Towards Integrability of Topological Strings

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Usually the most interesting questions in Physics are connected with the dynamics. How the system evolves in time, what are the equilibrium states, how the known structures are dynamically generated. There are a lot of interesting questions about the dynamics. But underneath there are more fundamental kinematic questions - what variables should we use to describe the system in the most natural way. Obviously the choice of the kinematic structure precedes the discussion of the dynamics. And the search for the appropriate structure is obviously not an easy task in general.

There are indications that we do not know the right variables in the string theory. Thus the most important problem is to uncover the corresponding underlying kinematic structure. From this point of view the string theories supposedly describing our world are very complicated and it might be extremely difficult to decipher what are the right variables in this case. The problem is that the theories are highly non-local from the point of view of the space-time. In the string geometry the points are considered to be close if there is a curve passing through them. So in a sense all points coincide and it is not clear what is the right way to describe this situation. Obviously this is not a disadvantage but actually an advantage of the theory - it clearly indicates that string theory is not about smooth manifolds or anything like standard geometry. Unfortunately it is not easy to say something meaningful in this situation.

Thus one should look for more manageable yet non-trivial examples of string theories. It seems that the appropriate set of such examples is given by the topological string theories. The main advantage is the huge reduction of the degrees of freedom. Most commonly discussed topological string theories are of two types - Type A and Type B. In mathematical terms the first one is connected with the symplectic geometry and Gromov-Witten invariants and the second one is described in terms of the variations of the Hodge structure in complex geometry. There is some redundancy in this description because Type A theory on a manifold M can be equivalent to Type B theory on another manifold $\tilde{M}$ (if we substitute the notion of the manifold by an appropriate generalization we get the equivalence of Type A and Type B topological strings).

The Type A topological strings are intrinsically non-local. The points on M are considered to be close to each other if there is a holomorphic curve passing through them. Due to the rigidity of the holomorphic maps this case is intermediate between local theories and "fully" non-local string theories. In particular it is difficult to get the explicit formula for Gromov-Witten invariants for a generic manifold.

Type B topological string theory is the most simple case from this point of view - it is a local quantum field theory. So one might suspect that it could give us nothing to help to understand the fundamental degrees of freedom behind the strings. Fortunately it seems not the case. The formulation of these theories in appropriate variables conjecturally leads to a drastic simplification of the theory. It becomes a "free" theory with quadratic action functionals. This phenomena is well known for a simple class of Type B topological strings. I would like to argue that this is a *general* phenomena for Type B topological strings leading to explicit solution of the theory.

Below I am going to describe some results in this direction obtained in collaboration with

Samson Shatashivili [GS1], [GS2]. Some parts were also previously presented in [G]. Let us also remark that the list of references is reduced to the absolute minimum (especially what concerns the next section).

1 Simplest example

To illustrate the ideas we consider the most simple example of the Type B topological string theory - the theory with the target-space being the point $M = pt$ (see [W] for the details). What we mean by the solution of the theory is the set of correlation functions which in this particular case can be defined as follows. Let $\overline{\mathcal{M}}_{g,n}$ be a compactified moduli space of the curves of genus g with n punctures. There are canonical cohomology classes $c_1(\mathcal{L}_i)$ on $\overline{\mathcal{M}}_{g,n}$ (the first Chern classes of the line bundles of the cotangent space at the marked point) associated with each puncture. Then the correlation functions are defined in terms of the generating function of the intersection numbers:

$$<< \mathcal{O}_{k_1} \cdots \mathcal{O}_{k_n} >>= \sum_{p=0}^{\infty} g^{2p-2} < c_1(\mathcal{L}_1)^{k_1} \cdots c_1(\mathcal{L}_n)^{k_n}, [\overline{\mathcal{M}}_{g,n}] > . \tag{1}$$

The whole set of correlation functions may be written as a generating function

$$\log \mathcal{Z}(t) = \sum_{n_1,n_2,\cdots} \frac{t_1^{n_1}}{n_1!} \frac{t_2^{n_2}}{n_2!} \cdots << (\mathcal{O}_1)^{n_1} (\mathcal{O}_2)^{n_2} \cdots >> . \tag{2}$$

