INVOLUTIVE DISTRIBUTIONS OF OPERATOR-VALUED EVOLUTIONARY VECTOR FIELDS

ARTHEMY V. KISELEV* AND JOHAN W. VAN DE LEUR

Abstract. We define and classify matrix linear total differential operators whose images in the Lie algebra $\mathfrak{g}$ of evolutionary vector fields on the infinite jet space over a fibre bundle are subject to the collective commutation closure,

$$\Bigl[\sum_{i=1}^{N} \operatorname{im} A_i, \sum_{j=1}^{N} \operatorname{im} A_j\Bigr] \subseteq \sum_{k=1}^{N} \operatorname{im} A_k.$$

As a by-product, we obtain a convenient criterion under which such $\mathbb{Z}_2$-graded operators are Hamiltonian. We prove that, under a change of coordinates in the common domain Ω of the operators, the arising bi-differential structural constants $\Gamma_{ij}^{k}\colon \Omega\times\Omega \to \Omega$ are transformed by the direct analogue of the reparametrization rule for Christoffel symbols. We show that the operators A with involutive images determine flat connections in the triads $\Omega \xrightarrow{A} \mathfrak{g}$ that consist of the two Lie algebras and the morphism.

Introduction. We begin with three examples from geometry of integrable systems, which serve as a motivation for the constructions in this paper.

Example 1. Let A be a Hamiltonian total differential operator, that is, a skew-adjoint linear matrix operator in total derivatives which determines a Poisson bracket on the space $\bar{H}$ of Hamiltonian functionals [2, 17, 25]. The image of A in the Lie algebra $\mathfrak{g}$ of evolutionary vector fields, which are of the form $\partial_\varphi = \varphi\,\frac{\partial}{\partial u} + \frac{\mathrm{d}}{\mathrm{d}x}(\varphi)\,\frac{\partial}{\partial u_x} + \cdots$, is closed under commutation. This is readily seen from the Jacobi identity for the variational Schouten bracket $[\![\,,\,]\!]$ and the representation of A as the variational Poisson bi-vector [9]. Indeed, the commutator of two Hamiltonian vector fields equals

$$\bigl[[\![A,\mathcal{H}_1]\!],[\![A,\mathcal{H}_2]\!]\bigr] = [\![A,[\![\mathcal{H}_2,[\![A,\mathcal{H}_1]\!]]\!]]\!] + [\![\mathcal{H}_2,[\![[\![A,\mathcal{H}_1]\!],A]\!]]\!],$$

whence the Poisson bracket $[\![\mathcal{H}_2,[\![A,\mathcal{H}_1]\!]]\!] = \{\mathcal{H}_1,\mathcal{H}_2\}_A$ of $\mathcal{H}_1,\mathcal{H}_2 \in \bar{H}$ appears in the right-hand side and the second summand contains $\mathrm{d}_A^2(\mathcal{H}_1) \equiv [\![A,[\![A,\mathcal{H}_1]\!]]\!] = 0$. The definition of the total derivatives [17, 25], which are written in local coordinates as $\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\partial}{\partial x} + u_x\,\frac{\partial}{\partial u} + \cdots$, implies the commutation closure of $\operatorname{im} A$ for all arguments of A, not necessarily exact (originating from a Hamiltonian functional).

For instance, the second Hamiltonian operator for the Korteweg–de Vries equation is $A = -\frac{1}{2}\,\frac{\mathrm{d}^3}{\mathrm{d}x^3} + u\,\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\circ u$. The image of A is closed under commutation, and the Lie algebra structure $[\,,\,]_A$ on its domain is related by the homomorphisms $\delta/\delta u$ and A to

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the Lie algebra $\bigl(\bar{H}, \{\,,\,\}_A\bigr)$ of Hamiltonians, endowed with the Poisson bracket, and to the Lie algebra $\bigl(\mathfrak{g}, [\,,\,]\bigr)$ of evolutionary vector fields, respectively (see [2]). It is readily seen that, for the above operator, the bracket $[\,,\,]_A$ on the domain of A is [28]

$$[p, q]_A = \partial_{A(p)}(q) - \partial_{A(q)}(p) + \tfrac{\mathrm{d}}{\mathrm{d}x}(p) \cdot q - p \cdot \tfrac{\mathrm{d}}{\mathrm{d}x}(q). \tag{1}$$

Example 2. The Noether symmetries of the scalar Liouville equation $\mathcal{E}_{\text{Liou}} = \bigl\{u_{xy} = \exp(2u)\bigr\}$, which is the open 2D Toda chain associated with the root system A_1, amount to $\varphi_{\mathcal{L}} = \square\bigl(\frac{\delta}{\delta w}\mathcal{H}(x, w, w_x, \ldots)\bigr)$ and $\overline{\varphi}_{\mathcal{L}} = \overline{\square}\bigl(\frac{\delta}{\delta \overline{w}}\overline{\mathcal{H}}(y, \overline{w}, \overline{w}_y, \ldots)\bigr)$, where $\square = u_x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}$ and $\overline{\square} = u_y + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}y}$ are total differential operators, the quantities $w = u_x^2 - u_{xx} \in \ker \frac{\mathrm{d}}{\mathrm{d}y}\bigr|_{\mathcal{E}_{\text{Liou}}}$ and $\overline{w} = u_y^2 - u_{yy} \in \ker \frac{\mathrm{d}}{\mathrm{d}x}\bigr|_{\mathcal{E}_{\text{Liou}}}$ are conserved on $\mathcal{E}_{\text{Liou}}$, and the functionals $\mathcal{H}$, $\overline{\mathcal{H}}$ are arbitrary (see [26]). As above, we obtain symmetries $\varphi = \square\bigl(\phi(x, w, w_x, \ldots)\bigr)$, $\overline{\varphi} = \overline{\square}\bigl(\overline{\phi}(y, \overline{w}, \overline{w}_y, \ldots)\bigr)$ of $\mathcal{E}_{\text{Liou}}$ for any ϕ and $\overline{\phi}$ even if they are not in the images of the respective variational derivatives, see [28].

The symmetries φ and $\overline{\varphi}$ generate two Lie subalgebras $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ in $\operatorname{sym} \mathcal{E}_{\text{Liou}} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$ such that $[\mathfrak{g}, \overline{\mathfrak{g}}] = 0$. At the same time, each of the two components is not Abelian. The commutator, say, on $\mathfrak{g}$ is transferred by $\square$ onto its domain, where it specifies the Lie algebra structure

$$[p, q]_\square = \partial_{\square(p)}(q) - \partial_{\square(q)}(p) + \tfrac{\mathrm{d}}{\mathrm{d}x}(p) \cdot q - p \cdot \tfrac{\mathrm{d}}{\mathrm{d}x}(q). \tag{2}$$

Example 3. In [12] we demonstrated that the dispersionless 3-component Boussinesq system of hydrodynamic type admits a two-parametric family of nontrivial finite deformations $[\,,\,]_{\boldsymbol{\epsilon}}$ for the standard bracket of its symmetries $\operatorname{sym} \mathcal{E}$. This is achieved by using two recursion differential operators $R_i \colon \operatorname{sym} \mathcal{E} \to \operatorname{sym} \mathcal{E}$, $i = 1, 2$, whose images are closed under commutation and which are *compatible* in this sense, spanning the two-dimensional space of the operators $R_{\boldsymbol{\epsilon}}$ with involutive images. The new brackets $[\,,\,]_{\boldsymbol{\epsilon}}$ are determined via $\bigl[R_{\boldsymbol{\epsilon}}(p), R_{\boldsymbol{\epsilon}}(q)\bigr] = R_{\boldsymbol{\epsilon}}\bigl([p, q]_{\boldsymbol{\epsilon}}\bigr)$ for $p, q \in \operatorname{sym} \mathcal{E}$, c.f. [8] and [24]. They admit the familiar deconposition in the two standard evolutionary terms and the bi-linear bi-differential bracket:

$$[p, q]_{\boldsymbol{\epsilon}} = \partial_{R_{\boldsymbol{\epsilon}}(p)}(q) - \partial_{R_{\boldsymbol{\epsilon}}(q)}(p) + \{\{p, q\}\}_{\boldsymbol{\epsilon}}.$$

Each example gives us operators whose images generate involutive distributions of evolutionary vector fields and which induce Lie algebra structures on their domains. In this paper we develop a systematic unification for these three empiric facts.

The three main sources of this problem, which locate it within geometry of integrable systems, are the Schouten–Gerstenhaber dual differential complexes for Poisson manifolds [2, 13, 14], the contractions of Lie algebras [8, 24], and the Riemannian geometry of the Dubrovin–Novikov–Ferapontov–Mokhov Hamiltonian operators for evolutionary systems of hydrodynamic type [5, 23], see [22] for references.

Let $\mathcal{P} \in \Gamma\bigl(\bigwedge^2(TF)\bigr)$ be a bi-vector field with vanishing Schouten bracket $[\![\mathcal{P}, \mathcal{P}]\!] = 0$ on a finite-dimensional smooth orientable real manifold F. Using the coupling $\langle\,,\,\rangle \colon \Gamma(T^*F) \times \Gamma(TF) \to C^\infty(F)$ and a nondegenerate Poisson bi-vector $\mathcal{P}$, one transfers the Lie algebra structure $[\,,\,]$ on $\Gamma(TF)$ to $[\,,\,]_{\mathcal{P}}$ on $\Gamma(T^*F) \ni \boldsymbol{p}, \boldsymbol{q}$ and obtains the Koszul–Dorfman–Daletsky–Karasëv bracket [2]

$$[\boldsymbol{p}, \boldsymbol{q}]_{\mathcal{P}} = \mathrm{L}_{\mathcal{P}\boldsymbol{p}}(\boldsymbol{q}) - \mathrm{L}_{\mathcal{P}\boldsymbol{q}}(\boldsymbol{p}) + \mathrm{d}_{\mathrm{dR}}\bigl(\mathcal{P}(\boldsymbol{p}, \boldsymbol{q})\bigr),$$

here L is the Lie derivative. The de Rham differential d_{dR} on $\bigwedge^\bullet(T^*F)$ is defined in the complex over the Lie algebra $\bigl(\Gamma(TF),[\,,\,]\bigr)$ by using Cartan's formula. If the Poisson bi-vector $\mathcal{P}$ has the inverse symplectic two-form $\mathcal{P}^{-1}$ such that

$$\mathcal{P}^{-1}[x,y]=[\mathcal{P}^{-1}x,\mathcal{P}^{-1}y]_{\mathcal{P}}, \tag{3}$$

then the differential d_{dR} is correlated with the Koszul–Schouten–Gerstenhaber bracket $[\![\,,\,]\!]_{\mathcal{P}}$ on $\bigwedge^\bullet(T^*F)$ by $\mathrm{d}_{\mathrm{dR}}=[\![\mathcal{P}^{-1},\cdot]\!]_{\mathcal{P}}$. The differential d_{dR} on $\bigwedge^\bullet(T^*F)$ is intertwined [14, 15] with the Poisson differential $\mathrm{d}_{\mathcal{P}}=[\![\mathcal{P},\cdot]\!]$ on $\bigwedge^\bullet(TF)$ by

$$\Bigl(\bigwedge\nolimits^{k+1}\mathcal{P}\Bigr)\bigl([\![\mathcal{P}^{-1},\Psi]\!]_{\mathcal{P}}\bigr)+[\![\mathcal{P},\bigl(\bigwedge\nolimits^{k}\mathcal{P}\bigr)(\Psi)]\!]=0,\qquad \forall\,\Psi\in\Gamma\Bigl(\bigwedge\nolimits^{k}(T^*F)\Bigr). \tag{4}$$

Trivial infinitesimal deformations of the Lie bracket $[\,,\,]$ on $\Gamma(TF)$ are obtained using the Nijenhuis structures $\mathrm{N}\colon\Gamma(TF)\to\Gamma(TF)$, see [2, 13]. Nontrivial deformations $[p,q]_0:=\lim_{\epsilon\to+0}R^{-1}(\epsilon)\bigl[R(\epsilon)\,p,R(\epsilon)\,q\bigr]$ of the standard bracket on the m-dimensional Lie algebras $\bigl(\Bbbk^m,[\,,\,]\bigr)\ni p,q$ are obtained, pointwise on F, through the continuous contractions by isomorphisms $R\colon\epsilon\in(0,1]\to GL(m)$ with a nontrivial analytic behaviour as $\epsilon\to+0$, see [8], [24] and references therein. In these terms, the correlation (3) enlarges the problem of contractions via the isomorphisms R to the problem of correlation for the two Lie algebra structures in the domains and images of the operators $\mathcal{P}\in\mathrm{Hom}_{C^\infty(F)}\bigl(\Gamma(T^*F),\Gamma(TF)\bigr)$, which leads to (4).

