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Abstract. Here, we give a detailed account of a proof for the estimates of Fourier coefficients of weakly holomorphic modular forms, which play an important role in the study of Borcherds lifts.

1 Introduction

1.1

Borcherds ([Bo1], [Bo2]) constructed an infinite product Ψ_f on an orthogonal group $G = O(2, n+2)$ attached to a weakly holomorphic modular form f of weight $-n/2$, and showed that Ψ_f (called the Borcherds lift of f) has a meromorphic continuation. The proof of meromorphic continuation in [Bo2] relies on the construction of Ψ_f by a regularlized theta integral of f, while the proof in [Bo1] is more direct and uses certain estimates of Fourier coefficients of f in an essential way. He gave a brief sketch of a proof of the estimates ([Bo1], Lemma 5.3) without giving a full detail. The aim of this note is to give a detailed account of a proof for the estimates. We hope that this note, a compilation of almost known results, is helpful for the study of Borcherds lifts.

The paper is organized as follows. In Section 2, after preparing notations and recalling the definition of vector valued weakly holomorphic modular forms, we state the main result of this note: the estimates of Fourier coefficients of weakly holomorphic modular forms (Theorem 2.2). The proof of Theorem 2.2 is given in Section 3.

1.2 Notation

As usual, we denote by $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ the set of natural numbers, the ring of rational integers, the field of rational numbers, the field of real numbers and the field of complex numbers respectively. We let $\mathbb{R}_{>0} := \{x \in \mathbb{R} \mid x > 0\}$ and $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \mid x \geq 0\}$. Let $\mathfrak{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ denote the upper half plane. Define an action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathfrak{H}$ and an automorphic factor $j\colon \mathrm{SL}_2(\mathbb{R}) \times \mathfrak{H} \to \mathbb{C}^\times$ by

$$g\langle\tau\rangle := \frac{a\tau + b}{c\tau + d},\ j(g, \tau) := c\tau + d \qquad \left(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \tau \in \mathfrak{H}\right)$$

as usual. Let $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. For $z \in \mathbb{C}$, we put $\mathbf{e}(z) := \exp(2\pi i z)$. For a symmetric matrix T of degree m and vectors $X, Y \in \mathbb{C}^m$, we put $T(X, Y) := {}^tXTY$ and $T[X] := T(X, X)$. Let δ_{ij} denote the Kronecker's delta. For $x \in \mathbb{R}$, we put $[x] := \max\{n \in \mathbb{Z} \mid n \leq x\}$.

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2 Main results

2.1 Lattices and quadratic forms

Throughout the paper, we fix a positive definite even integral symmetric matrix S of degree n. Let $L := \mathbb{Z}^n$ and $L' := S^{-1}L$. We put $S' := 2^{-1}S$ and

$$q(x) := -S'[x] \qquad (x \in \mathbb{C}^n).$$

2.2 The metaplectic group ([Bo2] Section 2 and [Br] 1.1)

Let $\mathrm{Mp}_2(\mathbb{R})$ be the metaplectic group. By definition, $\mathrm{Mp}_2(\mathbb{R})$ consists of (M, φ), where $M \in \mathrm{SL}_2(\mathbb{R})$ and φ is a holomorphic function on $\mathfrak{H}$ with $\varphi(z)^2 = j(M, z)$ with multiplication law

$$(M_1, \varphi_1(z))(M_2, \varphi_2(z)) = (M_1 M_2, \varphi_1(M_2\langle z\rangle)\,\varphi_2(z)).$$

In what follows, we take a square root $\sqrt{z}$ of $z \in \mathbb{C}^\times$ to be $-\pi/2 < \arg(\sqrt{z}) \leq \pi/2$. Let $\mathrm{Mp}_2(\mathbb{Z})$ be the inverse image of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ under the natural projection $\mathrm{Mp}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$. Then $\mathrm{Mp}_2(\mathbb{Z})$ is generated by

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right) \quad \text{and} \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{z}\right).$$

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we let $\widetilde{\gamma} := (\gamma, \sqrt{c\tau + d}) \in \mathrm{Mp}_2(\mathbb{Z})$.

