Zariski's multiplicity question for families of convenient Newton nondegenerate aligned singularities

! Old version !

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Abstract

We answer positively to Zariski's multiplicity question for families of convenient Newton nondegenerate aligned singularities.

Let $f\colon (\mathbf{C}^n \times \mathbf{C}, \{0\} \times \mathbf{C}) \to (\mathbf{C}, 0)$, $(z,t) \mapsto f(z,t) = f_t(z)$, with $n \geq 2$, be a germ (at the origin) of holomorphic function such that, for all t near 0, the germ f_t is reduced. Let ν_{f_t} be the multiplicity of f_t at 0, that is, the lowest degree in the power series expansion of f_t at 0. As we are assuming that f_t is reduced, ν_{f_t} is also the number of points of intersection, near 0, of $V_{f_t} := f_t^{-1}(0)$ with a generic (complex) line of $\mathbf{C}^n$ passing arbitrarily close to 0 but not through 0. Let μ_{f_t} denote the Milnor number of f_t at 0.

One says that $(f_t)_t$ is *topologically constant* if, for all t near 0, there is a germ of homeomorphism $\varphi_t\colon (\mathbf{C}^n, 0) \to (\mathbf{C}^n, 0)$ such that $\varphi(V_{f_t}) = V_{f_0}$. One says that $(f_t)_t$ is *μ-constant* if, for all t near 0, one has $\mu_{f_t} = \mu_{f_0}$. Notice that, in the special case where $(f_t)_t$ is a family of *isolated* singularities (i.e., when, for all t near 0, f_t has an isolated critical point at 0), if $n \neq 3$, then "topologically constant" is equivalent to "μ-constant" (cf. Lê [L], Teissier [Te] and Lê-Ramanujam [LR]). Finally, one says that $(f_t)_t$ is *equimultiple* if, for all t near 0, one has $\nu_{f_t} = \nu_{f_0}$.

In [Z], Zariski asked the following question: *if $(f_t)_t$ is topologically constant, then is it equimultiple?* More than thirty years later, the question is, in general, still unsettled (even for isolated hypersurface singularities). The answer is, nevertheless, known to be *yes* in several special cases the list of which can be found in the recent author's survey [Ey].

In this paper, we concentrate our attention on families of *convenient* germs f_t having a *nondegenerate Newton principal part* (see the Appendix for the definitions). Regarding this class of germs, Abderrahmane [A] proved the following theorem about *isolated* singularities

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(see also Saia-Tomazella [ST] for a related result).

THEOREM 1 (Abderrahmane [A, Theorem 1]). *We assume that, for all t near 0, the germ f_t has an isolated critical point at 0. Also we suppose that $(f_t)_t$ is μ-constant. If, for all t near 0, the germ f_t is convenient and has a nondegenerate Newton principal part with respect to a system of coordinates $z = (z_1, \ldots, z_n)$, then $(f_t)_t$ is equimultiple.*

In this paper, we extend this theorem to a special class of higher dimensional singularities, namely that of *aligned* singularities (see Section A2 of the Appendix for the definition). More precisely, we get the following result.

THEOREM 2. *We assume that, for all t near 0, the germ f_t has an s-dimensional aligned singularity at 0. Also we suppose that $(f_t)_t$ is topologically constant. Let $(t_k)_{k \in \mathbf{N}}$ be an infinite sequence of points in $\mathbf{C}$ tending to 0. Let $z = (z_1, \ldots, z_n)$ be an aligning set of coordinates, at 0, for f_0 and for f_{t_k}, for all $k \in \mathbf{N}$ (such a coordinates system always exists by Massey [M, Proof of Theorem 7.9]). If, for all t near 0, the germ f_t is convenient and has a nondegenerate Newton principal part with respect to the rotated coordinates $\tilde{z} = (z_{s+1}, z_{s+2}, \ldots, z_n, z_1, \ldots, z_s)$, then $(f_t)_t$ is equimultiple.*

For the definition of an "aligning set of coordinates", see Section A2 of the Appendix. Notice that, given an aligned singularity, aligning sets of coordinates are generic (in the inductive pseudo-Zariski topology).

