the paramodular theta group $\Gamma^{\mathrm{para}}(t;1,2)^I$ is a natural generalization of the theta group to nonprincipal polarizations and that when $t=1$, we have the equalities $\Gamma^{\mathrm{para}}(t;1,2) = \Gamma^{\mathrm{para}}(t;1,2)^I = \Gamma_2(1,2)$.

Definition 9. *For $t \in \mathbb{N}$, define the group*

$$G_t = \langle \Gamma_2(1,2)_\infty[t],\ \mu_t \Gamma_2(1,2)_\infty[t] \mu_t \rangle.$$

We will in due course show $G_t = \Gamma^{\mathrm{para}}(t;1,2)$. Compare this with the generators given by Gritsenko for $\Gamma^{\mathrm{para}}(t)$, see [9].

Lemma 10. *The following matrices are elements of G_t^I: $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$,*

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},\ g(\lambda, v, k, \ell) = \begin{pmatrix} 1 & 0 & 0 & tv \\ \lambda & 1 & v & kt+2\ell \\ 0 & 0 & 1 & -\lambda t \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$Jg(\lambda, v, k, \ell)J^{-1} = \begin{pmatrix} 1 & -\lambda t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -vt & 1 & 0 \\ -v & -kt-2\ell & \lambda & 1 \end{pmatrix},$$

whenever $k + \lambda v \in 2\mathbb{Z}$ and $k, \lambda, v, \ell \in \mathbb{Z}$.

Also, $i(A)$ and $j(A)$ for $A \in \Gamma_1(1,2)$, where

$$j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

Proof. We have $J = I_t^{-1}\mu_t i(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))\mu_t^{-1} i(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)) I_t \in G_t^I$ and $E_1 = I_t^{-1}\mu_t i(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))\mu_t^{-1} I_t \in G_t^I$. The element $g(k,\lambda,v,\ell)$ is in the Heisenberg part $w(H_e(\mathbb{Z}))^I \subseteq {\Gamma_2(1,2)_\infty}^I$ and the conjugate $Jg(k,\lambda,v,\ell)J^{-1}$ is therefore in G_t^I. Since $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ generate $\Gamma_1(1,2)$, we have $\forall A \in \Gamma_1(1,2)$, $j(A) \in G_t^I$. We already know $\forall A \in \Gamma_1(1,2)$, $i(A) \in G_t^I$. ☐

Proposition 11. *For $t \in \mathbb{N}$, $\Gamma^{\mathrm{para}}(t;1,2) = G_t$.*

Proof. Since $G_t \subseteq \Gamma^{\mathrm{para}}(t;1,2)$ is easily checked, we only prove the inclusion $\Gamma^{\mathrm{para}}(t;1,2)^I \subseteq G_t^I$. Take any $\gamma \in \Gamma^{\mathrm{para}}(t;1,2)^I$. Recall the notation $(\gamma)_2 = (u_1,u_2,u_3,u_4)$ to mean the second column of γ written as a row 4-tuple. Since γ is integral of determinant one, we know that $\gcd(u_1,u_2,u_3,u_4) = 1$.

STEP 1: $\exists \beta_0 \in G_t^I : (\beta_0\gamma)_2 = (x_1,x_2,x_3,x_4)$ and $x_4 \neq 0$.
Note at least one of the u_i must be nonzero. If $u_4 \neq 0$, then $\beta_0 = I$ works. If $u_4 = 0$ and $u_3 \neq 0$, then $\beta_0 = Jg(1,0,0,0)J^{-1}$ works. If $u_4 = 0$ and $u_1 \neq 0$, then $\beta_0 = Jg(1,0,0,0)J^{-1}\, i(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))$ works. Finally, if $u_4 = 0$ and $u_2 \neq 0$, then $\beta_0 = j(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))$ works. Note that we always have $\gcd(x_1,x_2,x_3,x_4) = 1$.

STEP 2: $\exists \beta_1 \in G_t^I : (\beta_1\beta_0\gamma)_2 = (y_1, y_2, y_3, y_4)$ and $\gcd(y_2, y_4) = 1$. Set $w = \gcd(x_2, x_4)$ and $w_2 = \gcd(x_1, x_3)$ and $w_3 = \gcd(x_4/w, w^{|x_4|})$. We make the following comments: $w \neq 0$ because $x_4 \neq 0$. There are $a, b \in \mathbb{Z}$ such that $w_2 = ax_1 + bx_3$. Finally, $\gcd(x_4/(ww_3), w) = 1$ and for any prime p, $p|w_3$ implies $p|w$.

Let $\beta_1 = g(\lambda, v, k, \ell)$ with $\lambda = ax_4/(ww_3)$, $v = bx_4/(ww_3)$, $k = -\lambda v$ and $\ell = 0$ so that

$$\beta_1 = g(\lambda, v, k, \ell) = \begin{pmatrix} 1 & 0 & 0 & \frac{bx_4t}{ww_3} \\ \frac{ax_4}{ww_3} & 1 & \frac{bx_4}{ww_3} & kt \\ 0 & 0 & 1 & -\frac{ax_4t}{ww_3} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $(\beta_1\beta_0\gamma)_2 = (y_1, y_2, y_3, y_4)$ where

$$y_4 = x_4 \text{ and } y_2 = x_2 + \gcd(x_1, x_3)\frac{x_4}{ww_3} + ktx_4.$$

It is already the case that $\gcd(y_2, y_4) = 1$. Consider any prime $p|y_4$.
Case $p|w$: If $p|w$ then $p|x_2$, and $p \not| \gcd(x_1, x_3)$ since $\gcd(x_1, x_2, x_3, x_4) = 1$. But also $p \not| \frac{x_4}{ww_3}$ since $\gcd(x_4/(ww_3), w) = 1$ and so $p \not| y_2$.
Case $p \not| w$: If $p \not| w$, then $p \not| x_2$ and $p| \frac{x_4}{w}$ since $p|x_4$ and $p \not| w$. Furthermore $p \not| w_3$ (else $p|w$) so that $p|\frac{x_4}{ww_3}$. Then $p \not| y_2$. In either case $p \not| \gcd(y_2, y_4)$ and thus $\gcd(y_2, y_4) = 1$.

STEP 3: $\exists \beta_2 \in G_t^I$ such that $(\beta_2\beta_1\beta_0\gamma)_2 = (z_1, 1, z_3, 0)$.
Note that if one of y_2, y_4 is even (and hence the other is odd) then we can find $A \in \Gamma_1(1, 2)$ such that $(j(A)\beta_1\beta_0\gamma)_2 = (z_1, 1, z_3, 0)$. In this case we may take $\beta_2 = j(A)$. If both y_2, y_4 are odd, then (y_1, y_2, y_3, y_4) being the second column of a $\Gamma^{\text{para}}(t; 1, 2)^I$ matrix forces $t|y_1$, $t|y_3$ and $y_1y_3/t + y_2y_4 \equiv 0 \mod 2$ which forces y_1, y_3 to be odd as well. Then $g(1, 0, 0, 0)\beta_1\beta_0\gamma$ satisfies $(g(1, 0, 0, 0)\beta_1\beta_0\gamma)_2 = (y_1, y_1+y_2, y_3-ty_4, y_4)$. Then $y_1 + y_2$ is even and y_4 is still odd, so that $\beta_2 = j(A)g(1, 0, 0, 0)$ suffices by the first argument.

STEP 4: $\exists \beta_3 \in G_t^I$ such that $(\beta_3\beta_2\beta_1\beta_0\gamma)_2 = (0, 1, z_3, z_1z_3/t)$.
From $\beta_2\beta_1\beta_0\gamma \in \Gamma^{\text{para}}(t; 1, 2)^I$ we see that $z_1z_3/t \equiv 0 \mod 2$, and that $t|z_1$ and $t|z_3$. Then $\beta_3 = g(z_1/t, 0, 0, 0)$ gives us $(\beta_3\beta_2\beta_1\beta_0\gamma)_2 = (0, 1, z_3, z_1z_3/t)$.