2.3 Metaplectic representations ([Bo2] Section 4 and [Br] 1.1)

Let $\mathbb{C}[L'/L] = \sum_{\alpha \in L'/L} \mathbb{C}\mathfrak{e}_\alpha$ be the group ring with $\mathfrak{e}_\alpha \mathfrak{e}_{\alpha'} = \mathfrak{e}_{\alpha+\alpha'}$. We define a representation r_L of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$ by

$$\begin{aligned} r_L(T)\mathfrak{e}_\alpha &= \mathbf{e}(q(\alpha))\mathfrak{e}_\alpha, \\ r_L(S)\mathfrak{e}_\alpha &= \frac{\sqrt{i}^n}{\sqrt{|L'/L|}} \sum_{\beta \in L'/L} \mathbf{e}(S(\alpha, \beta))\mathfrak{e}_\beta \end{aligned} \tag{2.1}$$

Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{C}[L'/L]$ given by

$$\langle \sum_{\alpha \in L'/L} \lambda_\alpha \mathfrak{e}_\alpha, \sum_{\alpha \in L'/L} \mu_\alpha \mathfrak{e}_\alpha \rangle = \sum_{\alpha \in L'/L} \lambda_\alpha \overline{\mu_\alpha} \qquad (\lambda_\alpha, \mu_\alpha \in \mathbb{C}).$$

Then r_L is a unitary representation of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$.

2.4 Weakly holomorphic modular forms ([Bo2] Sections 2, 4 and [Br] 1.1)

Let $k \in 2^{-1}\mathbb{Z}$. For a $\mathbb{C}[L'/L]$-valued holomorphic function f on $\mathfrak{H}$ and $(\gamma, \varphi) \in \mathrm{Mp}_2(\mathbb{Z})$, we put

$$(f|_k(\gamma, \varphi))\,(\tau) := \varphi(\tau)^{-2k} r_L(\gamma, \varphi)^{-1} f(\gamma\langle\tau\rangle).$$

Let $f(\tau) = \sum_{\alpha \in L'/L} f_\alpha(\tau)\mathfrak{e}_\alpha$, where f_α is a $\mathbb{C}$-valued function on $\mathfrak{H}$. Suppose that $f|_k T = f$. Then $f_\alpha(\tau+1) = \mathbf{e}\left(q(\alpha)\right) f_\alpha(\tau)$ and hence f admits the Fourier expansion

$$f(\tau) = \sum_{\alpha \in L'/L} \sum_{l \in \mathbb{Z}+q(\alpha)} c_f(\alpha, l)\mathfrak{e}_\alpha(l\tau), \tag{2.2}$$

where we put $\mathfrak{e}_\alpha(\tau) := \mathfrak{e}_\alpha\, \mathbf{e}(\tau) \ \ (\tau \in \mathfrak{H})$.

Suppose that $k \leq 0$. Let $\mathcal{W}_{k,\, r_L}$ be the space of holomorphic functions $f \colon \mathfrak{H} \to \mathbb{C}[L'/L]$ satisfying the following conditions:

(1) For any $(\gamma, \varphi) \in \mathrm{Mp}_2(\mathbb{Z})$, we have $f|_k(\gamma, \varphi) = f$.

(2) There exists a positive integer M such that $c_f(\alpha, m) = 0$ for any $(\alpha, m) \in L'/L \times (\mathbb{Z}+q(\alpha))$ with $m < -M$.

We call $\mathcal{W}_{k,\, r_L}$ the space of *weakly holomorphic* modular forms on $\mathrm{Mp}_2(\mathbb{Z})$ of weight k with respect to r_L.

