Conjectural Arithmetic
Riemann - Roch - Hirzebruch - Grothendieck
Theorem

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In the seminar: Arithmetic Riemann-Roch-Hirzebruch at MPI für Math., Bonn, we attempted to understand the arithmetic Riemann-Roch-Hirzebruch-Grothendieck theorem, after Gillet and Soulé. During this seminar, I found some counterexamples to the arithmetic Riemann-Roch-Grothendieck theorem stated in [GS 89]. Hence to achieve a version of arithmetic Riemann-Roch-Hirzebruch-Grothendieck theorem, we still have to do additional work, based on the very important pioneer works of Arakelov, Faltings, Bismut, Gillet, Soulé, etc..

The first thing to do is to guess the precise correct formulation of the theorem. In this direction, Gillet and Soulé offered a first formulation [GS 91]. Unfortunately, they neglected a certain constant term in their calculation, and so their conjectural form is false. The main idea for us to guess the conjectural arithmetic theorem comes from the projective space $\mathbf{P}^n$, $\mathcal{O}_{\mathbf{P}^n}$ with Fubini-Study metric, in which case we find an additional term. Historically, this is the way Hirzebruch found the Todd genus and his Riemann-Roch theorem. Once we find the proper conjecture, we then need to find philosophical support for it. In algebraic geometry, we know that the additional term is more or less a purely topological invariant of the manifold in question [H 56]. But in Arakelov theory, we could not find such support.

In Bismut, Gillet and Soulé's work [BGS 90], a 'proof' of the theorem proceeds in two steps, just as in the classical case, namely for an immersion and for a projection. On the other hand, the proof of the theorem itself for an immersion depends strongly on a certain condition (about the metrics), which is usually called the Bismut assumption (A). Just as Bismut and Lebeau [BL 90] pointed out, for one point embedded into a curve, the metric in question is nothing but the Arakelov admissible metric; but for two points embedded into a curve, the metric in question is not the Arakelov one. For immersions, it is clear what metric is the right one, but in general, it is not quite clear. Assuming that the problem about such metrics has satisfactory answers, we still need to deal with projections and put those two parts together carefully.

This paper is an extension of my seminar talk with the title: The constant occurring in Deligne's Riemann-Roch theorem. The aim here is to offer a conjectural arithmetic Riemann-Roch-Hirzebruch-Grothendieck theorem, based on the work of Gillet and Soulé.

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Throughout this paper, we will use the notation as in Bismut, Gillet and Soulé's series of papers [BGS 88], [BGS 90], [GS 88], [GS 89], [GS 90], [GS 91] with certain modifications about sign and constant factors, such that when reducing the general theory to arithmetic surfaces, the final results in Bismut, Gillet and Soulé are normalized exactly the same as in Lang [L 88].

The first remark is about the Quillen metric. Usually, as in [GS 91], people take the Quillen metric to be

$$h_{\mathrm{L}^2} \exp(-\tau(\mathcal{E})^0).$$

But this definition is not satisfactory. For example, in [Bo 87], for the case of relative dimension one, one finds

$$\zeta_1(0) = \frac{(1-g)}{3} r(\mathcal{E}) + \frac{d(\mathcal{E})}{2} - h^0(\mathcal{E}).$$

Thus if we take any constant multiple of the Laplace-Beltrami operator, we find that the above Quillen metric with respect to the new operator is usually not smooth, as $h^0(\mathcal{E})$ may jump. For this reason, we will use the definition of **Quillen metric** in the paper by Freed [F 87], §1. The key point in the definition of Quillen metric used in [F 87] is that for $a > 0$, the function,

$$\zeta^{(a)}(s) = \sum_{\lambda > a} \frac{1}{\lambda^s},$$

as in (1.20) [F 87], is smooth with respect to the base, over which the relative morphism defined. We will denote the Quillen metric by h_{Q}.

Next we introduce a new notation as follows. Let $\mathcal{E}$ be a holomorphic bundle on a complex compact manifold V. There exists a unique element $A(\mathcal{E}) \in H^{\mathrm{ev}}(V)$ in the even complex cohomology of V, characterized by the following properties:

A1. For any morphism f, we have $A(f^*\mathcal{E}) = f^*A(\mathcal{E})$.

