What does Lech's inequality for some deformation of a singularity imply for base changes?

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Abstract

Let (B_0, n_0) be a local singularity corresponding to a point of the Hilbert scheme (with respect to some formal embedding), which is Cohen-Macaulay. Then the Lech inequality $e_0(R) \leq e_0(S)$ for the versal deformation $(R, M) \longrightarrow (S, N)$ of $B_0^\wedge$ implies already the analogous inequality $e_0(A) \leq e_0(B)$ for every deformation $(A, m) \longrightarrow (B, n)$ of B_0.

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Introduction

In 1959 C. Lech [Le 59] stated the problem whether the multiplicities of local rings (A, m) and (B, n) being base, respectively total space, of a deformation $(A, m) \longrightarrow (B, n)$ of a local ring $B_0 = B/mB$ satisfy the inequality

$$e_0(A) \leq e_0(B). \tag{1}$$

Note that the only condition on such a homomorphism to be a deformation is its flatness.

A generalization of this is the analogous inequality

$$H_A^{d+i} \leq H_B^i \tag{2}$$

between sum transforms of the Hilbert series (d denotes the dimension of the fiber B_0). Here a sum transform is defined by

$$H_A^j := (1 - T)^{-j} \cdot H_A^0,$$

where H_A^0 is the usual Hilbert series

$$H_A^0 := \sum_{l=0}^{\infty} \dim_{A/m} m^l/m^{l+1} \cdot T^l.$$

The inequality between two formal power series $H = \sum_{l=0}^{\infty} H(l)\cdot T^l$ and $H' = \sum_{l=0}^{\infty} H'(l)\cdot T^l$ is always to be understood in its total sense, i.e. $H(l) \leq H'(l)$ for all l.

In 1970 H. Hironaka [Hi 70] asked whether inequality (2) is always true with $i = 1$, since that would simplify his proof of the existence of a resolution of singularities in characteristic zero [Hi 64].

Unfortunately, this paper does not deal with that problem, but only with the inequality (1) between the multiplicities. But also this inequality is established in very few cases, only. The most interesting result in that direction is due to Lech himself. It says that

$$H_A^1 \leq H_B^1$$

in the case, that the special fiber B_0 is a zero dimensional complete intersection [Le 64]. B. Herzog generalized this to the situation that B_0 corresponds to a regular point $[B_0]$ of the Hilbert scheme [He 90]. This includes all complete intersections and all singularities with embedding dimension less than 3 [Fo]. A further generalization, the inequality $e_0(A) \leq e_0(B)$ in the case that $[B_0]$ has regular reduction and is Cohen-Macaulay itself, is given in [J 92].

On the other hand, Larfeldt and Lech [LL] showed that the general problem (1) of Lech is equivalent to the following statement:

For every local ring A and every coheight one prime P in A the inequality

$$e_0(A_P) \leq e_0(A) \tag{3}$$

is true.

This one, its immediate corollaries and the analogous inequalities for Hilbert series are usually referred as Bennett's inequality. Note that these problems can not be easy, since they generalize, at least in the Hilbert series version

$$H_{A_P}^1 \leq H_A^0,$$

Serre's result [Se], that the localization of a regular local ring by a prime ideal is again regular. They are solved in the case A is excellent ([Be], [Si]).

We note, that there is also a completely different approach to the Lech-Hironaka problem. One can consider singularities with tangentially flat deformations only as in [He 91]. A generalization of that may be found in the doctoral thesis of the author [J 90].

In this paper we will follow the philosophy of [He 90] proving Lech's inequality, when B_0 corresponds to a mild singularity of the Hilbert scheme. Concretely, we require, that the base of the formal versal deformation of B_0 [Schl] is Cohen-Macaulay (and that the versal deformation fulfills Lech's inequality). To say the truth, we also give in Theorem 1.3 a more general condition, but that does not seem to be very easy.

We shall use the conventions and notations of commutative algebra as in [Ma]. Further all local rings are assumed to be Noetherian. An A-algebra is a homomorphism of the ring A into some ring, a homomorphism of A-algebras is a commutative triangle. k will always denote a fixed ground field. Note that we use "local k-algebra" for algebras $k \longrightarrow (A, m)$, where (A, m) is local and $k \longrightarrow A/m$ is an isomorphism. By a deformation of a local k-algebra B_0 we mean a flat local homomorphism of local k-algebras with special fiber B_0.

