ON SPATIOTEMPORAL CHAOS

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Abstract. The aim of this paper is to investigate the connection between various definitions of chaos for topological dynamical systems (i.e., continuous surjective maps of infinite compact metric spaces without isolated points into itself). Particular attention is paid to a very recent definition of spatiotemporal chaos which is based on Li–Yorke pairs and has some common features with sensitivity. We show that all topologically mixing systems, weakly mixing minimal systems, proximal systems and also some classes of recurrent systems are spatiotemporally chaotic.

0. Introduction and Definitions

The term 'chaos' in connection with a map was introduced by Li and Yorke [LY], although without a formal definition. Today there are various definitions of what it means for a map to be chaotic, some of them working reasonably only in particular phase spaces. Although the present article mainly deals with one of these definitions which was introduced in [BGKM], namely spatiotemporal chaos or simly ST chaos, it is also a survey on various definitions of chaos. Our main purpose is to incorporate the notion of ST chaos into the general frame of Topological Dynamics.

Let (X, T) be a topological dynamical system, with X a compact metric space with metric ϱ, and T a surjective continuous map from X to itself. Since topological transitivity is usually either a part of the definition of a chaos or transitivity implies or is implied by the chaos (at least in some spaces), we mention here some classes of transitive systems, some kinds of chaos and indicate their connections.

Let X be a metric space and $T : X \to X$ continuous. Consider the following two conditions:

(TT) for every pair of nonempty open sets U and V in X, there is a positive integer n such that $T^{-n}(V) \cap U \neq \emptyset$,

(DO) there is a point $x \in X$ such that the orbit of x is dense in X.

As usual, we adopt the condition (TT) as the definition of topological transitivity, but note that some authors take (DO) instead. Any point with dense orbit is called a *transitive point*. A point which is not transitive is called *intransitive*. The set of transitive or intransitive points of (X, T) will be denoted by X_{tr} or X_{intr} respectively.

The two conditions are independent in general. Nevertheless, under some additional assumptions on the phase space (or on the map) the two definitions (TT)

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and (DO) are equivalent. In fact, we have: if X has no isolated point then (DO) implies (TT); if X is separable and second category then (TT) implies (DO) [Silv].

Let us also remark that in general compact metric spaces, (TT) and (DO) are equivalent for onto maps. If T is transitive (i.e.,it satisfies (TT)) then T is onto. If a compact metric space admits a transitive map (i.e., if there exists a continuous selfmap T of X satisfying (TT)) then X has no isolated point if and only if it is infinite.

Convention: Throughout the paper, a couple (X, T) will denote a (*topological*) *dynamical system*. This means that X is an infinite compact metric space without isolated points and $T : X \to X$ is a surjective continuous map.

Let us note that a system (X, T) is transitive if and only if for every pair of non-empty open sets U and V in X the sequence of visit times $n(U, V) := \{n \geq 0 : T^{-n}(V) \cap U \neq \emptyset\}$ is infinite. If $n \in \mathbb{Z}_+$ and T^n is transitive then trivially T is transitive. The converse is not true in general. (X, T) is called *totally transitive* if the system (X, T^n) is transitive for all $n \in \mathbb{Z}_+$.

The most fundamental dynamical systems are the minimal ones. These are systems which have no nontrivial subsystems. The fact discovered by G. D. Birkhoff is that any topological dynamical system (X, T) has minimal subsystems (M, T). The sets M such that (M, T) is minimal are called the minimal sets of (X, T). So, a subset M of X is *minimal* if it is closed, nonempty, invariant (i.e., $T(M) \subseteq M$) and contains no proper subset with these three properties. A nonempty closed set $M \subseteq X$ is minimal if and only if the orbit of every point of M is dense in M. A point is called *minimal* or *almost periodic* if it belongs to a minimal set, and a dynamical system (X, T), or the map T, is called minimal if the set X is minimal. The set of all minimal points of (X, T) we define by X_{min}.

Given a dynamical system (X, T), the *ω-limit set* of a point $x \in X$ under T, $\omega_T(x)$, is the set of all limit points of the trajectory of x, i.e., $y \in \omega_T(x)$ if and only if $T^{n_k}(x) \to y$ for some sequence of integers $n_k \nearrow \infty$.

A point $x \in X$ is called *recurrent* if for some sequence of integers $n_i \nearrow \infty$, $\lim T^{n_i}x = x$; clearly $x \in \omega_T(x)$ and every transitive point is recurrent. A dynamical system (X, T), or the map T, is called *recurrent* if any point x is recurrent; clearly any minimal dynamical system is also recurrent.

In a system (X, T) there are the following possibilities:

(a) $X_{tr} = \emptyset$, $X_{intr} = X$,
(b) X_{tr} is dense G_δ and
  (b1) $X_{intr} = \emptyset$ (minimality) or
  (b2) X_{intr} is dense.

and no other possibility exists ([Ki], see also [Ak], [AAB], [KS] and [DY]). In other words if (X, T) is a transitive system then the set of points that are not transitive is either empty or dense (equivalently: if the set of transitive points has nonempty interior then the system is minimal).

We will use some important families of transitive systems. Recall their definitions and some known facts. Let (X, T) be a transitive system. We say that the system (X, T) is

(1) an F-*system* if it is totally transitive and the set of periodic points is dense in X;
(2) a P-*system* if it is transitive and the set of periodic points is dense in X;

(3) a ToP-*system* if it is non-minimal and each point of X is either transitive or periodic;
(4) an M-*system* if it is non-minimal transitive and the set X_{min} is dense in X;
(5) a ToM-*system* if it is non-minimal and each point of X is either transitive or minimal;
(6) an E-*system* if it has a T-invariant probability measure on X, which is positive on every non-empty open set.