The notion of aligned singularities was introduced by Massey in [M, Chapter 7]. Regarding this class of singularities, Massey proved the following reduction theorem.

THEOREM 3 (Massey [M, Theorem 7.9]). *The following are equivalent:*

(i) *for all $n \geq 4$, the answer to Zariski's multiplicity question is positive for families $(f_t)_t$ of reduced analytic hypersurfaces with isolated singularities;*

(ii) *for all $n \geq 4$, there exists an integer s such that the answer to Zariski's multiplicity question is positive for families $(f_t)_t$ of reduced analytic hypersurfaces with s-dimensional aligned singularities (i.e., for all t near 0, f_t has a s-dimensional aligned singularity at 0);*

(iii) *for all $n \geq 4$, for all integer s, the answer to Zariski's multiplicity question is positive for families $(f_t)_t$ of reduced analytic hypersurfaces with s-dimensional aligned singularities.*

With Theorem 1 in hand, the proof of Theorem 2 goes similar to the proof of Theorem 3. We will sketch it in Section A3 of the Appendix. Notice nevertheless that Theorem 2 cannot be obtained by a simple direct application of Theorem 3.

Theorem 2 provides a positive answer to Zariski's question for a large class of *nonisolated* singularities, without any assumption on the topological constancy, that is, without any assumption on the homeomorphism φ_t involved in the definition of "topologically equivalent". Under some additional hypotheses on φ_t, positive answers to Zariski's question exist. For example, it is known that the multiplicity is an embedded C^1 invariant (cf. Ephraim [Ep] and Trotman [Tr]) and an embedded "right-left bilipschitz" invariant (cf. Risler-Trotman [RT]). For a complete list, see [Ey].

APPENDIX

A1. NEWTON POLYHEDRA. In this section, we recall the basic material about Newton polyhedra introduced by Kouchnirenko in [K]. See also Oka [O].

We consider the complex space $\mathbf{C}^n$ with fixed coordinates $z = (z_1, \ldots, z_n)$. Let $f: (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a germ of holomorphic function defined by a convergent power series $\sum_\alpha a_\alpha z^\alpha$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, $a_\alpha \in \mathbf{C}$, and $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$. The *Newton polyhedron* $\Gamma_+(f; z)$ of f at 0, with respect to the coordinates $z = (z_1, \ldots, z_n)$, is the convex hull in $\mathbf{R}_+^n$ of the set

$$\bigcup_{a_\alpha \neq 0} (\alpha + \mathbf{R}_+^n),$$

where $\mathbf{R}_+^n = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \ ; \ x_i \geq 0 \text{ for } 1 \leq i \leq n\}$. The *Newton boundary* $\Gamma(f; z)$ of f at 0, with respect to z, is the union of the *compact* faces of the boundary of $\Gamma_+(f; z)$. The polynomial $\sum_{\alpha \in \Gamma(f;z)} a_\alpha z^\alpha$ is called the *Newton principal part* of f at 0 with respect to z. For a face Δ of $\Gamma(f; z)$, one defines the *face function* f_Δ by $f_\Delta(z) := \sum_{\alpha \in \Delta} a_\alpha z^\alpha$. One says that f is *Newton nondegenerate* on Δ if the equations

$$\frac{\partial f_\Delta}{\partial z_1}(z) = \ldots = \frac{\partial f_\Delta}{\partial z_n}(z) = 0$$

have no common solution on $z_1 \ldots z_n \neq 0$. When f is nondegenerate on every face Δ of $\Gamma(f; z)$, one says that f has a *nondegenerate Newton principal part* with respect to z. One says that f is *convenient*, with respect to z, if the intersection of $\Gamma(f; z)$ with each coordinate axis is nonempty, that is, if, for $1 \leq i \leq n$, the monomial $z_i^{\alpha_i}$, $\alpha_i \geq 1$, appears in the expression $f(z) = \sum_\alpha a_\alpha z^\alpha$ with a non-zero coefficient.