In this paper we are mainly concerned with the case of $k = -n/2$, since a Borcherds lift is constructed from $f \in \mathcal{W}_{-n/2,\, r_L}$ (cf. [Bo2]). We include the case of $n = 0$, in which case $\mathcal{W}_{-n/2,\, r_L}$ is the ring $\mathbb{C}[J]$ generated by the modular invariant J over $\mathbb{C}$. Here J is the Γ-invariant holomorphic function on $\mathfrak{H}$ with $J(\tau) = q^{-1} + \sum_{m=1}^{\infty} c_m q^m \ \ (q := \mathbf{e}(\tau))$.

2.5 Kloosterman sums ([Br] 1.3)

Let $c \in \mathbb{Z} \setminus \{0\}, \alpha, \beta \in L'/L, l \in \mathbb{Z}+q(\alpha)$ and $m \in \mathbb{Z}+q(\beta)$. We define the Kloosterman sum by

$$H_c(\beta, m, \alpha, l) := \frac{e^{\pi i\, \mathrm{sgn}(c) n/4}}{|c|} \sum_d \langle \mathfrak{e}_\beta, r_L\left(\widetilde{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}\right) \mathfrak{e}_\alpha \rangle\, \mathbf{e}\left(\frac{ma+ld}{c}\right), \tag{2.3}$$

where d runs over $(\mathbb{Z}/|c|\mathbb{Z})^\times$ and $a, b \in \mathbb{Z}$ are chosen such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. When $n = 0$, we write $H_c(\beta, m, \alpha, l)$ for $H_c(m, l) := |c|^{-1} \sum_d \mathbf{e}\left(c^{-1}(ma+ld)\right)$ by abuse of notation.

2.6 Modified Bessel functions ([OLBC] 10.25)

For $\nu \in \mathbb{R}_{>0}$ and $z \in \mathbb{C} \setminus (-\infty, 0]$, we define the modified Bessel functions by

$$I_\nu(z) := \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{k!\, \Gamma(\nu+k+1)}. \tag{2.4}$$

Here z^a for $z \in \mathbb{C} \setminus (-\infty, 0]$ and $a \in \mathbb{C}$ means the principal branch (see [OLBC], 4.2 (iv)). We put $I_\nu(0) = 1$. Later we need the following asymptotic formulas for I_ν (see [OLBC] 10.30).

Lemma 2.1. *We have*

$$I_\nu(y) \sim \begin{cases} \dfrac{(2^{-1}y)^\nu}{\Gamma(\nu+1)} & (y \to 0), \\[2ex] \dfrac{e^y}{\sqrt{2\pi y}} & (y \to \infty). \end{cases} \tag{2.5}$$

2.7 Estimates of Fourier coefficients

THEOREM 2.2. *Let $f \in \mathcal{W}_{-n/2, r_L}$ and*

$$f(\tau) = \sum_{\alpha \in L'/L} \sum_{l \in \mathbb{Z}+q(\alpha)} c_f(\alpha, l)\mathfrak{e}_\alpha(l\tau)$$

be the Fourier expansion of f.

(1) *For $\alpha \in L'/L$ and $l \in \mathbb{Z}+q(\alpha)$ with $l > 0$, we have*

$$\begin{aligned} c_f(\alpha, l) = \pi \sum_{\beta \in L'/L} \sum_{m \in \mathbb{Z}+q(\beta),\, m<0} & c_f(\beta, m) \left|\frac{m}{l}\right|^{(n+2)/4} \\ & \sum_{c \in \mathbb{Z}\setminus\{0\}} H_c(\beta, m, \alpha, l) I_{1+n/2}\left(\frac{4\pi}{|c|}\sqrt{|lm|}\right). \end{aligned} \tag{2.6}$$

(2) *There exist a positive integer A and positive real numbers δ, C such that*

$$|c_f(\alpha, l)| \le Ce^{\delta\sqrt{l}} \tag{2.7}$$

holds for any $(\alpha, l) \in L'/L \times (\mathbb{Z}+q(\alpha))$ with $l \ge A$.