It is known that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (4) $\Longrightarrow$ (6) (see [GW]). Of course, it is clear also that (3) $\Longrightarrow$ (2), (5) $\Longrightarrow$ (4) (see [DY]).

A dynamical system (X, T) is – an *ergodic system* if it has an ergodic T-invariant measure μ on X with full support (i.e., supp$\mu = X$); – a *syndetic system* if for each nonempty open set U in X the sequence of return times $n(U,U)$ is syndetic (i.e., $n(U,U)$ has uniformly bounded gaps in $\mathbb{Z}_+$).

An example in [W] shows that an E-system need not to be ergodic (in fact, the example is a P-system). Any minimal systems are ergodic systems and syndetic systems (see [GW], [GW2]).

Two topological dynamical systems (X, T) and (Y, S) are called *disjoint* if there exists no proper closed $T \times S$–invariant subset of the cartesian product $X \times Y$ with projections X and Y on the two coordinates; such a subset is called a *joining*. Recall that when two systems are disjoint, at least one of them is minimal.

A pair $\{x, y\} \subseteq X$ such that $\liminf_{n\to\infty} \varrho(T^n x, T^n y) > 0$ is said to be *distal*, and one for which $\lim_{n\to\infty} \varrho(T^n x, T^n y) = 0$ is said to be *asymptotic*. A pair $\{x, y\}$ is *proximal* if there exists a sequence n_i with $\lim_{i\to\infty} \varrho(T^{n_i} x, T^{n_i} y) = 0$. A pair $\{x, y\} \subseteq X$ is a Li–Yorke pair if simultaneously

$$\liminf_{n\to\infty} \varrho(T^n x, T^n y) = 0 \quad \text{and} \quad \limsup_{n\to\infty} \varrho(T^n x, T^n y) > 0.$$

Let us repeat these definitions for the case when X is a compact Hausdorff space. We denote by Δ the diagonal $\{(x, x) \in X^2 : x \in X\}$ of X^2. A pair $\{x, y\} \subseteq X$ such that $\omega_{T\times T}(x, y) \cap \Delta = \emptyset$ is distal, and one for which $\omega_{T\times T}(x, y) \subseteq \Delta$ is asymptotic. A pair $\{x, y\}$ is proximal if the set $\omega_{T\times T}(x, y) \cap \Delta$ is non-empty. A pair $\{x, y\} \subseteq X$ is a Li–Yorke pair if the set $\omega_{T\times T}(x, y) \cap \Delta$ is non-empty and $\omega_{T\times T}(x, y) \nsubseteq \Delta$. Thus a pair $\{x, y\}$ is a Li–Yorke pair if and only if it is proximal but not asymptotic. The sets of distal pairs, Li–Yorke pairs and asymptotic pairs partition X^2.

One of deep theorems of Topological Dynamics says: *any point x of a topological dynamical system (X, T) is proximal to some minimal point y* ([Au1], [El]). As we will see later, what we have asked for a system to be spatiotemporally chaotic: *any point x is proximal and not asymptotic to points y which are arbitrarily close to x* has a direct connection to the theorem. On the other hand, spatiotemporal chaotic maps are expected to have very little in common with distal maps.

Denote the set of proximal pairs by PR(X, T), asymptotic pairs by AS(X, T) and the set of points which are asymptotic to a point x by AS(x, T). Then PR$(X, T) = \cap_{n=1}^{\infty} \left(\cup_{i=1}^{\infty} (T \times T)^{-i}(\Delta_n)\right)$ and AS$(X, T) = \cap_{n=1}^{\infty} \left(\cup_{k=1}^{\infty} \left(\cap_{i=k}^{\infty} (T \times T)^{-i}(\overline{\Delta}_n)\right)\right)$, where $(\Delta_n)_{n=1}^{\infty}$ is a countable basis of neighborhoods of the diagonal Δ. Hence the set PR(X, T) is a G_δ subset of X^2 and in any transitive system the sets AS(X, T) and AS(x, T) are sets of the first category in X^2 and in X, respectively (see [HY2]).

Let (X,T) be a transitive system. We say that (X,T) is – a *distal system* if every non–diagonal pair $\{x,y\}$ (i.e., $x \neq y$) is distal; – an *almost distal system* if it is without Li-Yorke pairs; – a *point distal system* if there is a point which is distal with any other point; – a *proximal system* if it is without distal pairs.

The paper is organized as follows. In Section 1, after introducing general definitions and background, we survey some existing definitions of chaos, explaining what is known about them and about their relations.

In Section 2 we consider a very recent definition of spatiotemporal chaos which is based on Li–Yorke pairs and has some common features with sensitivity. We show that all topologically mixing systems (Theorem 2.1), weakly mixing minimal systems (Corollary 2.4), proximal systems (Proposition 2.5) and also some classes of recurrent systems (Theorems 2.6 and Proposition 2.8) are spatiotemporally chaotic.

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1. On Various Definitions of Chaos

1.1 Lyapunov Instability of Trajectories, Sensitivity

One family of definitions of chaos is based on the instability of trajectories. Given $\varepsilon > 0$, the map T is called *Lyapunov ε–unstable at $x \in X$* if for every neighborhood U of x, there is $y \in U$ and $n \geq 0$ with $\varrho(T^n x, T^n y) > \varepsilon$; T is called *unstable at x* (or the point x itself is called unstable) if there is $\varepsilon > 0$ such that T is Lyapunov ε-unstable at x.

If a dynamical system is pointwise unstable sometimes there exists no $\varepsilon > 0$ such that all points are ε–unstable. But if in addition the system is transitive then pointwise instability (in fact the instability of a point with dense orbit) implies uniform pointwise instability, i.e., the existence of such a universal $\varepsilon > 0$; this last property is also called *sensitive dependence on initial conditions* (*sensitivity* for short).