STEP 5: $\exists \beta_4 \in G_t^I$ such that $(\beta_4\beta_3\beta_2\beta_1\beta_0\gamma)_2 = (z_3, 1, 0, 0)$.
Note that z_1z_3/t is even and $\beta_4 = i(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right))j(\left(\begin{smallmatrix} 1 & 0 \\ -z_1z_3/t & 1 \end{smallmatrix}\right)) \in G_t^I$ works.

STEP 6: $\exists \beta_5 \in G_t^I$ such that $(\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0\gamma)_2 = (0, 1, 0, 0)$.
Use $\beta_5 = g(z_3/t, 0, 0, 0) \in G_t^I$.

By Lemma 5, we have $\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0\gamma \in G\Gamma_\infty(\mathbb{Z}) \cap \Gamma^{\text{para}}(t; 1, 2)^I$. Now, $G\Gamma_\infty(\mathbb{Z}) \cap \Gamma^{\text{para}}(t; 1, 2)^I \subseteq (G\Gamma_\infty(\mathbb{Q}) \cap \Gamma^{\text{para}}(t; 1, 2))^I$ and we have

$G\Gamma_\infty(\mathbb{Q}) \cap \Gamma^{\mathrm{para}}(t;1,2) = \Gamma_\infty(\mathbb{Q}) \cap \Gamma^{\mathrm{para}}(t;1,2) = \Gamma_2(1,2)_\infty[t]$, where the last equality is by Lemma 8. Thus $\beta_5\beta_4\beta_3\beta_2\beta_1\beta_0\gamma \in \Gamma_2(1,2)_\infty[t]^I \subseteq G_t^I$ and $\gamma \in G_t^I$. $\square$

Lemma 12. *We have* $\mathrm{Sp}_2(\mathbb{Z}) \subseteq \Gamma^{\mathrm{para}}(t)\, U\, \Gamma_\infty(\mathbb{Q})$, *where*

$$U = \left\{ \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -c & 1 \end{pmatrix} : c \in \mathbb{Z} \right\}.$$

Proof. Take any $\alpha \in \mathrm{Sp}_2(\mathbb{Z})$. Since the second column of α must have relatively prime entries, by a similar argument to Steps 1 and 2 of the proof to Proposition 11 we can find a $\beta_1 = \begin{pmatrix} 1 & 0 & 0 & v \\ \lambda & 1 & v & k \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^{\mathrm{para}}(t) \cap \mathrm{Sp}_2(\mathbb{Z})$ such that $(\beta_1\alpha)_2 = (y_1, y_2, y_3, y_4)$ where $\gcd(y_2, y_4) = 1$. Let $g = \gcd(ty_2, y_4) = aty_2 + by_4$ for some $a, b \in \mathbb{Z}$. Note $g|t$. Then let $\beta_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b/t \\ 0 & 0 & 1 & 0 \\ 0 & -y_4t/g & 0 & ty_2/g \end{pmatrix} \in \Gamma^{\mathrm{para}}(t)$ so that $(\beta_2\beta_1\alpha)_2 = (y_1, g/t, y_3, 0)$. Next let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} y_1 \\ y_3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)$ where $z = \gcd(y_1, y_3)$. Then let $\beta_3 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0/t \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^{\mathrm{para}}(t)$ so that $(\beta_3\beta_2\beta_1\alpha)_2 = (z, g/t, 0, 0)$. Finally, let $u = \begin{pmatrix} 1 & zt/g & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -zt/g & 1 \end{pmatrix}$ so that $(u^{-1}\beta_3\beta_2\beta_1\alpha)_2 = (0, g/t, 0, 0)$. The $\frac{zt}{g}$ are the integers c in the statement of the Lemma. Calling $\gamma = u^{-1}\beta_3\beta_2\beta_1\alpha$, then $\gamma \in \mathrm{Sp}_2(\mathbb{Q})$ and $(\gamma)_2 = (0, g/t, 0, 0)$ forces $\gamma \in \Gamma_\infty(\mathbb{Q})$ by Lemma 5. Then $\alpha = \beta_1^{-1}\beta_2^{-1}\beta_3^{-1}u\gamma$ says that $\alpha \in \Gamma^{\mathrm{para}}(t)\, U\, \Gamma_\infty(\mathbb{Q})$. $\square$

Lemma 13. *For* $\Gamma^{\mathrm{para}}(t;1,2)\backslash\Gamma^{\mathrm{para}}(t)$, *a complete list of right coset representatives can be taken to be*

$$C_1 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix},\ C_2 = \begin{pmatrix} 1&0&1&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix},\ C_3 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&1/t \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix},\ C_4 = \begin{pmatrix} 1&0&1&0 \\ 0&1&0&1/t \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix},$$

$$C_5 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 1&0&1&0 \\ 0&0&0&1 \end{pmatrix},\ C_6 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&1/t \\ 1&0&1&0 \\ 0&0&0&1 \end{pmatrix},\ C_7 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&t&0&1 \end{pmatrix},\ C_8 = \begin{pmatrix} 1&0&1&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&t&0&1 \end{pmatrix},$$

$$C_9 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 1&0&1&0 \\ 0&t&0&1 \end{pmatrix},\ C_{10} = \begin{pmatrix} 1&0&0&1 \\ 0&1&1&0 \\ 1&0&1&1 \\ 0&t&t&1 \end{pmatrix}$$

for t *odd, and we can omit* C_{10} *for* t *even.*

Proof. It is a straightforward calculation to check that the set of cosets $\{\Gamma^{\mathrm{para}}(t;1,2)C_i\}_{i=1}^{10}$ is stable under right multiplication by the following set of generators for $\Gamma^{\mathrm{para}}(t)$:

$$\alpha := \begin{pmatrix} 1&0&1&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix},\ \beta := \begin{pmatrix} 1&0&0&0 \\ 0&1&0&1/t \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \gamma := \begin{pmatrix} 1&0&0&1 \\ 0&1&1&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \delta := \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1/t \\ -1&0&0&0 \\ 0&-t&0&0 \end{pmatrix},$$

which shows that $\Gamma^{\mathrm{para}}(t) = \bigcup_{i=1}^{10} \Gamma^{\mathrm{para}}(t;1,2)C_i$. It is another simple calculation to see that $C_i C_j^{-1} \notin \Gamma^{\mathrm{para}}(t;1,2)$ when $i \neq j$ except in the case when t is even and $\{i,j\} = \{9,10\}$; this shows that the coset representatives are nonredundant except that we can omit C_{10} when t is even. One can check that the permutations of cosets induced by the right multiplication of these generators as cycles in S_{10} (or in S_9 for even t) are: $\bar{\alpha} = (12)(34)(78)$ and $\bar{\beta} = (13)(24)(56)$ and $\bar{\delta} = (25)(37)(49)(68)$ in either case, whereas $\bar{\gamma}$ is the identity for even t and $(56)(78)(9\,10)$ for odd t. □

Let $\Delta_2(\mathbb{F}) = \{ \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in \mathrm{Sp}_2(\mathbb{F}) \}$.

Proposition 14. *Let $t \in \mathbb{N}$.*

(1) *For t odd, we have*

$$\mathrm{Sp}_2(\mathbb{Z}) \subseteq \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q}).$$

(2) *For t even but with $t/2$ odd, we have*

$$\begin{aligned}\mathrm{Sp}_2(\mathbb{Z}) \subseteq\ & \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q}) \\ & \cup \Gamma^{\mathrm{para}}(t;1,2)\mu_t^{-1}\Gamma_\infty(\mathbb{Z})\mu_t\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q}).\end{aligned}$$

Proof. From Lemma 12 and Lemma 13, we have that

$$\mathrm{Sp}_2(\mathbb{Z}) \subseteq \bigcup_{i=1}^{10} \Gamma^{\mathrm{para}}(t;1,2)C_i U \Gamma_\infty(\mathbb{Q}),$$

where U is as defined in Lemma 12. It is clear that $C_i \subseteq \Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})$ for $i = 1, \dots, 6$, and so we have the inclusion $\Gamma^{\mathrm{para}}(t;1,2)C_i U\Gamma_\infty(\mathbb{Q}) \subseteq \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q})$ for these i.

