THE ATIYAH–HITCHIN BRACKET FOR THE CUBIC NONLINEAR SCHRÖDINGER EQUATION. IV. THE SCATTERING POTENTIALS.

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Abstract. This is the last in a series of four papers on Poisson formalism for the cubic nonlinear Schrödinger equation with repulsive nonlinearity. In this paper we consider scattering potentials.

1. Introduction

1.1. **Statement of the problem.** In this paper we consider the cubic NLS with repulsive nonlinearity[1]

$$i\psi^\bullet = -\psi'' + 2|\psi|^2\psi,$$

where $\psi = \psi(x,t)$ is a complex decaying function on the entire line, *i.e.* $x \in \mathbb{R}^1$.

The standard assumption on the decaying potential is that it is summable ($\psi \in L^1(\mathbb{R}^1)$). Such potentials are called scattering potentials. In this paper we assume that the phase space $\mathcal{M}$ consists of all Schwartz functions. This would imply, in particular, an existence of the infinite series of integrals of motion, *etc.* Most of our considerations without difficulty can be extended to the case of summable functions.

The cubic Schrödinger equation is a Hamiltonian system

$$\psi^\bullet = \{\psi, \mathcal{H}\},$$

with the bracket

$$\{A,B\} = 2i \int\limits_{\mathbb{R}^1} \frac{\delta A}{\delta\overline{\psi}(x)}\frac{\delta B}{\delta\psi(x)} - \frac{\delta A}{\delta\psi(x)}\frac{\delta B}{\delta\overline{\psi}(x)}\, dx; \tag{1.1}$$

and Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int\limits_{\mathbb{R}^1} |\psi'|^2 + |\psi|^4\, dx.$$

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[1]Prime $'$ signifies the derivative in the variable x and dot $\bullet$ the derivative with respect to time.

The equation arises as a compatibility condition for the commutator relation for some specially chosen differential operators. This leads to an auxiliary linear spectral problem for the Dirac operator

$$\mathfrak{D}\boldsymbol{f} = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} i\partial_x + \begin{pmatrix} 0 & -i\overline{\psi} \\ i\psi & 0 \end{pmatrix} \right] \boldsymbol{f} = \frac{\lambda}{2}\boldsymbol{f}, \tag{1.2}$$

acting in the space of vector-functions $\boldsymbol{f}^T = (f_1, f_2)$.

The relation of the Hamiltonian formalism and complex geometry of the Dirac operator with general continuous potentials was considered by us in [11]. The Dirac spectral problem for rapidly decaying potentials can be treated by the methods of scattering theory. In this paper we extend our general approach introduced in [11] to these scattering potentials and exploit some specific features that arise in this case.

1.2. **Description of results.** In their seminal paper devoted to the periodic KdV problem Dubrovin and Novikov [3] wrote
We would like to emphasize the difficulty of construction of angle type variables in the periodic case in comparison with the scattering case.

The situation is similar for the nonlinear Schrödinger equation. Two cases, the periodic and the scattering substantially vary in difficulty. The construction of the action–angle variables for the scattering potentials of NLS equation is known for more then thirty years since the work of Zakharov and Manakov, [14]. The action–angle variable for the periodic NLS problem were considered in [6]. They are constructed by introducing the Dirac spectral curve and the divisor on it. The flows are linearized on the (extended) Jacobian by using a version of the Abel map.

This paper eliminates the differences between the two cases. We utilize the previous approaches of Venakides, [13], and Ercolani and McKean, [1]. The analogs of spectral curve, divisor and the Abel map are introduced and studied now in the scattering situation.

Specifically, we consider Γ a two sheeted covering of the complex plane of the spectral parameter cut along the real line. We introduce the meromorphic function $\Pi(x, Q)$ on the spectral cover Γ. With the help of this function we construct the so–called scattering divisor. Then using the scattering divisor we construct the continuous analog of the Abel map which linearizes the flow. Finally, we compute a closed form of the Poisson bracket for the function $\Pi(x, Q)$.

1.3. **Thanks.** We conclude the introduction expressing thanks to A. Its, H. McKean and I. Krichever for stimulating discussions.

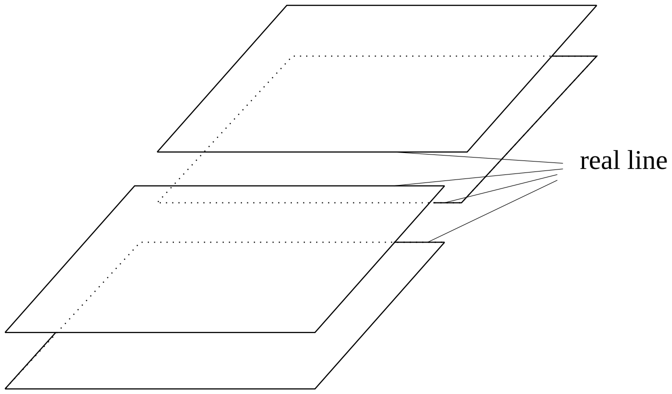

FIGURE 1. The spectral cover.

2. The Spectral Problem

2.1. **The NLS hierarchy.** We consider the NLS equation

$$i\psi^{\bullet} = -\psi'' + 2|\psi|^2\psi, \tag{2.1}$$

on the line, *i.e.* $x \in \mathbb{R}^1$. We assume that the function $\psi = \psi(x,t)$ belongs to the Schwartz' space $S(\mathbb{R}^1)$ of complex rapidly decreasing infinitely differentiable functions such that $\sup_x |(1+x^2)^n \psi^{(m)}(x)| < \infty, \quad m, n = 0, 1, \ldots$.

The NLS is a Hamiltonian system $\psi^{\bullet} = \{\psi, \mathcal{H}\}$, with Hamiltonian $\mathcal{H} = \frac{1}{2}\int\limits_{\mathbb{R}^1} |\psi'|^2 + |\psi|^4\, dx =$ *energy* and the bracket

$$\{A, B\} = 2i \int\limits_{\mathbb{R}^1} \frac{\delta A}{\delta\overline{\psi}(x)}\frac{\delta B}{\delta\psi(x)} - \frac{\delta A}{\delta\psi(x)}\frac{\delta B}{\delta\overline{\psi}(x)}\, dx. \tag{2.2}$$

The NLS equation is a compatibility condition for the zero curvature relation $[\partial_t - V_3, \partial_x - V_2] = 0$, with[2]

$$V_2 = -\frac{i\lambda}{2}\sigma_3 + Y_0 = \begin{pmatrix} -\frac{i\lambda}{2} & 0 \\ 0 & \frac{i\lambda}{2} \end{pmatrix} + \begin{pmatrix} 0 & \overline{\psi} \\ \psi & 0 \end{pmatrix}$$

and

$$V_3 = \frac{\lambda^2}{2} i\sigma_3 - \lambda Y_0 + |\psi|^2 i\sigma_3 - i\sigma_3 Y_0'.$$

We often omit the lower index and write $V = V_2$.

[2]Here and below σ denotes the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The NLS Hamiltonian $\mathcal{H} = \mathcal{H}_3$ is one in the infinite series of commuting integrals of motion

$$\begin{aligned}
\mathcal{H}_1 &= \frac{1}{2}\int\limits_{\mathbb{R}^1} |\psi|^2 dx, \\
\mathcal{H}_2 &= \frac{1}{2i}\int\limits_{\mathbb{R}^1} \overline{\psi}\psi' dx, \\
\mathcal{H}_3 &= \frac{1}{2}\int\limits_{\mathbb{R}^1} |\psi'|^2 + |\psi|^4\, dx, \quad \mathit{etc.}
\end{aligned}$$

Hamiltonians produce an infinite hierarchy of flows $e^{tX_m},\ m = 1, 2, \ldots$. Each flow e^{tX_m} of the hierarchy can be written as the zero curvature condition with the suitable operator $\partial_t - V_m$.

2.2. **Jost solutions.** In this section we introduce the classical Jost solutions. These asymptotically normalized solutions play the central role in analysis of the scattering spectral problem. Most of this material is standard but we present it here to fix notations. We omit the time variable t in all formulas of this section.

The auxiliary linear problem $(\partial_x - V)\boldsymbol{f} = 0$, can be written as an eigenvalue problem for the symmetric Dirac operator 1.2. The transition matrix $M(x, y, \lambda),\ x \geq y$; satisfies the differential equation

$$M'(x, y, \lambda) = V(x, \lambda)M(x, y, \lambda),$$

and the boundary condition $M(y, y, \lambda) = I$. The transition matrix is given by

$$M(x, y, \lambda) = \exp \int_y^x V(\xi, \lambda) d\xi.$$

The matrix $M(x, y, \lambda)$ is unimodular because V is traceless.

Also, we introduce the reduced transition matrix $T(x, y, \lambda)$, $x \geq y$; by the formula

$$T(x, y, \lambda) = E^{-1}\left(\frac{\lambda x}{2}\right) M(x, y, \lambda) E^{-1}\left(-\frac{\lambda y}{2}\right), \tag{2.3}$$

where $E(\frac{\lambda x}{2}) = \exp\left(-\frac{i\lambda x}{2}\sigma_3\right)$ is a solution of the linear problem with $\psi \equiv 0$. The matrix $T(x, y, \lambda)$ solves the equation

$$T'(x, y, \lambda) = Y_0(x)E(\lambda x)T(x, y, \lambda),$$

and satisfies the boundary condition $T(y, y, \lambda) = I$. Now the spectral parameter enters multiplicatively into the RHS of the differential equation. The solution is given by the formula

$$T(x,y,\lambda) = \exp \int_y^x Y_0(\xi) E(\lambda\xi) d\xi. \tag{2.4}$$

The symmetry of the matrix Y_0 : $\sigma_1 Y_0(x) \sigma_1 = \overline{Y_0(x)}$ is inherited by the reduced transition matrix :

$$\sigma_1 T(x,y,\overline{\lambda}) \sigma_1 = \overline{T(x,y,\lambda)}.$$

For real λ formula 2.4 and rapid decay of the potential imply an existence of the limit

$$T(\lambda) = \lim T(x,y,\lambda) = \begin{pmatrix} a(\lambda) & \overline{b}(\lambda) \\ b(\lambda) & \overline{a}(\lambda) \end{pmatrix}, \quad \text{when} \quad y \to -\infty \quad \text{and} \quad x \to +\infty;$$

and $|a(\lambda)|^2 - |b(\lambda)|^2 = 1$. Note, that $b(\lambda) \in S(\mathbb{R}^1)$ since $\psi \in S(\mathbb{R}^1)$.