Auslander and Yorke call chaotic a system that is both sensitive and transitive [AY]. The Devaney's definition of chaos is Auslander and Yorke's definition with one extra assumption of the existence of a dense set of periodic points [D]. Recently this definition became rather popular because of the following theorem (see [Silv], [BBCDS] and [GW]): Let X be an infinite metric space and $f : X \to X$ be continuous. If f is transitive and has dense periodic points then f has sensitive dependence on initial conditions. (Thus, one of the three conditions in the Devaney's definition is redundant.)

A point $x \in X$ is called an *equicontinuity point* (or (X,T) is *equicontinuous at x*) if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $y \in X$ satisfies $\varrho(x,y) < \delta$ then $\varrho(T^n(x), T^n(y)) < \varepsilon$ for all $n \geq 0$. Of course if X is compact and every $x \in X$ is an equicontinuity point then (X,T) is equicontinuous. A topologically transitive system is called *almost equicontinuous* if it has at least one equicontinuity point. If a transitive system (X,T) has no equicontinuous points then it is sensitive. Moreover a minimal system is either equicontinuous or sensitive (see [AY], [AAB] and [GW] for details).

Now it is known also that any non–invertible transitive systems, or non-minimal E-systems (in particular, M-systems) are chaotic in the sense of Auslander and Yorke (see [GW], [AAB]).

1.2 Li-Yorke Pairs

Now consider definitions of chaos based on Li–Yorke pairs. The definition of Li–Yorke chaos is based on ideas in [LY]. Recall that a pair of points $\{x, y\} \subseteq X$ is said to be a *Li–Yorke pair* (with modulus δ) if one has simultaneously

$$\limsup_{n\to\infty} \varrho(T^n x, T^n y) = \delta > 0 \text{ and } \liminf_{n\to\infty} \varrho(T^n x, T^n y) = 0.$$

A set $S \subseteq X$ is called *scrambled* if any pair of distinct points $\{x, y\} \subseteq S$ is a Li–Yorke pair. Finally, a system (X, T) is called *chaotic in the sense of Li and Yorke* (or shortly *Li-Yorke chaotic*) if X contains an uncountable scrambled set. A system (X, T) is called *densely Li-Yorke chaotic* if X contains a dense, uncountable scrambled set. The first motivation for studying this notion comes from the theory of interval transformations. For such maps the existence of a Li–Yorke pair implies the existence of an uncountable scrambled set [KuS].

Recently it was proved that the transitive systems with at least one periodic point (in particular, P-systems) are Li–Yorke chaotic [HY2].

A dynamical system (X, T) is called *generically* or *densely chaotic* if the set of Li–Yorke pairs is residual or dense in $X \times X$, respectively. These notions were studied first on the interval. Say, in [Sn1] several conditions equivalent to generic chaos have been found. One of them shows that, in a sense, on the interval there is not a big difference between generic chaos and transitivity (see also [Mu]). Here only recall that topological transitivity on the interval implies generic chaos but the converse is not true, since there are generically chaotic maps arbitrarily close to a constant map. Dense chaos does not imply generic chaos (even on the interval). Nevertheless, in [Sn2] it is shown that in a class of interval maps which contains all piecewise monotone maps the notion of dense chaos and that of generic chaos coincide.

Recently it was shown that any transitive generically chaotic system (X, T) is Li–Yorke chaotic [HY2].

It is interesting to compare Li–Yorke and Auslander–Yorke chaoticity; neither implies the other.

Sturmian systems provide the simplest example of minimal almost distal systems which are sensitive [BGKM]. Let Y be the 1–torus and S_α be the rotation by the irrational number $\alpha \in Y$. Code the orbits of (Y, S_α) according to the closed cover $\mathcal{C} = \{[0, 1-\alpha], [1-\alpha, 1]\}$ of Y. Call X the closed set of all bi–infinite $\mathcal{C}$–names and let σ be the shift. All points of Y have only one $\mathcal{C}$–name in X except 0 and all points of its orbit, which have two each; on the other hand there is a continuous map $\varphi: X \to Y$ sending an infinite name to the point of Y it was constructed from, and $S_\alpha \circ \varphi = \varphi \circ \sigma$.

It is not hard to check that if $x \neq y \in Y$, then the pair (x, y) is distal unless $\varphi(x) = \varphi(y)$; in the last case (x, y) is asymptotic under the actions of σ and σ^{-1}: (X, σ) is a finite–to–one asymptotic extension of the isometry (Y, S_α).

On the other hand, there are scattering systems (see Subsection 1.4) which are Li-Yorke chaotic but not sensitive to initial conditions [AG]. Let us also remark that, since every non sensitive minimal map is equicontinuous, in the minimal case it is obvious that Li–Yorke chaos implies Auslander–Yorke chaos.

1.3 Mixing

A system (X, T) is said to be *topologically mixing* if for any pair of nonempty open sets U and V in X there is a $k \in \mathbb{Z}_+$ such that the sequence of visit times $n(U, V)$ contains the sequence $k, k+1, k+2, \ldots$.

Let us also recall the definition of so-called specification property which was introduced by R. Bowen. (X, T) satisfies the *strong specification property* or is said to be an *SSP-system* if the following holds: for any $\varepsilon > 0$ there exists an integer $M(\varepsilon)$ such that for any $k \geq 2$, for any k points $x_1, x_2, \ldots, x_k \in X$, for any integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_i - b_{i-1} \geq M(\varepsilon)$ for $2 \leq i \leq k$, and for any integer p with $p \geq M(\varepsilon) + b_k - a_1$, there exists a point $x \in X$ with $T^p(x) = x$ such that $\varrho(T^n(x), T^n(x_i)) \leq \varepsilon$ for $a_i \leq n \leq b_i$, $1 \leq i \leq k$. (X, T) satisfies the *specification property* or is said to be an SP-*system* if the above condition holds for $k = 2$. D. Ruelle defines also a *weak specification property*: the same as strong, except that x is not required to be periodic (hence no condition on p).