A2. For any exact sequence $0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \longrightarrow 0$ of vector sheaves on V, we have

$$A(\mathcal{E}) = A(\mathcal{E}_1) + A(\mathcal{E}_2).$$

A3. When $\mathcal{L}$ is a line sheaf on V with $x = c_1(\mathcal{L}) \in H^2(V)$, then

$$A(\mathcal{L}) = \sum_{m \text{ odd}, m \geq 1} \left[2\zeta'(-m) + \zeta(-m)\left(1 + \frac{1}{2} + \ldots + \frac{1}{m}\right) \right] \frac{x^m}{m!}$$

$$+ \left(1 - \frac{x}{2}\right) \log 2 \sum_{m \text{ odd}, m \geq 1} \zeta(-m) \frac{x^m}{m!} - \frac{x}{4} \log 2$$

where $\zeta(s)$ is the Riemann zeta function.

Remark 1. The difference between $A(\mathcal{L})$ and the element $R(\mathcal{L})$ in [GS 91] is that we add the log 2 terms. More precisely, in [GS 91], Gillet and Soulé introduce the unique element $R(\mathcal{E})$ in $H^{\text{ev}}(V)$, such that $R(\mathcal{E})$ satisfies **A1**, **A2**; and for a line bundle $\mathcal{L}$, $R(\mathcal{L})$ is the first sum above, namely

$$R(\mathcal{L}) = \sum_{m \text{ odd}, m \geq 1} \left[2\zeta'(-m) + \zeta(-m)(1 + \frac{1}{2} + \ldots + \frac{1}{m}) \right] \frac{x^m}{m!}.$$

Now we define the **arithmetic Todd genus** of a Hermitian vector sheaf $(\mathcal{E}, h)$ on an arithmetic variety X by the formula,

$$\mathrm{Td}_{\mathrm{Ar}}(\mathcal{E}, h) = \mathrm{td}_{\mathrm{Ar}}(\mathcal{E}, h)(1 - a(A(\mathcal{E}_\infty)),$$

where $\mathrm{td}_{\mathrm{Ar}}(\mathcal{E}, h)$ is the arithmetic Todd characteristic class defined in [GS 90], and the map a sends forms to elements of the arithmetic Chow group, that is for any complex conjugate invariant form α , $a(\alpha) = (0, \alpha)$ in the arithmetic Chow-ring as in [GS 88].

Remark 2: We use $A(\mathcal{E}_\infty)$ instead of $R(\mathcal{E}_\infty)$ in the definition of the arithmetic Todd genus as in [GS 91]. The reason is that Gillet and Soulé quote the result of Ikeda and Taniguchi [IT 13] about the spectrum of the $\mathbf{P}^n(\mathbf{C})$. This result is for the real Laplace operator. But Gillet and Soulé use the Laplace-Beltrami operator, which is one half of the real Laplace operator.

With this, we may state our conjecture as

Conjectural Arithmetic Riemann-Roch-Hirzebruch-Grothendieck Theorem. Let $f : X \longrightarrow Y$ be a projective smooth morphism between arithmetic varieties. Assume that there is a Hermitian metric $h_{X/Y}$ on the relative tangent sheaf $\mathcal{T}_{X/Y}$ of f, such that this metric induces a Kahler metric on the fibers at infinity. Let α be an element of the arithmetic K-group $K_0(X)_{\mathrm{Ar}}$. Then in $\mathrm{CH}_{\mathrm{Ar}}(Y) \otimes \mathbf{Q}$,

$$ch_{\mathrm{Ar}}(f_K(\alpha)) = f_{\mathrm{CH}}(ch_{\mathrm{Ar}}(\alpha)\mathrm{Td}_{\mathrm{Ar}}(\mathcal{T}_{X/Y}, h_{X/Y})),$$

here f_K (resp.f_{CH}) denotes the push-out of the arithmetic K-group (resp. the arithmetic Chow-ring).