We introduce *Jost solutions* $J_\pm(x,\lambda)$ as a matrix solutions of the differential equation

$$J'_\pm(x,\lambda) = V(x,\lambda) J_\pm(x,\lambda),$$

that also have prescribed asymtotics at the spatial infinity

$$J_\pm(x,\lambda) = E\left(\frac{\lambda x}{2}\right) + o(1), \quad \text{when} \quad x \to \pm\infty.$$

An existence and analytic properties of the Jost solutions follow from the integral representations

$$J_+(x,\lambda) = E\left(\frac{x\lambda}{2}\right) + \int_x^{+\infty} \Gamma_+(x,\xi) E\left(\frac{\lambda\xi}{2}\right) d\xi,$$

$$J_-(x,\lambda) = E\left(\frac{x\lambda}{2}\right) + \int_{-\infty}^{x} \Gamma_-(x,\xi) E\left(\frac{\lambda\xi}{2}\right) d\xi.$$

The kernels $\Gamma_\pm$ are unique and infinitely smooth in both variables. Introducing the notation $J_\pm = \left[\boldsymbol{j}_\pm^{(1)}, \boldsymbol{j}_\pm^{(2)}\right]$ we see from the integral representations that $\boldsymbol{j}_-^{(1)}(x,\lambda)$, $\boldsymbol{j}_+^{(2)}(x,\lambda)$ are analytic in λ in the upper half-plane and continuous up to the boundary. Also, the columns $\boldsymbol{j}_-^{(2)}(x,\lambda)$, $\boldsymbol{j}_+^{(1)}(x,\lambda)$ are analytic in the lower half–plane and continuous up to the boundary.

Now we describe analytic properties of the coefficient $a(\lambda)$ of the matrix $T(\lambda)$. The monodromy matrix $M(x,y,\lambda)$ can be written in the form

$$M(x,y,\lambda) = J_+(x) J_+^{-1}(y) = J_-(x) J_-^{-1}(y).$$

Therefore,

$$J_+^{-1}(x)\, M(x,y,\lambda)\, J_-(y) = J_+^{-1}(y) J_-(y) = J_+^{-1}(x) J_-(x).$$

The variables x and y separate and the above expression does not depend on x or y at all. By passing to the limit with $x \to +\infty$, $y \to -\infty$ we have

$$T(\lambda) = J_+^{-1}(y)J_-(y) = J_+^{-1}(x)J_-(x).$$

Therefore $a(\lambda) = {\boldsymbol{j}_-^{(1)}}^T(\lambda)J\boldsymbol{j}_+^{(2)}(\lambda)$. The properties of Jost solutions imply that

- $a(\lambda)$ is analytic in the upper half-plane and continuous up to the boundary;
- $a(\lambda)$ is root-free;
- $|a(\lambda)| \geq 1$ and $|a(\lambda)|^2 - 1 \in S(\mathbb{R}^1)$ for λ real, $a(\lambda) = 1 + o(1)$ as $|\lambda| \longrightarrow \infty$.

Let $p_\infty(\lambda)$ be such that $a(\lambda) = \exp(-i2p_\infty(\lambda))$ for λ in the upper half-plane. From the properties of $a(\lambda)$ it follow that

- $p_\infty(\lambda)$ is analytic in the upper half-plane and continuous up to the boundary;
- $\Im p_\infty(\lambda) \geq 0$ for $\Im\lambda \geq 0$;
- $p_\infty(\lambda) = o(1)$ for $|\lambda| \to \infty$; for real λ, the density of the measure $d\mu_\infty(\lambda) = \Im p_\infty(\lambda)\, d\lambda$ belongs to $S(\mathbb{R}^1)$.

The function $p_\infty(\lambda)$ can be written in the form

$$p_\infty(\lambda) = \frac{1}{\pi}\int \frac{d\mu_\infty(t)}{t-\lambda}.$$

Expanding the denominator in inverse powers of λ, we obtain:

$$p_\infty(\lambda) = -\sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}}\frac{1}{\pi}\int_{-\infty}^{+\infty} t^k d\mu_\infty(t) = -\frac{\mathcal{H}_1}{\lambda} - \frac{\mathcal{H}_2}{\lambda^2} - \frac{\mathcal{H}_3}{\lambda^3} - \dots. \tag{2.5}$$

where $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ are the integrals introduced above. The expansion has an asymptotic character for $\lambda: \ \delta \leq \arg\lambda \leq \pi - \delta, \ \delta > 0$.

To describe asymptotic behavior in x of the Jost solutions $\boldsymbol{j}_+^{(2)}(x,\lambda)$ and $\boldsymbol{j}_-^{(1)}(x,\lambda)$ we assume that λ is real and fixed. Then, we have the scattering rule

$$\begin{array}{ccc} & x \longrightarrow -\infty & x \longrightarrow +\infty \\ \boldsymbol{j}_+^{(2)} & a(\lambda)\boldsymbol{f}_\rightarrow(x,\lambda) - \bar{b}(\lambda)\boldsymbol{f}_\leftarrow(x,\lambda) & \boldsymbol{f}_\rightarrow(x,\lambda) \\ \boldsymbol{j}_-^{(1)} & \boldsymbol{f}_\leftarrow(x,\lambda) & a(\lambda)\boldsymbol{f}_\leftarrow(x,\lambda) + b(\lambda)\boldsymbol{f}_\rightarrow(x,\lambda), \end{array}$$

where

$$\boldsymbol{f}_\leftarrow(x,\lambda) = \begin{bmatrix} e^{-i\frac{\lambda}{2}x} \\ 0 \end{bmatrix}, \quad \boldsymbol{f}_\rightarrow(x,\lambda) = \begin{bmatrix} 0 \\ e^{i\frac{\lambda}{2}x} \end{bmatrix}$$

are solutions of the free equation. Similar for the Jost solutions $\boldsymbol{j}_-^{(2)}(x,\lambda)$ and $\boldsymbol{j}_+^{(1)}(x,\lambda)$ analytic in the lower half plane we assume that λ is real and fixed. Then, we have the scattering rule

$$
\begin{array}{ccc}
 & x \to -\infty & x \to +\infty \\
\boldsymbol{j}_+^{(1)} & \bar{a}(\lambda)\boldsymbol{f}_{\leftarrow}(x,\lambda) - b(\lambda)\boldsymbol{f}_{\rightarrow}(x,\lambda) & \boldsymbol{f}_{\leftarrow}(x,\lambda) \\
\boldsymbol{j}_-^{(2)} & \boldsymbol{f}_{\rightarrow}(x,\lambda) & \bar{a}(\lambda)\boldsymbol{f}_{\rightarrow}(x,\lambda) + \bar{b}(\lambda)\boldsymbol{f}_{\leftarrow}(x,\lambda).
\end{array}
$$

Remark 2.1. If $\boldsymbol{f}(x,\lambda)$ is a solution of the auxiliary problem $(\partial_x - V(x,\lambda))\boldsymbol{f} = 0$ corresponding to λ, then $\hat{\boldsymbol{f}} = \sigma_1 \bar{\boldsymbol{f}}$ is a solution of $(\partial_x - V(x,\overline{\lambda}))\hat{\boldsymbol{f}} = 0$ corresponding to $\overline{\lambda}$. For example, $\hat{\boldsymbol{f}_{\leftarrow}}(\lambda) = \boldsymbol{f}_{\rightarrow}(\bar{\lambda})$.

We also introduce the matrix BA function

$$H_+(\lambda) = \left[\boldsymbol{j}_-^{(1)}(\lambda), \boldsymbol{j}_+^{(2)}(\lambda)\right] \quad \text{and} \qquad H_-(\lambda) = \left[\boldsymbol{j}_+^{(1)}(\lambda), \boldsymbol{j}_-^{(2)}(\lambda)\right]$$

analytic in the upper/lower hulf-plane respectively. They are connected by the gluing condition

$$H_-(x,\lambda) = H_+(x,\lambda)S(\lambda), \tag{2.6}$$

where $\lambda \in \mathbb{R}^1$ and the scattering matrix $S(\lambda)$

$$S(\lambda) = \frac{1}{a}\begin{bmatrix} 1 & \bar{b} \\ -b & 1 \end{bmatrix}.$$

The gluing condition easily follows from the scattering rule.

Any Jost solution satisfies $[J\partial_x - JV]\,\boldsymbol{j} = 0$. One can easily prove that $\boldsymbol{j}^T$ satisfies $\boldsymbol{j}^T\,[J\partial_x - JV] = 0$. The matrices H_+^+ and H_-^+ are defined as

$$H_+^+(\lambda) = \sigma_1 H_+^T(\lambda) = \begin{bmatrix} \boldsymbol{j}_+^{(2)\,T} \\ \boldsymbol{j}_-^{(1)\,T} \end{bmatrix}, \qquad H_-^+(\lambda) = \sigma_1 H_-^T(\lambda) = \begin{bmatrix} \boldsymbol{j}_-^{(2)\,T} \\ \boldsymbol{j}_+^{(1)\,T} \end{bmatrix}.$$

Extending a into the lower hulf-plane by the formula $a^*(\lambda) = \overline{a(\bar{\lambda})}$ we define

$$H_+^*(\lambda) = -\frac{\sigma_3}{a(\lambda)}H_+^+(\lambda), \qquad \text{for} \qquad \Im\lambda > 0;$$

and

$$H_-^*(\lambda) = -\frac{\sigma_3}{a^*(\lambda)}H_-^+(\lambda), \qquad \text{for} \qquad \Im\lambda < 0.$$

Then the dual gluing condition holds

$$H_-^*(x,\lambda) = S^{-1}(\lambda)H_+^*(x,\lambda), \qquad \text{where} \qquad \lambda \in \mathbb{R}^1, \tag{2.7}$$

and

$$S^{-1}(\lambda) = \frac{1}{a^*}\begin{bmatrix} 1 & -\bar{b} \\ b & 1 \end{bmatrix}.$$

Next lemma provides explicitly a few terms of the asymptotic expansion of Jost solutions.