This seems to be a very strong condition, but it is satisfied by many examples. All systems with the specification property are topologically mixing, E-systems, F-systems and the topological entropy of T is positive (see [DGS] for details).

A system (X, T) is *topologically weakly mixing* if its cartesian square $(X \times X, T \times T)$ is transitive. The next two very useful properties characterize weak mixing systems: T is (topologically) weakly mixing if and only if given nonempty open sets U and V of X there is $n \geq 0$ such that $T^{-n}(U) \cap U \neq \emptyset$ and $T^{-n}(V) \cap U \neq \emptyset$ [Pet]; (X, T) is weakly mixing if and only if for every pair of nonempty open sets U and V in X the sequence of visit times $n(U, V)$ is thick (i.e., $n(U, V)$ contains arbitrary long intervals) [Fu].

The implications between various mixing conditions implying transitivity may be summarized as follows:

$$\text{Mixing} \Longrightarrow \text{Weakly mixing} \Longrightarrow \text{Totally transitive} \Longrightarrow \text{Transitive.}$$

All of the implications in the diagram are elementary to check except the fact that weakly mixing maps are totally transitive (a proof can be found in [B]). Although the existence of plenty of maps which are weakly mixing but not mixing is well known (see for instance [GM], [AAB], [JS], [Cr]), sometimes they are difficult to exhibit for some classes of dynamical systems. Let us also remark that an F-system is weakly mixing ([Fu], [B]).

A subset E of X is called *independent* for (X, T) if for any finite collection $x_1, x_2, \ldots, x_n$ of distinct points of E the point $(x_1, x_2, \ldots, x_n)$ is transitive in the product system $(X, T)^n$ $((X \times \ldots \times X, T \times \ldots \times T)$, n times). Obviously any independent set is also scrambled.

(Weak) mixing is a rather strong chaotic property. In every weakly mixing system there exists an uncountable independent set ([Iw], using the main theorem in [M]). It implies generic (hence Li–Yorke), and Auslander–Yorke chaos (see [BGKM]). None of the converse implications is true. Sturmian systems are Auslander–Yorke chaotic without being weakly mixing. In [Sn1] there are examples of interval maps showing that weak mixing is not implied by generic chaos either, even assuming transitivity. Another example can be found in [BGKM]: Let (X, T) be the full shift on 2 letters and (Y, S) be an irrational rotation of the unit circle; let (Z, R) be the factor of $(X \times Y, T \times S)$ obtained by collapsing all elements of the form (x_0, y) where x_0 is the fixed point on 0. One can check that (Z, R) is transitive but not weakly mixing and has a dense set of Li–Yorke pairs with modulus $\delta > 0$, which is G_δ; on the other hand since it contains a fixed point it is Li–Yorke chaotic by [HY2].

1.4 Topological Chaos and Scattering

Positive topological entropy and scattering can be considered as a next family of chaotic properties, based on the complexity of covers.

The *topological entropy* of a map $T \in C(X)$, denoted by $h(T)$, was introduced in [AKM] as an invariant of topological conjugacy and an analogue to the notion of metric (measure-theoretical) entropy (see, e.g., [Wal]). Consider a topological dynamical system (X, T). For any finite cover $\mathcal{C}$ of the compact space X, let $\mathcal{N}(\mathcal{C})$ be the minimal cardinality of a sub-cover of $\mathcal{C}$. The *topological complexity function* of the finite cover $\mathcal{C}$ of (X, T) is the non-decreasing function

$$c_T(\mathcal{C}, n) = \mathcal{N}(\mathcal{C}^n),$$

where $\mathcal{C}^n = \mathcal{C} \bigvee T^{-1}\mathcal{C} \bigvee ... \bigvee T^{-(n-1)}\mathcal{C}$. The exponential growth rate of this function is the topological entropy of $\mathcal{C}$, and the *topological entropy* of (X, T), $h(X, T)$, is the supremum of the topological entropies of finite open covers (see [AKM]).

We give also here one of the definitions which were introduced by R. Bowen and E.I. Dinaburg (see, e.g., [Wal]) and are equivalent to the original one.

So let, as usual, (X, ϱ) be a compact metric space and let T be a continuous map from X into itself. A subset E of X is called *(n, ε)-separated* if for every two different points $x, y \in E$ there exists $0 \le j < n$ with $\varrho(T^j(x), T^j(y)) > \varepsilon$. Let $s_n(T, \varepsilon)$ be the maximal possible cardinality of an (n, ε)-separated set. Then the (topological) entropy of T is defined by

$$h(T) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(T, \varepsilon) \ .$$

We see that the topological entropy represents the exponential growth rate for the number of orbit segments distinguishable with arbitrarily fine but finite precision. A dynamical system (X, T) is called *topologically chaotic* if its topological entropy is positive. A dynamical system (X, T) is called *scattering* if any finite cover $\mathcal{C}$ by non–dense open sets has unbounded complexity, i.e., $c_T(\mathcal{C}, n) \to \infty$, *2–scattering* if the same condition holds but for 2–set covers only.

A system is scattering if and only if its cartesian product with any minimal system is transitive [BHM]. Weak mixing implies scattering; scattering is strictly weaker than weak mixing [AG], but the two properties are equivalent in the minimal case [BHM].

Scattering, actually even 2–scattering, implies transitivity [BHM], Li–Yorke chaos and generic chaos [HY2].

Scattering does not imply that there exist no equicontinuity points [AG]. The same example shows that scattering systems are not necessarily sensitive to initial conditions: a transitive sensitive system has no equicontinuity points [AAB].