For the case where t is odd, we have

$$C_7 = \begin{pmatrix} 1 & -t & 0 & 0 \\ -1 & 1+t & 0 & 0 \\ -1-t & 0 & 1+t & 1 \\ -2t & t(t+1) & t & 1 \end{pmatrix} \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 1&0&1&0 \\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} t+1 & t & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -t & t+1 \end{pmatrix}$$

and $C_8 = C_7 \begin{pmatrix} 1&0&1&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$ so that $C_7, C_8 \in \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})$. And we have

$$C_9 = \begin{pmatrix} 1 & -t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -t & 1 & 0 \\ -t & t & t & 1 \end{pmatrix} \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 1&0&1&0 \\ 0&0&0&1 \end{pmatrix} \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{pmatrix}$$

and $C_{10} = C_9 \begin{pmatrix} 1&0&0&1 \\ 0&1&1&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}$ so that $C_9, C_{10} \in \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})$. Then $\Gamma^{\mathrm{para}}(t;1,2)C_i U\Gamma_\infty(\mathbb{Q}) \subseteq \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q})$ for $i = 7, 8, 9, 10$ as well, which proves item (1).

For t even (and $t/2$ odd), we use that

$$C_7 = \mu_t^{-1} C_5 \mu_t$$

so that $C_7, C_8 \in \mu_t^{-1}\Gamma_\infty(\mathbb{Z})\mu_t\Delta_2(\mathbb{Q})$. The final case is C_9 when t is even. We will manipulate C_9u for any $u \in U$. Any $u \in U$ is of the form $u = \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -c & 1 \end{pmatrix}$ with $c \in \mathbb{Z}$. So $C_9u = \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & c & 1 & 0 \\ 0 & t & -c & 1 \end{pmatrix}$. For the case where c is odd, Let $g = \gcd(c-t, c) = a(c-t) + bc$ for some $a, b \in \mathbb{Z}$. We can verify that

$$C_9u = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & t(c+1) & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{c-t}{g} & 0 & -b & 0 \\ 0 & 1 & 0 & 0 \\ \frac{c}{g} & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & g & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -g & 1 \end{pmatrix} \begin{pmatrix} a+b & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-t}{g} & 0 & \frac{c-t}{g} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and this proves $C_9u \in \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q})$ when c is odd. For the case where c is even, we can verify that

$$\begin{aligned} C_9u = &\begin{pmatrix} 1 & tc/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -tc/2 & 1 \end{pmatrix} \begin{pmatrix} 1-t/2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \\ &\mu_t^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1+c^2/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mu_t \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -t/2 & 0 & 1-t/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

One important note is that we are assuming that $t/2$ is odd so that $\begin{pmatrix} 1-t/2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \Gamma^{\mathrm{para}}(t;1,2)$, and so the above proves $C_9u \in \Gamma^{\mathrm{para}}(t;1,2)\mu_t^{-1}\Gamma_\infty(\mathbb{Z})\mu_t\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q})$ when c is even. Thus

$$\begin{aligned} \Gamma^{\mathrm{para}}(t;1,2)C_9U\Gamma_\infty(\mathbb{Q}) \subseteq\ &\Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q}) \\ &\cup \Gamma^{\mathrm{para}}(t;1,2)\mu_t^{-1}\Gamma_\infty(\mathbb{Z})\mu_t\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q}). \end{aligned}$$

The proof of item (2) is now complete. □

3. Hecke Algebras

We recall the abstract Hecke algebra $\mathcal{H}_R(U,S)$. Let $U \subseteq S$ be a group contained in a semigroup inside of some larger group. For a ring R, let $\mathcal{L}_R(U,S)$ be the free R-module of finite linear combinations of the basis $U\backslash S$. A right action of U on $\mathcal{L}_R(U,S)$ is defined by $(Us)u \mapsto U(su)$, extended R-linearly. The invariant R-module is denoted

$$\mathcal{H}_R(U,S) = \{T \in \mathcal{L}_R(U,S) : \forall u \in U,\, Tu = T\}.$$

The right invariance of $\mathcal{H}_R(U,S)$ under U allows us to define a product $\mathcal{H}_R(U,S) \times \mathcal{L}_R(U,S) \to \mathcal{L}_R(U,S)$ by $\left(\sum_\alpha c_\alpha U s_\alpha\right) Us = \sum_\alpha U s_\alpha s$ for $c_\alpha \in R$ and $s_\alpha \in S$. The restriction of this product to $\mathcal{H}_R(U,S) \times \mathcal{H}_R(U,S) \to \mathcal{H}_R(U,S)$ makes $\mathcal{H}_R(U,S)$ an associative R-algebra and $\mathcal{H}_R(U,S)$ also acts on $\mathcal{L}_R(U,S)$ from the left.

Lemma 15. *Let $U_0 \subseteq S_0$ and $U \subseteq S$ be groups contained in semigroups inside of some larger groups. Let $i : (U_0, S_0) \to (U, S)$ be a relative homomorphism. Let R be a ring. If*

(L1) *There exists a subgroup $H \subseteq U$ such that $U = i(U_0)H$,*
(L2) *For all $s \in i(S_0)$, we have $sHs^{-1} \subseteq U$,*

then there is an R-algebra homomorphism $j : \mathcal{H}_R(U_0, S_0) \to \mathcal{H}_R(U, S)$ such that $j(\sum_\alpha c_\alpha U_0 x_\alpha) = \sum_\alpha c_\alpha U i(x_\alpha)$ for $c_\alpha \in R$ and $x_\alpha \in S_0$. Furthermore, if i is injective and

(L3) *$i(S_0)i(S_0)^{-1} \cap U \subseteq i(U_0)$,*

then j is injective.

Proof. Since $i(U_0) \subseteq U$ we may define a R-linear map $j : \mathcal{L}_R(U_0, S_0) \to \mathcal{L}_R(U, S)$ by $\sum_\alpha c_\alpha U_0 x_\alpha \mapsto \sum_\alpha c_\alpha U i(x_\alpha)$. To show that j restricts to an R-linear map on the Hecke algebras, select $T = \sum_\alpha c_\alpha U_0 x_\alpha \in \mathcal{H}_R(U_0, S_0)$. The right invariance of T under U_0 implies that $j(T)$ is right invariant under $i(U_0)$: $j(T)i(u_0) = j(Tu_0) = j(T)$. The right invariance of $j(T)$ under $h \in H$ follows from $(L2)$: $j(T)h = \sum_\alpha c_\alpha U i(x_\alpha)h = \sum_\alpha c_\alpha U\,(i(x_\alpha)h\,i(x_\alpha)^{-1})\,i(x_\alpha) = \sum_\alpha c_\alpha U i(x_\alpha) = j(T)$. Since $U = i(U_0)H$ by $(L1)$, we have $j(T) \in \mathcal{H}_R(U, S)$.