At some point we use the technical concept of tangential flatness. A local homomorphism $f : (A, m) \longrightarrow (B, n)$ of local rings such that the induced homomorphism $gr(A) \longrightarrow gr(B)$ of the associated graded rings

$$gr(A) := \bigoplus_{l=0}^{\infty} m^l/m^{l+1}$$

makes $gr(B)$ into a flat $gr(A)$ -module is called *tangentially flat*. The fundamental facts about tangential flatness may be found in [He 91].

At the end of the introduction the following principal remark: We consider only deformations $f : (A, m) \longrightarrow (B, n)$ of local k-algebras, where A and B are equicharacteristic and f is residually rational. Using Cohen's structure theory one could really generalize that, at least one can replace "residually rational" by "residually separable". We will omit the proof for that, since it does not seem to make sense to consider the abstract situation, when almost nothing is known in the "geometric case".

1 A condition on the versal deformation implying Lech's inequality

1.1 This is the fundamental statement of this paper and implies ewerything what follows. Note that it is a direct generalization of [He 90], Theorem 6.
Proposition. *Let the commutative diagram*

$$\begin{array}{ccc} (R, M) & \longrightarrow & (S, N) \\ \downarrow & & \downarrow \\ (A, m) & \longrightarrow & (B, n) \end{array}$$

of local rings and local homomorphisms be cartesian, i.e. $B \cong A \otimes_R S$, *and assume the following conditions to be fulfilled.*

1. R contains a field and the homomorphism $R \longrightarrow A$ is residually rational (i.e. induces an isomorphism of the residue fields).

2. The special fiber of $R \longrightarrow S$ has minimal dimension, i.e.

$$\dim S = \dim R + \dim S/MS$$

(e.g. $R \longrightarrow S$ is flat).

3. There exists a system of parameters $\{x_1, \dots, x_r\}$ of R such that

$$\ell(R/(x_1, \dots, x_r)) \leq e_0(S).$$

Then there is a natural number c (depending on S only) such that

$$H_A^{d+r+1}(n) \leq H_B^{r+1}(n + c)$$

for all n ($d := \dim B/mB$). In particular,

$$e_0(A) \leq e_0(B).$$

Proof. First we note that $B/mB = S \otimes_R A \otimes_A A/m = S \otimes_R R/M = S/MS$ and $\dim S/MS = d$.

Now we can even specify the constant c, for which we will prove the assertion above: *c is the minimal natural number such that*

$$H_S^1(n+c) \geq e_0(S) \cdot (1-T)^{-(d+r+1)}(n)$$

for all n.

The existence of such a c is the claim of Lemma 1.4 below. Note that $\dim S = d + r$.

After these preliminaries we start the essential part of the proof with two reduction steps.

First step. *We may assume A to be an Artin local ring.*

For proving

$$H_A^{d+r+1}(n) \leq H_B^{r+1}(n+c)$$

for an arbitrarily given n the local rings A and B can be replaced by A/m^{n+c+1} and $B/m^{n+c+1}B$, respectively.

Second step. *We may assume, that the local rings B, R and S are complete.*

Replace the local rings of the diagram above by their completions. Since $A \otimes N$ is an n-primary ideal in $B = A \otimes_R S$, the canonical topology of B is that as a finite S-module. Therefore

$$B^\wedge = (A \otimes_R S)^\wedge = A \otimes_{R^\wedge} S^\wedge.$$

Note that Cohen's structure theory ([Ma], Theorem 28.3 or [EGA IV$_0$], §19) implies now, that R contains its residue field $R/M =: k$.

