THE GENERALIZED TRACE MAP FOR WALDHAUSEN'S K-THEORY OF SPACES, AND APPLICATIONS. PART I

by

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Introduction

In [W2], Waldhausen constructs a map $W : A(X) \longrightarrow \Omega^\infty\Sigma^\infty(X_+)$ (where $A(X)$ denotes the (Waldhausen) K-theory of the space X) and showed that evaluation on the image of $M : \Omega^\infty\Sigma^\infty(X_+) \longrightarrow A(X)$ induced by the inclusion of monomial matrices produced a self-map $W \circ M : \Omega^\infty\Sigma^\infty(X_+) \longrightarrow \Omega^\infty\Sigma^\infty(X_+)$ homotopic to the identity by a homotopy natural in X. This yielded a splitting of $\Omega^\infty\Sigma^\infty(X_+)$ off of $A(X)$ (as well as it's stabilization $A^S(X)$), and this fact plays a key role in the proof of the fundamental theorem of Waldhausen relating $A(X)$ to pseudo-isotopy theory ([W2], [WM], [W]):

<u>Thm</u> [Waldhausen] $A(X) \simeq \Omega^\infty\Sigma^\infty(X_+) \times WH^{\mathrm{Diff}}(X)$ where $\Omega^2 Wh^{\mathrm{Diff}}(X) \simeq \mathcal{P}(X)$ = the stable pseudo-isotopy space of X (as defined by Hatcher-Wagoner-Igusa).

The construction of W is done in stages. Waldhausen first shows that fibre $(A(S^n \wedge X_+) \longrightarrow A(X))$ can be described through a certain range of dimensions (approximately 2n) in terms of a "cyclic" bar construction. On this cyclic bar construction he constructs a map to $\Omega^\infty\Sigma^\infty(X_+)$ compatible with stabilization. The result is a map $A^S(X) \longrightarrow \Omega^\infty\Sigma^\infty(X_+)$ natural in X, and precomposition with the stabilization map $A(X) \longrightarrow A^S(X)$ yields W. In this sequence of papers we construct a generalization of Waldhausen's map W and investigate it's properties. Specifically let X and Y be pointed simplicial sets, X connected. Then there exists a <u>generalized Waldhausen trace map</u> (2.2.8):

$$(*)\qquad Tr_X(Y) : \varinjlim_n \Omega^n \mathrm{fibre}(\bar{A}(\Sigma(X \vee \Sigma^n Y)) \longrightarrow \bar{A}(\Sigma X)) \longrightarrow \Omega^\infty\Sigma^\infty(\Sigma(\bigvee_{q\geq 1} | X^{[q-1]} \wedge Y|)).$$

This map is natural in X and Y. The first application of this is to prove a conjecture due to T. Goodwillie:

<u>Thm A</u> For connected X there is a weak equivalence $\tilde{\rho} = \prod_{q\geq 1} \tilde{\rho}_q : \Omega^\infty\Sigma^\infty(\Sigma(\bigvee_{q\geq 1} E\mathbb{Z}/q \wedge_{\mathbb{Z}/q} |X|^{[q]})) \xrightarrow{\simeq} \bar{A}(\Sigma X)$, natural in X.

The action of $\mathbb{Z}/p$ on $|X|^{[q]}$ is given by cyclic permutation, and as above $\bar{A}(Z)$ denotes fibre $(A(Z) \longrightarrow A(*))$. Theorem A has been announced previously in [CCGH] as well as by myself in [O1]. Unfortunately both of these papers contain serious mistakes. The proof of theorem A we give here follows the line of argument attempted in [CCGH], with technical modification along the lines of [W2]. An outline is as follows: in chapter 1 we recall the necessary results from [W2] and Goodwillie's Calculus of Functors [G1], and in this context define the maps $\tilde{\rho}_q$ used in the proof of theorem A. In chapter 2, we follow the arguments of [W2] in constructing the trace map $\overline{\mathrm{Tr}}_X(Y)$ and in section 2.3 we complete the proof of Theorem A by using $\overline{\mathrm{Tr}}_X(Y)$ to explitely compute the 1^{st} derivative of $\tilde{\rho}_q$ at a connected space X (this 1^{st} derivative is in the sense of Goodwillie. A crucical ingredient here is the computation, due to Goodwillie, of the derivatives of $A(\Sigma\_)$ on connected spaces, generalizing Waldhausen's proof of the equivalence $A^S(X) \xrightarrow{\simeq} \Omega^\infty\Sigma^\infty(X_+)$ [WM]). Tom Goodwillie has also been able to prove Theorem A by applying results of G. Carlsson to study the Goodwillie "Taylor series" for the functor $A\Sigma(\_)$. In chapter 3, we include some results (initially circulated as the preprint [O2]), concerning splittings of homotopy functors.