(3) *For any $\epsilon > 0$, there exist positive integers A, N and a positive real number C such that the following estimate holds for any $(\alpha, l) \in L'/L \times (\mathbb{Z}+q(\alpha))$ with $l \ge A$:*

$$\begin{aligned} &\left|c_f(\alpha, l) - \pi \sum_{\beta \in L'/L} \sum_{m \in \mathbb{Z}+q(\beta),\, m<0} c_f(\beta, m) \left|\frac{m}{l}\right|^{(n+2)/4} \sum_{0<|c|<N} H_c(\beta, m, \alpha, l) I_{1+n/2}\left(\frac{4\pi}{|c|}\sqrt{|ml|}\right)\right| \\ &\le C\, e^{\epsilon\sqrt{l}}. \end{aligned} \tag{2.8}$$

REMARK 2.3. The first assertion of Theorem 2.2 is essentially due to Rademacher and Zuckerman ([RaZu] and [Ra]). The third assertion of Theorem 2.2 is stated in [Bo1] as Lemma 5.3 with a brief sketch of the proof. It is noted that in [Bo1] there is no mention on the uniformity of the estimates of $c_f(\alpha, l)$ with respect to l, which is crucial to the proof of meromorphic continuation of the Borcherds lifts.

3 The proof of Theorem 2.2

3.1 Whittaker functions ([OLBC] 13.14)

For $\nu, \mu \in \mathbb{C}$ and $z \in \mathbb{C}$, set

$$\begin{aligned} M_{\nu,\mu}(z) &:= e^{-z/2} z^{\mu+1/2} \mathbf{M}\left(\mu - \nu + \frac{1}{2}, 2\mu+1, z\right), \\ W_{\nu,\mu}(z) &:= \frac{\Gamma(-2\mu)}{\Gamma(1/2-\mu-\nu)} M_{\nu,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma(1/2+\mu-\nu)} M_{\nu,-\mu}(z), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned}\mathbf{M}(a,b,z) &:= \sum_{l=0}^{\infty} \frac{(a)_l}{(b)_l l!} z^l, \\ (a)_l &:= \begin{cases} a(a+1)\cdots(a+l-1) & \text{if } l \geq 1, \\ 1 & \text{if } l = 0. \end{cases}\end{aligned} \tag{3.2}$$

Then $M_{\nu,\mu}(z)$ and $W_{\nu,\mu}(z)$ are linearly independent solutions of the Whittaker differential equation

$$\frac{d^2 w}{dz^2} + \left(-\frac{1}{4} + \frac{\nu}{z} - \frac{\mu^2 - 1/4}{z^2}\right) w = 0.$$

For $s \in \mathbb{C}$ and $y \in \mathbb{R}_{>0}$, we put

$$\begin{aligned}\mathcal{M}_s(y) &= y^{n/4} M_{n/4,\, s-1/2}(y), \\ \mathcal{W}_s(y) &= y^{n/4} W_{-n/4,\, s-1/2}(y).\end{aligned} \tag{3.3}$$

3.2 Estimates of Kloosterman sums

Lemma 3.1. *Let $\alpha, \beta \in L'/L, l \in \mathbb{Z} + q(\alpha)$ and $m \in \mathbb{Z} + q(\beta)$.*

(1) *We have*

$$|H_c(\beta, m, \alpha, l)| \leq 1$$

for any $c \in \mathbb{Z} \setminus \{0\}$.

(2) *Suppose that $n = 0$. Then there exist $\lambda > 0$ and $C > 0$ such that*

$$|H_c(\beta, m, \alpha, l)| \leq C|c|^{-\lambda}$$

holds for any $c \in \mathbb{Z} \setminus \{0\}$.