Third step. *The key step. We will prove that $B = A \otimes_R S$ is a factor of $B' := A \otimes_k S$ in a very nice way.*

Note that B' is Noetherian as a finitely generated S-algebra. Let n' be a maximal ideal in B'. Then $(m) := m \otimes S \subseteq n'$, since m is nilpotent, and $n'/(m)$ is maximal in $B'/(m) = A \otimes_k S/m \otimes S \cong S$. Therefore,

$$n' = m \otimes S + A \otimes N,$$

which shows B' to be local. Further we have

$$\begin{aligned} B &\cong A \otimes_R S \\ &\cong A \otimes_k R/(x_1, \dots, x_r) \otimes_{R/(x_1,\dots,x_r)} R/M \otimes_R S \\ &\cong A \otimes_k R/(x_1, \dots, x_r) \otimes_R R/M \otimes_R S, \end{aligned}$$

where $\{x_1, \dots, x_r\}$ is the system of parameters of R, required in assumption 3. Using the commutativity of the tensor product, one obtains, denoting $R/(x_1, \dots, x_r)$ by $\overline{R}$

$$\begin{aligned} B &\cong A \otimes_k S \otimes_R \overline{R} \otimes_R \overline{R}/\overline{M} \\ &\cong A \otimes_k S/(x_1, \dots, x_r) \otimes_{\overline{R}} \overline{R}/\overline{M} \\ &\cong B'/(x_1, \dots, x_r) \otimes_{\overline{R}} \overline{R}/\overline{M}. \end{aligned}$$

Here a direct computation of the dimensions of B' and $B'/(x_1, \dots x_r)$ shows that $\{x_1, \dots x_r\}$ is a subset of a system of parameters for B'. However we will not need that.

Fourth step. *Now we are in the position to complete the proof comparing the Hilbert series of B' and its factors.*

Let

$$C_i := B'/(x_1, \dots, x_i).$$

Then by Lemma 1.5

$$H^0_{C_i} \leq H^1_{C_{i+1}}$$

and by Lemma 1.6

$$H^1_{C_r} = H^1_{B'/(x_1,\dots x_r)} \leq \ell(\overline{R}) \cdot H^1_B.$$

Altogether we find

$$H^0_{B'} \leq \ell(\overline{R}) \cdot H^r_B. \tag{11}$$

By completeness we may assume the local k-algebra A to be $k[[X]]/J$ for some finite set X of indeterminates and some ideal J in $k[[X]]$. The canonical imbedding $k \longrightarrow S$ is trivially tangentially flat, hence so is

$$k[[X]] \longrightarrow S[[X]], \qquad X_i \longmapsto X_i.$$

Therefore, by [He 91], Remark (1.4.i), the induced homomorphism

$$A = k[[X]]/J \longrightarrow S[[X]]/J \cdot S[[X]] \cong (k[[X]]/J) \otimes_k S = A \otimes_k S = B'$$

is tangentially flat, too. Note that $S[[X]]/J \cdot S[[X]]$ may be written as the tensor product above, since the ring A is Artin. Now the reformulation of tangential flatness in terms of Hilbert series (see [He 91], Theorem (1.2.ii.e)) implies

$$H^1_{B'} = H^0_A \cdot H^1_S.$$

Writing down that explicitly one sees

$$\begin{aligned}
H^1_{B'}(n+c) &= \sum_{j=0}^{n+c} H^0_A(j) \cdot H^1_S(n+c-j) \\
&\geq \sum_{j=0}^{n} H^0_A(j) \cdot H^1_S(n+c-j) \\
&\geq e_0(S) \cdot H^0_A \cdot (1-T)^{-(d+r+1)}\,(n) \\
&= e_0(S) \cdot H^{d+r+1}_A(n)
\end{aligned}$$

by the construction of the constant c. Comparing that with (11) we get

$$e_0(S) \cdot H^{d+r+1}_A(n) \leq \ell(\overline{R}) \cdot H^{r+1}_B(n+c).$$

But assumption 3 states just $\ell(\overline{R}) \leq e_0(S)$. This yields

$$H_A^{d+r+1}(n) \leq H_B^{r+1}(n+c)$$

for all n, being just the claim.

$\square$

1.2 Remark. The following Theorem is, in some sense, the main result of this paper. But the condition, we assume the formal versal deformation of a singularity to satisfy, does not seem to be very clear. In the next section we will show, that the Theorem can be applied to interesting cases, for instance to singularities corresponding to a point of the Hilbert scheme, which is Cohen-Macaulay.