Lemma 2.2. *(i) For fixed x the following formulas hold*

$$\boldsymbol{j}_+^{(2)}(x,\lambda)=e^{+i\frac{\lambda}{2}x}\sum_{s=0}^{\infty}\begin{bmatrix} g_s \\ k_s \end{bmatrix}\lambda^{-s},$$

and

$$\boldsymbol{j}_-^{(1)}(x,\lambda)=e^{-i\frac{\lambda}{2}x}\sum_{s=0}^{\infty}\begin{bmatrix} h_s \\ f_s \end{bmatrix}\lambda^{-s},$$

The expansion has an asymptotic character for $\lambda:\ \delta\le\arg\lambda\le\pi-\delta,\ \delta>0$.

(ii) For fixed x the following formulas hold

$$\boldsymbol{j}_+^{(1)}(x,\lambda)=e^{-i\frac{\lambda}{2}x}\sum_{s=0}^{\infty}\begin{bmatrix} \bar{k}_s \\ \bar{g}_s \end{bmatrix}\lambda^{-s},$$

and

$$\boldsymbol{j}_-^{(2)}(x,\lambda)=e^{+i\frac{\lambda}{2}x}\sum_{s=0}^{\infty}\begin{bmatrix} \bar{f}_s \\ \bar{h}_s \end{bmatrix}\lambda^{-s}.$$

The expansion has an asymptotic character for $\lambda:\ -\delta\ge\arg\lambda\ge-\pi+\delta,\ \delta>0$.

(iii) The coefficients g's and k's are given by the formulas

$$g_0=0,\qquad\qquad g_1=-i\overline{\psi},$$

and

$$k_0=1,\qquad\qquad k_1=i\int_x^{+\infty}|\psi(x')|^2dx'.$$

The coefficients h's and f's are given by the formulas

$$f_0=0,\qquad\qquad f_1=i\psi,$$

and

$$h_0=1,\qquad\qquad h_1=i\int_{-\infty}^{x}|\psi(x')|^2dx'.$$

Lemma 2.3. *The scattering map*

$$\psi(x),\ x\in\mathbb{R}^1\quad\longrightarrow\quad b(\lambda),\ \lambda\in\mathbb{R}^1$$

is injective.

Proof. [2]. Assume that there are two different potentials with the same function $b(\lambda)$. Then the difference $\Delta\boldsymbol{j}(x,Q),\ Q=(\lambda+i0,+)$; of the corresponding Jost solutions does not vanish identically in the variable λ for some x. Using the scattering rule and Remark 2.1 we have

$$\hat{\boldsymbol{j}}(x,Q)=-\frac{b}{a}\boldsymbol{j}(x,Q)+\frac{1}{a}\boldsymbol{j}(x,\epsilon_\pm Q).$$

This identity produces

$$\sigma_1 \Delta \bar{\boldsymbol{j}}(x, Q) + \frac{b}{a} \Delta \boldsymbol{j}(x, Q) = \frac{1}{a} \Delta \boldsymbol{j}(x, \epsilon_\pm Q).$$

Multiplying on $\Delta \boldsymbol{j}^T(x, Q)\sigma_1$ from the left

$$|\Delta \boldsymbol{j}(x, Q)|^2 + \frac{b}{a} \Delta \boldsymbol{j}^T(x, Q)\sigma_1 \Delta \boldsymbol{j}(x, Q) = \frac{1}{a} \Delta \boldsymbol{j}^T(x, Q)\sigma_1 \Delta \boldsymbol{j}(x, \epsilon_\pm Q).$$

For arbitrary fixed x the RHS is analytic in the upper half plane and decay there as $O(|\lambda(Q)|^{-2})$. Therefore by the Cauchy theorem

$$\int d\lambda \, |\Delta \boldsymbol{j}(x, Q)|^2 + \int d\lambda \, \frac{b}{a} \Delta \boldsymbol{j}^T(x, Q)\sigma_1 \Delta \boldsymbol{j}(x, Q) = 0.$$

Since

$$\frac{|b(\lambda)|}{|a(\lambda)|} < 1,$$

the second term can not balance the first. The contradiction implies the result. $\square$

We conclude our discussion of the Jost solutions with the following lemma

Lemma 2.4. *The variational derivatives of the Jost solution* $\boldsymbol{j}_+^{(2)}$ *are given by the formulas*[3]

$$\begin{aligned}
\frac{\delta \boldsymbol{j}_+^{(2)}(x)}{\delta \psi(y)} &= \frac{\delta \boldsymbol{j}_+^{(2)}(x)}{\delta \overline{\psi}(y)} = 0, && y < x; \\
\frac{\delta \boldsymbol{j}_+^{(2)}(x)}{\delta \psi(y)} &= -\frac{j_-^1 j_+^1(y)}{a} \boldsymbol{j}_+^{(2)}(x) + \frac{j_+^1 j_+^1(y)}{a} \boldsymbol{j}_-^{(1)}(x), && y > x; \\
\frac{\delta \boldsymbol{j}_+^{(2)}(x)}{\delta \overline{\psi}(y)} &= +\frac{j_+^2 j_-^2(y)}{a} \boldsymbol{j}_+^{(2)}(x) - \frac{j_+^2 j_+^2(y)}{a} \boldsymbol{j}_-^{(1)}(x), && y > x.
\end{aligned}$$

The variational derivatives of the Jost solution $\boldsymbol{j}_-^{(1)}$ *are given by the formulas*

$$\begin{aligned}
\frac{\delta \boldsymbol{j}_-^{(1)}(x)}{\delta \psi(y)} &= \frac{\delta \boldsymbol{j}_-^{(1)}(x)}{\delta \overline{\psi}(y)} = 0, && x < y; \\
\frac{\delta \boldsymbol{j}_-^{(1)}(x)}{\delta \psi(y)} &= -\frac{j_-^1 j_+^1(y)}{a} \boldsymbol{j}_-^{(1)}(x) + \frac{j_-^1 j_-^1(y)}{a} \boldsymbol{j}_+^{(2)}(x), && y < x; \\
\frac{\delta \boldsymbol{j}_-^{(1)}(x)}{\delta \overline{\psi}(y)} &= +\frac{j_+^2 j_-^2(y)}{a} \boldsymbol{j}_-^{(1)}(x) - \frac{j_-^2 j_-^2(y)}{a} \boldsymbol{j}_+^{(2)}(x), && y < x.
\end{aligned}$$

[3] We abuse the notations denoting the components of the Jost solutions by upper indexes.

Proof. We give complete proof for the first set of formulas. The second can be proved in the same way.

Let ${\boldsymbol{j}_+^{(2)}}^\bullet$ be a variation of $\boldsymbol{j}_+^{(2)}$ in response to the variation of ψ. Then

$$ {\boldsymbol{j}_+^{(2)}}^{\bullet\prime} = V {\boldsymbol{j}_+^{(2)}}^\bullet + V^\bullet \boldsymbol{j}_+^{(2)}, $$

where $V^\bullet$ is a variation of V. Then, it can be readily verified that

$$ {\boldsymbol{j}_+^{(2)}}^\bullet = - \int\limits_x^{+\infty} d\xi \, H(x) H^{-1}(\xi) V(\xi) \boldsymbol{j}_+^{(2)}(\xi), $$

where $H = H_+(x, \lambda) = \left[\boldsymbol{j}_-^{(1)}, \boldsymbol{j}_+^{(2)}\right]$. From this one reads,

$$ \frac{\delta \boldsymbol{j}_+^{(2)}(x)}{\delta \psi(y)} = -H(x) H^{-1}(y) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \boldsymbol{j}_+^{(2)}(y), \qquad y > x; $$

and

$$ \frac{\delta \boldsymbol{j}_+^{(2)}(x)}{\delta \overline{\psi}(y)} = -H(x) H^{-1}(y) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \boldsymbol{j}_+^{(2)}(y), \qquad y > x. $$

Using the scattering rule one computes $\det H(x, \lambda) = a(\lambda)$ and inverts the matrix $H(y)$. The straightforward computation produces the result. □

3. The spectral cover and meromorphic functions on it.

3.1. **The spectral cover.** The spectral cover and the Weyl functions for general potentials were considered in [11]. In this section we introduce two other functions Π and Υ which important in the discussion of the scattering case.

Evidently the Jost solutions are particular case of the general Weyl solutions normalized not at some finite point but asymptotically when $x \longrightarrow \pm\infty$. Therefore, the construction of the Weyl function on the spectral cover can be carried for the scattering case without changes. We present this construction here because it will be also needed for the construction of the functions Π and Υ.

For each point of the complex plane of the spectral parameter λ with nonzero imaginary part there are two Jost solutions. To make the Jost solution a single valued function of a point we introduce $\Gamma = \Gamma_+ \cup \Gamma_-$, a two sheeted covering of the complex plane (Figure 2). Each sheet Γ_+ or Γ_- is a copy of the complex plane cut along the real line. Each point of the cover Γ is a pair $Q = (\lambda, \pm)$ where λ is a point of the complex plane and the sign $\pm$ specifies the sheet. We denote by P_+ or P_- the infinity corresponding to the sheet Γ_+ or Γ_-.

Let us introduce two components of the spectral cover

$$ \Gamma_R = \{Q \in \Gamma_+, \ \Im\lambda(Q) > 0\} \cup \{Q \in \Gamma_-, \ \Im\lambda(Q) < 0\} $$

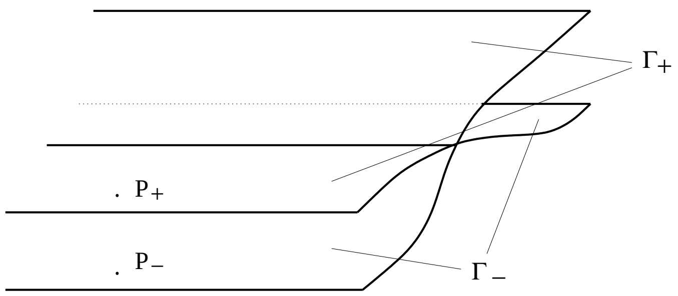

FIGURE 2. Two sheets of the spectral cover.

and

$$\Gamma_L = \{Q \in \Gamma_+,\ \Im\lambda(Q) < 0\} \cup \{Q \in \Gamma_-,\ \Im\lambda(Q) > 0\}.$$

Evidently $\Gamma = \Gamma_R \cup \Gamma_L$.