Proof. The first assertion is obvious. For the second one, see for example [Ra] (5.3). □

3.3 Poincaré series ([Br] 1.3)

For $\beta \in L'/L, m \in \mathbb{Z} + q(\beta)$ with $m < 0$, define

$$F_{\beta,m}(\tau, s) = \frac{1}{2\Gamma(2s)} \sum_{(\gamma,\phi) \in \widetilde{\Gamma}_\infty \backslash \mathrm{Mp}_2(\mathbb{Z})} \left(\mathcal{M}_s(4\pi|m|y)\mathfrak{e}_\beta(mx)\right)|_{-n/2}(\gamma, \phi) \qquad (\tau = x + iy \in \mathfrak{H}),$$

where $\widetilde{\Gamma}_\infty = \langle T \rangle$. The Poincaré series $F_{\beta,m}(\tau, s)$ is absolutely convergent for $\tau \in \mathfrak{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, and continued to a meromorphic function of s on $\mathbb{C}$.

Lemma 3.2 ([Br] Proposition 1.12). *For $f \in \mathcal{W}_{-n/2, r_L}$, we have*

$$f(\tau) = \frac{1}{2} \sum_{\beta \in L'/L} \sum_{m \in \mathbb{Z} + q(\beta),\, m < 0} c_f(\beta, m) F_{\beta,m}\left(\tau, 1 + \frac{n}{4}\right).$$

Lemma 3.3 ([Br] Proposition 1.10). *For $\beta \in L'/L, m \in \mathbb{Z}+q(\beta)$ with $m<0$, we have*

$$\begin{aligned} F_{\beta,m}\left(\tau, 1+\frac{n}{4}\right) &= \mathfrak{e}_\beta(m\tau)+\mathfrak{e}_{-\beta}(m\tau) \\ &\quad + \sum_{\alpha\in L'/L}\ \sum_{l\in\mathbb{Z}+q(\alpha),\, l\geq 0} b(\alpha,l)\mathfrak{e}_\alpha(l\tau) \\ &\quad + \sum_{\alpha\in L'/L}\ \sum_{l\in\mathbb{Z}+q(\alpha),\, l<0} b(\alpha,l)\mathcal{W}_{1+n/4}(4\pi l y)\mathfrak{e}_\alpha(lx). \end{aligned}$$

Here the Fourier coefficients $b(\alpha,l)$ are given by

$$b(\alpha,l) := \begin{cases} 2\pi\left|\dfrac{m}{l}\right|^{(n+2)/4}\sum_{c\in\mathbb{Z}\setminus\{0\}} H_c(\beta,m,\alpha,l) I_{1+n/2}\left(\dfrac{4\pi}{|c|}\sqrt{|lm|}\right) & \text{if } l>0, \\[2ex] \dfrac{(2\pi)^{2+n/2}|m|^{1+n/2}}{\Gamma(n/2+2)}\sum_{c\in\mathbb{Z}\setminus\{0\}} |c|^{-n/2-1} H_c(\beta,m,\alpha,0) & \text{if } l=0, \\[2ex] -\Gamma(n/2+1)^{-1}\delta_{l,m}\left(\delta_{\alpha,\beta}+\delta_{\alpha,-\beta}\right) & \\[2ex] +\dfrac{2\pi}{\Gamma(n/2+1)}\left|\dfrac{m}{l}\right|^{(n+2)/4}\sum_{c\in\mathbb{Z}\setminus\{0\}} H_c(\beta,m,\alpha,l) I_{1+n/2}\left(\dfrac{4\pi}{|c|}\sqrt{|lm|}\right) & \text{if } l<0. \end{cases} \tag{3.4}$$

3.4 Estimates for certain infinite sums

Let $\lambda \in \mathbb{R}_{\geq 0}$ and $\nu \in \mathbb{R}_{>0}$. For $\mu \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$, we put

$$\begin{aligned} S_{\lambda,\nu}(\mu) &:= \sum_{c\in\mathbb{Z}\setminus\{0\}} |c|^{-\lambda} I_\nu\left(\frac{\mu}{|c|}\right), \\ S^N_{\lambda,\nu}(\mu) &:= \sum_{c\in\mathbb{Z}\setminus\{0\},\, |c|\geq N} |c|^{-\lambda} I_\nu\left(\frac{\mu}{|c|}\right). \end{aligned}$$

The following fact is elementary, though we give its proof for completeness.