1.3 Theorem. *Let (B_0, n_0) be a local k-algebra. We assume that the formal versal deformation*

$$(R, M) \longrightarrow (S, n)$$

of its completion $B_0^\wedge$ satisfies the following condition.

There exists a system of parameters $\{x_1, \ldots, x_r\}$ of R such that

$$\ell(R/(x_1, \ldots, x_r)) \leq e_0(S).$$

Then for every deformation

$$(A, m) \longrightarrow (B, n)$$

of the local k-algebra B_0 the Lech inequality

$$e_0(A) \leq e_0(B)$$

is true.

Proof. We will prove the following better assertion.

There exists a natural number c (depending on S only) such that

$$H_A^{d+r+1}(n) \leq H_B^{r+1}(n+c)$$

for all n. Here d denotes the dimension of B_0.

First step. *We may assume the local k-algebras B_0, A and B to be complete.*

Replace A and B by their completions. Then the induced homomorphism $A^\wedge \longrightarrow B^\wedge$ is again flat and its fiber is $B^\wedge/m^\wedge B^\wedge = B_0^\wedge$. Of course, there is no effect on the Hilbert series.

Second step. *$A \longrightarrow B$ is a base change of the formal versal deformation of B_0.*

Schlessinger [Schl] calls the morphism $h_R \longrightarrow D_{B_0/k}$, induced by $R \longrightarrow S$ the "pro-representable hull" of the deformation functor

$$D_{B_0/k} : \{\text{Artin local } k\text{-algebras (with residue field } k)\} \longrightarrow \{\text{Sets}\}$$

of B_0. By [Schl], Remark (2.4), the induced morphism

$$h_R^\wedge = \mathrm{Hom}_{\mathrm{local}\, k-\mathrm{alg}}(R, \,.\,) \longrightarrow D_{B_0/k}^\wedge$$

between the canonical prolongations to

$$\{\text{complete (Noetherian) local } k\text{-algebras (with residue field } k)\}$$

is objectwise surjective.

Down the earth this means nothing but the existence of a cartesian diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ A & \longrightarrow & B. \end{array}$$

So the claim comes from the Proposition above.

□

1.4 Lemma. *Let S be a local ring of dimension s. Then there exists a natural number c such that*

$$H_S^1(n+c) \geq e_0(S) \cdot (1-T)^{-(s+1)}\ (n).$$

for all n.

Proof. The term on the right hand side is simply $e_0(S) \cdot \binom{n+s}{s}$, i.e. a polynomial of degree s with leading coefficient $\frac{1}{s!} \cdot e_0(S)$. For $n \gg 0$ we know that $H_S^1(n)$ is a polynomial of the same type. So their difference is of degree $s-1$ and its leading coefficient can be majorized by that of $H_S^1(n+c) - H_S^1(n)$ for c large enough. So we get the inequality for all n, except finitely many.

If $s \geq 1$, then $H_S^1(n)$ tends, monotonically increasing, to infinity. For $c \gg 0$ the inequality becomes true also for the exceptional set above.

If $s = 0$, then S is Artin and $H_S^1(n) = e_0(S) \cdot (1-T)^{-1}\ (n)$ for all $n \gg 0$.

□

1.5 Lemma. *Let (A, m) be a local ring and $x \in m$ be an element. Then*

$$H_A^0 \leq H_{A/xA}^1.$$

Proof. This statement is elementary and easily proved by the reader. Alternatively, see [Si], Theorem 1.

□

1.6 Lemma. *Let $f : (A, m) \longrightarrow (B, n)$ be a local homomorphism of local rings and assume A to be Artin. Then*

$$H_B^1 \leq \ell(A) \cdot H_{B \otimes_A A/m}^1.$$

Proof. Note that this is an easy special case of Theorem (1.2.i) of [He 91]. To give a direct proof, it will be sufficient to show $\ell(M) \leq \ell(A) \cdot \ell(M \otimes_A A/m)$ for every A-module M. But this is clear, since $\ell(M \otimes_A A/m) = \mu_A(M)$ is the minimal number of generators of M.