For simplicity, we now assume that f is smooth and we look at the difference of both side of the above conjecture for Hermitian vector sheaf $(\mathcal{E}, h)$ in the level one of the arithmetic Chow ring $\mathrm{CH}_{\mathrm{Ar}}(Y) \otimes \mathbf{Q}$. So let

$$\delta(\mathcal{E}) = c_1(\lambda(\mathcal{E}), h_Q)_{\mathrm{Ar}} - f_{\mathrm{CH}}(ch_{\mathrm{Ar}}(\mathcal{E}, h)\mathrm{Td}_{\mathrm{Ar}}(\mathcal{T}_{X/Y}, h_{X/Y}))^{(1)}.$$

Then with the same proof as Theorem 1.4 [GS 91], we have the following

Theorem 1. With the same assumption as in our conjecture, we have

1. The difference $\delta(\mathcal{E})$ is independent of h and $h_{X/Y}$, and

$$\delta(\mathcal{E}) \in a(H(Y)) \subset \mathrm{CH}_{\mathrm{Ar}}(Y) \otimes \mathbf{Q}.$$

2. For any short exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0,$$

we have

$$\delta(\mathcal{E}) = \delta(\mathcal{E}_1) + \delta(\mathcal{E}_2).$$

3. Let $\mathcal{E}'$ be any vector sheaf on Y. Then

$$\delta(\mathcal{E} \otimes f^*\mathcal{E}') = r(\mathcal{E}')\delta(\mathcal{E}).$$

4. Let $d = 1$, and assume there exists a real imbedding for the ground field. Then

$$\delta(\mathcal{E}) = c(g)r(\mathcal{E}),$$

where $c(g) \in \mathbf{R}$ is a constant which depends only on the genus of the fibres.

Remark 1.1. The constant $c(g)$ is more or less related with the value of the constant $a(g)$ occurring in Deligne's Riemann-Roch theorem for semistable arithmetic surfaces [De87], [So 89]. When I tried to use Gillet and Soulé's result to compute the value of $a(g)$, I found that the value for a(g) in [So 89] is not correct, and neither is the arithmetic Riemann-Roch-Grothendieck Theorem stated in [GS 89]. And then, after checking step by step, I found that there was a problem about the constant factor 2 in the Laplace operator used in [GS 91]. By the way, now we know that

$$a(0) = 48\zeta'(-1) - 2 - 8 \log 2,$$

and the value for $a(0)$ in [So 89] is not correct. If our conjecture is true, then, for all g, we have

$$a(g) = (1-g)a(0).$$

Remark 1.2. Note that nevertheless Theorem 1 and Theorem 1.4 of [GS 91] are both true, and so do not provide sufficient evidence for the conjecture.

We state two immediate consequences of the above conjecture.

Corollary 1. Conjectural Arithmetic Riemann-Roch-Hirzebruch Theorem for Relative Dimension 1. Let $f : X \to Y$ be a projective, smooth morphism of two arithmetic varieties of relative dimension 1. Suppose that the relative tangent sheaf $\mathcal{T}_{X/Y}$ is equipped with a Hermitian metric, which induces a Kähler metric on the fibre at infinity. Then for any Hermitian vector sheaf $(\mathcal{E}, h)$ on X,

$$c_1(\lambda(\mathcal{E}), h_Q)_{\mathrm{Ar}} = f_{\mathrm{CH}}(ch_{\mathrm{Ar}}(\mathcal{E}, h)\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_{X/Y}, h_{X/Y}))^{(1)}$$
$$-a((1-g)(4\zeta'(-1) - \frac{1}{6} - \frac{2}{3}\log 2)r(\mathcal{E}))$$

holds in $\mathrm{CH}_{\mathrm{Ar}}(Y) \otimes \mathbf{Q}$.

Corollary 2. Conjectural Normalized Constant in the Polyakov Measure. Let the assumption be as in Corollary 1. Let $\omega_{X/Y}$ be the dual of the Hermitian vector sheaf $\mathcal{T}_{X/Y}$. For every $j \geq 1$, there is an algebraic isomorphism

$$M : \lambda(\omega_{X/Y}^j) \simeq \lambda(\omega_{X/Y})^{6j^2-6j+1}$$

such that

$$h_Q(M(s), M(s)) = h_Q(s,s)\exp(a(g)(j-j^2)).$$

We view the next result as evidence for the general conjecture.