Next, we regard the manifold F as the fibre in the bundle $\pi\colon E^{m+n}\xrightarrow[F^m]{}B^n$ over a smooth n-dimensional orientable base manifold B (e.g., $\mathbb{S}^n$). This allows us to pass from Hamiltonian ODE on F to Hamiltonian PDE upon sections of this bundle.

For the dispersionless first-order Hamiltonian evolutionary systems of hydrodynamic type, the Poisson structures are determined by the Dubrovin–Novikov–Ferapontov–Mokhov operators [5, 23]. The profound relation of hydrodynamic type systems to the geometry of linear connections is the following: such first-order Hamiltonian operators are described by n-tuples of (pseudo)Riemannian metrics of constant curvature K (which requires the weak nonlocality of the operators if $K\neq0$). Note that both the metric tensors and the Christoffel symbols they specify are contained explicitly[1] in the operators.

The commutation closure for images of all Hamiltonian total differential operators for PDE is well-known [17, 25]. The analysis of Lie algebra structures on their domains Ω, which are constituted by the variational covectors, is performed in [2, 13]. In parallel with the finite-dimensional case, trivial infinitesimal deformations of the bracket in the algebra $\mathfrak{g}$ of evolutionary vector fields are obtained using Nijenhuis recursion operators $\mathrm{N}\colon\mathfrak{g}\to\mathfrak{g}$ (see [2, 13] and [6], where the compatibility conditions for the operators N and the variational Poisson bi-vectors $\mathcal{P}$ are formulated). A family of nontrivial finite deformations $[p,q]_{\boldsymbol{\epsilon}}=R_{\boldsymbol{\epsilon}}^{-1}\bigl(\bigl[R_{\boldsymbol{\epsilon}}(p),R_{\boldsymbol{\epsilon}}(q)\bigr]\bigr)$ of the commutators was obtained in [12] for symmetries of a dispersionless Boussinesq-type system $\mathcal{E}$. In that paper [12], the notion of linear compatible Noether operators $\operatorname{cosym}\mathcal{E}\to\operatorname{sym}\mathcal{E}$ and recursion operators $R_{\boldsymbol{\epsilon}}\colon\operatorname{sym}\mathcal{E}\to\operatorname{sym}\mathcal{E}$ with involutive images was proposed, here $\boldsymbol{\epsilon}\in\mathbb{R}^2$.

[1] This must not be confused with the concept of the bi-differential Christoffel operators Γ^k_{ij}, which is developed in this paper (in particular, for first-order Hamiltonian operators): the bi-differential symbols can not be contained in the differential operators. They determine a connection in a pair of Lie algebras, and it is always flat (see Theorem 7).

The problem of construction and classification of differential operators with involutive images, not imposing the requirement that the operators be isomorphisms or Hamiltonian structures, is the synthesis of the two problems: the contraction of Lie algebras and the correlation of brackets in the domains and images of the operators. This approach was proposed in [28], where it was motivated by the well-known structure of symmetry generators for open 2D Toda chains $\mathcal{E}_{\text{Toda}}$. It is remarkable that such non-Hamiltonian operators, which determine the symmetries, also induce the Poisson structures for the Drinfel'd–Sokolov KdV-type hierarchies [3] contained in $\operatorname{sym}\mathcal{E}_{\text{Toda}}$. This analysis was continued in [10, 11] for hyperbolic Euler–Lagrange systems of Liouville type.

We notice that the domains of the Noether or recursion operators for differential equations are uniquely determined by the system at hand. In this paper we define the domains of operators, whose images are subject to the collective commutation closure, without any reference to any underlying differential equation. Moreover, we deal with the general case of the N-tuples of such operators for arbitrary $N \geq 1$. We begin with $N = 1$ and hence with a unique operator A that satisfies (5), see below. We study this reduced case, $N = 1$, in more detail because it provides the Lie algebra structures on the domains of the operators A. This reduction is also achieved for many, $N \geq 2$, operators under the assumption they are *linear compatible.*

Let $\mathfrak{g}$ denote the Lie algebra of evolutionary vector fields on the infinite jet space $J^\infty(\pi)$ for a smooth fibre bundle $\pi\colon E \underset{F}{\longrightarrow} B$ over an orientable real manifold B. We consider linear total differential operators that take values in $\mathfrak{g}$ such that the sum of their images is closed under commutation. We associate bi-differential extensions Γ^k_{ij} of Christoffel symbols on the fibres F to N-tuples of such operators. We prove that the connections, constituted by operators A with involutive images in the triads $\operatorname{dom} A \xrightarrow{A} \mathfrak{g}$ of the two Lie algebras related by the morphism, are flat.

This paper consists of three parts. In section 1 we describe the domains Ω of the operators $A_i\colon \Omega \to \mathfrak{g}$. This requires *two* fibre bundles over the same base B, namely, π and the auxiliary bundle $\xi\colon I^{r+n} \underset{W^r}{\longrightarrow} B^n$. For example, the r-dimensional fibre W^r for a 2D Toda chain $\mathcal{E}_{\text{Toda}}$ corresponds to the differential generators $w^1, \ldots, w^r \in \ker \frac{\mathrm{d}}{\mathrm{d}y}\big|_{\mathcal{E}_{\text{Toda}}}$ of its conservation laws, with r being the rank of the semi-simple complex Lie algebra. Next, we consider the infinite jet bundle $\xi_\infty\colon J^\infty(\xi) \to B^n$ and the $C^\infty(J^\infty(\xi))$-module of sections $\Gamma\bigl(\xi^*_\infty(\xi)\bigr) = \Gamma(\xi) \otimes_{C^\infty(B)} C^\infty(J^\infty(\xi))$ of the induced fibre bundle, see [9, 17]. Also, we consider its dual module w.r.t. the coupling that takes values in the n-th horizontal cohomology group $\bar{H}^n(\xi) = \Gamma\bigl(\xi^*_\infty\bigl(\bigwedge^n TB^n\bigr)\bigr) \big/ \operatorname{im} \xi^*_\infty\bigl(\mathrm{d}_{\mathrm{dR}(B^n)}\bigr)$. These two modules act as the jet bundle analogues of the (co)tangent bundle to the fibre manifold, respectively (see [19]). Namely, sections $\varphi \in \Gamma\bigl(\xi^*_\infty(\xi)\bigr)$ yield evolutionary vector fields $\partial_\varphi = \varphi \cdot \partial/\partial w + \cdots$ on $J^\infty(\xi)$ and the sections in the dual bundle are transformed as the variational covectors $\psi = \delta(\cdot)/\delta w$. Note that $\mathfrak{g}(\pi) \equiv \bigl(\Gamma\bigl(\pi^*_\infty(\pi)\bigr), [\,,\,]\bigr)$, where $[\,,\,]$ is the standard commutator of the fields. Finally, we require the existence of a differential substitution $w\colon J^\infty(\pi) \to \Gamma(\xi)$ that converts $\Gamma\bigl(\xi^*_\infty(\xi)\bigr)$ to the $C^\infty(J^\infty(\pi))$-submodule of $\Gamma\bigl(\pi^*_\infty(\xi)\bigr) = \Gamma(\xi) \otimes_{C^\infty(B)} C^\infty(J^\infty(\pi))$, and similar for the dual module. From now on, we denote the coordinates w along W^r and these substitutions $w[u]$ by the same letter, which makes no confusion.

Having chosen one of the two modules $\mathfrak{f}$ of sections as above, we study total differential operators A_i on the image of w in $\mathfrak{f}$. Thus we set $\Omega(\xi_\pi) := \mathfrak{f}\big|_w$. In particular, if $\xi = \pi$ and $w = \mathrm{id}$, then we obtain the recursion operators, see [12]. If the sections of ξ are composed by evolution equations $\mathcal{E} = \{w^i[u] = 0,\ 1 \leq i \leq r\}$, then the sections of the dual module $\widehat{\Gamma\bigl(\xi^*_\infty(\xi)\bigr)}$ are the *cosymmetries* [9, 17, 25], and we deal with Noether operators $\Omega \to \operatorname{sym}\mathcal{E}$. At the same time, in the second (dual) case with $\pi \neq \xi$ and nontrivial nonlinear operators w, we describe higher symmetry algebras for the 2D Toda chains [10, 11].

If $N = 1$, there is a unique operator $A\colon \Omega(\xi_\pi) \to \mathfrak{g}(\pi)$ with the image closed w.r.t. the commutation,

$$[\operatorname{im} A, \operatorname{im} A] \subseteq \operatorname{im} A. \tag{5}$$

The operator A transfers the Lie algebra structure $[\,,\,]\big|_{\operatorname{im} A}$ to the skew-symmetric bracket $[\,,\,]_A$ in the quotient $\operatorname{dom} A/\ker A$,

$$\bigl[A(\boldsymbol{p}), A(\boldsymbol{q})\bigr] = A\bigl([\boldsymbol{p},\boldsymbol{q}]_A\bigr), \qquad \boldsymbol{p},\boldsymbol{q} \in \Omega(\xi_\pi). \tag{6}$$

By definition, the kernel $\ker A$ is an ideal in the Lie algebra $\bigl(\Omega(\xi_\pi), [\,,\,]_A\bigr)$. In section 1 we analyse further the standard decomposition (11) of $[\,,\,]_A$.

Example 4 ($N = 1$)**.** In Theorem 1 we show why the image of every $\mathbb{Z}_2$-graded Hamiltonian operator is closed under the commutation in $\mathfrak{g}$. This approach (see (12) and (15) on p. 9) provides the most convenient method for verification whether a given graded operator is Hamiltonian.[2] The proof of Theorem 1 makes it clear that the commutation closure is not superfluous but appears as an element of the construction.

Examples of non-Hamiltonian operators A that satisfy (5), and the brackets $[\,,\,]_A$, are contained in [28] and [12]. The brackets $[\,,\,]_A$ that describe the commutation relations in symmetry algebras for open 2D Toda chains are calculated in [11], see Remark 3.

Definition 1. Suppose that *each* of the $N \geq 2$ operators $A_1, \ldots, A_N$ with a common domain satisfies (5). We say that these operators are *linear compatible* if their linear combinations $A_{\boldsymbol{\lambda}} = \sum_{i=1}^N \lambda_i A_i$ retain the same property (5) for any $\boldsymbol{\lambda}$.