A2. ALIGNED SINGULARITIES. In this section, we recall the notion of aligned singularities introduced by Massey in [M]. Aligned singularities generalize isolated singularities and smooth one-dimensional singularities (in particular line singularities).

Let $f: (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a germ of holomorphic function. A *good stratification* for f at 0 is an analytic stratification of the germ V_f such that the smooth part of V_f is a stratum and so that the stratification satisfies Thom's a_f condition with respect to the complement of V_f (i.e., if $(p_k)_k$ is a sequence of points in the complement of V_f such that $p_k \to p \in S$, where S is a stratum, and the tangent space $T_{p_k} V_{f - f(p_k)}$ converges to some hyperplane T, then $T_p S \subset T$). Notice that good stratifications always exist (cf. Hamm-Lê [HL]). An *aligned* good stratification for f at 0 is a good stratification for f at 0 in which the closure of each stratum of the singular set of f is smooth. If such an aligned good stratification exists, and if the dimension (at 0) of the singular locus of f is s, one says that f has an *s-dimensional aligned singularity* at 0. If $\mathcal{S}$ is an aligned good stratification for f at 0, one says that a linear choice of coordinates $z = (z_1, \ldots, z_n)$ is an *aligning set of coordinates* for $\mathcal{S}$ if, for each $1 \leq i \leq n-1$, the $(n-i)$-plane of $\mathbf{C}^n$, defined by $z_1 = \ldots = z_i = 0$, intersects transversely the closure of each stratum of $\mathcal{S}$ of dimension $\geq i$. One says that a set of coordinates $z = (z_1, \ldots, z_n)$ is aligning for f at 0 if there exists an aligned good stratification for f at 0 with respect to which z is aligning.

A3. PROOF OF THEOREM 2. We essantially repeat Massey's proof of Theorem 3. Since $z = (z_1, \ldots, z_n)$ is an aligning set of coordinates for f_0 and for f_{t_k} at 0, all k, and $(f_t)_t$ is topologically constant, the Lê numbers (cf. [M, Definition 1.11]) $\lambda^i_{f_0,z}$ $(0 \leq i \leq n-1)$ of f_0 at 0 with respect to z are equal to the Lê numbers $\lambda^i_{f_{t_k},z}$ of f_{t_k} at 0 with respect to z, for all k large enough (cf. [M, Corollary 7.8]). By an inductive application of the Massey's generalized Iomdine-Lê formula (cf. [M, Theorem 4.5 and Corollary 4.6]), for all integers $j_1, \ldots, j_s$ such that $0 \ll j_1 \ll j_2 \ll \ldots \ll j_s$, the germs $f_0 + z_1^{j_1} + \ldots + z_s^{j_s}$ and $f_{t_k} + z_1^{j_1} + \ldots + z_s^{j_s}$ have an isolated singularity at 0 and the same Milnor number at 0, provided k is large enough; here, and hereafter, according to [M, Theorem 4.5 and Corollary 4.6], for the germ $f_t + z_1^{j_1} + \ldots + z_s^{j_s}$, all t, one always uses the rotated coordinates $\tilde{z} = (z_{s+1}, z_{s+2}, \ldots, z_n, z_1, \ldots, z_s)$. In particular, by the upper semicontinuity of the Milnor number, this implies that, for all t sufficiently close to 0, the germ $f_t + z_1^{j_1} + \ldots + z_s^{j_s}$ has an isolated singularity at 0 and the same Milnor number, at 0, as $f_0 + z_1^{j_1} + \ldots + z_s^{j_s}$. On the other hand, if the j_i's are sufficiently large, our hypothesis implies that, for all t sufficiently close to 0, the germ $f_t + z_1^{j_1} + \ldots + z_s^{j_s}$ is convenient and has a nondegenerate Newton principal part with respect to the coordinates system $\tilde{z}$. Hence, by Theorem 1, the multiplicity of $f_t + z_1^{j_1} + \ldots + z_s^{j_s}$ at 0 does not depend on t, provided t is sufficiently close to 0. Theorem 2 follows immediately.

REMARK. By contrast with Theorem 3, we do *not* assume $n \geq 4$.

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