If (X, T) is transitive but not sensitive, T has a property called uniform rigidity ([GW], [AAB]). It implies that T is a homeomorphism and that it has zero entropy. Therefore any topologically chaotic transitive system has sensitive dependence on initial conditions; it is thus Auslander–Yorke chaotic.

Recall that a minimal scattering system is weakly mixing, therefore chaotic in many senses. There are however many examples of weakly mixing systems with zero entropy (for example, any uniquely ergodic system which is, measure theoretically, weakly mixing and has simple spectrum) and even minimal and uniformly rigid ones [GM]. Topological chaos is another strong notion of chaoticity.

Very recently a positive answer to the question whether topological chaos implies Li–Yorke chaos was given in [BGKM]. It is worth mentioning that Li–Yorke pairs are not the only ones that are necessarily found in topological chaotic systems: there must also exist asymptotic pairs, that is, pairs (x, y) such that $d(T^n x, T^n y) \to 0$ as $n \to \infty$ [BHR]. By the way let us also remark that there are topologically chaotic (transitive) systems without distal pairs.

In [BGKM] the authors have also explored systems without Li–Yorke pairs. Surely distal systems have this property but, as we shall see, the class of such systems is much wider. However, it turns out that systems without Li–Yorke pairs share many of the basic properties of distal systems. We have already seen that they also have zero entropy; such systems are minimal when transitive, and any factor of a transitive system without Li–Yorke pairs has no Li–Yorke pairs (see [BGKM]). For these reasons the transitive systems without Li–Yorke pairs are called *almost distal*. One important difference with distality is that almost distal systems are not necessarily invertible. The Sturmian and Morse subshifts, when considered only on one–sided sequences, are non–invertible almost distal systems (see [BGKM]). Wide classes of almost distal systems which are almost one–to–one, not finite–to–one, asymptotic extensions of rotations on the 2-torus which are minimal point distal homeomorphisms (respectively minimal point distal noninvertible maps) on that torus are described in [R] (respectively [KST]).

As a generalisation of results in [Fu] and [BHM], in [BGKM] it was shown that almost distal minimal systems are also disjoint from any scattering system.

1.5 Comparing of the Definitions of Chaos

Recall that for interval maps many of the existing definitions of chaos more or less coincide (see for instance [KS]).

Below we summarize the relations between some definitions of chaos for transitive dynamical systems (no other implications work except of those which can be deduced using the transitivity of implication). We also add short remarks, inspired by the results mentioned above.

$$
\begin{tikzcd}
\text{Mixing} \arrow[r, Rightarrow] & \text{Weakly mixing} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Scattering} \arrow[r, Rightarrow] & \text{Generic chaos} \arrow[d, Rightarrow] \\
 & \text{Sensitivity} & \text{Topological chaos} \arrow[l, Rightarrow] \arrow[r, Rightarrow] & \text{Li–Yorke chaos}
\end{tikzcd}
$$

If a dynamical system is minimal, then it is known that some of the definitions of chaos are equivalent. Let us remark some very useful properties of minimal weakly mixing systems. Firstly, any minimal system which is not weakly mixing has a non-trivial equicontinuous factor (see [Au] for a simple proof due to McMahon). In [BHM] it was shown by using this property that for a minimal dynamical system scattering and weak mixing are equivalent. Secondly, for minimal systems the notion of weak mixing is equivalent to the statement that $RP = X \times X$, where RP is the regionally proximal relation on X (the *regionally proximal relation* $RP(X,T)$ is the set of all points (x,y) of X^2 such that for each $\varepsilon > 0$ and each open neighborhoods U of x and V of y there are $x' \in U$, $y' \in V$ and $n \geq 0$ with $\varrho(T^n(x'), T^n(y')) < \varepsilon$) (see [Pet]). Recently in [HY2] it was shown more. Namely that for minimal systems the notions of weak mixing, generic chaos and dense Li-Yorke chaos are equivalent. In the below diagram we summarize the relations between the chaotic systems described in the previous diagram (moreover, we add dense Li–Yorke chaos).

$$\begin{tikzcd}
\text{Mixing} \arrow[r, Rightarrow] & \text{Weakly mixing} \arrow[r, equal] & \text{Scattering} \arrow[r, equal] & \text{Generic chaos} \arrow[d, equal] \\
& \text{Sensitivity} & \text{Li–Yorke chaos} \arrow[l, Rightarrow] & \text{Dense Li–Yorke chaos} \arrow[l, Rightarrow] \\
& & \text{Topological chaos} \arrow[u, Rightarrow] &
\end{tikzcd}$$

In the below table, we indicate some transitive dynamical systems and their belonging to (denoted as $\subsetneq$), or non-empty intersection with (denoted as $\cap$) the sets of chaotic systems which was described in the above diagrams: 1 - topologically chaotic; 2 - topologically mixing; 3 - weakly mixing; 4 - scattering; 5 - generically chaotic; 6 - Li-Yorke chaotic; 7 - systems with sensitive dependence on initial conditions. As usual the symbol ? means that the author does not know anything about the connection.

	1	2	3	4	5	6	7
SP-systems	$\subsetneq$	$\subsetneq$	$\subsetneq$	$\subsetneq$	$\subsetneq$	$\subsetneq$	$\subsetneq$
F-systems	$\cap$	$\subsetneq$	$\subsetneq$	$\subsetneq$	$\subsetneq$	$\subsetneq$	$\subsetneq$
P-systems	$\cap$	$\cap$	$\cap$	$\cap$	$\cap$	$\subsetneq$	$\subsetneq$
M-systems	$\cap$	$\cap$	$\cap$	$\cap$	$\cap$	?	$\subsetneq$
ToP-systems	$\cap$	?	?	?	?	$\subsetneq$	$\subsetneq$
ToM-systems	$\cap$	?	?	?	?	?	$\subsetneq$

2. ST Chaos

Other definitions, introduced in [BGKM], based on Li–Yorke pairs but having some common features with sensitivity may also be considering. We say that a dynamical system (X,T) has *chaotic dependence on initial conditions* if there is $\delta > 0$ such that for any $x \in X$ and every neighborhood U of x there is $y \in U$ such that the pair (x,y) is Li–Yorke with modulus δ.