In Theorem 2 we prove that the bracket, induced on the domain of a linear combination $A_{\boldsymbol{\lambda}}$ of the linear compatible operators, is equal to the sum of their 'individual' brackets.

Definition 2. We say that the $N \geq 2$ operators $A_i\colon \Omega(\xi_\pi) \to \mathfrak{g}(\pi)$ are *strong compatible* if the sum of their images is closed under commutation in $\mathfrak{g}(\pi)$,

$$\Bigl[\sum_i \operatorname{im} A_i, \sum_j \operatorname{im} A_j\Bigr] \subseteq \sum_k \operatorname{im} A_k, \qquad 1 \leq i,j,k \leq N. \tag{7}$$

The involutivity (7) gives rise to the bi-differential operators $\mathbf{c}^k_{ij}\colon \Omega(\xi_\pi)\times\Omega(\xi_\pi) \to \Omega(\xi_\pi)$ through

$$\bigl[A_i(\boldsymbol{p}), A_j(\boldsymbol{q})\bigr] = \sum_k A_k\bigl(\mathbf{c}^k_{ij}(\boldsymbol{p},\boldsymbol{q})\bigr), \qquad \boldsymbol{p},\boldsymbol{q} \in \Omega(\xi_\pi). \tag{8}$$

[2] In principle, it is possible to obtain comparably transparent and equally algorithmic verification formulas by re-deriving the entire construction of [6] in the $\mathbb{Z}_2$-graded setup; that concept is based on the use of variational polyvectors which are already endowed with their own grading.

The structural constants $\mathbf{c}_{ij}^k$ absorb the bi-differential action on $\boldsymbol{p}, \boldsymbol{q}$ under the commutation in the images of the operators (c.f. [7, §III.3]).

Example 5 ($N = 2$)**.** The Magri scheme [21] for the restriction of two compatible[3] Hamiltonian operators A_1, A_2 onto the commutative hierarchy of the descendants $\mathcal{H}_i \in \bar{H}^n(\pi)$ of the Casimirs $\mathcal{H}_0$ for A_1 gives an example of (7) with $N = 2$ and $\mathbf{c}_{ij}^k \equiv 0$. We conclude section 1 asking whether the converse is true and, consequently, helps to generate completely integrable systems at $n \geq 1$.

We do not assume that *each* operator A_i *alone* satisfies (5), therefore it may well occur that $\mathbf{c}_{ii}^k \neq 0$ for some $k \neq i$.

In section 2 we analyse the standard structure (21) of the decompositions (8). We extract the components $\Gamma_{ij}^k \in \mathcal{C}\mathrm{Diff}\bigl(\Omega(\xi_\pi) \times \Omega(\xi_\pi) \to \Omega(\xi_\pi)\bigr)$ from $\mathbf{c}_{ij}^k$ that act by total differential operators on both arguments $\boldsymbol{p}, \boldsymbol{q} \in \Omega(\xi_\pi)$ at once. Our main result, Theorem 4, states that, under a change of coordinates in the domain, the symbols Γ_{ij}^k are transformed by a proper analogue (23) of the classical rule $\Gamma \mapsto g\,\Gamma g^{-1} + \mathrm{d}g\,g^{-1}$ for the connection 1-forms Γ and reparametrizations g. We note (Corollary 5) that the bi-differential symbols Γ_{ij}^k are symmetric in lower indices if the domain $\Omega(\xi_\pi)$ of the operators A_i is $\Gamma\bigl(\widehat{\xi_\infty^*(\xi)}\bigr)\bigr|_{w[u]}$ and hence its elements acquire an additional grading.

In section 3 we confirm the geometric interpretation of Γ_{ij}^k as Christoffel symbols for a connection, although not in a fibre bundle. For transparency, we consider linear compatible operators, whence we deal with points $A \in \bigoplus_{i=1}^N \mathbb{R} \cdot A_i$. We recognize the symbols Γ_{ij}^k as the coefficients of a connection in the triad $\bigl(\Omega(\xi_\pi), [\,,\,]_A\bigr) \xrightarrow{A} \bigl(\mathfrak{g}, [\,,\,]\bigr)$ of the two Lie algebras and the morphism.

Let us recall that, for a commutative associative unital $\Bbbk$-algebra $\mathcal{A}$ and an $\mathcal{A}$-algebra $\mathcal{B}$ related by a $\Bbbk$-homomorphism $\imath\colon \mathcal{A} \hookrightarrow \mathcal{B}$, the definition of connections ∇ in the triads $\mathcal{A} \xrightarrow{\imath} \mathcal{B}$ was proposed in [16] as follows:

$$\mathrm{Der}(\mathcal{A}) \hookrightarrow \mathrm{Der}(\mathcal{A}, \mathcal{B}) \xrightarrow{\nabla} \mathrm{Der}(\mathcal{B}, P), \qquad P \text{ is a } \mathcal{B}\text{-module.}$$

In its turn, this is the algebraic counterpart of our initial geometric picture $\mathcal{A} = C^\infty(B^n)$, $\mathcal{B} = C^\infty(E^{m+n})$, and $\imath\colon \mathcal{A} \hookrightarrow \mathcal{B}$ for the bundle $\pi\colon E^{m+n} \xrightarrow[F^m]{} B^n$. We notice that this understanding of connections admits a tautological analogue for the spaces of *inner* derivations of Lie algebras. Namely, we let $\nabla^A\colon \mathrm{Der}_{\mathrm{In}}\bigl(\Omega(\xi_\pi), \mathfrak{g}(\pi)\bigr) \to \mathrm{Der}\bigl(\mathfrak{g}(\pi), P\bigr)$ be the map $A \circ [\psi, \cdot]_A \mapsto \bigl[A(\psi), \cdot\bigr]$ for each $\psi \in \Omega(\xi_\pi)$ and any $\mathfrak{g}(\pi)$-module P. In Theorem 7 we prove that such connections ∇^A are always flat.

1. Compatible differential operators

We begin with some notation; the standard reference in geometry of integrable systems is [25], see also [4, 17, 18]. In the sequel, everything is real and C^∞-smooth.

Let B^n be an n-dimensional orientable manifold, and let $\pi\colon E^{m+n} \xrightarrow[F^m]{} B^n$ be a bundle over B^n with m-dimensional fibres $F^m \ni u = (u^1, \ldots, u^m)$. By $J^\infty(\pi)$ we denote the infinite jet space over π, and we set $\pi_\infty\colon J^\infty(\pi) \to B^n$. We denote by u_σ,

[3]The Hamiltonian operators are (*Poisson*) *compatible* if their linear combinations remain Hamiltonian, see Theorem 3 below.

$|\sigma| \geq 0$, its fibre coordinates. Then $[u]$ stands for the differential dependence on u and its derivatives up to some finite order, and we put $\mathcal{F}(\pi) := C^\infty(J^\infty(\pi))$, understanding it as the inductive limit of filtered algebras.

We recall that two $\mathcal{F}(\pi)$-modules are canonically associated with the jet space $J^\infty(\pi)$. First, we have the $\mathcal{F}(\pi)$-module $\Gamma(\pi_\infty^*(\pi)) = \Gamma(\pi) \otimes_{C^\infty(B^n)} \mathcal{F}(\pi)$. The shorthand notation is $\varkappa(\pi) \equiv \Gamma(\pi_\infty^*(\pi))$. Its sections $\varphi \in \varkappa(\pi)$ describe the π-vertical evolutionary derivations $\partial_\varphi = \sum_\sigma \frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}x^\sigma}(\varphi) \cdot \partial/\partial u_\sigma$ on $J^\infty(\pi)$. For all ψ such that $\partial_\varphi(\psi)$ makes sense, the linearizations $\ell_\psi^{(u)}$ are defined by $\ell_\psi^{(u)}(\varphi) = \partial_\varphi(\psi)$, where $\varphi \in \varkappa(\pi)$.

Second, let $\bar{\Lambda}^n(\pi)$ be the module of highest π-horizontal forms on $J^\infty(\pi)$, $\bar{\mathrm{d}}$ be the horizontal π_∞^*-lifting of the de Rham differential on the base B^n, and $\bar{H}^n(\pi)$ be the n-th horizontal cohomology. Then we denote by $\widehat{\varkappa}(\pi) = \mathrm{Hom}_{\mathcal{F}(\pi)}\bigl(\varkappa(\pi), \bar{H}^n(\pi)\bigr)$ the $\mathcal{F}(\pi)$-module dual to $\varkappa(\pi)$.

Likewise, let $\xi\colon I^{r+n} \xrightarrow[W^r]{} B^n$ be another bundle over the same base B. Then we consider the $\mathcal{F}(\pi)$-module $\mathfrak{h} = \Gamma(\pi_\infty^*(\xi)) = \Gamma(\xi) \otimes_{C^\infty(B^n)} \mathcal{F}(\pi)$ of sections of the induced bundle over M^n. We denote by $\widehat{\mathfrak{h}} = \mathrm{Hom}_{\mathcal{F}(\pi)}\bigl(\mathfrak{h}, \bar{H}^n(\pi)\bigr)$ the dual of $\mathfrak{h}$. The standard example of $\widehat{\mathfrak{h}}$ is the module of 'cosymmetries' dual to the module $\mathfrak{h}$ of differential equations[4] that are imposed upon sections of the bundle π.

We suppose further that there is a differential substitution $w\colon J^\infty(\pi) \to \Gamma(\xi)$. Obviously, the substitution converts $\mathcal{F}(\xi)$-modules to the submodules of $\mathcal{F}(\pi)$-modules. We continue denoting the fibre coordinates in ξ and the nonlinear differential operators $w[u]$ that take values in $\Gamma(\xi)$ by the same letter w.

The main object of our study are total differential operators (that is, linear matrix differential operators in total derivatives) which take values in $\mathfrak{g}(\pi)$. By definition, the domain $\mathfrak{f}$ of the operators is one of the following: we have that either

$$\mathfrak{f} = \varkappa(\xi)\big|_{w\colon J^\infty(\pi)\to\Gamma(\xi)} \quad \text{or} \quad \mathfrak{f} = \widehat{\varkappa}(\xi)\big|_{w\colon J^\infty(\pi)\to\Gamma(\xi)}.$$

We refer to the operators with such domains as *first* and *second* kind, respectively. In particular, we have $\xi = \pi$ for the recursion operators $\varkappa(\pi) \to \varkappa(\pi)$ and it is standard to identify $\mathfrak{f} = \widehat{\varkappa}(\pi)$ for the Hamiltonian operators (see above); here we set $w = \mathrm{id}\colon \Gamma(\pi) \to \Gamma(\xi)$ in both cases.

Under any differential reparametrizations $\tilde{u} = \tilde{u}[u]\colon J^\infty(\pi) \to \Gamma(\pi)$ and $\tilde{w} = \tilde{w}[w]\colon J^\infty(\xi) \to \Gamma(\xi)$, the operators A of first kind are transformed according to

$$A \mapsto \tilde{A} = \ell_{\tilde{u}}^{(u)} \circ A \circ \ell_w^{(\tilde{w})}\Big|_{\substack{w=w[u]\\ u=u[\tilde{u}]}}. \tag{9a}$$

Respectively, the operators of second kind obey

$$A \mapsto \tilde{A} = \ell_{\tilde{u}}^{(u)} \circ A \circ \bigl(\ell_{\tilde{w}}^{(w)}\bigr)^*\Big|_{\substack{w=w[u]\\ u=u[\tilde{u}]}}. \tag{9b}$$

For an N-tuple of operators A_i with a common domain $\mathfrak{f}$ we set

$$\Omega(\xi_\pi) = \mathfrak{f}\Big/\bigcap_{i=1}^N \ker A_i. \tag{10}$$

[4]We stress that reparametrizations of the equations $\{w^i[u] = 0$, with $1 \leq i \leq r\}$ that constitute an r-component system are independent from any changes of the coordinates u in the bundle π.