On Γ we define an involution $\epsilon_\pm$ permuting sheets by the rule

$$\epsilon_\pm : \quad (\lambda, \pm) \longrightarrow (\lambda, \mp).$$

Obviously, the involution permutes infinities $\epsilon_a : P_+ \longrightarrow P_-$ but it leaves Γ_R and Γ_L invariant. On Γ we also define an involution ϵ_a by the rule

$$\epsilon_a : \quad (\lambda, \pm) \longrightarrow (\overline{\lambda}, \mp).$$

The involution permutes infinities $\epsilon_a : P_+ \longrightarrow P_-$ and commutes with $\epsilon_\pm$.

Now we lift Jost solutions from the λ–plane of the spectral parameter to the spectral cover, where they become single valued function of a point. We define for $Q \in \Gamma_+$:

$$\boldsymbol{j}(x, Q) = \begin{cases} \boldsymbol{j}_+^{(2)}(x, \lambda) & \text{if } \Im\lambda(Q) > 0, \\ \boldsymbol{j}_-^{(2)}(x, \lambda) & \text{if } \Im\lambda(Q) < 0; \end{cases}$$

for $Q \in \Gamma_-$:

$$\boldsymbol{j}(x, Q) = \begin{cases} \boldsymbol{j}_-^{(1)}(x, \lambda) & \text{if } \Im\lambda(Q) > 0, \\ \boldsymbol{j}_+^{(1)}(x, \lambda) & \text{if } \Im\lambda(Q) < 0. \end{cases}$$

Let $\boldsymbol{f}^T$ denote the transposition of the vector $\boldsymbol{f}$ and let $\boldsymbol{f}^*$ denote the adjoint of the vector $\boldsymbol{f}$. Let $\boldsymbol{L}^2(a, b)$ be a space of vector functions with the property

$$\int_a^b \boldsymbol{f}^*(x, \lambda)\boldsymbol{f}(x, \lambda)\, dx < \infty.$$

As it follows from the construction[4] $\boldsymbol{j}(x, Q) \in \boldsymbol{L}^2[y, +\infty)$ when $Q \in \Gamma_R$ and $\boldsymbol{j}(x, Q) \in \boldsymbol{L}^2(-\infty, y]$ when $Q \in \Gamma_L$.

[4]The parameter y is an arbitrary real number.

For general potentials considered in [10] a rule connecting Weyl solutions on different banks of the cut is not specified. For the periodic potentials considered in [11] the suitably normalized Weyl (Floquet) solutions can be analytically extended across the real line. On the contrary, in the scattering case this can not be done, but the gluing condition 2.6 connects different branches of the Jost function $\boldsymbol{j}(x,Q)$.

The next Lemma shows that the properties of $\boldsymbol{j}(x,Q)$ resemble properties of the Floquet solutions for periodic potentials (see Lemma 2.5, [11]).

Lemma 3.1. *The following identity holds*

$$\boldsymbol{j}(x,\epsilon_a Q)=\sigma_1\overline{\boldsymbol{j}(x,Q)}. \tag{3.1}$$

In the vicinity of infinities the function $\boldsymbol{j}(x,Q)$ *has the asymptotic,*

$$\boldsymbol{j}(x,Q)=e^{\pm i\frac{\lambda}{2}x}\left[\boldsymbol{j}_0/\hat{\boldsymbol{j}}_0+o\left(1\right)\right], \qquad Q\in(P_\pm),\quad \lambda=\lambda(Q), \tag{3.2}$$

and

$$\boldsymbol{j}_0=\begin{bmatrix}0\\1\end{bmatrix}, \qquad \hat{\boldsymbol{j}}_0=\begin{bmatrix}1\\0\end{bmatrix}.$$

Proof. The asymptotics 3.2 follows from Lemma 2.2.

To prove 3.1 consider the solution $\sigma_1\overline{\boldsymbol{j}(x,Q)}$. Due to Remark 2.1 it must be a linear combination of two Jost solutions at the points above $\lambda(\epsilon_a Q)$. Using 3.2 and comparing the asymptotics for $x\to\pm\infty$ we obtain the result. □

As in [10] we define the Weyl function by the formula

$$\mathcal{X}(x,Q)=\frac{\boldsymbol{j}_2(x,Q)}{\boldsymbol{j}_1(x,Q)}, \qquad Q\in\Gamma.$$

This is the same Weyl function that we introduced using general Weyl solution. It follows from 3.1 that

$$\mathcal{X}(x,\epsilon_a Q)=\frac{1}{\overline{\mathcal{X}(x,Q)}}.$$

Thus we constructed the map

$$\mathcal{M}\quad\longrightarrow\quad(\Gamma,\mathcal{X})$$

which is called the direct spectral transform.

3.2. **The functions Π and Υ.** In this section we construct meromorphic functions Π and Υ, following ideas of [11, 12].

First let us introduce the dual Jost solution $\boldsymbol{j}^+(x,Q)$ by the formula

$$\boldsymbol{j}^+(x,Q)=\boldsymbol{j}^T(x,\epsilon_\pm Q).$$

The gluing condition 2.7 connect various branches of the dual Jost solution.

For any Jost solution or dual Jost solution we define

$$\mathrm{j}(x,Q)=\mathbf{1}^T\boldsymbol{j}(x,Q), \qquad \mathrm{j}^+(x,Q)=\boldsymbol{j}^+(x,Q)\mathbf{1},$$

where $\mathbf{1}^T = (1,1)$. Due to the fact that the Dirac spectral problem is symmetric the function $\mathsf{j}(x,Q)$ or $\mathsf{j}^+(x,Q)$ never vanishes for any x and Q with $\Im\lambda(Q) \neq 0$.

Let us introduce the Wronskian function $\mathcal{W}(Q)$ by the formula

$$\mathcal{W}(Q) = \boldsymbol{j}^+(x,Q)J\boldsymbol{j}(x,Q).$$

It can be proved easily $\mathcal{W}(\epsilon_\pm Q) = -\mathcal{W}(Q)$ and $\mathcal{W}(\epsilon_a Q) = -\overline{\mathcal{W}(Q)}$. From the asymptotic formulas for $Q \in \Gamma_+$ with $\Im\lambda(Q) > 0$ we have $\mathcal{W}(Q) = a(\lambda)$. It can be computed on other parts of the cover using involutions.

Let us introduce the function $\mathcal{P}(x,Q)$ by

$$\mathcal{P}(x,Q) = \mathsf{j}_+(x,Q)\mathsf{j}_-(x,\epsilon_\pm Q).$$

Apparently, $\mathcal{P}(x,\epsilon_\pm Q) = \mathcal{P}(x,Q)$ and it depends on $\lambda = \lambda(Q)$ only; $\mathcal{P}(x,\epsilon_a Q) = \overline{\mathcal{P}(x,Q)}$.

We introduce the function $\Pi(x,Q)$ by the formula:

$$\Pi(x,Q) = \frac{\mathcal{P}(x,Q)}{\mathcal{W}(Q)}.$$

We have $\Pi(x,\epsilon_\pm Q) = -\Pi(x,Q)$ and $\Pi(x,\epsilon_a Q) = -\overline{\Pi(x,Q)}$. Since dependence on the sheet is very simple we introduce the function $\Pi(x,\lambda)$ which coincides with $\Pi(x,Q)$ on Γ_+. Apparently,

$$\Pi(x,\lambda) = \frac{\mathsf{j}_+(x,Q)\mathsf{j}_-(x,\epsilon_\pm Q)}{a(\lambda)}, \qquad \Im\lambda = \lambda(Q) > 0;$$

and

$$\Pi(x,\lambda) = \frac{\mathsf{j}_+(x,Q)\mathsf{j}_-(x,\epsilon_\pm Q)}{a^*(\lambda)}, \qquad \Im\lambda = \lambda(Q) < 0.$$

The function $\Pi(x,\lambda)$ is defined on the complex plane cut along the real line. Its' properties follow from Lemmas 2.2, and 2.1.

- The holomorphic function $\Pi(x,\lambda)$ never vanishes on the complex plane cut along the real line. It satisfies the identity

$$\Pi(x,\overline{\lambda}) = \overline{\Pi(x,\lambda)}. \tag{3.3}$$

- It has the following asymptotics for λ such that $\delta \le \arg\lambda \le \pi - \delta,\ \delta > 0$,

$$\Pi(x,\lambda) = 1 + \frac{i\psi(x) - i\overline{\psi}(x)}{\lambda} + \ldots,$$

  and in the angle $-\delta > \arg\lambda > -\pi + \delta,\ \delta > 0$,

$$\Pi(x,\lambda) = 1 + \frac{i\psi(x) - i\overline{\psi}(x)}{\lambda} + \ldots.$$

We deal with $\Pi(x,Q)$ or $\Pi(x,\lambda)$ depending on the situation.

We introduce the function $\Upsilon(x,Q)$ by the formula

$$\Upsilon(x,Q)=\frac{\mathfrak{j}(x,Q)}{\mathfrak{j}(x,\epsilon_\pm Q)}, \qquad Q\in\Gamma.$$

Evidently,

$$\Upsilon(x,\epsilon_\pm Q)=\frac{1}{\Upsilon(x,Q)}$$

and

$$\Upsilon(x,\epsilon_a Q)=\overline{\Upsilon(x,Q)}.$$

The function $\Upsilon(x,\lambda)$ is defined on the complex plane cut along the real line and coincides with $\Upsilon(x,Q)$ on Γ_+. It has the following properties

- The holomorphic function $\Upsilon(x,\lambda)$ never vanishes on the complex plane for $\Im\lambda\neq 0$. It satisfies the identity
$$\Upsilon(x,\overline{\lambda})=\frac{1}{\overline{\Upsilon(x,\lambda)}}. \tag{3.4}$$
- It has the following asymptotics[5] for λ such that $\delta\le\arg\lambda\le\pi-\delta,\ \delta>0$,
$$\Upsilon(x,\lambda)=e^{i\lambda x}\left[1+\frac{-i\psi(x)-i\overline{\psi}(x)-2iD^{-1}|\psi|^2(x)}{\lambda}+\dots\right],$$
and in the angle $-\delta>\arg\lambda>-\pi+\delta,\ \delta>0$,
$$\Upsilon(x,\lambda)=e^{i\lambda x}\left[1+\frac{-i\psi(x)-i\overline{\psi}(x)-2iD^{-1}|\psi|^2(x)}{\lambda}+\dots\right].$$

Since $\Pi(x,\lambda)$ and $\Upsilon(x,\lambda)$ are holomophic in the open upper and lower half planes and do not have zeros there, we can define Ω and Ξ to be the logarithm of these functions:

$$\begin{aligned}\Upsilon &= e^{\Omega}=e^{\omega+i\tilde{\omega}},\\ \Pi &= e^{\Xi}=e^{\tilde{\xi}+i\xi}.\end{aligned}$$

The functions $\Pi(x,\lambda)$ and $\Upsilon(x,\lambda)$ take different values when λ approaches the real axis from above and below. The identities 3.3 and 3.4 immediately imply

$$\omega(x,\lambda-i0)=-\omega(x,\lambda+i0), \qquad \tilde{\omega}(x,\lambda-i0)=\tilde{\omega}(x,\lambda+i0);$$

and

$$\xi(x,\lambda-i0)=-\xi(x,\lambda+i0), \qquad \tilde{\xi}(x,\lambda-i0)=\tilde{\xi}(x,\lambda+i0).$$

[5]The anti-derivative D^{-1} is defined by the formula $D^{-1}f(x)=\frac{1}{2}\left[\int\limits_{-\infty}^{x}dyf(y)-\int\limits_{x}^{+\infty}dyf(y)\right]$.