For this generating function the most explicit representation was given by Kontsevich [K] in terms of the matrix Airy function

$$Z(t) = \int DX \exp\left(i \frac{1}{3} trX^3 - i\, tr\Lambda X \right), \tag{3}$$

where the integral is over hermitian $(N \times N)$ matrices and $t_n(\Lambda) = 2^{-(2n+1)/3}(2n-1)!!\ Tr(\Lambda^{-n-1/2})$. Explicitly we have

$$\mathcal{Z}(t(\Lambda)) = \frac{Z(\Lambda)}{Z_{cl}(\Lambda)}, \tag{4}$$

where $Z_{cl}(\Lambda)$ is the value of the integral in the quasi-classical approximation. It is natural to organize the parameters t_n in the generating series

$$\varphi(z) = \sum_{n=0}^{\infty} t_n z^n. \tag{5}$$

Thus the generation function is a functional of $\varphi(z)$. This has clear interpretation in terms of 2D Quantum Field Theory (QFT) on the disk D

$$\mathcal{Z}(t) = \int \mathcal{D}\phi(z,\overline{z})\ e^{-\frac{i}{g^2} S[\phi]}, \quad \phi|_{\partial D} = \varphi(z). \tag{6}$$

The action functional S of the two-dimensional field theory is a prototype of the action functionals in string/field theories. In the full-fledged string theories it includes metric tensor, gauge fields and the other fields corresponding to the different modes of the fluctuating strings.

Unfortunately even in the simplest case of Type B theory for $M = pt$ the action functional is a polynomial function of the higher derivatives of the fields. So the corresponding functional theory seems very difficult to calculate explicitly. However the existence of the general solution (3) shows that something missing in this formulation.

Indeed it appears that the change of the variables such that

$$\psi = e^{i\phi}, \qquad \psi^* = e^{-i\phi}, \quad \psi^*\psi = d\phi, \tag{7}$$

saves the case. The relations (7) might look strange (as a change of the variables) but one should take into account regularization of the functional integral in QFT. In the new variables the resulting action has much more simple structure which schematically may be written in the following form

$$S(\psi) = S_D(\psi) + S_b(\psi)) = \int_D d^2z\,(\psi^*\overline{\partial}\psi + c.c.) + (\oint_{\partial D} dz\,\psi^*\partial^3\psi + c.c.). \tag{8}$$

Note that the action $S(\psi)$ is at most quadratic (actually linear) over the fields ψ^* and ψ and thus the value of the functional integral is given by the infinite-dimensional determinant. The theory turns out to be very simple being written in one set of variables and complicated being written in another set. What is striking is that the variables that provide the simplest way to solve the theory are not the natural string variables ϕ!

The following simple analogy might help understanding what these new variables are about. Let $\mathbf{g}$ be a Lie algebra. Consider the collection of the finite-dimensional representations V_a i.e.

$$\pi_a : \mathbf{g} \to End(V_a) = V_a \otimes V_a^*.$$

Thus the image of the generators of the Lie algebra in a given representation can be represented as an element of the tensor product of the representation and its dual. In other terms knowing the category of representations of a group one can reconstruct the group itself. Basically this is what was used above (see the last relation in (7)). We transform the natural string variables ϕ parameterizing string algebra into the variables ψ describing its representation. And this turns out to be not a simple tautology but drastically simplifies the description of the theory.

The meaning of the deformation S_b of the standard action S_D in (8) can be illustrated by the following simple algebraic problem. Consider the Weyl algebra of the differential operators in one variable $\mathbb{C}[z, w \equiv \partial_z]$. The following transformation

$$\begin{aligned} Z &= z + w^2, \\ W &= w, \end{aligned} \tag{9}$$

is an automorphism of the algebra. Remark that in the the classical approximation the corresponding automorphism of the Poisson algebra is generated by the canonical transformation with the generating function

$$\mathcal{F}(w) = \frac{1}{3}w^3, \tag{10}$$

transforming the linear Lagrangian submanifolds into non-linear. The integral kernel of the transformation (9) is easy to find

$$K(z, Z) = e^{\frac{1}{3}\partial_z^3}\delta(z - Z) = \int_{-\infty}^{\infty} dp\;\; e^{i(\frac{1}{3}p^3 + p(z-Z))}. \tag{11}$$

This is an integral representation of the Airy function that appears in the Kontsevich's formula and the reason for the appearance of the third derivative term in (8) is the same as in last formula.