Remark 1. The recursion and Noether operators with involutive images, addressed in [12], are examples of the operators of first and second kind, respectively. The operators that yield symmetries of 2D Toda chains are of second kind [10, 11, 28].

At the same time, there is no coordinate-independent understanding for the family of operators $A = \frac{\mathrm{d}}{\mathrm{d}x} \circ \prod_{j=1}^{s} \bigl(\frac{\mathrm{d}}{\mathrm{d}x} + \mu(j, \mu(j'))\, u\bigr)$, $s \geq 1$, $\mu\bigl(j, \mu(j')\bigr)) \in \mathbb{N}$, $j' < j$, that was discovered in [28] and proved to be infinite in [27] for a fixed system of coordinates (x, u) in π. The 'chain rule' for the brackets on *different* domains of such operators, divisible one by another, is described in [27] and [12].

Let us consider in more detail the case $N = 1$ of only one total differential operator $A\colon \Omega(\xi_\pi) \to \mathfrak{g}(\pi)$ that satisfies (5). By the Leibniz rule, two sets of summands appear in the bracket of evolutionary vector fields $A(\boldsymbol{p}), A(\boldsymbol{q})$ that belong to the image:

$$\bigl[A(\boldsymbol{p}), A(\boldsymbol{q})\bigr] = A\bigl(\partial_{A(\boldsymbol{p})}(\boldsymbol{q}) - \partial_{A(\boldsymbol{q})}(\boldsymbol{p})\bigr) + \bigl(\partial_{A(\boldsymbol{p})}(A)(\boldsymbol{q}) - \partial_{A(\boldsymbol{q})}(A)(\boldsymbol{p})\bigr).$$

In the first summand we have used the permutability of evolutionary derivations and total derivatives. The second summand hits the image of A by construction. Consequently, the Lie algebra structure $[\,,\,]_A$ on the domain of A equals

$$[\boldsymbol{p}, \boldsymbol{q}]_A = \partial_{A(\boldsymbol{p})}(\boldsymbol{q}) - \partial_{A(\boldsymbol{q})}(\boldsymbol{p}) + \{\{\boldsymbol{p}, \boldsymbol{q}\}\}_A. \tag{11}$$

The bracket $[\,,\,]_A$ contains the two standard summands and the bi-differential skew-symmetric part $\{\{\,,\,\}\}_A \in \mathcal{C}\mathrm{Diff}\bigl(\Omega(\xi_\pi) \times \Omega(\xi_\pi) \to \Omega(\xi_\pi)\bigr)$ that generally does not satisfy the Jacobi identity.

Remark 2. The bracket $\{\{\,,\,\}\}_A$ for Hamiltonian operators A can be derived explicitly from the Jacobi identity $[\![A, A]\!] = 0$ for the Lie algebra $\bigl(\bar{H}^n(\pi), \{\,,\,\}_A\bigr)$ of the Hamiltonian functionals endowed by A with the Poisson bracket. Following [17], we put[5] $\ell_{A,\psi}(\varphi) := \bigl(\partial_\varphi(A)\bigr)(\psi)$ for any $\varphi \in \varkappa(\pi)$, $\psi \in \mathfrak{f}$, and a total differential operator $A \in \mathcal{C}\mathrm{Diff}(\mathfrak{f}, \varkappa(\pi))$. Note that $\ell_{A,\psi}$ is an operator in total derivatives w.r.t. its argument φ and w.r.t. ψ (but not w.r.t. the coefficients of A), and hence the adjoint $\ell^\dagger_{A,\psi}$ is well defined.

A skew-adjoint operator $A = \bigl\| A^{ij}_{\boldsymbol{\tau}} \cdot \frac{\mathrm{d}^{|\boldsymbol{\tau}|}}{\mathrm{d}x^{\boldsymbol{\tau}}} \bigr\|$ is Hamiltonian if and only if the relation

$$\ell_{A,\boldsymbol{p}}(A(\boldsymbol{q})) - \ell_{A,\boldsymbol{q}}(A(\boldsymbol{p})) = A\bigl(\ell^\dagger_{A,\boldsymbol{q}}(\boldsymbol{p})\bigr)$$

holds for all $\boldsymbol{p}, \boldsymbol{q} \in \mathfrak{f}$. This formula provides the bracket $\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_A = \ell^\dagger_{A,\boldsymbol{p}}(\boldsymbol{q})$ explicitly, c.f. [12, 17, 25] and Theorem 1 below; in coordinates, the k-th ($1 \leq k \leq m$) component of $\{\{\,,\,\}\}_A$ equals

$$\{\{\boldsymbol{p}, \boldsymbol{q}\}\}^k_A = \sum_{|\boldsymbol{\sigma}|,|\boldsymbol{\tau}|\geq 0} \sum_{i,j=1}^{m} \Bigl(\frac{\mathrm{d}^{|\sigma|}}{\mathrm{d}x^\sigma}\Bigr)^\dagger \Bigl[q_i \cdot \frac{\partial A^{ij}_{\boldsymbol{\tau}}}{\partial u^k_{\boldsymbol{\sigma}}} \cdot \frac{\mathrm{d}^{|\boldsymbol{\tau}|}}{\mathrm{d}x^{\boldsymbol{\tau}}}(p_j)\Bigr], \tag{12}$$

where $\dagger$ denotes the adjoint. Formula (12) is extended straightforwardly onto the super-setup of bosonic super-fields and parity-preserving Hamiltonian operators that endow bosonic functionals with Poisson brackets. Now the multi-indices $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ can run through the super-derivations as well, and the partial derivatives $\partial/\partial u^k_{\boldsymbol{\sigma}}$ in (12) act according to the graded Leibniz rule.

[5]We emphasize that the notation $\ell_{A,\psi}$ is *not* the same as the linearization $\ell^{(u)}_{A(\psi)}$.

Example 6. Let $\boldsymbol{u} = u_0(x,t)\cdot\mathbf{1} + \theta_1\cdot u_1(x,t) + \theta_2\cdot u_2(x,t) + \theta_1\theta_2\cdot u_{12}(x,t)$ be a scalar bosonic super-field, that is, a mapping of $\mathbb{R}^2 \ni (x,t)$ to the four-dimensional Grassmann algebra generated over $\mathbb{C}$ by θ_1 and θ_2 satisfying $\theta_i\theta_j = -\theta_j\theta_i$. By definition, put $\mathcal{D}_i = \partial/\partial\theta_i + \theta_i\cdot \mathrm{d}/\mathrm{d}x$, here $1\leq i,j\leq 2$ and it is readily seen that $\mathcal{D}_i\mathcal{D}_j + \mathcal{D}_j\mathcal{D}_i = 2\delta_{ij}\cdot\mathrm{d}/\mathrm{d}x$.

Consider the super-operator $\boldsymbol{A}_2$ that comes from the N=2 classical super-conformal algebra [1] and yields the second Hamiltonian structure for the triplet of integrable N=2 supersymmetric Korteweg–de Vries equations [20],

$$\boldsymbol{A}_2 = \mathcal{D}_1\mathcal{D}_2\tfrac{\mathrm{d}}{\mathrm{d}x} + 2\boldsymbol{u}\tfrac{\mathrm{d}}{\mathrm{d}x} - \mathcal{D}_1(\boldsymbol{u})\mathcal{D}_1 - \mathcal{D}_2(\boldsymbol{u})\mathcal{D}_2 + 2\boldsymbol{u}_x. \tag{13}$$

Let the bosonic super-sections $\boldsymbol{p},\boldsymbol{q}\in\widehat{\varkappa}(\pi)$ be two arguments of $\boldsymbol{A}_2$. Then formula (12) yields their skew-symmetric bracket

$$\{\{\boldsymbol{p},\boldsymbol{q}\}\}_{\boldsymbol{A}_2} = 2\big(\tfrac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{p}\cdot\boldsymbol{q} - \boldsymbol{p}\cdot\tfrac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{q}\big) - \mathcal{D}_1(\boldsymbol{p})\cdot\mathcal{D}_1(\boldsymbol{q}) - \mathcal{D}_2(\boldsymbol{p})\cdot\mathcal{D}_2(\boldsymbol{q}), \tag{14}$$

and the validity of (11) confirms that the super-operator $\boldsymbol{A}_2$ is indeed Hamiltonian.

The purely bosonic setup of Remark 2 and the N=2 supersymmetry invariance in Example 6 are particular cases in the general $\mathbb{Z}_2$-graded framework of $(m_0+n \mid m_1)$-dimensional fibre bundles π and Hamiltonian operators $A\colon\widehat{\varkappa}(\pi)\to\varkappa(\pi)$ for bosonic Hamiltonian functionals.

Let $\langle\,,\,\rangle$ denote the standard coupling $\widehat{\varkappa}(\pi)\times\varkappa(\pi)\to\bar{H}^n(\pi)$ and define $\langle\,|\,\rangle$ by setting $\langle\boldsymbol{p}\mid\boldsymbol{q}\rangle := \langle\boldsymbol{q},\boldsymbol{p}\rangle$. Namely, if $\boldsymbol{p}=(\boldsymbol{p}^0,\boldsymbol{p}^1)$ and $\boldsymbol{q}=(\boldsymbol{q}^0,\boldsymbol{q}^1)$ are decomposed to even and odd-graded components, then $\langle\boldsymbol{p},\boldsymbol{q}\rangle = \boldsymbol{p}^0\cdot\boldsymbol{q}^0 + \boldsymbol{p}^1\cdot\boldsymbol{q}^1$ and $\langle\boldsymbol{p}\mid\boldsymbol{q}\rangle = \boldsymbol{p}^0\cdot\boldsymbol{q}^0 - \boldsymbol{p}^1\cdot\boldsymbol{q}^1$. The definition of adjoint graded operators implies $\langle\boldsymbol{p},A(\boldsymbol{q})\rangle = \langle\boldsymbol{q},A^\dagger(\boldsymbol{p})\rangle = \langle A^\dagger(\boldsymbol{p})\mid\boldsymbol{q}\rangle$.

Theorem 1. *A $\mathbb{Z}_2$-graded parity-preserving skew-adjoint total differential operator $A\colon\widehat{\varkappa}(\pi)\to\varkappa(\pi)$ is Hamiltonian if and only if its image is closed under the commutation and, for all $\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}\in\widehat{\varkappa}(\pi)$, the bracket $\{\{\,,\,\}\}_A$ in (11) satisfies the equality*

$$\big\langle A\big(\{\{\boldsymbol{p},\boldsymbol{q}\}\}_A\big)\mid\boldsymbol{r}\big\rangle = {:}\,\big\langle\boldsymbol{p},\partial_{\underleftarrow{A}(\underrightarrow{\boldsymbol{r}})}(A)(\boldsymbol{q})\big\rangle\,{:}, \tag{15}$$

where the normal order $:\ :$ suggests that all derivations are thrown off $A(\boldsymbol{r})$ by the graded Green formula and the arrows indicate that first $A(\boldsymbol{r})$ is moved right w.r.t. $\boldsymbol{q}$, and then the operator A is pushed left w.r.t. $\boldsymbol{p}$ by Green's formula again. The arising argument of the skew-adjoint operator A is the bracket $\{\{\boldsymbol{p},\boldsymbol{q}\}\}_A$.