□

2 Applications

2.1 Remark. We will apply the results of the first section to three special situations, each one is contained in the next. Concretely we consider the cases that the base of the formal versal deformation of some singularity is

(a) regular,

(b) permissible in the sense of [J 92] (e.g. Cohen-Macaulay with regular reduction) and

(c) Cohen-Macaulay.

Note that the applications (a) and (b) give results being already known from [He 90] and [J 92].

2.2 Corollary (Application (a)). *Let (B_0, n_0) be a local k-algebra and assume the base R of the formal versal deformation $R \longrightarrow S$ of $B_0^\wedge$ to be regular. Then for every deformation $(A, m) \longrightarrow (B, n)$ of the local k-algebra B_0 the Lech inequality*

$$e_0(A) \leq e_0(B)$$

is true.

Proof. Choose $\{x_1, \ldots, x_r\}$ to be a regular system of parameters for R. Then

$$1 = \ell(R/(x_1, \ldots x_r)) \leq e_0(S).$$

and Theorem 1.3 above gives the assertion.

□

2.3 Remark. This is a result due to B. Herzog [He 90]. He can even show the inequality

$$H_A^{d+1} \leq H_B^1,$$

where d denotes the dimension of B_0. Note however that our proof is substancially easier, since we did not use Bennett's inequality here.

2.4 Corollary (Application (b)). *Let (B_0, n_0) be a local k-algebra and assume the base R of the formal versal deformation $R \longrightarrow S$ of $B_0^\wedge$ to be permissible, i.e. there exists a system of parameters $\{x_1, \ldots, x_r\}$ for R such that*

$$\ell(R/(x_1, \ldots, x_r)) \leq \ell(R_p)$$

for all minimal primes p in R. Then for every deformation $(A, m) \longrightarrow (B, n)$ of the local k-algebra B_0 the Lech inequality

$$e_0(A) \leq e_0(B)$$

is true.

Proof. Let P be a minimal prime of S. Then $R_p \longrightarrow S_P$ with $p := P \cap R$ is flat and local, therefore $\ell(R_p) \leq \ell(S_P)$. Further, S is complete, such that we have Bennett's inequality (cf. the introduction to the present paper)

$$\ell(S_P) = e_0(S_P) \leq e_0(S).$$

Using the assumption one obtains

$$\ell(R/(x_1, \ldots, x_r)) \leq e_0(S)$$

and Theorem 1.3 can be applied.

□

2.5 Remark. This is the result of [J 92].

2.6 Proposition. (Application (c)) *Assume the base field k to be infinite and let the commutative diagram*

$$\begin{array}{ccc} (R, M) & \longrightarrow & (S, N) \\ \downarrow & & \downarrow \\ (A, m) & \longrightarrow & (B, n) \end{array}$$

of local k-algebras and local homomorphisms be cartesian, i.e. $B \cong A \otimes_R S$, *where*

$$(R, M) \longrightarrow (S, N)$$

is flat.

Then, if R is Cohen-Macaulay, the inequality $e_0(R) \leq e_0(S)$ *implies already that there exists a natural number c such that*

$$H_A^{d+r+1}(n) \leq H_B^{r+1}(n + c)$$

for all n ($d := \dim B/mB$, $r := \dim R$). *In particular,*

$$e_0(R) \leq e_0(S).$$

Remark. The same result is true over a finite base field, but the author can prove that only by reduction to the infinite case. Since that is completely another idea, we will do it separately (Proposition 2.8).

Proof of the Proposition. To apply Proposition 1.1 we have to show that there exists a system of parameters $\{x_1, \ldots, x_r\}$ for R such that

$$\ell(R/(x_1, \ldots, x_r)) \leq e_0(R).$$

This is a part of the following Fact.

□

2.7 Fact. (a) *Let* (R, M) *be a local ring. Then for every system of parameters* $\{x_1, \ldots, x_r\}$ *of R*

$$\ell(R/(x_1, \ldots, x_r)) \geq e_0(R).$$

(b) *Assume the field R/M to be infinite. Then the following statements are equivalent:*

(i) *There exists a system of parameters* $\{x_1, \ldots, x_r\}$ *such that*

$$\ell(R/(x_1, \ldots, x_r)) = e_0(R).$$

(ii) *R is Cohen-Macaulay.*

Proof. (a) We have $\ell(R/(x_1, \dots, x_r)) \geq e((x_1, \dots, x_r))$ by [Ma], Theorem 14.10 and $e((x_1, \dots, x_r)) \geq e(M) = e_0(R)$ by [Ma], Formula 14.4.