2 Kodaira-Spencer theory on Calabi-Yau three-folds

The example discussed in the previous section might look too simple to make a decision on the fate of the strings in general. However the main ideas can be generalized to more general cases. Thus there are a lot of examples of more complex topological theories (various Landau-Ginzburg theories including the mirror counterparts of Type A topological theories on Fano varieties) that can be explicitly solved in a similar fashion.

Let us consider the most geometrical case of the Type B topological string on Calabi-Yau (CY) threefold. In the classical approximation (that is taking only the contributions of the genus zero curves) one can express the generating function in terms of the critical values of the functional integral as follows [BCOV].

Let M be a compact CY threefold and $\mathcal{M}_M$ be the moduli space of the deformations of M as a CY manifold. Let $\widehat{\mathcal{M}}_M$ be the moduli space of the compact calibrated CY manifolds (i.e. CY manifolds M supplied with a holomorphic $(3,0)$-form Ω_*). The deformation of the complex structure can be parametrized by the Beltrami differential $\mu \in \Omega^{-1,1}(M)$

$$\overline{\partial} \to \overline{\partial}_\mu = \overline{\partial} + \mu\partial, \tag{12}$$

subjected to the integrability condition $\overline{\partial}_\mu^2 = 0$ (Kodaira-Spencer equation):

$$(\overline{\partial} + \mu\partial)^2 = 0.$$

The deformations are considered to be equivalent if they are connected by the action of the global vector fields $v^{0,1}$. Given a holomorphic $(3,0)$-form Ω_* we could identify $\Omega^{-p,q}(M)$ with $\Omega^{3-p,q}(M)$ using the inner product:

$$\Omega^{-p,q}(M) \to \Omega^{3-p,q}(M), \tag{13}$$

$$A \to A^\vee = A \vdash \Omega_*. \tag{14}$$

Define the following operations on the forms:

$$A^\vee \circ B^\vee = (A \wedge B) \vdash \Omega_*,$$

$$< A^\vee, B^\vee, C^\vee >= A^\vee \wedge (B^\vee \circ C^\vee),$$

Then one can rewrite the integrability condition as follows:

$$\begin{aligned} \overline{\partial}\mu^\vee + \tfrac{1}{2}\partial(\mu^\vee \circ \mu^\vee) &= 0, \\ \partial\mu^\vee &= 0, \end{aligned} \tag{15}$$

where now $\mu^\vee \in \Omega^{1,2}$ and we consider the solutions modulo the action of the vector $v^{0,1}$ leaving the holomorphic three-form Ω_* invariant. From the last equation we have $\mu^\vee = x + \partial b$ where x is an element of $H^{1,2}$. Now the critical points of the following functional

$$S_{KS} = \int_M (\frac{1}{2}\partial b \wedge \overline{\partial} b + \frac{1}{6} < (x + \partial b), (x + \partial b), (x + \partial b) >), \tag{16}$$

satisfy (15). Notice that the two-form b here plays the role very similar to the field ϕ in the previous discussion. Therefore the hypothetical representation for the correlation functions would be the functional integral in Kodaira-Spencer theory [BCOV]

$$\mathcal{Z}(x) = \int Db\, e^{-\frac{1}{g^2}S_{KS}(b)}. \tag{17}$$

This functional integral in the classical approximation (evaluated at the critical points) indeed gives a generating function of the correlation functions of the Type B topological strings. We discuss its relevance to the full quantum theory below.

One can derive this representation using the quantum geometry of the moduli space of CY manifolds. Consider the infinite-dimensional space $\mathcal{N}$ of the real three-forms on M supplied with the symplectic structure:

$$\omega^{symp} = \int_M \delta\Omega \wedge \delta\Omega. \tag{18}$$

The phase space $H^3(M,\mathbb{R})$ (which can be identified with the open part of the moduli space $\mathcal{M}_M$) is then obtained by the reduction of the infinite-dimensional symplectic space with respect to the first class constraint:

$$d\Omega = 0. \tag{19}$$

There are two natural polarizations on the symplectic manifold $\mathcal{N}$. Given a complex structure on M let

$$\Omega_{\mathbb{C}} = \Omega^{3,0} \oplus \Omega^{2,1} \oplus \Omega^{1,2} \oplus \Omega^{0,3}, \tag{20}$$

be the Hodge decomposition of the complex three-forms on M. The real forms are singled out by the reality condition: $\Omega^{0,3} = \overline{\Omega^{0,3}}$, $\Omega^{1,2} = \overline{\Omega^{2,1}}$. The subspaces $\Omega^{3,0}\oplus\Omega^{2,1}$ and $\Omega^{1,2}\oplus\Omega^{0,3}$ define complementary linear complex Lagrangian sub-manifolds in the space of complex three-forms and thus a linear complex polarization of $\mathcal{N}$.