Proof. Let us expand each of the three summands of the Jacobi identity,

$$\sum_{\circlearrowleft}\partial_{A(\boldsymbol{p})}(\langle\boldsymbol{q},A(\boldsymbol{r})\rangle) = 0,$$

by using the Leibnitz rule. We obtain

$$\sum_{\circlearrowleft}\Big[\langle\partial_{A(\boldsymbol{p})}(\boldsymbol{q}),A(\boldsymbol{r})\rangle + \langle\boldsymbol{q},\partial_{A(\boldsymbol{p})}(A)(\boldsymbol{r})\rangle + \langle\boldsymbol{q},A(\partial_{A(\boldsymbol{p})}(\boldsymbol{r}))\rangle\Big] = 0. \tag{16}$$

Consider the third term in (16) and, by the substitution principle [25], suppose that $\boldsymbol{r}$ is the variational derivative of a Hamiltonian functional, whence the linearization $\ell_{\boldsymbol{r}}$ is self-adjoint in the graded sense. Consequently,

$$\begin{aligned}\langle\boldsymbol{q},A(\partial_{A(\boldsymbol{p})}(\boldsymbol{r}))\rangle &= -\langle A(\boldsymbol{q})\mid\partial_{A(\boldsymbol{p})}(\boldsymbol{r})\rangle = -\langle A(\boldsymbol{q})\mid\ell_{\boldsymbol{r}}(A(\boldsymbol{p}))\rangle = -\langle A(\boldsymbol{p})\mid\ell_{\boldsymbol{r}}^\dagger(A(\boldsymbol{q}))\rangle\\ &= -\langle A(\boldsymbol{p})\mid\ell_{\boldsymbol{r}}(A(\boldsymbol{q}))\rangle = -\langle\ell_{\boldsymbol{r}}(A(\boldsymbol{q})),A(\boldsymbol{p})\rangle = -\langle\partial_{A(\boldsymbol{q})}(\boldsymbol{r}),A(\boldsymbol{p})\rangle.\end{aligned}$$

Substituting this back in (16) and taking the sum over the cyclic permutations, we cancel 3×2 terms, except for

$$\langle \boldsymbol{q}, \partial_{A(\boldsymbol{p})}(A)(\boldsymbol{r})\rangle + \langle \boldsymbol{r}, \partial_{A(\boldsymbol{q})}(A)(\boldsymbol{p})\rangle + \langle \boldsymbol{p}, \partial_{A(\boldsymbol{r})}(A)(\boldsymbol{q})\rangle = 0. \tag{17}$$

Now we consider separately the first and second summands in (17), paying due attention to the order of graded objects and the directions the derivations act in. First, applying the even vector field $\partial_{A(\boldsymbol{p})}$ to the equality $\langle \boldsymbol{q}, A(\boldsymbol{r})\rangle = \langle A^\dagger(\boldsymbol{q}) \mid \boldsymbol{r}\rangle$ and using $A^\dagger = -A$, we conclude that

$$\langle \boldsymbol{q}, \partial_{A(\boldsymbol{p})}(A)(\boldsymbol{r})\rangle = -\langle \partial_{A(\boldsymbol{p})}(A)(\boldsymbol{q}) \mid \boldsymbol{r}\rangle.$$

Likewise, the second summand in (17) gives

$$\langle \boldsymbol{r}, \partial_{A(\boldsymbol{q})}(A)(\boldsymbol{p})\rangle = \langle \partial_{A(\boldsymbol{q})}(A)(\boldsymbol{p}) \mid \boldsymbol{r}\rangle.$$

Hence from (17) we obtain

$$\langle \partial_{A(\boldsymbol{p})}(A)(\boldsymbol{q}) \mid \boldsymbol{r}\rangle - \langle \partial_{A(\boldsymbol{q})}(A)(\boldsymbol{p}) \mid \boldsymbol{r}\rangle = \langle \boldsymbol{p}, \partial_{A(\boldsymbol{r})}(A)(\boldsymbol{q})\rangle.$$

Integrating the right-hand side by parts, we move the skew-adjoint operator A off $\boldsymbol{r}$ and obtain the bracket $\{\{\boldsymbol{p},\boldsymbol{q}\}\}_A$ as its argument.

We have shown that if the bracket induced on the domain of a given graded skew-adjoint operator A with involutive image, see (11), coincides with the bracket $\{\{\,,\,\}\}_A$ emerging from (15), then A is indeed Hamiltonian, and *vice versa*. This concludes the proof. $\square$

Example 7. Writing the super-operator (13) in components, now with $p_i = \delta\mathcal{H}/\delta u_i$, whence p_0 and p_{12} are even and p_1, p_2 are odd, we obtain the (4×4)-matrix operator [1]

$$\hat{P}_2 = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}x} & -u_2 & u_1 & 2u_0\frac{\mathrm{d}}{\mathrm{d}x} + 2u_{0;x} \\ -u_2 & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + u_{12} & -2u_0\frac{\mathrm{d}}{\mathrm{d}x} - u_{0;x} & 3u_1\frac{\mathrm{d}}{\mathrm{d}x} + 2u_{1;x} \\ u_1 & 2u_0\frac{\mathrm{d}}{\mathrm{d}x} + u_{0;x} & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + u_{12} & 3u_2\frac{\mathrm{d}}{\mathrm{d}x} + 2u_{2;x} \\ 2u_0\frac{\mathrm{d}}{\mathrm{d}x} & -3u_1\frac{\mathrm{d}}{\mathrm{d}x} - u_{1;x} & -3u_2\frac{\mathrm{d}}{\mathrm{d}x} - u_{2;x} & \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^3 + 4u_{12}\frac{\mathrm{d}}{\mathrm{d}x} + 2u_{12;x} \end{pmatrix}. \tag{18}$$

The application of Theorem 1 is particularly transparent since the coefficients of (18) are linear functions. The right-hand side of (15) yields the four components of the skew-symmetric bracket $\{\{\boldsymbol{p},\boldsymbol{q}\}\}_{A_2}$,

$$\begin{aligned}
\{\{\boldsymbol{p},\boldsymbol{q}\}\}^0_{A_2} &= 2(p_{0;x}q_{12} - p_{12}q_{0;x}) - (p_{1;x}q_2 + p_2q_{1;x}) + (p_{2;x}q_1 + p_1q_{2;x}),\\
\{\{\boldsymbol{p},\boldsymbol{q}\}\}^1_{A_2} &= 2(p_{1;x}q_{12} - p_{12}q_{1;x}) + (p_0q_2 - p_2q_0) + (p_{12;x}q_1 - p_1q_{12;x}),\\
\{\{\boldsymbol{p},\boldsymbol{q}\}\}^2_{A_2} &= 2(p_{2;x}q_{12} - p_{12}q_{2;x}) + (p_1q_0 - p_0q_1) + (p_{12;x}q_2 - p_2q_{12;x}),\\
\{\{\boldsymbol{p},\boldsymbol{q}\}\}^{12}_{A_2} &= 2(p_{12;x}q_{12} - p_{12}q_{12;x}) - p_1q_1 - p_2q_2.
\end{aligned}$$

This is the component expansion of (14).

We note that Poisson compatible Hamiltonian operators are linear compatible, and *vice versa*, because formula (12), c.f. (15), is linear in the coefficients of A. This manifests a general property of the *linear compatible* operators, each subject to (5) and same for their linear combinations.

Theorem 2 ([12])**.** *The bracket* $\{\{\,,\,\}\}_{A_{\boldsymbol{\lambda}}}$ *on the domain of the combination* $A_{\boldsymbol{\lambda}}$ *of linear compatible operators* A_i *is*

$$\{\{\,,\,\}\}_{\sum\limits_{i=1}^{N}\lambda_i A_i} = \sum_{i=1}^{N} \lambda_i \cdot \{\{\,,\,\}\}_{A_i}.$$

The pairwise linear compatibility implies the collective linear compatibility of $A_1, \ldots, A_N$.

Let us revisit the classical Magri scheme [21] for completely integrable bi-Hamiltonian systems in $n \geq 1$ dimensions and focus our attention on its element which is often omitted but becomes particularly clear in the cohomological formulation. We recall that the Schouten bracket [6] of the variational bi-vectors satisfies the Jacobi identity

$$[\![[\![A_1, A_2]\!], A_3]\!] + [\![[\![A_2, A_3]\!], A_1]\!] + [\![[\![A_3, A_1]\!], A_2]\!] = 0. \tag{19}$$

Hence the original Jacobi identity $[\![A, A]\!](\psi_1, \psi_2, \psi_3) = 0$ for the arguments of A implies that $\mathrm{d}_A = [\![A, \cdot\,]\!]$ is a differential, giving rise to the Poisson cohomology $H^k_A(\pi)$. Obviously, the Casimirs $\mathcal{H}_0 \in \bar{H}^n(\pi)$ such that $[\![A, \mathcal{H}_0]\!] = 0$ for a Hamiltonian operator A constitute $H^0_A(\pi)$.

Theorem 3 ([2, 21])**.** *Suppose* $[\![A_1, A_2]\!] = 0$, $\mathcal{H}_0 \in H^0_{A_1}(\pi)$ *is a Casimir of* A_1, *and the first Poisson cohomology w.r.t.* $\mathrm{d}_{A_1} = [\![A_1, \cdot\,]\!]$ *vanishes. Then for any* $k > 0$ *there is a Hamiltonian* $\mathcal{H}_k \in \bar{H}^n(\pi)$ *such that*

$$[\![A_2, \mathcal{H}_{k-1}]\!] = [\![A_1, \mathcal{H}_k]\!]. \tag{20}$$

Put $\varphi_k := A_1\big(\delta/\delta u(\mathcal{H}_k)\big)$. *The Hamiltonians* $\mathcal{H}_i$, $i \geq 0$, *pairwise Poisson commute w.r.t. either* A_1 *or* A_2, *the densities of* $\mathcal{H}_i$ *are conserved on any equation* $u_{t_k} = \varphi_k$, *and the evolutionary derivations* ∂_{φ_k} *pairwise commute for all* $k \geq 0$.

Standard proof. The main homological equality (20) is established by induction on k. Starting with a Casimir $\mathcal{H}_0$, we obtain

$$0 = [\![A_2, 0]\!] = [\![A_2, [\![A_1, \mathcal{H}_0]\!]]\!] = -[\![A_1, [\![A_2, \mathcal{H}_0]\!]]\!] \mod [\![A_1, A_2]\!] = 0,$$

using the Jacobi identity (19). The first Poisson cohomology $H^1_{A_1}(\pi) = 0$ is trivial by an assumption of the theorem, and hence the closed element $[\![A_2, \mathcal{H}_0]\!]$ in the kernel of $[\![A_1, \cdot]\!]$ is exact: $[\![A_2, \mathcal{H}_0]\!] = [\![A_1, \mathcal{H}_1]\!]$ for some $\mathcal{H}_1$. For $k \geq 1$, we have

$$[\![A_1, [\![A_2, \mathcal{H}_k]\!]]\!] = -[\![A_2, [\![A_1, \mathcal{H}_k]\!]]\!] = -[\![A_2, [\![A_2, \mathcal{H}_{k-1}]\!]]\!] = 0$$

using (19) and by $[\![A_2, A_2]\!] = 0$. Hence $[\![A_2, \mathcal{H}_k]\!] = [\![A_1, \mathcal{H}_{k+1}]\!]$ by $H^1_{A_1}(\pi) = 0$, and we thus proceed infinitely. □

We see now that the inductive step — the existence of the $(k + 1)$-st Hamiltonian in involution — is possible if and only if H_0 is a Casimir,[6] and therefore the operators A_1 and A_2 are restricted onto the linear subspace that is spanned in $\widehat{\varkappa}(\pi)$ by the Euler derivatives of the descendants of $\mathcal{H}_0$, the Hamiltonians of the hierarchy. In fact, the image under A_2 of a generic section from $\widehat{\varkappa}(\pi)$ can not be resolved w.r.t. A_1 by (20). On the other hand, the *strong* compatibility of the restrictions of Poisson compatible

[6]The Magri scheme starts from any two Hamiltonians $\mathcal{H}_{k-1}, \mathcal{H}_k \in \bar{H}^n(\pi)$ that satisfy (20), but we operate with maximal subspaces of $\bar{H}^n(\pi)$ such that the sequences of Hamiltonians can not be extended with $k < 0$.

operators A_1 and A_2 onto the hierarchy follows from Theorem 3 tautologically, since their images are commutative Lie algebras.