3.3. **The function Π and the resolvent.** In this subsection we alternatively define the function Π using the resolvent of the symmetric operator $\mathfrak{D}$. Indeed, $\mathfrak{D}-\frac{\lambda}{2}$ can be inverted for λ, $\Im\lambda \neq 0$, *i.e.* if $\left(\mathfrak{D}-\frac{\lambda}{2}\right)\boldsymbol{f}=\boldsymbol{e}$, then $\boldsymbol{f}=\mathfrak{R}(\lambda)\boldsymbol{e}$, where $\mathfrak{R}=\left(\mathfrak{D}-\frac{\lambda}{2}\right)^{-1}$ is the resolvent.

The resolvent is an integrable operator given by the formula

$$\boldsymbol{f}(x)=\mathfrak{R}(\lambda)\boldsymbol{e}=\int_{-\infty}^{+\infty} R(x,y,\lambda)\boldsymbol{e}(y)dy.$$

For λ with $\Im\lambda>0$ and $Q\in\Gamma_+$ above this λ the kernel $R(x,y,\lambda)$ is given by the formulas

$$R(x,y,\lambda)=\frac{\boldsymbol{j}(x,Q)\boldsymbol{j}^T(y,\epsilon_\pm Q)i\sigma_1}{a(\lambda)}, \qquad \text{for} \qquad y\leq x;$$

$$R(x,y,\lambda)=\frac{\boldsymbol{j}(x,\epsilon_\pm Q)\boldsymbol{j}^T(y,Q)i\sigma_1}{a(\lambda)}, \qquad \text{for} \qquad y\geq x.$$

For λ with $\Im\lambda<0$ and $Q\in\Gamma_+$ above this λ the kernel $R(x,y,\lambda)$ is given by the formulas

$$R(x,y,\lambda)=-\frac{\boldsymbol{j}(x,\epsilon_\pm Q)\boldsymbol{j}^T(y,Q)i\sigma_1}{a^*(\lambda)}, \qquad \text{for} \qquad y\leq x;$$

$$R(x,y,\lambda)=-\frac{\boldsymbol{j}(x,Q)\boldsymbol{j}^T(y,\epsilon_\pm Q)i\sigma_1}{a^*(\lambda)}, \qquad \text{for} \qquad y\geq x.$$

Let us introduce the vector-function $\boldsymbol{\delta}_x$ as $\boldsymbol{\delta}_x^T(y)=(\delta_1(x-y),\delta_2(x-y))$. The formula for the resolvent kernel implies[6]

$$i\Pi(x,\lambda)=<\boldsymbol{\delta}_x,\mathfrak{R}(\lambda)\boldsymbol{\delta}_x>, \qquad \Im\lambda>0,$$

and

$$-i\Pi(x,\lambda)=<\boldsymbol{\delta}_x,\mathfrak{R}(\lambda)\boldsymbol{\delta}_x>, \qquad \Im\lambda<0.$$

This formula is quite suggestive. It was shown in [8] the Poisson bracket for the function similar to Π can be computed in terms of the function itself. Here the situation is more complicated as it will be demonstrated in Theorem 5.1.

Remark 3.2. These formulas imply that

$$\pm\frac{\pi}{2}+\xi(x,\lambda)=\Im\log<\boldsymbol{\delta}_x,\mathfrak{R}(\lambda)\boldsymbol{\delta}_x>, \qquad \Im\lambda>0/\Im\lambda<0.$$

Similar formula was used by Krein, [5], to define by means of the resolvent the, so–called, spectral shift function. See also [7] for the trace formulas using the Krein's spectral shift.

[6] The sign $<\ ,\ >$ signifies the inner product in $L^2(\mathbb{R}^1)$.

4. The Scattering divisor and the Abel map.

4.1. **Useful identities.** If it is not mentioned otherwise we always assume that λ lies on the upper bank, *i.e.* $\lambda = \lambda + i0$. We derive two identities that will be used in calculations.

Theorem 4.1. *For $\lambda = \lambda + i0$ and any $x \in \mathbb{R}$ the following identities hold:*

$$|a|e^{-i\xi} = -be^{i\tilde{\omega}} + e^{-\omega}, \tag{4.1}$$

$$|a|e^{-i\xi} = e^{\omega} + \bar{b}e^{-i\tilde{\omega}}, \tag{4.2}$$

where $a = a(\lambda)$, $\xi = \xi(x, \lambda)$, etc.

Proof. Let $Q = (\lambda + i0, +)$. Using the scattering rule and Remark 2.1 we have

$$\hat{\boldsymbol{j}}(x, Q) = -\frac{b}{a}\boldsymbol{j}(x, Q) + \frac{1}{a}\boldsymbol{j}(x, \epsilon_{\pm}Q).$$

Therefore,

$$\bar{\mathfrak{j}}(x, Q) = -\frac{b}{a}\mathfrak{j}(x, Q) + \frac{1}{a}\mathfrak{j}(x, \epsilon_{\pm}Q).$$

Multiplying on $\mathfrak{j}(x, Q)$ we obtain

$$|\mathfrak{j}(x, Q)|^2 = -\frac{b}{a}\mathfrak{j}^2(x, Q) + \frac{\mathfrak{j}(x, Q)\mathfrak{j}(x, \epsilon_{\pm}Q)}{a}. \tag{4.3}$$

The formula

$$\frac{\mathfrak{j}^2(x, Q)}{a(\lambda)} = \Pi(x, Q)\Upsilon(x, Q) = e^{\tilde{\xi}+i\xi+\omega+i\tilde{\omega}}$$

implies, in particular, $|\mathfrak{j}(x, Q)|^2 = |a|e^{\tilde{\xi}+\omega}$. From these we obtain 4.1.

The proof of 4.2 is similar. Using the scattering rule and Remark 2.1 we have

$$\hat{\boldsymbol{j}}(x, \epsilon_{\pm}Q) = \frac{\bar{b}}{a}\boldsymbol{j}(x, \epsilon_{\pm}Q) + \frac{1}{a}\boldsymbol{j}(x, Q).$$

Therefore,

$$\bar{\mathfrak{j}}(x, \epsilon_{\pm}Q) = \frac{\bar{b}}{a}\mathfrak{j}(x, \epsilon_{\pm}Q) + \frac{1}{a}\mathfrak{j}(x, Q).$$

Multiplying on $\mathfrak{j}(x, \epsilon_{\pm}Q)$

$$|\mathfrak{j}(x, \epsilon_{\pm}Q)|^2 = \frac{\bar{b}}{a}\mathfrak{j}^2(x, \epsilon_{\pm}Q) + \frac{\mathfrak{j}(x, Q)\mathfrak{j}(x, \epsilon_{\pm}Q)}{a},$$

and using the formula

$$\frac{\mathfrak{j}^2(x, \epsilon_{\pm}Q)}{a} = \frac{\Pi(x, Q)}{\Upsilon(x, Q)} = e^{\tilde{\xi}+i\xi-\omega-i\tilde{\omega}}$$

which implies, in particular, $|\mathfrak{j}(x, \epsilon_{\pm}Q)|^2 = |a|e^{\tilde{\xi}-\omega}$ we obtain 4.2. $\square$

The formulas 4.1–4.2 contain three functions ω, ξ and $\tilde{\omega}$. Eliminating any of this functions leads to three important identities.

The first relation for two functions ω and ξ was derived first by Ercolani and McKean, [1]:

$$|a|\cos\xi = \cosh\omega. \tag{4.4}$$

It is obtained by eliminating $\tilde{\omega}$ from the identities 4.1–4.2. Since cosh and cos are even functions formula 4.4 holds for the functions ξ and ω from both sides of the cut.

The second important identity can be obtained by eliminating ω from the formulae 4.1–4.2. Taking sum

$$|a|e^{-i\xi} = \cosh\omega - i|b|\sin(\tilde{\omega} + \text{phase}\, b).$$

Computing the imaginary part of this formula we have

$$|a|\sin\xi = |b|\sin(\tilde{\omega} + \text{phase}\, b).$$

After simple transformations we arrive at the formula

$$\text{phase } b = -\tilde{\omega} + \sin^{-1}\left|\frac{a}{b}\right|\sin\xi. \tag{4.5}$$

This is a continuum version of the Abel map, first obtained by Venakides, [13]. More "geometric" form of this formula will be given in 4.2.

The last third identity can be obtained by eliminating ξ from the formulas 4.1–4.2. Subtracting one from another

$$e^{\omega} - e^{-\omega} + \bar{b}e^{-i\tilde{\omega}} + be^{i\tilde{\omega}} = 0,$$

and

$$\sinh\omega + |b|\cos(\tilde{\omega} + \text{phase}\, b) = 0.$$

After simple transformations we arrive at the formula

$$\text{phase } b = -\tilde{\omega} + \cos^{-1} -\frac{1}{|b|}\sinh\omega.$$

Evidently, with the aid of 4.4 changing the last term, we obtain the Venakides formula.