In the specific case of the three-forms in six dimensions there is another polarization [H]. It can be constructed using the decomposition of the generic real three-form Ω into the sum of two decomposable forms $\Omega = \Omega_+ + \Omega_-$. Let us consider the open subset in the space of three-forms $\mathcal{N}$ where the following decomposition holds

$$\Omega = \Omega_+ + \Omega_- = E^1 \wedge E^2 \wedge E^3 + \overline{E}^1 \wedge \overline{E}^2 \wedge \overline{E}^3, \tag{21}$$

with E^i being complex one-forms. It is easy to see that the subspace of the decomposable forms defines the Lagrangian family and gives rise to a *non-linear* polarization.

One can construct the canonical transformation U relating the two polarizations. Basically the generating function of this canonical transformation coincides with the value of the action at the critical points of the integral (17)

$$S^*_{KS}(x) = \frac{1}{6}\int_M < \mu^\vee, \mu^\vee, \mu^\vee > . \tag{22}$$

The Kodaira-Spencer functional integral (17) is equal then to the result of the application of the corresponding quantum version of the canonical transformation $\hat{U}$ to the simple wave function $\mathcal{Z}_0$ defined in the non-linear polarization

$$\mathcal{Z} = \hat{U}\mathcal{Z}_0. \tag{23}$$

Here $\mathcal{Z}_0$ is a simple distribution supported on the fixed section of the $\mathbb{C}^*$-bundle $\widehat{\mathcal{M}} \to \mathcal{M}$.

One can reformulate the relation (23) between the different wave functions in terms of the higher dimensional field theory. Quantization of the space of three-forms on the six-dimensional manifold M with the imposed constraint (19) can be formulated in terms of the the seven-dimensional field theory with the action

$$S^7(\Omega) = \int_{M \times I} \Omega \; d \; \Omega + S_b(\Omega), \tag{24}$$

where $I = [0, T]$ is an interval and $S_b(\Omega) = S_b^{t=0}(\Omega) + S_b^{t=T}(\Omega)$ is a boundary term defining the polarization at $t = 0$ and $t = T$. To reproduce the relation (23) one can chose the linear polarization for the wave function at $t = 0$ and the nonlinear polarization at $t = T$. The most nontrivial part is the boundary term defining the nonlinear polarization which is expressed in terms of the (generalized) Hitchin functional

$$e^{-S_b^{t=T}} = e^{-\int_M \sqrt{-\lambda(\Omega)}} \sim \int DK \; e^{-\int_M \left(\frac{1}{\sqrt{-\frac{1}{6}trK^2}} \Omega \wedge i_K \Omega + \sqrt{-\frac{1}{6}trK^2} \right)}, \tag{25}$$

where $\sqrt{-\lambda(\Omega)}$ is a volume element constructed using three-form Ω (see [H]) and $K \in End(TM)$ is an operator acting in the tangent space (see [H, GS1]). Note that the last equality holds for the critical point contribution of the integral over K.

The important point is that the quantum unitary transformation $\widehat{U}$ in (23) is not uniquely defined by its classical counterpart U. Thus the corresponding Kodaira-Spencer (KS) action (16) is unambiguously defined only in the classical approximation and may need corrections to reproduce the exact result beyond the classical approximation. Below we will argue that this is what actually happens in KS theory. Moreover proper account of these corrections presumably allows one to solve the theory exactly. Thus we have the full analogy with the simplest example $M = pt$ discussed in the first section.