Regarding the converse statement as a potential generator of multi-dimensional completely integrable systems, we formulate the open problem: Is the strong compatibility of linear compatible Hamiltonian operators achieved *only* for their restrictions onto the hierarchies of Hamiltonians in involution so that the bi-differential constants $\mathbf{c}_{ij}^k$ necessarily vanish ?

2. Bi-differential Christoffel symbols

In this section we consider *strong compatible* total differential operators $A_i\colon \Omega(\xi_\pi) \to \mathfrak{g}(\pi)$, $1 \leq i \leq N$, whose images in the Lie algebra $\mathfrak{g}(\pi)$ of evolutionary vector fields on $J^\infty(\pi)$ are subject to the collective commutation closure (7).

Let us extract the total bi-differential parts of the operators $\mathbf{c}_{ij}^k$ in (8), similar to (11) now for $N \geq 1$. We have

$$\mathbf{c}_{ij}^k = \partial_{A_i(\boldsymbol{p})}(\boldsymbol{q}) \cdot \delta_j^k - \partial_{A_j(\boldsymbol{q})}(\boldsymbol{p}) \cdot \delta_i^k + \Gamma_{ij}^k(\boldsymbol{p}, \boldsymbol{q}), \qquad \boldsymbol{p}, \boldsymbol{q} \in \Omega(\xi_\pi), \tag{21}$$

where $\Gamma_{ij}^k \in \mathcal{C}\mathrm{Diff}\bigl(\Omega(\xi_\pi) \times \Omega(\xi_\pi) \to \Omega(\xi_\pi)\bigr)$. By definition, the three indices in Γ_{ij}^k match the respective operators A_i, A_j, A_k in (8). Obviously, the convention $\Gamma_{11}^1 = \{\{\,,\,\}\}_{A_1}$ holds if $N = 1$. At the same time, for fixed i, j, k, the symbol Γ_{ij}^k remains a (class of) matrix differential operator in each of its two arguments $\boldsymbol{p}, \boldsymbol{q} \in \Omega(\xi_\pi)$. Hence the total number of the indices is much greater than three; we note that the upper or lower location of the omitted indices depends on the (co)vector nature of the domain $\Omega(\xi_\pi)$. The symbol Γ_{ij}^k represents a class of bi-differential operators because they are not uniquely defined. Indeed, they are gauged by the conditions

$$\sum_{k=1}^N A_k\Bigl(\partial_{A_j(\boldsymbol{q})}(\boldsymbol{p})\delta_i^k - \partial_{A_i(\boldsymbol{p})}(\boldsymbol{q})\delta_j^k + \Gamma_{ij}^k(\boldsymbol{p}, \boldsymbol{q})\Bigr) = 0, \qquad \boldsymbol{p}, \boldsymbol{q} \in \Omega(\xi_\pi). \tag{22}$$

The zero in the r.h.s. of (22) appears if $\bigl[A_k(\psi), \sum_{\ell=1}^N \operatorname{im} A_\ell\bigr] = 0$ implies $\psi \in \ker A_k$; for this it is sufficient that the sum of the images of A_ℓ in $\mathfrak{g}(\pi)$ be semi-simple.

Example 8. Consider the Liouville equation $\mathcal{E}_{\text{Liou}} = \{u_{xy} = \exp(2u)\}$. The differential generators $w \in \ker \frac{\mathrm{d}}{\mathrm{d}y}\bigr|_{\mathcal{E}_{\text{Liou}}}$, $\bar{w} \in \ker \frac{\mathrm{d}}{\mathrm{d}x}\bigr|_{\mathcal{E}_{\text{Liou}}}$ of its conservation laws are $w = u_x^2 - u_{xx}$ and $\bar{w} = u_y^2 - u_{yy}$. The operators[7] $\square = u_x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}$ and $\bar{\square} = u_y + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}y}$ determine higher symmetries $\varphi, \bar{\varphi}$ (in particular, Noether symmetries $\varphi_{\mathcal{L}}, \bar{\varphi}_{\mathcal{L}}$) of $\mathcal{E}_{\text{Liou}}$, which are

$$\varphi = \square\bigl(\phi(x, [w])\bigr), \quad \varphi_{\mathcal{L}} = \square\Bigl(\frac{\delta\mathcal{H}(x, [w])}{\delta w}\Bigr); \quad \bar{\varphi} = \bar{\square}\bigl(\bar{\phi}(y, [\bar{w}])\bigr), \quad \bar{\varphi}_{\mathcal{L}} = \bar{\square}\Bigl(\frac{\delta\bar{\mathcal{H}}(y, [\bar{w}])}{\delta\bar{w}}\Bigr)$$

for any smooth $\bar{\phi}, \phi \in \widehat{\varkappa}(\xi)\bigr|_{w[u]}$ and $\bar{\mathcal{H}}, \mathcal{H} \in \bar{H}^1(\xi)\bigr|_{w[u]}$. The images of $\square$ and $\bar{\square}$ are closed w.r.t. the commutation; for instance, the bracket (11) for $\square$ contains $\{\{p, q\}\}_\square = \frac{\mathrm{d}}{\mathrm{d}x}(p) \cdot q - p \cdot \frac{\mathrm{d}}{\mathrm{d}x}(q)$, and similar for $\bar{\square}$. The two summands in the symmetry algebra $\operatorname{sym}\mathcal{E}_{\text{Liou}} \simeq \operatorname{im}\square \oplus \operatorname{im}\bar{\square}$ commute between each other, $[\operatorname{im}\square, \operatorname{im}\bar{\square}] \doteq 0$ on $\mathcal{E}_{\text{Liou}}$. The

[7] We denote the operators by $\square$ and $\bar{\square}$, following the notation of [28] and [10, 11].

operators $\Box$, $\bar{\Box}$ generate the bi-differential symbols

$$\Gamma^{\Box}_{\Box\Box} = \{\!\{\,,\,\}\!\}_{\Box} = \tfrac{\mathrm{d}}{\mathrm{d}x} \otimes \mathbf{1} - \mathbf{1} \otimes \tfrac{\mathrm{d}}{\mathrm{d}x}, \qquad \Gamma^{\bar{\Box}}_{\bar{\Box}\bar{\Box}} = \{\!\{\,,\,\}\!\}_{\bar{\Box}} = \tfrac{\mathrm{d}}{\mathrm{d}y} \otimes \mathbf{1} - \mathbf{1} \otimes \tfrac{\mathrm{d}}{\mathrm{d}y},$$
$$\Gamma^{\Box}_{\Box\bar{\Box}} = \tfrac{\mathrm{d}}{\mathrm{d}y} \otimes \mathbf{1}, \qquad \Gamma^{\bar{\Box}}_{\Box\bar{\Box}} = -\mathbf{1} \otimes \tfrac{\mathrm{d}}{\mathrm{d}x}, \qquad \Gamma^{\Box}_{\bar{\Box}\Box} = -\mathbf{1} \otimes \tfrac{\mathrm{d}}{\mathrm{d}y}, \qquad \Gamma^{\bar{\Box}}_{\bar{\Box}\Box} = \tfrac{\mathrm{d}}{\mathrm{d}x} \otimes \mathbf{1},$$

where the notation is obvious. Note that $\Gamma^{\Box}_{\Box\bar{\Box}}(p,q) \doteq \Gamma^{\bar{\Box}}_{\Box\bar{\Box}}(p,q) \doteq \Gamma^{\Box}_{\bar{\Box}\Box}(q,p) \doteq \Gamma^{\bar{\Box}}_{\bar{\Box}\Box}(q,p) \doteq 0$ on $\mathcal{E}_{\text{Liou}}$ for any $p(x,[w])$ and $q(y,[\bar{w}])$.

The matrix operators $\Box$, $\bar{\Box}$ of second kind are well-defined [10, 11] for each 2D Toda chain $\mathcal{E}_{\text{Toda}}$ associated with a semi-simple complex Lie algebra. They exhibit the same properties as above.

Remark 3 (on representability). The main result of [11] leads to the open problem: Let, as above, the operator $\Box$ be of second kind and suppose that the substitution $w = w[u]$ is such that the velocity $\partial_{\Box(\boldsymbol{p})}(w)$ is a differential function of w for any $\boldsymbol{p} \in \mathfrak{f}\big|_w$, and same for the coefficients of the bracket $\{\!\{\,,\,\}\!\}_{\Box}$. In view of [10, 11, 28], such assumptions are natural. Now, when is there a *Hamiltonian* operator $A\colon \mathfrak{f} \to \varkappa(\xi)$ that endows the $\mathcal{F}(\xi)$-module $\mathfrak{f}$ of variational covectors with the same Lie algebra structure as $\Box$ does, *i.e.*, such that $\{\!\{\,,\,\}\!\}_{\Box} = \{\!\{\,,\,\}\!\}_A$? [For example, the identity $\partial_{\Box(p)}(w) = \bigl(-\frac{1}{2}\frac{\mathrm{d}^3}{\mathrm{d}x^3} + u\,\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x} \circ u\bigr)(p)$ implies the equality of the brackets in (1) and (2).]

The operators $\Box$, $\bar{\Box}$ yield the involutive distributions of evolutionary vector fields that are tangent to the *integral manifolds*, the 2D Toda differential equations. But generally there is no Frobenius theorem for such distributions.