Lemma 3.4. *Assume that $\lambda+\nu>1$.*

(1) *For any $\delta>1$, there exist $M>0$ and $C>0$ such that*

$$|S_{\lambda,\nu}(\mu)| \leq Ce^{\delta\mu}$$

holds for any $\mu \geq M$.

(2) *For any $\epsilon>0$, there exist $N_0\in\mathbb{N}$, $M>0$ and $C>0$ such that*

$$\left|S^N_{\lambda,\nu}(\mu)\right| \leq Ce^{\epsilon\mu}$$

holds for any $N\geq N_0$ and $\mu\geq M$.

Proof. By Lemma 2.1, there exist positive real numbers C_1, C_2 such that

$$I_\nu(y) \le \begin{cases} C_1 y^{-1/2} e^y & \text{if } y \ge 1, \\ C_2 y^\nu & \text{if } y \le 1. \end{cases} \tag{3.5}$$

Let $\mu \ge 1$. By (3.5), we have

$$\begin{aligned} 2^{-1} S_{\lambda,\nu}(\mu) &= \sum_{c=1}^{\infty} c^{-\lambda} I_\nu\left(\frac{\mu}{c}\right) \\ &\le \sum_{c=1}^{[\mu]} c^{-\lambda}\, C_1 \frac{e^{\mu/c}}{\sqrt{\mu/c}} + \sum_{c=[\mu]+1}^{\infty} c^{-\lambda} C_2 \left(\frac{\mu}{c}\right)^\nu \\ &= C_1\, \mu^{-1/2} \sum_{c=1}^{[\mu]} c^{1/2-\lambda} e^{\mu/c} + C_2\, \mu^\nu \sum_{c=[\mu]+1}^{\infty} c^{-\lambda-\mu} \\ &\le C_1\, \mu^{\max\{1/2,1-\lambda\}} e^{\mu} + C_2\, \zeta(\lambda+\mu)\mu^\nu, \end{aligned}$$

from which the first assertion of the lemma follows. Here $\zeta(s)$ denotes the Riemann zeta function. To prove the second assertion, choose $N_0 \in \mathbb{N}$ such that $N_0 \ge 2\epsilon^{-1}$. If $N \ge N_0$, we have

$$\begin{aligned} 2^{-1} S^N_{\lambda,\nu}(\mu) &= \sum_{c=N}^{[\mu]} c^{-\lambda}\, C_1 \frac{e^{\mu/c}}{\sqrt{\mu/c}} + \sum_{c=[\mu]+1}^{\infty} c^{-\lambda}\, C_2 \left(\frac{\mu}{c}\right)^\nu \\ &= C_1 \mu^{-1/2} \sum_{c=N}^{[\mu]} c^{1/2-\lambda} e^{\mu/c} + C_2\, \mu^\nu \sum_{c=[\mu]+1}^{\infty} c^{-\lambda-\nu} \\ &\le C_1 \mu^{-1/2} \sum_{c=1}^{[\mu]} c^{1/2-\lambda}\, e^{\mu/N} + C_2\, \zeta(\lambda+\mu)\mu^\nu \\ &\le C_1\, \mu^{\max\{1/2,1-\lambda\}} e^{2^{-1}\epsilon\mu} + C_2 \zeta(\lambda+\nu)\mu^\nu, \end{aligned}$$

which implies the second assertion of the lemma.

□

3.5 Proof of Theorem 2.2

The equality (2.6) is immediate from Lemma 3.2 and Lemma 3.3. We obtain the estimates (2.7) and (2.8) by combining (2.6), Lemma 3.1 and Lemma 3.4.

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