To show that $j : \mathcal{H}_R(U_0, S_0) \to \mathcal{H}_R(U, S)$ is a homomorphism it suffices to prove the commutativity of the following diagram:

$$\begin{array}{ccc}
\mathcal{H}_R(U_0, S_0)\times\mathcal{L}_R(U_0, S_0) & \xrightarrow{\times} & \mathcal{L}_R(U_0, S_0) \\
j \times j \downarrow & & \downarrow j \\
\mathcal{H}_R(U, S)\times\mathcal{L}_R(U, S) & \xrightarrow{\times} & \mathcal{L}_R(U, S).
\end{array}$$

We have

$$j(T(U_0x)) = \sum_\alpha c_\alpha U i(x_\alpha x) = \left(\sum_\alpha c_\alpha U i(x_\alpha)\right)(Ui(x)) = j(T)j(U_0x).$$

To show the injectivity of j given the injectivity of i and $(L3)$, write $T = \sum_\alpha c_\alpha U_0 x_\alpha \in \mathcal{H}_R(U_0, S_0)$ with distinct cosets $U_0 x_\alpha$. It suffices to show that the cosets $j(U_0 x_\alpha) = U i(x_\alpha)$ are distinct. If $Ui(x_1) = Ui(x_2)$ then $i(x_1)i(x_2)^{-1} \in i(S_0)i(S_0)^{-1} \cap U \subseteq i(U_0)$ by $(L3)$ and we conclude $U_0 x_1 = U_0 x_2$ by the injectivity of i. $\square$

We will apply Lemma 15 with the following choices:

Lemma 16. *Consider the Hecke pairs* (U_0, S_0) *and* (U, S)*:*

$U_0 = \Gamma_1(1,2)$,

$S_0 = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in G\,\mathrm{Sp}_1^+(\mathbb{Z}) : ac, bd \text{ even}, ad - bc \in \mathbb{N}\}$;

$U = \Gamma_2(1,2)_\infty$,

$S = \{(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}) \in G\,\mathrm{Sp}_2^+(\mathbb{Z}) : A'C, B'D \text{ even matrices}, AD' - BC' \in \mathbb{N}I\}$

and the relative injection $i : (U_0, S_0) \to (U, S)$ *given in Lemma 6. We have* $i(S_0)i(S_0)^{-1} \cap U = i(U_0)$*. Let* $H \subseteq U$ *be the subgroup given by* $H = \pm w(H_e(\mathbb{Z}))$*.* H *is a normal subgroup with* $U = i(U_0)H$*. In fact, for all* $s \in i(S_0)$*, we have* $sHs^{-1} \subseteq H$*. Therefore,* (U_0, S_0) *and* (U, S) *satisfy* $(L1)$*,* $(L2)$ *and* $(L3)$ *of Lemma 15.*

Proof. That H is a normal subgroup of $U = \Gamma_2(1,2)_\infty$ with $U = i(U_0)H$ follows from Lemma 6. We have

$$H = \left\{ \rho \begin{pmatrix} 1 & 0 & 0 & v \\ \lambda & 1 & v & k \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} : v, k, \lambda \in \mathbb{Z}, \rho = \pm 1, k + v\lambda \text{ even} \right\} \tag{3}$$

For condition (L2), take any $s = i((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})) \in i(S_0)$. So $ad - bc \in \mathbb{N}$ and ac, bd are even. Take a general $h \in H$ as in (3). Then

$$sHs^{-1} = \rho \begin{pmatrix} 1 & 0 & 0 & -b\lambda + av \\ d\lambda - cv & 1 & -b\lambda + av & (ad - bc)k \\ 0 & 0 & 1 & -d\lambda + cv \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For this to be in H, we need that the following is even:

$$(ad - bc)k + (d\lambda - cv)(-b\lambda + av).$$

But this can be rearranged to

$$(ad - bc)(k + v\lambda) + 2bcv\lambda - acv^2 - bd\lambda^2$$

which is even because $(k + v\lambda), ac, bd$ are all even. Thus $sHs^{-1} \subseteq H$, and condition (L2) of Lemma 15 is also satisfied.

We now show $i(S_0)i(S_0)^{-1} \cap U = i(U_0)$ even though it is easy. The general element $u \in U$ may be written

$$u = \begin{pmatrix} a & 0 & b & g \\ e & f & h & j \\ c & 0 & d & r \\ 0 & 0 & 0 & n \end{pmatrix} \in \Gamma_2(1,2)$$

where the conditions are $ad - bc = 1$, $fn = 1$, $de - ch + fr = 0$, $be + fg - ah = 0$ and $ab, cd, eh + fj$ are even. The alternate parity conditions are ac, bd and $gr+jn$ even. If this u is in $i(S_0)i(S_0)^{-1} \subseteq i(\mathrm{GL}_2^+(\mathbb{Q}))$ then e, g, h, j, and r vanish while $n=1$ and $f>0$. From the equation $1=\det(u)=(ad-bc)f$ and from $a,b,c,d,f \in \mathbb{Z}$ we get $(ad-bc)=f=1$. Therefore $u=i((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})) \in i(U_0)$. $\square$

Definition 17. *For each $m \in \mathbb{N}$, consider the operator*

$$T_m^{(1)} = \sum U_0 \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \mathcal{L}_{\mathbb{Z}}(U_0, S_0),$$

where the sum is over $a,b,d \in \mathbb{N}$ with $ad=m$, $0 \le b < 2d$, and $a,(b+d)$ both odd.

Lemma 18. *For each $m \in \mathbb{N}$, $T_m^{(1)} \in \mathcal{H}_{\mathbb{Z}}(U_0, S_0)$.*

Proof. First note that the left cosets in the above sum are disjoint because

$$\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} a_2 & b_2 \\ 0 & d_2 \end{smallmatrix}\right)^{-1} = \begin{pmatrix} \frac{a}{a_2} & \frac{b}{d_2} - \frac{b_2}{d} \\ 0 & \frac{a_2}{a} \end{pmatrix},$$

and the only way that this could be in U_0 is if $a=a_2$, hence $d=d_2$, and $\frac{b-b_2}{d}$ is even, which means that $b-b_2$ would have to be a multiple of $2d$. Next, we will show that $T_m^{(1)}$ is right invariant by elements from U_0. Since $U_0 = \Gamma_1(1,2)$ is generated as a group by the two elements $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, we only need to show right invariance by these two elements. In fact, because the above cosets are disjoint, we only need to show that a coset representative multiplied on the right by these generators always land in another of the cosets above.

First, we can easily calculate that

$$\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2\ell \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b+2a-2\ell d \\ 0 & d \end{smallmatrix}\right).$$

By picking $\ell \in \mathbb{Z}$ such that $0 \le b+2a-2\ell d < 2d$ and noting that $(b+2a-2\ell d)$ has the same parity as b, then $\left(\begin{smallmatrix} a & b+2a-2\ell d \\ 0 & d \end{smallmatrix}\right)$ is one of the coset representatives.

Second, let $u=\gcd(b,d)$, so that u is odd. Let $x,y \in \mathbb{Z}$ such that $bx+dy=u$. Since u is odd, we can choose x,y such that b,x have the same parity and d,y have the same parity. (Just replace by $b(x+d)+d(y-b)=u$ if necessary.) Let $A = \begin{pmatrix} -b/u & y \\ -d/u & -x \end{pmatrix}$. Note $A \in U_0$. One can easily verify that

$$\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = A\left(\begin{smallmatrix} u & -ax \\ 0 & ad/u \end{smallmatrix}\right) = A\left(\begin{smallmatrix} 1 & 2\ell \\ 0 & 1 \end{smallmatrix}\right)\begin{pmatrix} u & -ax-2\ell ad/u \\ 0 & ad/u \end{pmatrix}$$

where we choose $\ell \in \mathbb{Z}$ so that $0 \le -ax-2\ell ad/u < 2ad/u$. Since ax has the same parity as b, and since $\frac{ad}{u}$ has the same parity as d, then $\begin{pmatrix} u & -ax-2\ell ad/u \\ 0 & ad/u \end{pmatrix}$ is one of the coset representatives, and we have shown that $T_m^{(1)}$ is right invariant by U_0. $\square$

So we may apply Lemma 15 to conclude the operator $j(T_m^{(1)})$ is in $\mathcal{H}_{\mathbb{Z}}(U,S)$. We denote $T_m = j(T_m^{(1)})$; namely, we have proven the following Corollary:

Corollary 19. *We may define*

$$T_m = j(T_m^{(1)}) = \sum \Gamma_2(1,2)_\infty i\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right) \in \mathcal{H}_{\mathbb{Z}}(U,S)$$

where the sum is over $a,b,d \in \mathbb{N}$ *with* $ad = m$, $0 \le b < 2d$, *and* $a,(b+d)$ *both odd.*

4. Jacobi forms and Siegel forms and the lift

For $r \in \mathbb{Q}$ and $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_g(\mathbb{R})$ and $\Omega \in \mathcal{H}_g$, we set

$$(f|_r\gamma)(\Omega) = \det(C\Omega + D)^{-r} f(\gamma \circ \Omega),$$

for the choice of holomorphic root on $\mathcal{H}_g$ determined by the condition that $\det(\Omega/i)^r > 0$ for $\Omega = iY$ with $Y \in \mathcal{P}_g(\mathbb{R})$. Let Γ be a subgroup commensurable with Γ_g. A holomorphic function $f : \mathcal{H}_g \to \mathbb{C}$ is a modular form of weight r with respect to Γ and a map $v : \Gamma \to \mathbb{C}^*$ if

$$\forall \gamma \in \Gamma, \forall \Omega \in \mathcal{H}_g, \quad (f|_r\gamma)(\Omega) = v(\gamma)\, f(\Omega),$$

and if additionally, for all $\gamma \in \Gamma_g$ and for all $Y_0 \in \mathcal{P}_g(\mathbb{R})$, $f|\gamma$ is bounded on domains of type $\{\Omega \in \mathcal{H}_g : \mathrm{Im}\,\Omega > Y_0\}$. By a result of Koecher, this boundedness condition is redundant for $g \ge 2$. We denote by $M_r(\Gamma, v)$ the vector space of such functions and use the notation $M_r(\Gamma)$ when the map v is identically 1. The space $M_r(\Gamma, v)$ is trivial unless $\mu(\gamma,\Omega) = \det(C\Omega + D)^r v(\gamma)$ is a factor of automorphy; that is, $\mu : \Gamma \times \mathcal{H}_g \to \mathbb{C}^*$ satisfies the cocycle condition: $\mu(\gamma_1\gamma_2, \Omega) = \mu(\gamma_1, \gamma_2 \circ \Omega)\mu(\gamma_2, \Omega)$. For integral weights k, $\det(C\Omega + D)^k$ is already a factor of automorphy and hence $v : \Gamma \to \mathbb{C}^*$ is a character.

The transformation formula for the theta function, see pages 176 and 182 of [12],

$$\exists v_\theta^{(g)} : \Gamma_g(1,2) \to e\,(1/8) : \forall \gamma \in \Gamma_g(1,2), \quad \theta[0]|_{1/2}\gamma = v_\theta^{(g)}(\gamma)\theta[0],$$

gives an example of a Siegel modular form of weight $1/2$; the standard thetanull $\theta[0](0,\Omega)$ gives an element of $M_{1/2}(\Gamma_g(1,2), v_\theta^{(g)})$. We write $v_\theta = v_\theta^{(g)}$ when the degree g is clear from the context.

For holomorphic $f : \mathcal{H}_g \to \mathbb{C}$ we define

$$\Phi(f)(\Omega_1) = \lim_{\lambda \longrightarrow +\infty} f\begin{pmatrix} \Omega_1 & 0 \\ 0 & i\lambda \end{pmatrix}$$

when this limit exists for all $\Omega_1 \in \mathcal{H}_{g-1}$. In particular, this operator maps $M_r(\Gamma_g)$ to $M_r(\Gamma_{g-1})$ and $M_r(\Gamma_g(1,2))$ to $M_r(\Gamma_{g-1}(1,2))$, see [7] for details. A modular form is a cusp form if $\forall \gamma \in \Gamma_g$, $\Phi(f|_r\gamma) = 0$. We shall denote by $S_r(\Gamma, v)$ the subspace of cusp forms and use the notation $S_r(\Gamma)$ when v is identically 1. We let $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$.

Definition 20. *Let $k, m \in \mathbb{Q}$. Let $\Gamma \subseteq \Gamma_\infty(\mathbb{Z})$ and fix a map $v : \Gamma \to \mathbb{C}^*$. The Jacobi forms with respect to Γ and v, denoted $J_{k,m}(\Gamma, v)$, are the vector space of holomorphic $\phi : \mathcal{H}_1 \times \mathbb{C} \to \mathbb{C}$ such that for all $\gamma \in \Gamma$, we have $\tilde{\phi}|_k\gamma = v(\gamma)\, \tilde{\phi}$, where we define*

$$\tilde{\phi}(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)) = \phi(\tau, z)e(mw),$$

and for all $\gamma \in \Gamma_\infty(\mathbb{Z})$, we have that the Fourier expansion for $\tilde{\phi}|_k\gamma$ is supported on semidefinite index matrices, namely $(\tilde{\phi}_k\gamma)(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)) = \sum_{s \geq 0} c(s)e(\mathrm{tr}(s\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)))$, where $s \geq 0$ indicates s is summed over only semidefinite 2×2 matrices. Furthermore, we say ϕ is a Jacobi cusp form and write $\phi \in J^{\mathrm{cusp}}_{k,m}(\Gamma, v)$ if for all $\gamma \in \Gamma_\infty(\mathbb{Z})$, we have that the Fourier expansion for $\tilde{\phi}|_k\gamma$ has no nonzero coefficients at indefinite index matrices, namely $(\tilde{\phi}_k\gamma)(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)) = \sum_{s > 0} c(s)e(\mathrm{tr}(s\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)))$, where $s > 0$ indicates s is summed over only positive definite 2×2 matrices. When v is identically 1, we write $J_{k,m}(\Gamma) = J_{k,m}(\Gamma, v)$ and similarly for cusp forms.

We study $J_{k,t/2}(\Gamma_2(1,2)_\infty)$ in this article. Note that $\Gamma_2(1,2)_\infty$ contains translation matrices of the form $\left(\begin{smallmatrix} I & S \\ 0 & I \end{smallmatrix}\right)$ where S is symmetric integral with even diagonal entries. This implies that $\phi \in J^{\mathrm{cusp}}_{k,t/2}(\Gamma_2(1,2)_\infty)$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n, r \in \mathbb{Z}: \\ tn - r^2 > 0,\, n > 0}} c(n, r)e(\tfrac{1}{2}n\tau + rz).$$

For $g = 2$, the Fourier Jacobi expansion of $\theta[0]$

$$\theta[0]\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \theta[0](0, \tau) + 2\theta[0](z, \tau)e\left(\omega/2\right) + \ldots$$

shows that $\theta[0](z, \tau)$ is automorphic with respect to $\Gamma_2(1,2) \cap \Gamma_\infty(\mathbb{Z}) = \Gamma_2(1,2)_\infty$ of weight $1/2$ and index $1/2$. Thus $\theta[0](z, \tau)$ gives an element of $J_{\frac{1}{2},\frac{1}{2}}(\Gamma_2(1,2)_\infty, v_\theta)$.