Theorem 4 (Transformations of Γ^k_{ij})**.** *Let $\tilde{w} = \tilde{w}[w]$ be a nondegenerate change of fibre coordinates in the bundle ξ. Recall that the sections $\boldsymbol{p}, \boldsymbol{q} \in \mathfrak{f}\big|_w$ in the domains of strong compatible operators are reparametrized by $\boldsymbol{p} \mapsto \tilde{\boldsymbol{p}} = g\boldsymbol{p}$ and $\boldsymbol{q} \mapsto \tilde{\boldsymbol{q}} = g\boldsymbol{q}$, where $g = \ell^{(w)}_{\tilde{w}}$ for the operators* (9a) *of first kind and $g = \bigl[\bigl(\ell^{(w)}_{\tilde{w}}\bigr)^*\bigr]^{-1}$ for the operators* (9b) *of second kind. In this notation, the operators $A_1, \ldots, A_N\colon \Omega(\xi_\pi) \to \mathfrak{g}(\pi)$ with a common domain $\Omega(\xi_\pi) = \mathfrak{f}\big|_w / \bigcap_i \ker A_i$ are transformed by $A_i \mapsto \tilde{A}_i = A_i \circ g^{-1}\big|_{w=w[\tilde{w}]}$.*

Then the bi-differential symbols $\Gamma^k_{ij} \in \mathcal{C}\mathrm{Diff}\bigl(\Omega(\xi_\pi) \times \Omega(\xi_\pi) \to \Omega(\xi_\pi)\bigr)$ obey the direct analogue of the standard rule $\Gamma \mapsto g\,\Gamma\,g^{-1} + \mathrm{d}g \cdot g^{-1}$ for the connection 1-forms Γ,

$$\Gamma^k_{ij}(\boldsymbol{p},\boldsymbol{q}) \mapsto \Gamma^{\tilde{k}}_{\tilde{\imath}\tilde{\jmath}}\bigl(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{q}}\bigr) = \bigl(g \circ \Gamma^{\tilde{k}}_{\tilde{\imath}\tilde{\jmath}}\bigr)\bigl(g^{-1}\tilde{\boldsymbol{p}}, g^{-1}\tilde{\boldsymbol{q}}\bigr) + \delta^{\tilde{k}}_{\tilde{\imath}} \cdot \partial_{\tilde{A}_{\tilde{\jmath}}(\tilde{\boldsymbol{q}})}(g)\bigl(g^{-1}\tilde{\boldsymbol{p}}\bigr) - \delta^{\tilde{k}}_{\tilde{\jmath}} \cdot \partial_{\tilde{A}_{\tilde{\imath}}(\tilde{\boldsymbol{p}})}(g)\bigl(g^{-1}\tilde{\boldsymbol{q}}\bigr). \tag{23}$$

Proof. Denote $A = A_i$ and $B = A_j$; without loss of generality assume $i = 1$ and $j = 2$. Let us calculate the commutators of vector fields in the images of A and B using two systems of coordinates in the domain. We equate the commutators straighforwardly, because the fibre coordinates in the images of the operators are not touched at all. So, we have, originally,

$$\bigl[A(\boldsymbol{p}),B(\boldsymbol{q})\bigr] = B\bigl(\partial_{A(\boldsymbol{p})}(\boldsymbol{q})\bigr) - A\bigl(\partial_{B(\boldsymbol{q})}(\boldsymbol{p})\bigr) + A\bigl(\Gamma^A_{AB}(\boldsymbol{p},\boldsymbol{q})\bigr) + B\bigl(\Gamma^B_{AB}(\boldsymbol{p},\boldsymbol{q})\bigr) + \sum_{k=3}^{N} A_k\bigl(\Gamma^k_{AB}(\boldsymbol{p},\boldsymbol{q})\bigr).$$

On the other hand, we substitute $\tilde{\boldsymbol{p}} = g\boldsymbol{p}$ and $\tilde{\boldsymbol{q}} = g\boldsymbol{q}$ in $\bigl[\tilde{A}(\tilde{\boldsymbol{p}}), \tilde{B}(\tilde{\boldsymbol{q}})\bigr]$, whence, by the Leibnitz rule, we obtain

$$\begin{aligned}\bigl[\tilde{A}(\tilde{\boldsymbol{p}}),&\tilde{B}(\tilde{\boldsymbol{q}})\bigr] = \tilde{B}\bigl(\partial_{\tilde{A}(\tilde{\boldsymbol{p}})}(g)(\boldsymbol{q})\bigr) + \bigl(\tilde{B}\circ g\bigr)\bigl(\partial_{\tilde{A}(\tilde{\boldsymbol{p}})}(\boldsymbol{q})\bigr) - \tilde{A}\bigl(\partial_{\tilde{B}(\tilde{\boldsymbol{q}})}(g)(\boldsymbol{p})\bigr) + \bigl(\tilde{A}\circ g\bigr)\bigl(\partial_{\tilde{B}(\tilde{\boldsymbol{q}})}(\boldsymbol{p})\bigr)\\ &+ \bigl(A\circ g^{-1}\bigr)\bigl(\Gamma^{\tilde{A}}_{\tilde{A}\tilde{B}}(g\boldsymbol{p}, g\boldsymbol{q})\bigr) + \bigl(B\circ g^{-1}\bigr)\bigl(\Gamma^{\tilde{B}}_{\tilde{A}\tilde{B}}(g\boldsymbol{p}, g\boldsymbol{q})\bigr) + \sum_{\tilde{k}=3}^{N}\bigl(A_{\tilde{k}}\circ g^{-1}\bigr)\bigl(\Gamma^{\tilde{k}}_{\tilde{A}\tilde{B}}(g\boldsymbol{p}, g\boldsymbol{q})\bigr).\end{aligned}$$

Therefore,

$$\begin{aligned}\Gamma^A_{AB}(\boldsymbol{p},\boldsymbol{q}) &= \bigl(g^{-1}\circ\Gamma^{\tilde{A}}_{\tilde{A}\tilde{B}}\bigr)(g\boldsymbol{p}, g\boldsymbol{q}) - \bigl(g^{-1}\circ\partial_{B(\boldsymbol{q})}(g)\bigr)(\boldsymbol{p}),\\ \Gamma^B_{AB}(\boldsymbol{p},\boldsymbol{q}) &= \bigl(g^{-1}\circ\Gamma^{\tilde{B}}_{\tilde{A}\tilde{B}}\bigr)(g\boldsymbol{p}, g\boldsymbol{q}) + \bigl(g^{-1}\circ\partial_{A(\boldsymbol{p})}(g)\bigr)(\boldsymbol{q}),\\ \Gamma^k_{AB}(\boldsymbol{p},\boldsymbol{q}) &= \bigl(g^{-1}\circ\Gamma^{k}_{\tilde{A}\tilde{B}}\bigr)(g\boldsymbol{p}, g\boldsymbol{q}) \qquad \text{for } k\geq 3.\end{aligned}$$

Acting by g on these equalities and expressing $\boldsymbol{p} = g^{-1}\tilde{\boldsymbol{p}}$, $\boldsymbol{q} = g^{-1}\tilde{\boldsymbol{q}}$, we conclude the proof. □

Remark 4. Within the Hamiltonian formalism, it is very productive to postulate that the arguments of Hamiltonian operators, the cosymmetries, are *odd*[8], see [9]. Indeed, in this particular situation they can be conveniently identified with Cartan 1-forms times the pull-back of the volume form $\mathrm{d}\,\mathrm{vol}(B^n)$ for the base of the jet bundle.

We preserve this $\mathbb{Z}$-grading for domains $\Omega(\xi_\pi)$ of operators (9b) of second kind. Hence if π and ξ are super-bundles with Grassmann-valued sections, then the operators A are bi-graded. Their proper new $\mathbb{Z}$-grading is $|A|_{\mathbb{Z}} = -1$ because the images in $\mathfrak{g}(\pi)$ have degree zero; the $\mathbb{Z}_2$-degree, if any, can be arbitrary for the operators A.

Corollary 5. For any $i, j, k \in [1, \ldots, N]$ and for arguments $\boldsymbol{p}, \boldsymbol{q} \in \Omega(\xi_\pi)$ of $\mathbb{Z}$-degree 1 for strong compatible operators of second kind, we have that

$$\Gamma^k_{ij}(\boldsymbol{p},\boldsymbol{q}) = -\Gamma^k_{ji}(\boldsymbol{q},\boldsymbol{p}) = (-1)^{|\boldsymbol{p}|_{\mathbb{Z}}\cdot|\boldsymbol{q}|_{\mathbb{Z}}}\cdot\Gamma^k_{ji}(\boldsymbol{q},\boldsymbol{p}) \tag{24}$$

due to the skew-symmetry of the commutators in (7). Hence the symbols Γ^k_{ij} are symmetric w.r.t. the $\mathbb{Z}$-grading in this case.

Proposition 6. If, additionally, two strong compatible operators A_i and A_j are linear compatible, then their 'individual' brackets Γ^i_{ii} and Γ^j_{jj} are

$$\{\{\boldsymbol{p},\boldsymbol{q}\}\}_{A_i} = \Gamma^j_{ij}(\boldsymbol{p},\boldsymbol{q}) + \Gamma^j_{ji}(\boldsymbol{p},\boldsymbol{q}) \quad\text{and}\quad \{\{\boldsymbol{p},\boldsymbol{q}\}\}_{A_j} = \Gamma^i_{ij}(\boldsymbol{p},\boldsymbol{q}) + \Gamma^i_{ji}(\boldsymbol{p},\boldsymbol{q})$$

for any $\boldsymbol{p}, \boldsymbol{q} \in \Omega(\xi_\pi)$.

Proof. For brevity, denote $A = A_i$, $B = A_j$ and consider the linear combination $\mu A + \nu B$, which satisfies (5). By Theorem 2, we have

$$\begin{aligned}\bigl(\mu A + \nu B\bigr)&\bigl(\{\{\boldsymbol{p},\boldsymbol{q}\}\}_{\mu A+\nu B}\bigr) =\\ &= \mu^2 A\bigl(\{\{\boldsymbol{p},\boldsymbol{q}\}\}_A\bigr) + \mu\nu\cdot A\bigl(\{\{\boldsymbol{p},\boldsymbol{q}\}\}_B\bigr) + \mu\nu\cdot B\bigl(\{\{\boldsymbol{p},\boldsymbol{q}\}\}_A\bigr) + \nu^2 B\bigl(\{\{\boldsymbol{p},\boldsymbol{q}\}\}_A\bigr).\end{aligned}$$

[8] Here we assume for simplicity that all fibre coordinates in π and ξ are permutable.

On the other hand,

$$\begin{aligned}&\Big[\big(\mu A+\nu B\big)(\boldsymbol{p}),\big(\mu A+\nu B\big)(\boldsymbol{q})\Big]\\&\qquad=\mu^2\big[A(\boldsymbol{p}),A(\boldsymbol{q})\big]+\mu\nu\big[A(\boldsymbol{p}),B(\boldsymbol{q})\big]-\mu\nu\big[A(\boldsymbol{q}),B(\boldsymbol{p})\big]+\nu^2\big[B(\boldsymbol{p}),B(\boldsymbol{q})\big].\end{aligned}$$

Taking into account (21) and equating the coefficients of $\mu\nu$, we obtain

$$A\big(\{\{\boldsymbol{p},\boldsymbol{q}\}\}_B\big)+B\big(\{\{\boldsymbol{p},\boldsymbol{q}\}\}_A\big)=A\big(\Gamma^A_{AB}(\boldsymbol{p},\boldsymbol{q})\big)+B\big(\Gamma^B_{AB}(\boldsymbol{p},\boldsymbol{q})\big)-A\big(\Gamma^A_{AB}(\boldsymbol{q},\boldsymbol{p})\big)-B\big(\Gamma^B_{AB}(\boldsymbol{q},\boldsymbol{p})\big).$$

Using the formulas $\Gamma^A_{AB}(\boldsymbol{q},\boldsymbol{p})=-\Gamma^A_{BA}(\boldsymbol{p},\boldsymbol{q})$ and $\Gamma^B_{AB}(\boldsymbol{q},\boldsymbol{p})=-\Gamma^B_{BA}(\boldsymbol{p},\boldsymbol{q})$, see (24), we isolate the arguments of the operators and obtain the assertion. □

3. Flat connections in the triads $\Omega(\xi_\pi)\xrightarrow{A}\mathfrak{g}(\pi)$

In this section we consider *linear compatible* operators, which span the linear space $\mathcal{A}=\bigoplus_{k=1}^N A_k\cdot\mathbb{R}$ of operators that satisfy (5) at each point $A\in\mathcal{A}$.