(b) "(i) $\Longrightarrow$ (ii)" By (a) we get necessarily $\ell(R/(x_1, \dots, x_r)) = e((x_1, \dots, x_r))$, which implies that R is Cohen-Macaulay by [Ma], Theorem 17.11.

"(ii) $\Longrightarrow$ (i)" [Ma], Theorem 14.14 shows that there exists a system of parameters $\{x_1, \dots, x_r\}$ for R such that $e((x_1, \dots, x_r)) = e(M) = e_0(R)$. (Recall R/M is infinite!) On the other hand $\ell(R/(x_1, \dots, x_r)) = e((x_1, \dots, x_r))$ by [Ma], Theorem 17.11.

□

2.8 Now we release the restriction to the base field to be infinite.

Proposition. *Let the commutative diagram*

$$\begin{array}{ccc} (R,M) & \longrightarrow & (S,N) \\ \downarrow & & \downarrow \\ (A,m) & \longrightarrow & (B,n) \end{array}$$

of local k-algebras and local homomorphisms be cartesian, i.e. $B \cong A \otimes_R S$, and assume

$$(R,M) \longrightarrow (S,N)$$

to be flat.

Then, if R is Cohen-Macaulay, the inequality $e_0(R) \leq e_0(S)$ implies already that there exists a natural number c such that

$$H_A^{d+r+1}(n) \leq H_B^{r+1}(n+c)$$

for all n ($d := \dim B/mB$, $r := \dim R$). In particular,

$$e_0(R) \leq e_0(S).$$

Proof. In the case k is infinite this is Proposition 2.6. So assume k to be finite.

First note that we may assume A to be Artin. Take an infinite field extension K/k. Then

$$\begin{array}{ccc} R \otimes_k K & \longrightarrow & S \otimes_k K \\ \downarrow & & \downarrow \\ A \otimes_k K & \longrightarrow & B \otimes_k K \end{array}$$

is again cartesian, since $B \otimes_k K = A \otimes_R (R \otimes_k K) \otimes_{R \otimes_k K} S \otimes_k K = A \otimes_k K \otimes_{R \otimes_k K} S \otimes_k K$. Note that $A' := A \otimes_k K$ is local with maximal ideal $m \otimes_k K$ (since m is nilpotent) and Artin. We claim that the commutative diagram of local K-algebras and local homomorphisms

$$\begin{array}{ccccccc} R' & := & (R \otimes_k K)_{M \otimes K} & \longrightarrow & (S \otimes_k K)_{N \otimes K} & =: & S' \\ & & \downarrow & & \downarrow & & \\ A' & := & A \otimes_k K & \longrightarrow & (B \otimes_k K)_{n \otimes K} & =: & B' \end{array}$$

is also cartesian.

Indeed,

$$A' \otimes_{R'} S' = (A' \otimes_{R \otimes_k K} R') \otimes_{R'} S' = A \otimes_k K \otimes_{R \otimes_k K} (S \otimes_k K)_{N \otimes K} \longrightarrow (B \otimes_k K)_{n \otimes K} = B'$$

is simply a localization map and, on the other hand, $A' \otimes_{R'} S'$ is already local (with maximal ideal $m' \otimes S' + A' \otimes N'$, when one notes that m' is nilpotent). This implies

$$A' \otimes_{R'} S' = (B \otimes_k K)_p$$

for some prime $p \supseteq n \otimes K$ and therefore even $A' \otimes_{R'} S' = (B \otimes_k K)_{n \otimes K}$, since the ideal $n \otimes K$ is maximal.

Now a general remark: For every local k-algebra (C, o) we have the equality $H^1_C = H^1_{(C \otimes_k K)_{o \otimes K}}$. In fact

$$\begin{aligned} H^1_{(C \otimes_k K)_{o \otimes K}}(n) &= \ell((C/o^{n+1} \otimes_k K)_{o \otimes K}) \\ &= \ell(C/o^{n+1} \otimes_k K) \\ &= \ell(C/o^{n+1}) \\ &= H^1_C(n), \end{aligned}$$

where we used that $C/o^{n+1} \otimes_k K$ is local from the same reason as $A \otimes_k K$ is.