4.2. **The scattering curve and divisor.** The scattering curve Γ_∞ was introduced by McKean and Ercolani in [1] for the KdV equation. It is a continuum analog of the real ovals accommodating points of the divisor (poles of the Floquet solutions) in the periodic case [12]. This construction is considered here for the NLS.

We assume that the function $a(\lambda)$ is known. The function $h(\lambda) = \cos^{-1}|a(\lambda)|^{-1}$ determines the branch points of the scattering curve Γ_∞. The scattering curve itself has a continuum of the real ovals as it is shown on Figure 3. For each λ the

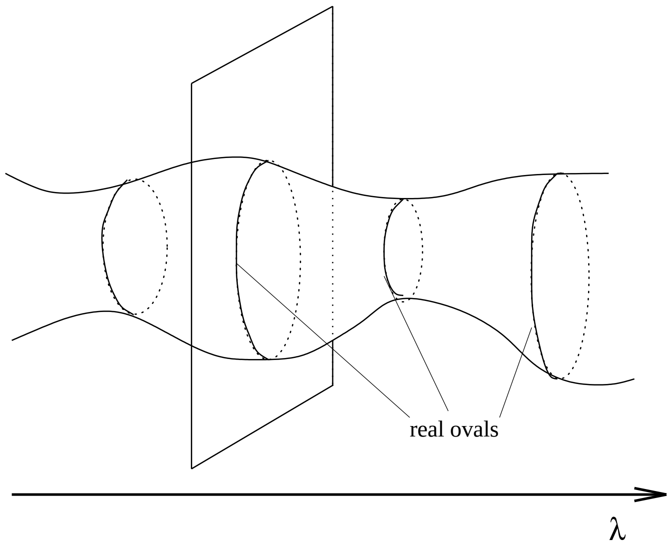

FIGURE 3. The continuum of real ovals.

real oval is two sheeted covering of the segment $[-h(\lambda),\ h(\lambda)]$. The identity 4.4 implies that the function $\xi(x,\lambda)$ satisfies the inequality

$$-h(\lambda) \leq \xi(x,\lambda) \leq h(\lambda).$$

The set of pairs

$$\gamma(\lambda) = (\xi(x,\lambda),\ \operatorname{sign}\omega(x,\lambda)), \qquad \lambda \in \mathbb{R}^1;$$

constitutes the, so–called, scattering S-divisor. The coordinate $\xi(x,\lambda)$ determines the projection, while the sign $\omega(x,\lambda)$ determines the "+" or "-" arch of the oval as on Figure 4. The situation is quite similar to the periodic case, [6].

Lemma 4.2. *Let us assume that the scattering curve (the function $|a(\lambda)|$) is fixed. The S-divisor determines the potential.*

Proof. Identity 4.4 determines the function $\omega(\lambda),\ \lambda \in \mathbb{R}^1$. Both harmonic functions ω and ξ can be recovered from their values on the real line. Then the conjugate functions $\tilde{\xi}$ and $\tilde{\omega}$ also can be found and the additive constants are determined from the asymptotic for the functions $\Pi(x,\lambda)$ and $\Upsilon(x,\lambda)$. Thus $\mathfrak{j}^2(x,Q)$ is known from Π and Υ on all parts of the spectral cover. The formula 4.3 for $Q = (\lambda + i0, +)$

$$|\mathfrak{j}(x,Q)|^2 = -\frac{b}{a}\mathfrak{j}^2(x,Q) + \Pi(x,\lambda),$$

provides the value of $b(\lambda)$ for all $\lambda \in \mathbb{R}^1$. Lemma 2.3 implies the result. $\square$

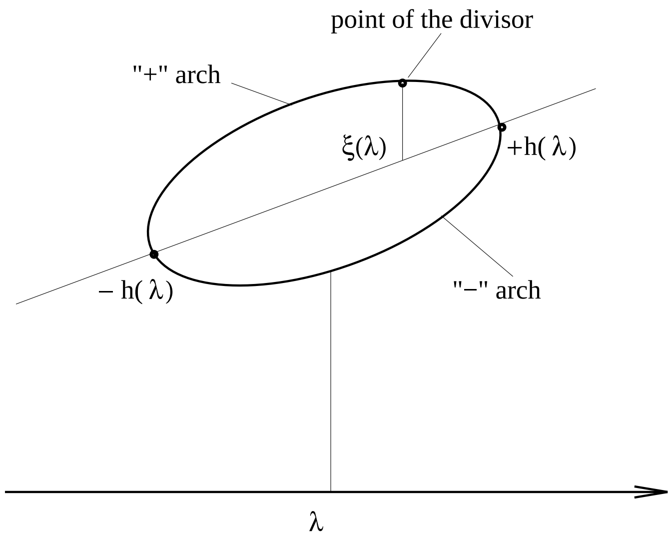

FIGURE 4. The point of the S-divisor on the real oval.

Now we put formula 4.5 into more geometric form. We fix some divisor $\gamma_0 = (\xi_0, \operatorname{sign}\omega_0)$ on the curve Γ_∞. The formula 4.4 implies for each x and λ

$$\omega - \omega_0 = \int_{\xi_0}^{\xi} \frac{d\cos\xi}{\sqrt{\cos^2\xi - \frac{1}{|a|^2}}}.$$

The sign of the radical is the same as the sign of ω. We replace the limits of integrations on γ assuming this agreement

$$\omega - \omega_0 = \int_{\gamma_0}^{\gamma} \frac{d\cos\xi}{\sqrt{\cos^2\xi - \frac{1}{|a|^2}}}.$$

Similar,

$$\sin^{-1}\left|\frac{a}{b}\right|\sin\xi - \sin^{-1}\left|\frac{a}{b}\right|\sin\xi_0 = \int_{\gamma_0}^{\gamma} \frac{d\sin\xi}{\sqrt{\cos^2\xi - \frac{1}{|a|^2}}}.$$

Therefore, denoting the Hilbert transform by H, we have

$$\operatorname{phase} b - \operatorname{phase} b_0 = -H\left[\int_{\gamma_0}^{\gamma} \frac{d\cos\xi}{\sqrt{\cos^2\xi - \frac{1}{|a|^2}}}\right] + \int_{\gamma_0}^{\gamma} \frac{d\sin\xi}{\sqrt{\cos^2\xi - \frac{1}{|a|^2}}}.$$

This form of 4.5 resembles the Abel sum in the Baker's form for the periodic case. The summation over open gaps is replaced by the singular integral of the Hilbert transform. The only difference is the second term. This term corresponds to the contribution into the Abel sum of the open gap which does not contains zero of the normalized differential, see [6].

4.3. **Action of the NLS flows on S–divisor.** The main results of this section are Theorems 4.3 and 4.5 which describe the action of the first two flows of the NLS hierarchy on the scattering divisor.

Theorem 4.3. *For any $x \in \mathbb{R}^1$ the action of the first vector field $X_1 = \{\bullet, \mathcal{H}_1\}$ on the S-divisor is given by the formula*

$$X_1\xi(x,\lambda) = \frac{1}{|a(\lambda)|e^{\tilde{\xi}(x,\lambda)}} \sinh \omega(x,\lambda). \tag{4.6}$$

Using formula 4.4 one can put the result in the form

$$X_1\xi(x,\lambda) = \pm e^{-\tilde{\xi}(x,\lambda)} \sqrt{\cos^2 \xi(x,\lambda) - \frac{1}{|a(\lambda)|^2}}.$$

The sign of the radical is the same as $\operatorname{sign} \omega(x,\lambda)$. The instant the function $\xi(x,\lambda)$ reaches the end (branch point) of the interval $[-h(\lambda),\ h(\lambda)]$ the signature of the radical changes and the direction of motion reverses.

We precede the proof with the following

Lemma 4.4. *The action of the first vector field $X_1 = \{\bullet, \mathcal{H}_1\}$ on the Jost solutions is given by the formulas*

$$\begin{aligned} X_1 j_+^1(x) &= ij_+^1(x), & X_1 j_-^1(x) = 0,\\ X_1 j_+^2(x) &= 0, & X_1 j_-^2(x) = -ij_-^2(x). \end{aligned}$$

Proof. We will present complete proof of the formulas for the action of X_1 on $\boldsymbol{j}_-(x)$.

$$X_1\boldsymbol{j}_-(x) = \{\boldsymbol{j}_-(x), \mathcal{H}_1\} = 2i \int_{-\infty}^{+\infty} \frac{\delta \boldsymbol{j}_-(x)}{\delta\overline{\psi}(y)}\frac{\delta\mathcal{H}_1}{\delta\psi(y)} - \frac{\delta \boldsymbol{j}_-(x)}{\delta\psi(y)}\frac{\delta\mathcal{H}_1}{\delta\overline{\psi}(y)} dy.$$

Substituting the gradients from Lemma 2.4 after simple algebra we obtain

$$\therefore = \frac{i\boldsymbol{j}_-(x)}{a(\lambda)} \int_{-\infty}^{x} \left(j_-^2 j_+^2 \overline{\psi}(y) + j_-^1 j_+^1 \psi(y)\right) dy - \frac{i\boldsymbol{j}_+(x)}{a(\lambda)} \int_{-\infty}^{x} \left(j_-^2 j_-^2 \overline{\psi}(y) + j_-^1 j_-^1 \psi(y)\right) dy.$$

Both expressions under the integral signs turn out to be a full derivatives

$$j_-^2 j_+^2 \overline{\psi} + j_-^1 j_+^1 \psi = (j_+^2 j_-^1)', \tag{4.7}$$

$$j_-^2 j_-^2 \overline{\psi} + j_-^1 j_-^1 \psi = (j_-^1 j_-^2)'. \tag{4.8}$$

Therefore, using the scattering rule

$$\int_{-\infty}^{x} \left(j_-^2 j_+^2 \overline{\psi}(y) + j_-^1 j_+^1 \psi(y)\right) dy = j_+^2 j_-^1 |_{-\infty}^{x} = j_+^2 j_-^1(x) - a(\lambda),$$

and

$$\int_{-\infty}^{x} \left(j_-^2 j_-^2 \overline{\psi}(y) + j_-^1 j_-^1 \psi(y)\right) dy = j_-^1 j_-^2(x).$$