A dynamical system (X,T) is said to be *spatiotemporally strongly chaotic* or *ST strongly chaotic* for short if it is transitive and depends chaotically on initial conditions.

ST strong chaos implies Auslander–Yorke chaos by definitions. It seems that the assumption of the existence of a universal $\delta > 0$ in the definition of ST strong chaos is a too strong one. So, we will say that a dynamical system (X, T) is *spatiotemporally chaotic* or *ST chaotic* for short if it is transitive and for any $x \in X$ and every neighborhood U of x there is $y \in U$ such that the pair (x, y) is Li–Yorke.

What are the connections between ST chaos and other definitions of chaos? What topological properties of a system imply ST chaos or are implied by the ST chaos?

Although topologically chaotic transitive systems are Li–Yorke chaotic and Auslander–Yorke chaotic, they do not imply ST chaos: there are point distal maps (which cannot be ST strong chaotic) with positive topological entropy.

If (X, T) is a dynamical system and S is an increasing sequence of positive integers, denote by $\mathrm{Prox}(x, T)$ the set of all points proximal to x and by $\mathrm{Prox}^S(x, T)$ the set of all points which are proximal to x with respect to the sequence S (i.e., $y \in \mathrm{Prox}^S(x, T)$ if and only if there is a subsequence $n_1 < n_2 < \dots$ of the sequence S such that $T^{n_i}(x)$ converges to x).

Lemma 2.0. *Let (X, T) be a dynamical system and $x \in X$. If $\mathrm{Prox}(x, T)$ is dense then there is an increasing sequence S of positive integers such that $\mathrm{Prox}^S(x, T)$ is G_δ-dense. Hence the set $\mathrm{Prox}(x, T)$ is residual.*

Proof. We have a dense set $\{p_\lambda : \lambda \in \Lambda\}$ (Λ being an index set) such that each of the points p_λ is proximal to x. Thus, for every λ there si an increasing sequence S_λ of positive integers such that the trajectory of p_λ relative to this sequence converges to x. Now let $n_1 < n_2 < \dots$ be the increasing sequence of positive integers containing all elements of all sequences S_λ, $\lambda \in \Lambda$. Denote this sequence by S. Obviously, all the points p_λ are proximal to x also with respect to the sequence S. Thus, the set $\mathrm{Prox}^S(x, T)$ is dense. To finish the proof we need to show that it is of type G_δ. This follows from the obvious fact that $\mathrm{Prox}^S(x, T) = \cap_{k=1}^{\infty}(\cup_{j=k}^{\infty} T^{-n_j}(W_k))$ where $(W_k)_{k=1}^{\infty}$ is a countable basis of neighborhoods of x. ■

Theorem 2.1. *Any topologically mixing system (X, T) is spatiotemporally chaotic.*

Proof. [1] Fix a point $x \in X$. Since X is compact, there are a point $x_\infty \in \omega_T(x)$ and a sequence $n_i \nearrow \infty$ such that $\lim_{i\to\infty} T^{n_i}(x) = x_\infty$. Denote the sequence $\{n_i\}_{i=1}^{\infty}$ by $N_x^{x_\infty}$ and for $k = 1, 2, \dots$ let V_k be the open ball with radius $1/k$ centered at x_∞. Let U be a non-empty open set in X. Since T is mixing, there is $n_{i_1} \in N_x^{x_\infty}$ such that $T^{-n_{i_1}}(V_1) \cap U \neq \emptyset$ and therefore there is a non-empty open set $U_1 \subset U$ such that $T^{n_{i_1}}(U_1) \subset V_1$. Similarly it is possible to find $n_{i_2} \in N_x^{x_\infty}$, $i_2 > i_1$, such that $T^{-n_{i_2}}(V_2) \cap U_1 \neq \emptyset$ and therefore there is a non-empty open set $U_2 \subset U_1$ such that $T^{n_{i_2}}(U_2) \subset V_2$. By induction we obtain a subsequence $\{n_{i_k}\}_{k=1}^{\infty}$ of $N_x^{x_\infty}$ and a nested sequence of non-empty open sets $\{U_k\}_{k=1}^{\infty}$ in U such that $T^{n_{i_k}}(U_k) \subset V_k$ for every k. Hence the set $\cap_{k=1}^{\infty} \overline{U}_k$ is non-empty and for every y in this set we have $\lim_{k\to\infty} T^{n_{i_k}}(y) = x_\infty$. In other words the pair $\{x, y\}$ is proximal.

Since U was an arbitrary open set, the set $\mathrm{Prox}(x, T)$ is dense in X. By Lemma 2.0 this set is also residual. Since, as we mentioned in Section 0, the set $\mathrm{AS}(x, T)$ (of points which are asymptotical to the point x) is the first category, the set of points y such that $\{x, y\}$ is a Li–Yorke pair is residual. In particular, such points exist in any neighborhood of x. ■

[1] One can prove this theorem also by using a result from [JS] (see Concluding Remarks).

Corollary 2.2. *Any system which has the specification property is spatiotemporally chaotic.*

Now a natural question is whether even all weakly mixing systems are spatiotemporally chaotic. We do not know the answer. Nevertheless, we are going to show that all weakly mixing minimal systems are necessarily spatiotemporally chaotic.