Theorem 2. The conjectural arithmetic Riemann-Roch-Hirzebruch-Grothendieck theorem is true for $f : \mathbf{P}^n \to \mathrm{Spec}(\mathbf{Z})$ with $\mathcal{E} = \mathcal{O}_{\mathbf{P}^n}$ and Fubini-Study metric h_{FS} on $\mathbf{P}^n(\mathbf{C})$.

Proof: The proof of Theorem 2.1.1 [GS 91] can be suitably modified to give

$$c_1(\lambda(\mathcal{E}),\ {}_{\Delta}h_Q)_{\mathrm{Ar}} = f_{\mathrm{CH}}(ch_{\mathrm{Ar}}(\mathcal{E},h)\mathrm{td}_{\mathrm{Ar}}(\mathcal{T}_{\mathbf{P}^n}, h_{\mathrm{FS}})(1-a(R(\mathcal{T}_{\mathbf{P}^n(\mathbf{C})}))))^{(1)}$$

where ${}_{\Delta}h_Q$ denotes the Quillen metric associated with real Laplace operator

$$2(\overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*).$$

What we want to do is to find $c_1(\lambda(\mathcal{E}), h_Q)_{\mathrm{Ar}}$, where h_Q denotes the Quillen metric associated with the Laplace-Beltrami operator $\Box = \Delta/2$. Let

$$\Box_n(s) = \sum_{q=0}^{n}(-1)^{(1+q)}q\zeta_q(s).$$

Obviously, we have

$$h_Q = {}_{\Delta}h_Q 2^{\Box_n(0)}$$

Thus we have to calculate $\Box_n(0)$. By [IT 78], we know that

$$\Box_n(s) = \sum_{q=0}^{n}(-1)^{(q+1)}\frac{d_{n,q}(k)}{(k(k+n+1-q))^s}$$

Formally as the case of Proposition 2.3.4 of [GS 91], we have

Lemma 1 (N. Skoruppa [S 90]). For any $a > 0$, any polynomial

$$p(x) = \sum\nolimits_{i=0}^{N} c_i x^i,$$

let

$$< \zeta, p > = \sum\nolimits_{i=1}^{N} c_i\zeta(-i),$$

and

$$\int p(x) = \sum_{i=1}^{N} c_i \frac{x^{i+1}}{i+1}.$$

Then

$$\sum_{k=0}^{\infty} p(k)(k(k+a))^{-s}|_{s=0} = <\zeta, p> - \frac{1}{2}\int p(-a)$$

Proof. Note that if $k > a$, $(1+a/k)^{-s}$ has a Taylor expansion, which then allows us to use the formal properties of the Riemann zeta function. The final result comes by using the functional equation of $\zeta(s)$ and the fact that there is only one simple pole of $\zeta(s)$. The proof takes a couple of pages.

By this lemma, using the properties of $d_{n,q}(k)$ stated in [GS 91], we have

$$\sum_{n=0}^{\infty} \Box_n(0)T^n = -\frac{d}{dT}(\frac{1}{2}\log(1-\mathrm{T}) + \frac{1}{1-\mathrm{T}} + \frac{1}{(1-\mathrm{T})\log(1-\mathrm{T})}).$$

Lemma 2 (N. Skoruppa [S 90]). Let $S(x)$ be the power series such that $\Box_n(0)$ is the coefficient of x^n in the expression

$$(n+1)(\frac{x}{1-e^{-x}})^{n+1}S(x).$$

Then

$$S(x) = \frac{1}{2}\frac{x}{e^x-1} - \frac{e^x}{e^x-1} + \frac{1}{x}.$$

Proof: This lemma is easily proved by the standard technique of generating functions.

Now note that

$$-\frac{e^x}{e^x-1} + \frac{1}{x} = \sum_{k\geq 0}\zeta(-k)\frac{x^k}{k!},$$

and

$$\frac{x}{e^x-1} = -\sum_{k\geq 0}\zeta(-k)\frac{x^{k+1}}{k!} + 1 - x,$$

whence we have our Theorem 2.