The definition of Jacobi form above is equivalent to the usual one. The group $\Gamma_1(1,2)^J = \Gamma_1(1,2) \ltimes H_e(\mathbb{Z})$ is isomorphic to $\Gamma_2(1,2)_\infty/\{\pm I\}$ by Lemma 6, and this shows the equivalence to the usual definition by taking generators of $\Gamma_1(1,2)$ and $H_e(\mathbb{Z})$. These transformations are

$$\forall\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(1,2),$$

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = v\left(i\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)(c\tau+d)^k e\left(\frac{cmz^2}{c\tau+d}\right)\phi(\tau,z),$$

$$\forall(\lambda, v, \kappa) \in H_e(\mathbb{Z}),$$

$$\phi(\tau, z+\lambda\tau+v) = v\left(w(\lambda,v,\kappa)\right) e\left(m(\lambda^2\tau + 2\lambda z + (\lambda v + \kappa))\right)\phi(\tau,z).$$

The first equation shows that if $\phi \in J_{k,m}(\Gamma, v)$ then $\phi(\tau, 0)$ gives an element of $J_{k,0}(\Gamma, v)$. Using the isomorphism $M_k(i^{-1}(\Gamma), i^*v) = J_{k,0}(\Gamma, v)$ we have $M_{k_1}(i^{-1}(\Gamma), i^*v_1)J_{k_2,m}(\Gamma, v_2) \subseteq J_{k_1+k_2,m}(\Gamma, v_1v_2)$. We use this containment in the statement of Corollary 2 to write

$$S_{k-\frac{1}{2}}(\Gamma_1(1,2), (v_\theta^{(1)})^{2k-1})\theta[0](z,\tau) \subseteq J^{\mathrm{cusp}}_{k,\frac{1}{2}}(\Gamma_1(1,2)^J, (v_\theta^{(2)})^{2k}).$$

Here one needs to check that $i^*(v_\theta^{(2)}) = v_\theta^{(1)}$ on $\Gamma_1(1,2)$. This can be done by restricting the theta function to diagonal $\left(\begin{smallmatrix} \tau & 0 \\ 0 & \omega \end{smallmatrix}\right) \in \mathcal{H}_2$.

Definition 21. *Fix $t \in \mathbb{N}$ and $k \in \mathbb{Z}$. For $\phi \in J^{\mathrm{cusp}}_{k,\frac{1}{2}t}(\Gamma_2(1,2)_\infty)$, define*

$$\tilde{\phi}\left(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)\right) = \phi(\tau,z)e(\tfrac{1}{2}tw).$$

Define a formal series F_ϕ by

$$F_\phi = \sum_{m=1}^{\infty} m^{2-k}(-1)^{m+1}\tilde{\phi}\,|_k\,T_m = \sum_{m=1}^{\infty}\sum_{a,b,d} m^{2-k}(-1)^{m+1}\tilde{\phi}\,|_k\,i\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right),$$

where the inner sum is over $a, b, d \in \mathbb{N}$ with $ad = m$, $0 \le b < 2d$, and $a, (b+d)$ both odd.

Proposition 22. *Let $\phi \in J^{\mathrm{cusp}}_{k,\frac{1}{2}t}(\Gamma_2(1,2)_\infty)$ have expansion*

$$\phi(\tau,z) = \sum_{n,r\in\mathbb{Z}:\, tn-r^2>0,\, n>0} c(n,r)e(\tfrac{1}{2}n\tau + rz).$$

Then the formal series $F_\phi(\Omega)$ may be rearranged to

$$F_\phi(\Omega) = \sum_{\substack{T=\left(\begin{smallmatrix} n & r \\ r & m \end{smallmatrix}\right): \\ mn-r^2>0,\, n>0, m>0 \\ t|m}} a(T)e(\tfrac{1}{2}\mathrm{tr}(T\Omega))$$

where the coefficients $a(T)$ are given by

$$a\left(\begin{pmatrix} n & r \\ r & m \end{pmatrix}\right) = (-1)^{(m/t+1)(n+1)} \sum_{\substack{a|(n,r,m/t) \\ a \text{ odd}}} a^{k-1}c\left(\frac{mn}{ta^2}, \frac{r}{a}\right).$$

Proof. Applying the action of T_m to $\tilde{\phi}$, we get

$$F_\phi((\begin{smallmatrix}\tau & z\\ z & w\end{smallmatrix})) = \sum_{m=1}^{\infty} m^{2-k}(-1)^{m+1} \sum_{\substack{ad=m\\ 0\le b<2d\\ a,b+d \text{ odd}}} m^{2k-3}d^{-k}\phi(\tfrac{a\tau+b}{d}, az)e(\tfrac{1}{2}tmw)$$

$$\begin{aligned}
&=\sum_{m=1}^{\infty} m^{k-1}(-1)^{m+1} \sum_{\substack{ad=m\\ 0\le b<2d\\ a,b+d \text{ odd}}} d^{-k}\cdot \\
&\qquad\qquad \sum_{tn-r^2>0} c(n,r)e(\tfrac{1}{2}n\tfrac{a\tau+b}{d} + raz + \tfrac{1}{2}tmw) \\
&=\sum_{m=1}^{\infty} \sum_{tn-r^2>0} \sum_{\substack{ad=m\\ a \text{ odd}}} m^{k-1}(-1)^{m+1}c(n,r)d^{-k}\cdot \\
&\qquad\qquad e(\tfrac{1}{2}n\tfrac{a\tau}{d} + raz + \tfrac{1}{2}tmw) \sum_{\substack{0\le b<2d\\ b+d \text{ odd}}} e(\tfrac{nb}{2d}).
\end{aligned}$$

If m is odd, then d is odd, and b must be even and we would have

$$\sum_{\substack{0\le b<2d\\ b+d \text{ odd}}} e(\tfrac{nb}{2d}) = \sum_{j=0}^{d-1} e(\tfrac{nj}{d}) = \begin{cases} d & \text{if } d|n \\ 0 & \text{otherwise.}\end{cases}$$

If m is even, then d is even and b must be odd and we would have

$$\begin{aligned}
\sum_{\substack{0\le b<2d\\ b+d \text{ odd}}} e(\tfrac{nb}{2d}) &= \sum_{j=0}^{d-1} e(\tfrac{n(2j+1)}{2d}) \\
&= e(\tfrac{1}{2}\tfrac{n}{d}) \sum_{j=0}^{d-1} e(\tfrac{nj}{d}) = \begin{cases} d(-1)^{\frac{n}{d}} & \text{if } d|n \\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

We can unify these two cases of m even or odd by

$$\sum_{\substack{0\le b<2d\\ b+d \text{ odd}}} e(\tfrac{nb}{2d}) = \begin{cases} d(-1)^{(m+1)\frac{n}{d}} & \text{if } d|n \\ 0 & \text{otherwise.}\end{cases}$$

Plugging this into the formula, we can make the substitution $n = dn_1$ where $n_1 \in \mathbb{Z}$ to get

$$F_\phi((\begin{smallmatrix}\tau & z\\ z & w\end{smallmatrix})) = \sum_{m=1}^{\infty} \sum_{\substack{ad=m\\ a \text{ odd}}} \sum_{tdn_1-r^2>0} m^{k-1}(-1)^{(m+1)}(-1)^{(m+1)n_1}\cdot$$
$$c(dn_1, r)d^{-k+1}e(\tfrac{1}{2}n_1 a\tau + raz + \tfrac{1}{2}tmw)$$
$$= \sum_{m=1}^{\infty} \sum_{\substack{ad=m\\ a \text{ odd}}} \sum_{t\frac{m}{a}n_1-r^2>0} a^{k-1}(-1)^{(m+1)(n_1+1)}c(\tfrac{m}{a}n_1, r)\cdot$$
$$e(\tfrac{1}{2}n_1 a\tau + raz + \tfrac{1}{2}tmw).$$