Recall that if (X,T) is weakly mixing, for every pair of nonempty open sets U and V in X the sequence of visit times $n(U,V)$ is thick (i.e., $n(U,V)$ contains arbitrary long intervals). A point x is minimal if it belongs to a minimal set of X. If x is minimal, for any its neighborhood U the sequence of return times $n(x,U) := \{n \geq 0 : T^n(x) \in U\}$ is syndetic (i.e., $n(x,U)$ has uniformly bounded gaps in $\mathbb{Z}_+$).

Lemma 2.3. *Let (X,T) be a weakly mixing system. Then for any minimal point x the set of points which form Li–Yorke pairs with x is dense in X.*

Proof. By G. D. Birkhoff any topological dynamical system (X,T) has a minimal set. Fix a minimal point $x \in \mathcal{M}$, where $\mathcal{M}$ is a minimal subset of X, and a nonempty open set U. We are going to find a point $y \in U$ which is proximal to x.

Let $\{V_k\}_{k=1}^{\infty}$ be the sequence of open balls with radius $1/k$ centered at x. Since T is weakly mixing and x is minimal, there is n_1 such that $T^{-n_1}(V_1) \cap U \neq \emptyset$ and $T^{n_1}(x) \in V_1$. Hence there is a non-empty open set $U_1 \subset U$ such that $T^{n_1}(U_1) \subset V_1$. Again thus it is possible to find $n_2 > n_1$ such that $T^{-n_2}(V_2) \cap U_1 \neq \emptyset$ and $T^{n_2}(x) \in V_2$. Therefore there is a non-empty open set $U_2 \subset U_1$ such that $T^{n_2}(U_2) \subset V_2$. By induction we obtain an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers and a nested sequence of non-empty open sets $\{U_k\}_{k=1}^{\infty}$ in U such that $T^{n_k}(U_k) \subset V_k$ for every k and $\lim_{k\to\infty} T^{n_k}(x) = x$. Hence the set $\cap_{k=1}^{\infty} \overline{U}_k$ is non-empty and for every y in this set we have $\lim_{k\to\infty} T^{n_k}(y) = x$. In other words, the pair $\{x,y\}$ is proximal.

We showed that the set $\mathrm{Prox}(x,T)$ of all points which are proximal to x is dense in X. By Lemma 2.0 this set is residual. Since the set $\mathrm{AS}(x,T)$ is the first category, the set of points y such that $\{x,y\}$ is a Li–Yorke pair is residual. In particular, such points exist in any neighborhood of x. ∎

Corollary 2.4. *Any weakly mixing minimal system (X,T) is spatiotemporally chaotic.*

Since for minimal systems the notions of weak mixing, scattering, generic chaos and dense Li-Yorke chaos are equivalent, minimal systems with any of these properties are also ST chaotic.

Recal that a transitive system is proximal if there are no distal pairs.

Proposition 2.5. *Any proximal transitive system is spatiotemporally chaotic.*

Proof. It is a direct corollary of the definition of a proximal system and the fact that for any transitive system (X,T) and any point $x \in X$ the set $\mathrm{AS}(x,T)$ of points which are asymptotical to the point x is of the first category. ∎

A system (X,T) is called *uniformly rigid* if there is a sequence $n_i \nearrow \infty$ such that the sequence $\{T^{n_i}\}$ tends uniformly to the identity map on X. In particular, it means that any point of X is recurrent. One can check that topological mixing systems are not uniformly rigid (see also [GM]). Although any uniformly rigid systems are zero topological entropy and without asymptotic pairs, some of them are weakly mixing systems (see [GW], [AAB]). We would like to note the following implications for transitive systems

$$(X,T) \text{ is topologically chaotic} \Longrightarrow \mathrm{AS}(X,T) \text{ is non-empty}$$
$$\Longrightarrow (X,T) \text{ is not uniformly rigid} \Longrightarrow (X,T) \text{ is Auslander–Yorke chaotic}$$

([BHR], [GM], [AAB]).

From [AAB] we know that there is a uniformly rigid system (X,T) with a fixed point $u \in X$ which is a unique minimal subset of X (the system is a subsystem of the so called Bebutov system (see also [N], [KW])).

Theorem 2.6. *Let (X,T) be a uniformly rigid transitive system. Suppose there is a fixed point u which is a unique minimal subset of X. Then (X,T) is spatiotemporally chaotic.*

Proof. Since the system is transitive by the assumption, we need to show only that for any $x \in X$ and every neighborhood U of x there is $y \in U$ such that the pair (x,y) is Li–Yorke.

Since (X,T) is uniformly rigid there is a sequence $n_i \nearrow \infty$ such that for any x the sequence $\{T^{n_i}(x)\}$ tends to x. Hence each point $(x,y) \in X^2$ is recurrent for the map $T \times T$. Since the ω-limit set of x is a non-empty closed ivariant set, by Birkhoff theorem there is a minimal subset in it and by the assumpion of the theorem it is the fixed point u.

Fix a point $x \in X$. If x coincides with the fixed point u then it is obvious that any pair $\{x,y\}$, where $x \neq y$, is Li–Yorke. Suppose $x \neq u$ and let U be a neighborhood of x. Since x is a recurrent point there exists a positive integer n such that $T^n(x) \in U$. Since $u \in \omega_T(x)$ and T is continuous, there exists a sequence $n_i \nearrow \infty$ such that $\lim_{i\to\infty}(T^{n_i}(x), T^{n+n_i}(x)) = (u,u)$. Therefore the pair $\{x, T^n(x)\}$ as a recurrent point of $T \times T$ is also Li–Yorke. ∎

Repeat that any topologically transitive system which is not sensitive is uniformly rigid and almost equicontinuous. An example of uniformly rigid systems in [AAB] is an almost equicontinuous system with a fixed point which is a unique minimal subset of X. Therefore there are spatiotemporally chaotic systems which are not sensitive.

