ON THE SPACE OF CUSP FORMS FOR $\mathrm{GL}(n)$

WERNER MÜLLER

(joint work with Birgit Speh)

Let $\mathbb{A}$ be the ring of adèles of $\mathbb{Q}$ and let $G = \mathrm{GL}(n)$. Let $Z = Z_G(\mathbb{R})^0$. Let R be the regular representation of $G(\mathbb{A})$ in $L^2(ZG(\mathbb{Q})\backslash G(\mathbb{A}))$, defined by

$$(R(g)f)(g') = f(g'g).$$

This is a unitary representation and one of the fundamental problems in the theory of automorphic forms is to study the decomposition of R into irreducible unitary representations of $G(\mathbb{A})$. Note that every irreducible representation of $G(\mathbb{A})$ is a rstricted tensor product

$$\pi = \otimes_{v\le\infty}\pi_v = \pi_\infty \otimes \otimes_p \pi_p$$

of irresucible representations of the local groups $\mathrm{GL}(n,\mathbb{Q}_v)$. By Langlands' theory of Eisenstein series there exists a decomposition

$$L^2(ZG(\mathbb{Q})\backslash G(\mathbb{A})) = L^2_d(ZG(\mathbb{Q})\backslash G(\mathbb{A})) \oplus L^2_c(ZG(\mathbb{Q})\backslash G(\mathbb{A})),$$

into invariant subspaces, where $L^2_d(ZG(\mathbb{Q})\backslash G(\mathbb{A}))$ is the smallest closed subspace containing all irresucible subrepresentations of R. Its orthogonal complement $L^2_c(ZG(\mathbb{Q})\backslash G(\mathbb{A}))$ can be described in terms of Eisenstein series. Let

$$L^2_{\mathrm{cusp}}(ZG(\mathbb{Q})\backslash G(\mathbb{A})) \subset L^2_d(ZG(\mathbb{Q})\backslash G(\mathbb{A}))$$

be the space of cusp forms. Then

$$L^2_d(ZG(\mathbb{Q})\backslash G(\mathbb{A})) = L^2_{\mathrm{cusp}}(ZG(\mathbb{Q})\backslash G(\mathbb{A})) \oplus L^2_{\mathrm{res}}(ZG(\mathbb{Q})\backslash G(\mathbb{A})).$$

The residual subspace has been completely determined by Moeglin and Waldspurger. Let

$$L^2_{\mathrm{cusp}}(ZG(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_{\pi\in\widehat{G}(\mathbb{A})} m(\pi)\mathcal{H}_\pi.$$

Then the main issue is to gain information about the multiplicities. The main toll to study cusp forms is the Arthur-Selberg trace formula. Let $G(\mathbb{A})^1$ be the intersection of the kernels of the maps $x \mapsto |\chi(x)|$, $x \in G(\mathbb{A})$, where χ ranges over the group $X(G)_{\mathbb{Q}}$ of characters of G

Date: June, 2001.

defined over $\mathbb{Q}$. Then the (noninvariant) trace formula of Arthur is an identidy

$$\sum_{\mathfrak{o}\in\mathfrak{O}} J_{\mathfrak{o}}(f) = \sum_{\chi\in\mathfrak{X}} J_\chi(f), \quad f \in C_0^\infty(G(\mathbb{A})^1),$$

between distributions on $G(\mathbb{A})^1$. The left hand side is the *geometric side* and the right hand side the *spectral side* of the trace formula. The distributions $J_{\mathfrak{o}}$ are parametrized by semisimple conjugacy in $G(\mathbb{Q})$ and are closely related to weighted orbital integrals on $G(\mathbb{A})^1$.
In this talk we study the spectral side of the trace formula which we try to use to study the asymptotic distribution of the multiplicities. The main result is the following theorem.

Theorem 0.1. *The spectral side of the Arthur trace formula for* $\mathrm{GL}(n)$ *is absolutely convergent.*

This theorem will facilitate the application of the trace formula and will also open up new directions for its application.
Of particular importance in the proof is the investigation of the analytic properties of the intertwining operators. Let $P \subset G$ be a maximal parabolic subgroup with $P = MN$ and

$$M = \mathrm{GL}(n_1) \times \mathrm{GL}(n_2).$$

Let $\pi = \pi_1 \otimes \pi_2$ be a cuspidal automorphic representation of $M(\mathbb{A})$. For $s \in \mathbb{C}$ let $\chi_s : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^\times$ be the character $\chi_s(a) = |a|^s$. Extend χ_s to $\mathrm{GL}(m,\mathbb{A})$ by $\chi_s(g) = \chi_s(\det(g))$. Let

$$\pi_s = \pi_1 \cdot \chi_s \otimes \pi_2\chi_{-s}.$$

Let $I_P^G(\pi_s)$ be the corresponding induced repersentation of $\mathcal{G}(\mathbb{A})$ acting in $\mathcal{H}_P(\pi_s)$. Let $\mathcal{H}_P^0(\pi_s)$ be the subspace of K-finite vectors and let

$$M(\pi, s) : \mathcal{H}_P^0(\pi_s) \to \mathcal{H}_{\overline{P}}^0(\pi_s)$$

be the standard intertwining operator. It is a meromorphic operator valued function of s. Let $L(s, \pi_1 \times \check{\pi}_2)$ and $\epsilon(s, \pi_1 \times \check{\pi}_2)$ be the Rankin-Selberg L-function and the ϵ-factor. Then

$$M(\pi, s) = \frac{L(s, \pi_1 \times \check{\pi}_2)}{L(1+s, \pi_1 \times \check{\pi}_2)\epsilon(s, \pi_1 \times \check{\pi}_2)} \bigotimes_{v\in S} N(\pi_v, s), \tag{0.1}$$

where S is a finite set of places containing the Archimedean one, and $N(\pi_v, s)$ are normalized local intertwining operators. Let $m(\pi, s)$ denote the L-factor in (0.1). Let $\| \pi \|$ be the norm of the infinitesimal character of π_∞. Then we have

Theorem 0.2. *For every pair of open compact subgroups* $K_{i,f} \subset \mathrm{GL}(m_i, \mathbb{A}_f)$, $i = 1, 2$, *there exists* $C > 0$ *such that*

$$\int_{\mathbb{R}} \left| \frac{m'}{m}(\pi, i\lambda) \right| (1 + \lambda^2)^{-2} \leq C \parallel \pi \parallel$$

for all $\pi = \pi_1 \times \pi_2$ *such that* $\pi_i \in \Pi_{\mathrm{cusp}}(\mathrm{GL}(m_i, \mathbb{A})$ *has a nonzero* K*-invariant vector.*

In fact we conjecture that the following holds

Conjecture. There exist $C > 0$, $k \in \mathbb{N}$, such that

$$\left| \frac{L'(1 + i\lambda, \pi_1 \times \check{\pi}_2)}{L(1 + i\lambda, \pi_1 \times \check{\pi}_2)} \right| \leq C(\log \lambda)^k$$

for $|\lambda| \geq 1$ and all $\pi_i \in \Pi_{\mathrm{cusp}}\, \mathrm{GL}(m_i, \mathbb{A})$.

Finally we establish a certain weak form of the Ramanujan conjecture.

Universität Bonn, Mathematisches Institut, Beringstrasse 1, D – 53115 Bonn, Germany
E-mail address: mueller@math.uni-bonn.de