3 On quantum completion and exact solution

Let us compare in more details the integral representation for the KS theory for three-dimensional CY manifold M with the representation for the simple case $M = pt$. In both cases we have the quadratic bulk action with the non-linear boundary term which is unambiguously defined only in the leading order over the coupling constant. The cubic dependence in (22) manifests the clear analogy with the classical cubic action in the case $M = pt$. In technical terms both are the first critical points in the corresponding hierarchies of the field theories. Let us recall that in the case $M = pt$ (complex dimension is zero) the representation for the generating function was written in terms of the Quantum Field Theory on the one-dimensional complex manifold. Following this analogy we should expect that the proper formulation of the KS theory on the complex compact three-dimensional manifold should be given in terms of the Quantum Field Theory on a complex *four-dimensional* non-compact manifold. This has perfect sense from the point of view of the underlying topological string theory. In general the correlation functions of the topological strings are described in terms of the cohomology classes of the compactified moduli space of the holomorphic maps of the two dimensional surfaces Σ into a given manifold M. The (open part of the) moduli space has a structure of a fibration over the moduli space of the holomorphic structures on Σ. The correlation functions are given by the cohomology classes of M pulled back to the moduli space and multiplied by the powers of $c_1(\mathcal{L})$. The possibility to multiply on powers of the cohomology classes $c_1(\mathcal{L})$ effectively makes the manifold one dimension higher. The fact that this additional dimension does not show up *explicitly* in the Kodaira-Spencer formulation is due to the specifics of the geometry of CY threefolds - one can consistently truncate the set of the correlation functions to the subset that does not include the fields corresponding to higher powers of $c_1(\mathcal{L})$. However to formulate the theory in a way that reveals its simplicity one should use the full set of the correlation functions. Thus to pursue the full analogy with the case of $M = pt$ one should reformulate KS theory in terms of the QFT on four-dimensional complex manifold. The *exact* quantum formulation of the Type B topological theory on CY threefold would be given in terms of the quadratic first derivative theory in eight dimensions deformed by the quadratic higher derivative boundary terms.

The simplest example of the eight-dimensional manifold in question is $\mathbb{C} \times M$. Its product structure is a manifestation of the fact that M is a CY threefold. In general it does not need to be so. One has the quadratic theories on $(d+1)$-dimensional manifold providing the description of Type B toplogical strings on d-dimensional manifold. Note that this can be considered as a complexified version of AdS/CFT correspondence [GS2].

Let us remark that the germ of this higher-dimensional formulation has already appeared at the end of the last section. The only thing that should be added is the one additional real dimension. The appearance of this additional dimension can be traced back to the geometrization of the multiplicative group $\mathbb{C}^*_{\mathbb{R}}$ which is the Tannaka-Krein dual to the category of the pure Hodge structures.

Thus from this point of view the Kodaira-Spencer action is an effective action that should be modified in such a way that the functional integral becomes quadratic in appropriate variables (actually by adding infinite number of additional variables to make it quadratic). Some confirmation of this representation comes from the duality considerations. By independent reasons one knows the expansion of the full generating function around degeneration locuses of the moduli space of some CY manifolds. Quite surprisingly we get various modular forms of the Borcherds type as generating functions. This implies that there is a hidden quadratic structure.

What is the geometrical meaning of the proposed description? Following the previous example one would say that the fundamental fields in this case are definitely not the strings (which are directly connected with variables μ in KS theory). There should be some underlying structure behind the differential forms that are used in KS theory. It seems the hint in the right direction is the following fundamental relation

$$\oplus_{p+q=n} H^p(M, \wedge^q \mathcal{T}_M) = Ext^n_{\mathcal{C}oh(M\times\overline{M})}(\mathcal{O}_\Delta, \mathcal{O}_\Delta) \tag{26}$$

where $\mathcal{T}_M$ is holomorphic tangent bundle to M, $\mathcal{C}oh(X)$ is the category of the coherent sheaves on X, $\overline{M}$ is M with the opposite complex structure and $\Delta \subset M \times M$ is a diagonal embedding. This interpretation of the de Rham cohomology can be translated into the relation between open and closed topological strings and is the conceptual reason for the change of the variables discussed in the first section. One of the indications that the reformulation in terms of the underlying category is relevant to the problem discussed above is the fact that for the semisimple Frobenius structures on the moduli space (corresponding to the simplest categories in the above formulation) the representation directly generalizing the representation discussed at the first section is known [Giv].

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