Finally, we obtain

$$X_1 \boldsymbol{j}_-(x) = \frac{i\boldsymbol{j}_-(x)}{a(\lambda)}(j_+^2 j_-^1(x) - a(\lambda)) - \frac{i\boldsymbol{j}_+(x)}{a(\lambda)} j_-^1 j_-^2(x).$$

From this formula using $\boldsymbol{j}_- J \boldsymbol{j}_+ = a(\lambda)$ we have

$$X_1 j_-^1(x) = \frac{i}{a}\left(j_-^1 j_+^2 j_-^1 - j_+^1 j_-^1 j_-^2\right) - ij_-^1 = \frac{i}{a} j_-^1\, \boldsymbol{j}_- J \boldsymbol{j}_+ - ij_-^1 = ij_-^1 - ij_-^1 = 0,$$

and

$$X_2 \boldsymbol{j}_-(x) = \frac{i}{a}\left(j_-^2 j_+^2 j_-^1 - j_+^2 j_-^1 j_-^2\right) - ij_-^2 = ij_-^2.$$

Now we give the proof of identities 4.7 and 4.8. To prove the first identity we take the first equation of the auxiliary linear system for $\boldsymbol{j}_-$

$$j_-^{1\,\prime} = -\frac{i\lambda}{2} j_-^1 + \overline{\psi} j_-^2$$

and the second equation written for $\boldsymbol{j}_+$

$$j_+^{2\,\prime} = \psi j_+^1 + \frac{i\lambda}{2} j_+^2.$$

Multiplying the first equation on j_+^2 and the second equation on j_-^1 and taking their sum we obtain 4.7.

To prove 4.8 we write equations for both components of $\boldsymbol{j}_-$

$$j_-^{1\,\prime} = -\frac{i\lambda}{2} j_-^1 + \overline{\psi} j_-^2,$$

$$j_-^{2\,\prime} = \psi j_-^1 + \frac{i\lambda}{2} j_-^2.$$

Multiplying the first equation on j_-^2 and the second equation on j_-^1 and taking their sum we obtain 4.8. We are done.

The second set of formulas for the action of X_1 on $\boldsymbol{j}_+(x)$ can be proved along these lines. $\square$

Now we are ready to prove Theorem 4.3.

Proof. Since $Xa(\lambda) = 0$ we have

$$\begin{aligned} X_1 \xi(x,\lambda) &= X_1 \Im \log \mathrm{j}_- \mathrm{j}_+(x,\lambda) \\ &= \frac{1}{2i}\left(\frac{X_1 \mathrm{j}_-}{\mathrm{j}_-} - \frac{X_1 \bar{\mathrm{j}}_-}{\bar{\mathrm{j}}_-}\right) + \frac{1}{2i}\left(\frac{X_1 \mathrm{j}_+}{\mathrm{j}_+} - \frac{X_1 \bar{\mathrm{j}}_+}{\bar{\mathrm{j}}_+}\right). \end{aligned}$$

Using the result of Lemma 4.4 for the first bracket we have

$$\therefore = \frac{1}{2}\left(\frac{-j_-^2\bar{j}_-^1 - |j_-^2|^2 - \bar{j}_-^2 j_-^1 - |j_-^2|^2}{|\mathrm{j}_-|^2}\right).$$

Using the identity[7]

$$|j_-^1|^2 - |j_-^2|^2 = \boldsymbol{j}_-^T J\hat{\boldsymbol{j}}_- = \boldsymbol{f}_\leftarrow^T J\hat{\boldsymbol{f}}_\leftarrow = 1 \tag{4.9}$$

we obtain

$$\therefore = \frac{-|\mathrm{j}_-|^2 + 1}{2\,|\mathrm{j}_-|^2}.$$

For the second bracket we have

$$\therefore = \frac{1}{2}\left(\frac{j_+^1\bar{j}_+^2 + |j_+^1|^2 + \bar{j}_+^1 j_+^2 + |j_+^1|^2}{|\mathrm{j}_+|^2}\right),$$

and using the identity

$$|j_+^1|^2 - |j_+^2|^2 = \boldsymbol{j}_+^T J\hat{\boldsymbol{j}}_+ = \boldsymbol{f}_\rightarrow^T J\hat{\boldsymbol{f}}_\rightarrow = -1 \tag{4.10}$$

we obtain

$$\therefore = \frac{|\mathrm{j}_+|^2 - 1}{2\,|\mathrm{j}_+|^2}.$$

Taking the sum we obtain

$$\frac{1}{2}\left(\frac{1}{|\mathrm{j}_-|^2} - \frac{1}{|\mathrm{j}_+|^2}\right).$$

The formulas

$$|a|e^{\tilde{\xi}+\omega} = |j_+|^2, \qquad |a|e^{\tilde{\xi}-\omega} = |j_-|^2;$$

follow from the definitions of Π and Υ allow to put the computation into the final form. ☐

Theorem 4.5. *The action of the second vector field $X_2 = \{\bullet, \mathcal{H}_2\}$ on the S-divisor is given by the formula*

$$X_2\xi(x,\lambda) = [-\lambda + 2\Im\psi(x)]X_1\xi(x,\lambda). \tag{4.11}$$

Note that up to the sign the relation between the first and the second NLS flows is the same as in the periodic case, [6]. We start with the analog of Lemma 4.4. This result is proved without computation of the gradients of Jost solutions.

Lemma 4.6. *The action of the second vector field $X_2 = \{\bullet, \mathcal{H}_2\}$ on the Jost solutions is given by the formulas*

$$X_2\boldsymbol{j}_+(x) \;=\; \boldsymbol{j}_+'(x), \qquad X_2\boldsymbol{j}_-(x) = \boldsymbol{j}_-'(x).$$

[7] This identity is obtained using Remark 2.1.

Proof. Suppose we are interested in the deformation of the operator $\partial_x - V(x,\lambda)$ defined by the formula

$$\partial_\tau V - \partial_x U + [U,V] = 0.$$

where the matrix $U = U(x,\lambda)$ and τ is the parameter corresponding to the deformation. How the solution $\boldsymbol{j}(x,\lambda)$ of the problem

$$(\partial_x - V)\,\boldsymbol{j} = 0$$

changes under such deformation? Differentiating this formula with respect to τ and substituting the expression for $\partial_\tau V$ we have after simple algebra

$$(\partial_x - V)\,\partial_\tau \boldsymbol{j} = (\partial_x - V)\,U\boldsymbol{j}.$$

This implies

$$\partial_\tau \boldsymbol{j} = U\boldsymbol{j} + \alpha \boldsymbol{j}_- + \beta \boldsymbol{j}_+,$$

with some constants $\alpha = \alpha(\lambda)$ and $\beta = \beta(\lambda)$. In the case of the Jost solution $\boldsymbol{j} = \boldsymbol{j}_\pm$ the constants can be determined from the asymptotic behavior when $x \longrightarrow \pm\infty$.

Let us demonstrate first how these arguments produce the first result of Lemma 4.4. In this case

$$\partial_\tau \boldsymbol{j}_+ = \frac{i}{2}\sigma_3 \boldsymbol{j}_+ + \alpha \boldsymbol{j}_- + \beta \boldsymbol{j}_+.$$

Now letting $x \longrightarrow +\infty$ and using the scattering rule we have

$$0 = \partial_\tau \boldsymbol{f}_\rightarrow = \frac{i}{2}\sigma_3 \boldsymbol{f}_\rightarrow + \alpha(a\boldsymbol{f}_\leftarrow + b\boldsymbol{f}_\rightarrow) + \beta \boldsymbol{f}_\rightarrow.$$

Now we see that $\alpha = 0$ and $\beta = \frac{i}{2}$. Therefore,

$$X_1 \boldsymbol{j}_+ = \partial_\tau \boldsymbol{j}_+ = \frac{i}{2}(\sigma_3 + I)\boldsymbol{j}_+.$$

In the case of the second flow the result follows in a similar way. Let us prove it for $\boldsymbol{j}_+$. The parameter τ consides with x and $U = V$. In this case

$$\partial_x \boldsymbol{j}_+ = \left(-\frac{i\lambda}{2}\sigma_3 + Y_0\right)\boldsymbol{j}_+ + \alpha \boldsymbol{j}_- + \beta \boldsymbol{j}_+.$$

Now letting $x \longrightarrow +\infty$ and using the scattering rule we have

$$\frac{i\lambda}{2}\boldsymbol{f}_\rightarrow = -\frac{i\lambda}{2}\sigma_3 \boldsymbol{f}_\rightarrow + \alpha(a\boldsymbol{f}_\leftarrow + b\boldsymbol{f}_\rightarrow) + \beta \boldsymbol{f}_\rightarrow.$$

This implies that $\alpha = \beta = 0$ and

$$X_2 \boldsymbol{j}_+ = \partial_\tau \boldsymbol{j}_+ = \boldsymbol{j}'_+.$$

□

Now we are ready to prove Theorem 4.5.