Remark 2.1. What Vojta really uses in his proof of the Mordell conjecture is the asymptotic arithmetic Riemann-Roch-Hirzebruch-Grothendieck theorem in the style of Theorem 1. Thus Vojta's proof still holds now. In fact, the proof can be carried out by using the following statement in the first version of Vojta [V 91]. With the notation as in [V 91], let $\mathcal{E} = \mathcal{L}_{d.r.b}$. Then by Theorem 1, statement 2, we know that δ is additive, i.e. δ is a linear function on the arithmetic K-group $K_0(W)_{\mathrm{Ar}} \otimes \mathbf{Q}$, where W denotes the arithmetic three-fold defined by letting W_{fin} be the desingularization of the product of two arithmetic surfaces, which correspond to the curve in question, over the base B. Also by Theorem 1, statement 1, we know that δ is independent of the metric chosen, so in fact δ is a function on the ordinary K-group $K_0(W)$. Now define $x \in K_0(W) \otimes \mathbf{Q}$ by

$$x = [\mathcal{L}_{d.r.b}] - [\mathcal{O}_W];$$

for some integer d_0 canceling all denominators in Y . Then

$$\delta(\mathcal{L}_{d.r.b}) = \delta([\mathcal{O}_W]) + \frac{d}{d_0}\delta(x) + \binom{\frac{d}{d_0}}{2}\delta(x^2) + \binom{\frac{d}{d_0}}{3}\delta(x^3).$$

Higher order terms vanish because x^n is composed of sheaves supported on the subsets of W of codimension n, and dim $W = 3$. Also by [B 81] and [KS 86], we know that $\mathrm{CH}^0(W)$ is a finite group. Thus note that $x \in \mathrm{CH}^0(W) \otimes \mathrm{Q}$, and we have

$$\delta(x^3) = 0.$$

Therefore

$$\delta(\mathcal{L}_{d.r.b}) = O(d^2)$$

for $d \to \infty$. In this way, Vojta's proof of Mordell conjecture still holds.

Remark 2.2. Taking Bismut's assumption (A) about metrics into consideration, we offer the following problem: Do there exist admissible metrics in the style of Bismut's assumption (A), having the following property. Let $(\mathcal{E}, h)$ be an admissible Hermitian vector sheaf on an arithmetic variety X. Then for $f : X \to Y$ as in the general conjecture, there exists an unique characteristic class $B(\mathcal{E}) \in H^{\mathrm{ev}}(X(\mathbf{C}))$ such that

B1. B commutes with pull-back;

B2. B is additive

B3. Let

$$\mathrm{Td}'_{\mathrm{Ar}}(\mathcal{E}, h) = \mathrm{td}_{\mathrm{Ar}}(\mathcal{E}, h)(1 - a(B(E_\infty))$$

in $\mathrm{CH}_{\mathrm{Ar}}(X) \otimes \mathbf{Q}$ and let $h_{X/Y}$ be the admissible metric. Then

$$c_1(\lambda(\mathcal{E}), h_Q)_{\mathrm{Ar}} = f_{\mathrm{CH}}(ch_{\mathrm{Ar}}(\mathcal{E}, h)\mathrm{Td}'_{\mathrm{Ar}}(\mathcal{T}_{X/Y}, h_{X/Y}))^{(1)}.$$

Furthermore, for a line sheaf $\mathcal{L}$, $B(\mathcal{L})$ may be determined by applying **B3** to

$$f : \mathbf{P}^n \to \mathrm{Spec}(\mathrm{Z}), \text{ and } \mathcal{E} = \mathcal{O}_{\mathbf{P}^n},$$

with certain admissible metric on $\mathbf{P}^n(\mathbf{C})$.

Remark 2.3. Note that when reducing to the theory of arithmetic surfaces, we find that the general conjecture and the above Remark 2.2 correspond to Deligne's Riemann-Roch theorem and Faltings' Riemann-Roch theorem. It is well-known that Faltings' Riemann-Roch theorem comes immediately from the definition of Arakelov admissible metric and Faltings metric. But for the proof of Deligne's Riemann-Roch theorem, one really needs to do additional hard work. Similarly, we think that more work will be needed to prove the final version of arithmetic Riemann-Roch-Hirzebruch-Grothendieck theorem.

Remark 2.4. Note that what Bismut-Lebeau proved in their paper [BL 90] is a similar result as the adjunction formula for arithmetic surfaces. For such a formula, if we only deal with a section of degree 1, we do not need the additional term with Greens function about the corresponding point in question. But if we deal with a multiple section, we do need such additional term, which will contribute to the additional work.

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