Let $A\colon\mathfrak{k}\to\varkappa(\pi)$ be such an operator. It provides the homomorphism of Lie algebras

$$A\colon\big(\Omega(\xi_\pi),[\,,\,]_A\big)\to\big(\mathfrak{g}(\pi),[\,,\,]\big). \tag{25}$$

Let P be a $\mathfrak{g}(\pi)$-module; for example, $P=\mathcal{F}(\pi)$ or any other horizontal $\mathcal{F}(\pi)$-module (including $\mathfrak{g}(\pi)$ itself). By the homomorphism A, the $\mathfrak{g}(\pi)$-module P is an $\Omega(\xi_\pi)$-module as well.

We recall that, due to the Jacobi identity, the adjoint action by an element of the Lie algebra $\Omega(\xi_\pi)$ is a derivation. We bear in mind that the inclusion $\mathrm{Der}_{\mathrm{In}}\big(\Omega(\xi_\pi),\mathfrak{g}(\pi)\big)\subseteq\mathrm{Der}\big(\Omega(\xi_\pi),\mathfrak{g}(\pi)\big)$ *is* strict whenever the constructions are defined on the empty jet space. Indeed, if $\operatorname{im}A\neq\mathfrak{g}(\pi)$, then the $\mathfrak{g}(\pi)$-valued derivation $\big[\varphi_0,A(\cdot)\big]$ does not belong to $\mathrm{Der}_{\mathrm{In}}\big(\Omega(\xi_\pi),\mathfrak{g}(\pi)\big)$ for any $\varphi_o\notin\operatorname{im}A$. Besides, we assume that $[\psi_1,\cdot]_A=[\psi_2,\cdot]_A$ implies $\psi_1=\psi_2$ in $\Omega(\xi_\pi)$. Both requirements are fulfilled if the image of A in $\mathfrak{g}(\pi)$ is semi-simple and Whitehead's lemma holds for it.

Now we define a *connection* ∇^A in the triad (25),

$$\nabla^A\colon\ \mathrm{Der}_{\mathrm{In}}\big(\Omega(\xi_\pi),\mathfrak{g}(\pi)\big)\to\mathrm{Der}\big(\mathfrak{g}(\pi),P\big).$$

This connection lifts *inner* $\mathfrak{g}(\pi)$-valued derivations of $\Omega(\xi_\pi)$ to P-valued derivations of $\mathfrak{g}(\pi)$. We set

$$\nabla^A\colon A\circ[\psi,\cdot]_A\mapsto[A(\psi),\cdot]. \tag{26}$$

The above definition is $\Omega(\xi_\pi)$-linear. Indeed, for a derivation $\Delta=[\psi,\cdot]_A$ we have that

$$\nabla^A_{f\times\Delta}=A(f)\times\nabla^A_\Delta,\qquad f\in\Omega(\xi_\pi),\quad\Delta\in\mathrm{Der}_{\mathrm{In}}\big(\Omega(\xi_\pi)\big), \tag{27}$$

where the multiplication $\times$ by f and by its image under A is the adjoint action.[9]

Remark 5. Of course, the connection (26) in the triads (25) is not the Cartan connection on $J^\infty(\pi)$. Indeed, they are defined in entirely different geometric setups. This is also readily seen from the fact that the evolutionary fields on $J^\infty(\pi)$ are π-vertical and are projected to zero vector fields on B^n under $\pi_{\infty,*}$. (Everything is projected to *the* point x_0 if the jet bundle amounts to the finite-dimensional manifold F^m and $\pi\colon F^m\to\{x_0\}$.)

[9]Note that the right-hand side of the analogue of (27) in a classical definition of the $C^\infty(B^n)$-linearity of connections in the fibre bundles $\pi\colon E^{m+n}\xrightarrow[F^m]{}B^n$ does contain the image $\imath(f)$ of the identical embedding $\imath\colon C^\infty(B^n)\hookrightarrow C^\infty(E^{m+n})$ and not f itself.

Theorem 7. *The connection* (26) *is flat*:

$$\Bigl(\nabla^A_{\boldsymbol{p}}\circ\nabla^A_{\boldsymbol{q}}-\nabla^A_{\boldsymbol{q}}\circ\nabla^A_{\boldsymbol{p}}-\nabla^A_{[\boldsymbol{p},\boldsymbol{q}]_A}\Bigr)(\varphi)=0,\quad \forall\boldsymbol{p},\boldsymbol{q}\in\Omega(\xi_\pi),\quad \varphi\in\mathfrak{g}(\pi). \tag{28}$$

Proof. The Jacobi identity for the bracket of evolutionary vector fields,

$$\bigl[A(\boldsymbol{p}),\bigl[A(\boldsymbol{q}),\varphi\bigr]\bigr]+\bigl[A(\boldsymbol{q}),\bigl[\varphi,A(\boldsymbol{p})\bigr]\bigr]+\bigl[\varphi,\bigl[A(\boldsymbol{p}),A(\boldsymbol{q})\bigr]\bigr]=0,$$

is the flatness condition (28). $\square$

Corollary 8. The bi-differential symbols Γ^k_{ij} determine symmetric flat connections $\nabla^{\boldsymbol{\lambda}}=\sum_k\lambda_k\nabla^{A_k}$ in the graded triads $\Omega(\xi_\pi)\xrightarrow{A_{\boldsymbol{\lambda}}}\mathfrak{g}(\pi)$ given by the operators $A_{\boldsymbol{\lambda}}=\sum_{i=1}^N\lambda_i\cdot A_i\in\mathcal{A}$ of second kind.

Remark 6. If the flows of a commutative hierarchy $\mathfrak{A}$ belong to the image of an operator $A\in\mathcal{A}$, then the hierarchy is a *geodesic* w.r.t. the connection (26). Indeed, for any curve $\psi(\tau)\colon\mathbb{R}\to\Omega(\mathfrak{A})$ located in the inverse image of $\mathfrak{A}$ under A, the covariant derivative $\nabla^A_{\psi(\tau)}A(\psi'(\tau))$ of the velocity $\psi'(\tau)$ vanishes along the curve.

Remark 7. The operators (25) induce the homomorphism $\Omega(\xi_\pi)\to\bigwedge^\bullet\mathfrak{g}(\pi)$ to the Schouten algebra of evolutionary polyvector fields, which is endowed with the Schouten bracket $[\![\,,\,]\!]$. The flat connection (26) in the triad (25) is naturally extended to the connection in $\Omega(\xi_\pi)\xrightarrow{A}\bigwedge^\bullet\mathfrak{g}(\pi)$, which remains flat in the graded sense. In particular, we thus obtain the connections in the triads $\Omega\xrightarrow{\mathcal{P}}\mathfrak{g}$ composed by the Poisson bi-vectors $\mathcal{P}\in\Gamma\bigl(\bigwedge^2TF\bigr)$, the Schouten algebra $\mathfrak{g}=\bigl(\bigwedge^\bullet TF,[\,,\,]\bigr)$, and the Gerstenhaber algebra $\Omega=\bigl(\bigwedge^\bullet T^*F,[\,,\,]_{\mathcal{P}}\bigr)$.

Twice in this paper, we imposed the requirements of vanishing for the zeroth and first Chevalley cohomology with values in $\mathfrak{g}(\pi)$ for the Lie subalgebra $\sum_{i=1}^N\operatorname{im}A_i\subseteq\mathfrak{g}(\pi)$ of the Lie algebra of evolutionary vector fields. Consider the operator (25) that makes $\Omega(\xi_\pi)$ the Lie algebra (isomorphic to its image under A). Using Cartan's formula, we associate the differential complex on the Chevalley cohomology $\operatorname{Hom}_{\mathbb{R}}\bigl(\bigwedge^k\Omega(\xi_\pi),\Omega(\xi_\pi)\bigr)$ with values in the Lie algebra itself. (Likewise, the values could be in the $\Omega(\xi_\pi)$-module $\mathfrak{g}(\pi)$, or the entire construction repeated for the Lie algebra $\operatorname{im}A$.) For any $k\geq0$ and $\omega_k\in\operatorname{Hom}_{\mathbb{R}}\bigl(\bigwedge^k\Omega(\xi_\pi),\Omega(\xi_\pi)\bigr)$, the differential $\mathbf{d}_A\colon\omega_k\mapsto\omega_{k+1}$ is defined by

$$\begin{aligned}\mathbf{d}_A\omega_k\bigl(\psi_0,\ldots,\psi_k\bigr)&=\sum_i(-1)^i\bigl[\psi_i,\omega_k\bigl(\psi_0,\ldots,\widehat{\psi_i},\ldots,\psi_k\bigr)\bigr]_A\\&\qquad+\sum_{i<j}(-1)^{i+j-1}\omega_k\bigl([\psi_i,\psi_j]_A,\psi_0,\ldots,\widehat{\psi_i},\ldots,\widehat{\psi_j},\ldots,\psi_k\bigr).\end{aligned}$$

Hence we obtain the analogue of the Gerstenhaber complex, see (3–4),

$$\begin{aligned}\Omega(\xi_\pi)\xrightarrow{\text{const}}\operatorname{Hom}_{\mathbb{R}}\bigl(\Omega(\xi_\pi),\Omega(\xi_\pi)\bigr)\longrightarrow\operatorname{Hom}_{\mathbb{R}}\Bigl(\bigwedge^2\Omega(\xi_\pi),\Omega(\xi_\pi)\Bigr)&\\\longrightarrow\operatorname{Hom}_{\mathbb{R}}\Bigl(\bigwedge^3\Omega(\xi_\pi),\Omega(\xi_\pi)\Bigr)\to\cdots.&\end{aligned}\tag{29}$$

The first inclusion in (29) consists of the commutations $[\psi_0,\cdot]_A$ with fixed elements $\psi_0\in\Omega(\xi_\pi)$, whence the zeroth cohomology $H^0\bigl(\Omega(\xi_\pi)\bigr)$ is described by the centre of $\Omega(\xi_\pi)$.

Likewise, the first cohomology group $H^1\bigl(\Omega(\xi_\pi)\bigr)$ is composed by the derivations of $\Omega(\xi_\pi)$ which are not inner.

Both requirements $H^0\bigl(\Omega(\xi_\pi)\bigr) = 0 = H^1\bigl(\Omega(\xi_\pi)\bigr)$, see above, are fulfilled for the data $\bigl(\xi, \pi, w[u], A\bigr)$ if the Lie algebra $\bigl(\Omega(\xi_\pi), [\,,\,]_A\bigr)$ is *semi-simple* and the Whitehead lemma holds for it (c.f. [7]). Several operators A can be admissible for given ξ, π, and w. With these input data, the A-dependent cohomology calculation problem for $\Omega\bigl(\xi_\pi\bigr)$ is open.

Discussion

Our concept confirms the well-established principle in mathematical physics: ordinary differential equations and related structures on a manifold F are converted to partial differential equations and differential operators, respectively, if F is realized as the *fibre*, but not the base in a bundle — that is, F becomes the target, but not the source space for the sections. Hence the practical approach to the (jet) bundles is to "widen the fibre" rather than "tower the base:" we 'blow up' the fibre points $u \in F$ along the base B^n in the bundle π. At the same time, it is generally impossible to extend the Christoffel symbols of a connection ∇ in the (co)tangent bundle over F to the bi-differential operators Γ^k_{ij}, preserving ∇ as the zero-order term.

We performed all the reasonings for local graded differential operators; all the structures were defined on the empty jet spaces. A rigorous extension of these objects to nonlocal operators on (noncommutative, upon maps to an associative unital algebra) differential equations is a separate problem for future research. In addition, the use of difference operators subject to (7) can be a fruitful idea *au début* for discretization of integrable systems with free functional parameters in the symmetries (e.g., Toda-like difference systems).

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