Making another substitution $R = ar$ and $N = an_1$, where we sum over R, N which must be multiples of a (or equivalently, we must only use a which divide all of m, R, N), we get

$$(4)\quad F_\phi((\begin{smallmatrix}\tau & z\\ z & w\end{smallmatrix})) = \sum_{m=1}^{\infty} \sum_{tmN-R^2>0} \sum_{\substack{a|(m,R,N)\\ a \text{ odd}}} a^{k-1}(-1)^{(m+1)(N+1)}\cdot$$
$$c(\tfrac{mN}{a^2}, \tfrac{R}{a})e(\tfrac{1}{2}N\tau + Rz + \tfrac{1}{2}tmw),$$

where we used the fact that N has the same parity as n_1 because a is odd when we replaced n_1 by N in the exponent of (-1). A final substitution $M = mt$ where M ranges over $\mathbb{N}$ with $t|M$ gives

$$F_\phi((\begin{smallmatrix}\tau & z\\ z & w\end{smallmatrix})) = \sum_{\substack{M,N\in\mathbb{N}, R\in\mathbb{Z}\\ MN-R^2>0\\ t|M}} \sum_{\substack{a|(M/t,R,N)\\ a \text{ odd}}} a^{k-1}(-1)^{(M/t+1)(N+1)}c(\tfrac{MN}{ta^2}, \tfrac{R}{a})\cdot$$
$$e(\tfrac{1}{2}N\tau + Rz + \tfrac{1}{2}Mw),$$

and this proves the proposition. □

Proposition 23. *Fix $t \in \mathbb{N}$ and $k \in \mathbb{Z}$. Let $\phi \in J^{\text{cusp}}_{k,\frac{1}{2}t}(\Gamma_2(1,2)_\infty)$. The series $F_\phi(\Omega)$ converges absolutely for all $\Omega \in \mathcal{H}_2$ and $F_\phi : \mathcal{H}_2 \to \mathbb{C}$ defines a holomorphic function. Also, for $(\begin{smallmatrix}\tau & z\\ z & w\end{smallmatrix}) \in \mathcal{H}_2$ we have*

$$F_\phi(\begin{pmatrix} tw & z\\ z & \frac{1}{t}\tau\end{pmatrix}) = F_\phi((\begin{smallmatrix}\tau & z\\ z & w\end{smallmatrix})).$$

Proof. Since ϕ has its Fourier coefficients $c(n, r; \phi)$ bounded by polynomial growth, so does F_ϕ have its Fourier coefficients $a((\begin{smallmatrix}n & r\\ r & m\end{smallmatrix}))$ bounded by polynomial growth. In more detail, the cusp form ϕ has a bound $|\phi(\tau, z)| \le M_\phi\, v^{-k/2}e^{\pi t y^2/v}$, where we write $z = x+iy$ and $\tau = u+iv$ for real x, y, u, v. This implies that the Fourier coefficients of ϕ have a polynomial bound $|c(n, r; \phi)| \le A_\phi\,(2tn-r^2)^{k/2}$ where $A_\phi = (2\pi e/kt)^{k/2}M_\phi$.

A crude estimate shows $|a\left(\left(\begin{smallmatrix} n & r \\ r & m \end{smallmatrix}\right)\right)| \leq A_\phi\, m^k(2tn-r^2)^{k/2}$. This suffices to show the absolute convergence of F_ϕ on compact subsets of $\mathcal{H}_2$. Note that in the above proof of Proposition 22, in equation 4, the expression is nearly symmetric in m and N. Thus switching m and N, we see that

$$F_\phi(\begin{pmatrix} tw & z \\ z & \frac{1}{t}\tau \end{pmatrix}) = F_\phi(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)).$$

$\square$

Proposition 24. *Fix $t \in \mathbb{N}$ and let ϕ and F_ϕ be as in Proposition 23. Then $F_\phi\,|_k\,\mu_t = (-1)^k F_\phi$.*

Proof. Note that $\tilde{\phi}\,|_k\, i(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)) = \tilde{\phi}$ implies $\phi(\tau,-z) = (-1)^k\phi(\tau,z)$ and so $c(n,-r;\phi) = (-1)^k c(n,r;\phi)$. We have

$$(F_\phi\,|_k\,\mu)(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right)) = F_\phi(\begin{pmatrix} tw & -z \\ -z & \frac{1}{t}\tau \end{pmatrix}) = (-1)^k F_\phi(\begin{pmatrix} tw & z \\ z & \frac{1}{t}\tau \end{pmatrix}) = (-1)^k F_\phi(\left(\begin{smallmatrix} \tau & z \\ z & w \end{smallmatrix}\right))$$

$\square$

The following Theorem completes the proof of Theorem 1 from the Introduction. The form of the Fourier coefficients has already been given in Proposition 22.

Theorem 25. *Let $t \in \mathbb{N}$ and $k \in \mathbb{Z}$. For $\phi \in J^{\mathrm{cusp}}_{k,\frac{1}{2}t}(\Gamma_2(1,2)_\infty)$, we have $F_\phi \in M_k(\Gamma^{\mathrm{para}}(t;1,2))$ and $F_\phi|\mu_t = (-1)^k F_\phi$. If $t \not\equiv 0 \mod 4$, then we have $F_\phi \in S_k(\Gamma^{\mathrm{para}}(t;1,2))$.*

Proof. We know that F_ϕ is holomorphic from Proposition 23. From Definition 21, we know that F_ϕ is invariant under $\Gamma_2(1,2)_\infty$ because the series defining it is term by term invariant. From the form of F_ϕ in Proposition 22, it is clear that F_ϕ is invariant under γ_t and so F_ϕ is invariant under $\Gamma_2(1,2)_\infty[t]$. From Proposition 24, we know $F_\phi\,|_k\,\mu_t = (-1)^k F_\phi$ and therefore F_ϕ is invariant under $G_t = \langle \Gamma_2(1,2)_\infty[t],\, \mu_t\Gamma_2(1,2)_\infty[t]\mu_t\rangle = \Gamma^{\mathrm{para}}(t;1,2)$ by Proposition 11.

We only need to prove that F_ϕ is a cusp form when $t \not\equiv 0 \mod 4$. Take any $\beta \in \mathrm{Sp}_2(\mathbb{Z})$. Since $t \not\equiv 0 \mod 4$, by Proposition 14, we have that $\beta = \alpha\gamma_1\delta\gamma_2$, or $\beta = \alpha\mu_t^{-1}\gamma_1\mu_t\delta\gamma_2$, where $\alpha \in \Gamma^{\mathrm{para}}(t;1,2)$, $\delta \in \Delta_2(\mathbb{Q})$ and $\gamma_1 \in \Gamma_\infty(\mathbb{Z})$ and $\gamma_2 \in \Gamma_\infty(\mathbb{Q})$. Then $F_\phi|\beta = F_\phi|\gamma_1\delta\gamma_2$ or $F_\phi|\beta = (-1)^k F_\phi|\gamma_1\mu_t\delta\gamma_2$.

Since F_ϕ has no nonzero indefinite coefficients in its Fourier expansion, and since $\gamma_1 \in \Gamma_\infty(\mathbb{Z})$, we have that $F_\phi|\gamma_1$ has no nonzero indefinite coefficients. Since δ and $\mu_t\delta$ are upper triangular, then $F_\phi|\gamma_1\delta$ and $F_\phi|\gamma_1\mu_t\delta$ have no nonzero indefinite coefficients either; these two cases can be unified together by saying that $F_\phi|\beta\gamma_2^{-1}$ has no nonzero indefinite coefficients.

Consider the Siegel operator $(\Phi_2 f)(\tau) = \lim_{s\to\infty} f(\left(\begin{smallmatrix} \tau & 0 \\ 0 & is \end{smallmatrix}\right))$ for a modular form f. Since $\gamma_2 \in \Gamma_\infty(\mathbb{Q})$ and $(f|\gamma_2)(\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right)) = (*)f(\left(\begin{smallmatrix} * & * \\ * & \omega+* \end{smallmatrix}\right))$ where the $*$ depend only on τ, z and not on ω, then $\Phi_2 f = 0$ would imply $\Phi_2(f|\gamma_2) = 0$. Thus

$$\Phi_2(F_\phi|\beta) = \Phi_2((F_\phi|\beta\gamma_2^{-1})|\gamma_2) = 0.$$

Since this is true for all $\beta \in \mathrm{Sp}_2(\mathbb{Z})$, F_ϕ is a cusp form. $\square$

When $t = 1$ and k is even, we get the following corollary which we state as a theorem because it is of particular interest for the degree two chiral superstring measure.