Proof. Since $Xa(\lambda) = 0$ we have using the result of Lemma 4.6

$$\begin{aligned}
X_2\xi(x,\lambda) &= X_2\Im\log \mathfrak{j}_-\mathfrak{j}_+(x,\lambda) \\
&= \frac{1}{2i}\left(\frac{X_2\mathfrak{j}_-}{\mathfrak{j}_-} - \frac{X_2\bar{\mathfrak{j}}_-}{\bar{\mathfrak{j}}_-}\right) + \frac{1}{2i}\left(\frac{X_2\mathfrak{j}_+}{\mathfrak{j}_+} - \frac{X_2\bar{\mathfrak{j}}_+}{\bar{\mathfrak{j}}_+}\right) \\
&= \frac{1}{2i}\left(\frac{\mathfrak{j}_-'}{\mathfrak{j}_-} - \frac{\bar{\mathfrak{j}}_-'}{\bar{\mathfrak{j}}_-}\right) + \frac{1}{2i}\left(\frac{\mathfrak{j}_+'}{\mathfrak{j}_+} - \frac{\bar{\mathfrak{j}}_+'}{\bar{\mathfrak{j}}_+}\right) \\
&= \frac{1}{2i}\frac{\mathfrak{j}_-'\bar{\mathfrak{j}}_- - \bar{\mathfrak{j}}_-'\mathfrak{j}_-}{|\mathfrak{j}_-|^2} + \frac{1}{2i}\frac{\mathfrak{j}_+'\bar{\mathfrak{j}}_+ - \bar{\mathfrak{j}}_+'\mathfrak{j}_+}{|\mathfrak{j}_+|^2}.
\end{aligned}$$

To compute the numerators we establish the identity:

$$\mathfrak{j}'\bar{\mathfrak{j}} - \bar{\mathfrak{j}}'\mathfrak{j} = (-i\lambda + 2i\Im\psi(x))\left(|j^1|^2 - |j^2|^2\right), \qquad \mathfrak{j} = \mathfrak{j}_\pm(x,\lambda). \tag{4.12}$$

The original system

$$j^{1\prime} = -\frac{i\lambda}{2}j^1 + \overline{\psi}j^2,$$
$$j^{2\prime} = \psi j^1 + \frac{i\lambda}{2}j^2;$$

implies

$$\mathfrak{j}' = -\frac{i\lambda}{2}(j^1 - j^2) + \overline{\psi}j^2 + \psi j^1.$$

After simple computaions we obtain the stated identity

$$\begin{aligned}
\mathfrak{j}'\bar{\mathfrak{j}} - \bar{\mathfrak{j}}'\mathfrak{j} &= 2i\Im\psi(x)\left(|j^1|^2 - |j^2|^2\right) - \frac{i\lambda}{2}\left[(j^1 - j^2)(\bar{j}^1 + \bar{j}^2) + (\bar{j}^1 - \bar{j}^2)(j^1 + j^2)\right] \\
&= (-i\lambda + 2i\Im\psi(x))\left(|j^1|^2 - |j^2|^2\right).
\end{aligned}$$

Using 4.12, 4.9 and 4.10, we have

$$\therefore = (-\lambda + 2\Im\psi(x)) \times \frac{1}{2}\left(\frac{1}{|\mathfrak{j}_-|^2} - \frac{1}{|\mathfrak{j}_+|^2}\right).$$

The second multiple is nothing but the action of the first flow on the divisor.

□

5. The Poisson bracket.

5.1. **The Poisson bracket for the Π function.** Surprisingly, the bracket for the function $\Pi(x,\lambda)$ corresponding two different values of the spectral parameter can be expressed in a closed form. We need to introduce some notations. Let λ,μ in the upper half–plane

$$\boldsymbol{j}_+ = \boldsymbol{j}_+^{(2)}(x,\lambda), \qquad \boldsymbol{j}_- = \boldsymbol{j}_-^{(1)}(x,\lambda),$$

and

$$\boldsymbol{g}_+ = \boldsymbol{g}_+^{(2)}(x,\mu), \qquad \boldsymbol{g}_- = \boldsymbol{g}_-^{(1)}(x,\mu).$$

Denote by

$$W\left[\boldsymbol{j}_\pm,\boldsymbol{g}_\pm\right] = \boldsymbol{j}_\pm(x,\lambda)J\boldsymbol{g}_\pm(x,\mu)$$

the Wronskian of two solutions.

Theorem 5.1. *For $\Pi(\lambda)=\Pi(x,\lambda)$, $\Upsilon(\mu)=\Upsilon(x,\mu)$, and λ, μ in the upper half–plane we have*

$$\begin{aligned}\{\Pi(\lambda),\Pi(\mu)\} &= \frac{2\Pi(\lambda)\Pi(\mu)}{a(\lambda)\,a(\mu)(\mu-\lambda)}\times \\ &\times \left(\Upsilon(\lambda)\Upsilon(\mu)W^2\left[\boldsymbol{j}_-,\boldsymbol{g}_-\right](x) - \Upsilon^{-1}(\lambda)\Upsilon^{-1}(\mu)W^2[\boldsymbol{j}_+,\boldsymbol{g}_+](x)\right).\end{aligned}$$

Proof.

$$\begin{aligned}\{\Pi(\lambda),\Pi(\mu)\} = \{\frac{\mathfrak{j}_+\mathfrak{j}_-}{a(\lambda)},\frac{\mathfrak{g}_+\mathfrak{g}_-}{a(\mu)}\} &= \{\mathfrak{j}_+,\mathfrak{g}_+\}\frac{\mathfrak{j}_-\mathfrak{g}_-}{a(\lambda)\,a(\mu)} \\ &+ \{\mathfrak{j}_-,\mathfrak{g}_-\}\frac{\mathfrak{j}_+\mathfrak{g}_+}{a(\lambda)\,a(\mu)} \\ &+ \{\mathfrak{j}_+\mathfrak{j}_-,\frac{1}{a(\mu)}\}\frac{\mathfrak{g}_+\mathfrak{g}_-}{a(\lambda)} \\ &+ \{\frac{1}{a(\lambda)},\mathfrak{g}_+\mathfrak{g}_-\}\frac{\mathfrak{j}_+\mathfrak{j}_-}{a(\mu)}.\end{aligned}$$

Step 1. Using Lemma 2.4 one can compute for the first term

$$\begin{aligned}\{\mathfrak{j}_+,\mathfrak{g}_+\}\frac{\mathfrak{j}_-\mathfrak{g}_-}{a(\lambda)\,a(\mu)} = \frac{2}{a^2(\lambda)a^2(\mu)(\mu-\lambda)}\times[&- \mathfrak{j}_+\mathfrak{j}_-\mathfrak{g}_+\mathfrak{g}_-\left(\boldsymbol{j}_+^TJ\boldsymbol{g}_+\right)\left(\boldsymbol{j}_-^TJ\boldsymbol{g}_-\right) \\ &+ \mathfrak{j}_+\mathfrak{j}_-\mathfrak{g}_-\mathfrak{g}_-\left(\boldsymbol{j}_+^TJ\boldsymbol{g}_+\right)\left(\boldsymbol{j}_-^TJ\boldsymbol{g}_+\right) \\ &+ \mathfrak{j}_-\mathfrak{j}_-\mathfrak{g}_+\mathfrak{g}_-\left(\boldsymbol{j}_+^TJ\boldsymbol{g}_+\right)\left(\boldsymbol{j}_+^TJ\boldsymbol{g}_-\right) \\ &- \mathfrak{j}_-\mathfrak{j}_-\mathfrak{g}_-\mathfrak{g}_-\left(\boldsymbol{j}_+^TJ\boldsymbol{g}_+\right)\left(\boldsymbol{j}_+^TJ\boldsymbol{g}_+\right)].\end{aligned}$$

Step 2. Similar one can compute for the second term

$$\{\mathfrak{j}_-, \mathfrak{g}_-\}\frac{\mathfrak{j}_+\mathfrak{g}_+}{a(\lambda)\,a(\mu)} = \frac{2}{a^2(\lambda)a^2(\mu)(\mu-\lambda)} \times [\begin{array}{ll} + & \mathfrak{j}_+\mathfrak{j}_-\mathfrak{g}_+\mathfrak{g}_- \left(\boldsymbol{j}_+^T J \boldsymbol{g}_+\right)\left(\boldsymbol{j}_-^T J \boldsymbol{g}_-\right) \\ - & \mathfrak{j}_+\mathfrak{j}_-\mathfrak{g}_+\mathfrak{g}_+ \left(\boldsymbol{j}_+^T J \boldsymbol{g}_-\right)\left(\boldsymbol{j}_-^T J \boldsymbol{g}_-\right) \\ - & \mathfrak{j}_+\mathfrak{j}_+\mathfrak{g}_+\mathfrak{g}_- \left(\boldsymbol{j}_-^T J \boldsymbol{g}_-\right)\left(\boldsymbol{j}_-^T J \boldsymbol{g}_+\right) \\ + & \mathfrak{j}_+\mathfrak{j}_+\mathfrak{g}_+\mathfrak{g}_+ \left(\boldsymbol{j}_-^T J \boldsymbol{g}_-\right)\left(\boldsymbol{j}_-^T J \boldsymbol{g}_-\right)]. \end{array}$$

Step 3. For the third term

$$\{\mathfrak{j}_+\mathfrak{j}_-, \frac{1}{a(\mu)}\}\frac{\mathfrak{g}_+\mathfrak{g}_-}{a(\lambda)} = \frac{2}{a^2(\lambda)a^2(\mu)(\mu-\lambda)} \times [\begin{array}{ll} + & \mathfrak{j}_+\mathfrak{j}_+\mathfrak{g}_+\mathfrak{g}_- \left(\boldsymbol{j}_-^T J \boldsymbol{g}_-\right)\left(\boldsymbol{j}_-^T J \boldsymbol{g}_+\right) \\ - & \mathfrak{j}_-\mathfrak{j}_-\mathfrak{g}_+\mathfrak{g}_- \left(\boldsymbol{j}_+^T J \boldsymbol{g}_+\right)\left(\boldsymbol{j}_+^T J \boldsymbol{g}_-\right)]. \end{array}$$

Step 4. For the fourth term

$$\{\frac{1}{a(\lambda)}, \mathfrak{g}_+\mathfrak{g}_-\}\frac{\mathfrak{j}_+\mathfrak{j}_-}{a(\mu)} = \frac{2}{a^2(\lambda)a^2(\mu)(\mu-\lambda)} \times [\begin{array}{ll} + & \mathfrak{j}_-\mathfrak{j}_+\mathfrak{g}_+\mathfrak{g}_+ \left(\boldsymbol{j}_-^T J \boldsymbol{g}_-\right)\left(\boldsymbol{j}_+^T J \boldsymbol{g}_-\right) \\ - & \mathfrak{j}_-\mathfrak{j}_+\mathfrak{g}_-\mathfrak{g}_- \left(\boldsymbol{j}_+^T J \boldsymbol{g}_+\right)\left(\boldsymbol{j}_-^T J \boldsymbol{g}_+\right)]. \end{array}$$

Step 5. Taking sum of the results of the previous four steps after cancellations we have

$$\{\Pi(\lambda), \Pi(\mu)\} = \frac{2}{a^2(\lambda)a^2(\mu)(\mu-\lambda)} \times [\mathfrak{j}_+^2\mathfrak{g}_+^2 \left(\boldsymbol{j}_-^T J \boldsymbol{g}_-\right)^2 - \mathfrak{j}_-^2\boldsymbol{g}_-^2 \left(\boldsymbol{j}_+^T J \boldsymbol{g}_+\right)^2].$$

After simple algebra we obtain the formula stated above.

□

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