Any minimal system is either equicontinuous or sensitive (see [AAB]). As a corollary of this fact we have

Proposition 2.7. *Any minimal spatiotemporally chaotic system is sensitive.*

Recall that a set is called *regular closed* or canonical closed if it is the closure of an open set (equivalently, if it is the closure of its interior). Suppose we have a collection $\mathcal{X} := \{X_0, X_1, ..., X_{n-1}\}$ of subsets of X such that $T(X_i) \subseteq X_{i+1(\mathrm{mod}\, n)}$ for $0 \neq i < n-1$. Clearly, $T^k(X_i) \subseteq X_{i+k(\mathrm{mod}\, n)}$ for all $k \geq 0$. A collection $\mathcal{X}$ is called *regular periodic decomposition* if the X_i are regular closed, $X_i \cap X_j$ is nowhere dense whenever $i \neq j$ and $\cup_{i=0}^{n-1} X_i = X$. From [B] we know that a map T is transitive if and only if T^n is transitive on each element of a regular periodic decomposition of T of length n and any transitive map T has at most one regular periodic decomposition of length n.

From [DY] we know that there are transitive non-minimal systems any non-transitive point of which is periodic, i.e. ToP-systems.

Proposition 2.8. *Any* ToP*-system is spatiotemporally chaotic.*

Proof. Let (X,T) be a ToP-system. Fix a point $x \in X$ and its neighborhood U. Suppose x is a periodic point of period n. Then by [B] there is a regular periodic decomposition $\{X_0, X_1, ..., X_{n-1}\}$ of period n. Since $x \in X_i$ for some $0 \le i < n-1$, the map $G := T^n : X_i \to X_i$ is transitive and x is a fixed point of G. Since X_i is a regular closed set, the set $U \cap X_i$ has non-empty interior. Hence a pair $\{x, y\}$, where $y \in U \cap X_i$ is a transitive point of G (consequently of T), is Li–Yorke.

Let x be a transitive point of (X,T) and U be its neighborhood. Since T is a ToP-system, there is a periodic point y of some period n. Then by [B] there is a regular periodic decomposition $\{X_0, X_1, ..., X_{n-1}\}$ of period n such that $x \in X_i$ and x is transitive for the map $G := T^n : X_i \to X_i$. Since the transitive map T has at most one regular periodic decomposition of length n, there is a periodic point $y_0 \in X_i$ of the period n (from the T-orbit of the periodic point y).

As we know X_i is a regular closed set. Hence $\mathrm{int}(U \cap X_i)$ is non-empty and by transitivity of G there is a positive integer m such that $G^m(x) \in U \cap X_i$. Since $y_0 \in \omega_G(x)$ and G is continuous, there exist a sequence $m_i \nearrow \infty$ such that $\lim_{i\to\infty}(G^{m_i}(x), G^{m+m_i}(x)) = (y_0, y_0)$. The pair $\{x, G^m(x)\}$ is also recurrent point of $G \times G$, because the point x is transitive. Hence $\{x, G^m(x)\}$ is Li–Yorke for the map T. ■

Concluding Remarks. Among other new results in Topological Dynamics we wish to mention two directions which have common features with the subject of the article.

(i) - *Stroboscopical property.* If $\mathcal{S}$ is an increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$, denote the set of all limit points of the sequence $T^{n_1}(x), T^{n_2}(x), T^{n_3}(x), \dots$ by $\omega_T(x, \mathcal{S})$ (in particular, if $\mathcal{S} = \mathbb{N}$ it is the ω-limit set of x with respect to T (i.e., $\omega_T(x) = \omega_T(x, \mathbb{N})$)). Note that since X is compact, $\omega_T(x, \mathcal{S})$ is nonempty. A system $(X;T)$ has the *(Misiurewicz) strong stroboscopical property* if, given an increasing sequence of positive integers $\mathcal{S}$ and a point $x \in X$, one can always find a dense set of points $y \in X$ with $x \in \omega_T(y, \mathcal{S})$. This and similar notions have very recently been studied in [JS]. Among many other results, it was shown that the notions of topological mixing and strong stroboscopical property are equivalent. We proved Theorem 2.1 by using only the definition of topological mixing. Nevertheless, the following proof which uses this results from [JS] is much shorter. Fix $x \in X$ and take a sequence $n_i \nearrow \infty$ such that $T^{n_i}(x)$ converges to a point $x_1 \in X$. Fix a point $x_2 \in X$, $x_2 \ne x_1$ (we may assume that $\varrho(x_2, x_1) \ge 1/2\ \mathrm{diam}(X)$). By [JS; Proposition 5], for $j = 1, 2$ there is a G_δ-dense set Y_j of points whose ω-limit set with respect to the sequence $\{n_i\}_{i=1}^{\infty}$ contains the point x_j. Put $Y = Y_1 \cap Y_2$. For any $y \in Y$, $\{x, y\}$ is obviously a Li-Yorke pair with modulus $\delta = \varrho(x_1, x_2)$ (thus, we may claim that $\delta \ge 1/2\ \mathrm{diam}X$). Since Y is G_δ-dense, the result follows. We proved even more - any topologically mixing system (X,T) is spatiotemporally strong chaotic.

(ii) - *The sequence of visit/return times.* As we mentioned, the definitions of transitive, weakly mixing and mixing systems are equivalent with saying that the sequences of visit times $n(U, V) := \{n \ge 0 : T^{-n}(V) \cap U \ne \emptyset\}$ are infinite, thick or contain the sequence $k, k+1, k+2, \dots$ for some positive integer k, respectively. It was very recently shown that for any E-system the sequence of visit times is syndetic [GW2], for any scattering system and strong scattering system (systems which are disjoint from any E-system) the sequence of visit times is a recurrent set and a Poincaré set, respectively (see [HY3] for details).

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