Theorem 26. Lifting to Degree Two Theta Group for even k. *Let $k \in \mathbb{N}$ be even and $\phi \in J^{\mathrm{cusp}}_{k,\frac{1}{2}}(\Gamma_2(1,2)_\infty)$. Then $F_\phi \in S_k(\Gamma_2(1,2))$.*

Corollary 27. *For $t = 1$, if $k \in \mathbb{N}$ is odd and $\phi \in J^{\mathrm{cusp}}_{k,\frac{1}{2}}(\Gamma_2(1,2)_\infty)$ then $F_\phi = 0$.*

Proof. Since k is odd, then by Proposition 24, $F_\phi|\mu_t = -F_\phi$. Let

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Note that both $g, \epsilon \in \Gamma_2(1,2)_\infty$ (see Lemma 6) and so $F_\phi|g = F_\phi$ and $F_\phi|\epsilon = F_\phi$. But it is straightforward to check that

$$\mu_1 g \epsilon \mu_1 g^{-1} \epsilon \mu_1 g = I$$

is the identity matrix. But $F_\phi|(\mu_1 g\epsilon\mu_1 g^{-1}\epsilon\mu_1 g) = (-1)^3 F_\phi = -F_\phi$ and $F_\phi|I = F_\phi$. This forces $F_\phi = 0$. $\square$

5. The Chiral String modular form in genus 2

Now we discuss the weight 15/2 form that gives $\Xi_2[0]$. We define here the variety of theta functions that we use. For $\Omega \in \mathcal{H}_g$, $z \in \mathbb{C}^g$ and $a, b \in \mathbb{R}^g$, define the first order theta function with characteristics a and b as a holomorphic function on $\mathbb{C}^g \times \mathcal{H}_g$ given by the series

$$\theta \begin{bmatrix} a \\ b \end{bmatrix} (z, \Omega) = \sum_{n\in\mathbb{Z}^g} e\left(\frac{1}{2}(n+a)'\Omega(n+a) + (n+a)'(n+z+b)\right).$$

For $r \in \mathbb{N}$, the r^{th} order theta functions $\theta_r[\nu] : \mathcal{H}_g \to \mathbb{C}$ are given by

$$\theta_r[\nu](\Omega) = \theta \begin{bmatrix} \nu/r \\ 0 \end{bmatrix} (0, r\Omega).$$

In $g=1$, we use the standard abbreviations $\theta_{ab}(\tau)=\theta\begin{bmatrix}a/2\\b/2\end{bmatrix}(0,\tau)$ for $a,b\in\{0,1\}$. In $g=2$, we use $\theta(\begin{smallmatrix}a_1&a_2\\b_1&b_2\end{smallmatrix})(\Omega)=\theta\begin{bmatrix}a_1/2&a_2/2\\b_1/2&b_2/2\end{bmatrix}(0,\Omega)$. The Dedekind eta function, mentioned only in connection with the $g=1$ chiral superstring measure in the Introduction, is the standard one. In the introduction, we have given $\Phi=\theta^{11}F_2-16\,\theta^7F_2^2\in S_{15/2}\left(\Gamma_0(4)^*,\tilde{v}_\theta^{15}\right)$ in terms of the generators of $\oplus_{\ell=0}^{\infty}M_{\ell/2}(\Gamma_0(4),\tilde{v}_\theta^{\ell})$:

$$\theta(\tau)=\theta_2[0](\tau)=\sum_{n\in\mathbb{Z}}q^{n^2}=1+2q+2q^4+2q^9+\dots$$

$$F_2(\tau)=\left(\frac{\theta_2[1](\tau)}{2}\right)^4=\sum_{n\in\mathbb{N}\colon n\text{ odd}}\sigma_1(n)q^n=q+4q^3+6q^5+8q^7+\dots$$

Here $\tilde{v}_\theta:\Gamma_0(4)^*\to\mathbb{C}^*$ is given explicitly by $\tilde{v}_\theta(\begin{smallmatrix}a&b\\c&d\end{smallmatrix})=v_\theta\bigl(\begin{smallmatrix}a&2b\\c/2&d\end{smallmatrix}\bigr)$. In these terms we can show directly that, for $W_4=\left(\begin{smallmatrix}0&\frac{1}{2}\\-2&0\end{smallmatrix}\right)$, we have

$$F_2|W_4=F_2-\frac{1}{2^4}\theta^4.$$

However, the following alternate expression immediately shows modularity with respect to the theta group.

$$\Phi(\tau/2)=\theta_{00}(\tau)^3\left(\frac{\theta_{00}(\tau)\theta_{01}(\tau)\theta_{10}(\tau)}{2}\right)^4\in S_{15/2}\left(\Gamma_1(1,2),v_\theta^{15}\right).$$

Consider a form $g\in S_{k-\frac{1}{2}}(\Gamma_1(1,2),v_\theta^{2k-1})$ whose Fourier expansion is $g(\tau)=\sum_{n\in\mathbb{N}}c(n;g)q^{n/2}$. Multiplication by $\theta[0]\in J_{1/2,1/2}(\Gamma_1(1,2)^J,v_\theta)$ whose Fourier expansion is $\theta[0](z,\tau)=\sum_{n\in\mathbb{N}}q^{n^2/2}\zeta^n$ produces a Jacobi form $\phi(\tau,z)=g(\tau)\theta[0](z,\tau)\in J^{\text{cusp}}_{k,1/2}(\Gamma_1(1,2)^J,v_\theta^{2k})$ whose Fourier expansion is $\phi(\tau,z)=\sum_{n\in\mathbb{N},r\in\mathbb{Z}}c(n-r^2;g)q^{n/2}\zeta^r$. Note that when $4|k$, ϕ has trivial character. In this case we have $c(n,r;\phi)=c(n-r^2;g)$ and the formula for the Fourier coefficients of the lift $L(\phi)$ is

$$a\left(\begin{pmatrix}n&r\\r&m\end{pmatrix};L(\phi)\right)=(-1)^{(m+1)(n+1)}\sum_{\substack{a|(n,r,m)\\a\text{ odd}}}a^{k-1}c(\frac{mn-r^2}{a^2};g).$$

This proves the formula for the Fourier coefficients of the chiral superstring form $\Xi_2[0]$ that was given at the conclusion of the Introduction.

6. Final Remarks

A final remark is that when $t \equiv 0 \mod 4$, we can prove that

$$\left(\begin{smallmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & t & -2 & 1 \end{smallmatrix}\right) \notin \Gamma^{\mathrm{para}}(t;1,2)\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q}) \\ \cup \Gamma^{\mathrm{para}}(t;1,2)\mu_t^{-1}\Gamma_\infty(\mathbb{Z})\mu_t\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q})$$

by showing that any matrix in the coset $\Gamma^{\mathrm{para}}(t;1,2)\left(\begin{smallmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & t & -2 & 1 \end{smallmatrix}\right)$ cannot have a 0 in the $(3,2)$ or $(4,2)$ entry but a matrix in $\Gamma_\infty(\mathbb{Z})\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q})$ must have a 0 in the $(4,2)$ entry and any matrix that happens to be in $\Gamma^{\mathrm{para}}(t;1,2)\mu_t^{-1}\Gamma_\infty(\mathbb{Z})\mu\Delta_2(\mathbb{Q})\Gamma_\infty(\mathbb{Q})$ must have a 0 in the $(3,2)$ entry. Thus the above method of proof in Theorem 25 that the lift is a cusp form does not work when $t \equiv 0 \mod 4$. It is conceivable that the lift of a Jacobi cusp form might not be a cusp form in general when $t \equiv 0 \mod 4$ but we don't know any examples of this. The intended case where $t/2$ is strictly half integral has been fully treated, as well as the slightly more general case when $t \not\equiv 0 \mod 4$.

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