Applying this to R and S we get $e_0(R') \leq e_0(S')$. Further, R' is, of course, Cohen-Macaulay, such that we can use Proposition 2.6 (Application (c)) above. One obtains $H^{d+r+1}_{A'}(n) \leq H^{r+1}_{B'}(n + c)$. So using our remark for A and B we find

$$H^{d+r+1}_A(n) \leq H^{r+1}_B(n + c),$$

being just what we had to prove.

□

2.9 Theorem. *Let (B_0, n_0) be a local k-algebra. We assume that the base R of the formal versal deformation*

$$(R, M) \longrightarrow (S, N)$$

of $B_0^\wedge$ is Cohen-Macaulay.

Then the inequality

$$e_0(R) \leq e_0(S)$$

for the versal deformation implies already the Lech inequality

$$e_0(A) \leq e_0(B)$$

for every deformation $(A, m) \longrightarrow (B, n)$ of the local k-algebra B_0.

Proof. Repeating the proof of Theorem 1.3 word for word we see, that it is sufficient to show the existence of a natural number c such that $H^{d+r+1}_A(n) \leq H^{r+1}_B(n + c)$ for all n, where d and r denote the dimensions of B_0 and R, respectively, that we may assume B_0, A and B to be complete and get a cartesian diagram

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ A & \longrightarrow & B. \end{array}$$

So Proposition 2.8 above gives the assertion.

□

References

[Ar] Artin, M.: *Lectures on deformations of singularities,* Tata Institute of Fundamental Research, Bombay 1976

[Be] Bennett, B. M.: *On the characteristic functions of a local ring,* Ann. Math. 91(1970)25-87

[Gr] Grothendieck, A.:*Techniques de construction et théorèmes d' existence en géométrie algébrique, IV: Les schémas de Hilbert,* Seminaire Bourbaki, Exp.195, 1959/60

[EGA] Grothendieck, A., Dieudonné, J.: *Eléments de géométrie algébrique,* Publ. Math. IHES, Paris 1960-68

[Fo] Fogarty, J.: *Algebraic families on an algebraic surface,* Amer. J. of Math. 90(1968)511-521

[He 90] Herzog, B.: *Lech-Hironaka inequalities for flat couples of local rings,* manuscripta math. 68(1990)351-371

[He 91] Herzog, B.: *Local singularities such that all deformations are tangentially flat,* Trans. Amer. Math. Soc. 324(1991)555-601

[Hi 64] Hironaka, H.: *Resolution of singularities of an algebraic variety over a field of characteristic zero,* Annals of Math. 79(1964)109-326

[Hi 70] Hironaka, H.: *Certain numerical characters of singularities,* J. Math. Kyoto Univ. 10-1(1970)151-187

[J 90] Jahnel, J.: *Tangentiale Flachheit und Lech- Hironaka- Ungleichungen,* Promotionsschrift (doctoral thesis, in German), Jena 1990

[J 92] Jahnel, J.: *The base of the formal versal deformation of a singularity and Lech inequalities,* Preprint, Bonn 1992

[LL] Larfeldt, T., Lech, C.: *Analytic ramifications and flat couples of local rings,* Acta Math. 146(1981)201-208

[Le 59] Lech, C.: *Note on multiplicities of ideals,* Ark. Math. 4(1959)63-86

[Le 64] Lech, C.: *Inequalities related to certain couples of local rings,* Acta Math. 112(1964)69-89

[Ma] Matsumura, H.: *Commutative ring theory,* Cambridge University Press 1986

[Schl] Schlessinger, M.: *Functors of Artin rings,* Trans. Amer. Math. Soc. 130(1968)208-222

[Se] Serre, J. P.: *Algèbre locale multiplicités,* Lecture Notes in Mathematics, Volume 11, Springer, Berlin 1965

[Si] Singh, B.: *Effect of a permissible blowing up on the local Hilbert Functions,* Invent. Math. 26(1974)201-212