As indicated by the title, this paper appears as part of a series. Part II, which will appear as joint work with W. Vogell, determines the effect on the decomposition of theorem A under the involution on $\overline{A}(\Sigma X)$ induced by a stable spherical fibration on ΣX. In Part III we use $\overline{\mathrm{Tr}}_X(Y)$ and extensions of the representation $\tilde{\rho}_q$ to determine the effect of non-suspension maps $f : \Sigma X \longrightarrow \Sigma Y$ on this decomposition; this can be used to gain information on $\overline{A}(X)$ for 1-connected spaces X not homotopy equivalent to a suspension.

In other installments, we hope to investigate the effect of reduced power operations and transfer, as well as the connection of $\overline{\mathrm{Tr}}_X(Y)$ to the 1^{st} derivative (at X, evaluated at Y) of Bökstedt's topological Dennis trace map and it's lift to the homotopy fixed point set $\Omega^\infty\Sigma^\infty(X_+^{S^1})^{hS^1}$.

I would like to thank R. Schwänzl and R. Vogt for helpful conversations, and the authors of [CCGH] for providing the skeleton of the proof of theorem A. Mainly I would like to thank T. Goodwillie and F. Waldhausen, whose work has provided the foundation as well as much of the motivation for this paper.

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CHAPTER I

§1.1 Background and Notation for A(X)

We recall the construction of $A(X)$ as given in [W2]. Let X be a pointed, connected simplicial set, GX it's Kan loop group. Let $H^n_k(|GX|)$ denote the total singular complex of the topological monoid $\mathrm{Aut}_{|GX|}(\overset{k}{\vee} S^n \wedge |GX|_+)$ of $|GX|$-equivariant self-homotopy equivalences of the free base-pointed $|GX|$-space $\overset{k}{\vee} S^n \wedge |GX|_+$. $H^k_n(|GX|)$ identifies naturally with a set $\bar{M}^n_k(|GX|_+)$ of path components of $M^n_k(|GX|_+) = \mathrm{Map}\,(\overset{k}{\vee} S^n, \overset{k}{\vee} S^n \wedge |GX|_+)$ under the inclusion $H^n_k(|GX|) \longrightarrow \mathrm{Map}_{|GX|}(\overset{k}{\vee} S^n \wedge |GX|_+, \overset{k}{\vee} S^n \wedge |GX|_+) \cong \mathrm{Map}(\overset{k}{\vee} S^n, \overset{k}{\vee} S^n \wedge |GX|_+)$. One has stabilitization maps $M^n_k(|GX|_+) \xrightarrow{\iota} M^n_{k+1}(|GX|_+)$ given by wedge product with the identity map, suspension maps $M^n_k(|GX|_+) \xrightarrow{\Sigma} M^{n+1}_k(|GX|_+)$ given by smash product with the identity, and pairing maps $M^n_k(|GX|_+) \times M^n_\ell(|GX|_+) \longrightarrow M^n_{k+\ell}(|GX|_+)$ induced by wedge-sum. This pairing restricted to $\{H^n_k(|GX|)\}_{k\geq 0}$ gives $\coprod_{k\geq 0} H^n_k(|GX|)$ the structure of a simplicial permutative category for all $n \geq 0$. These operations - wedge sum, suspension, stabilization - commute up to natural isomorphism. So letting $H_k(|GX|) = \varinjlim_{\Sigma} H^n_k(|GX|)$, $H(|GX|) = \varinjlim_{\iota} H^k(|GX|)$ we see that $\coprod_{k\geq 0} H_k(|GX|)$ is also a simplicial permutative category under wedge-sum. Waldhausen's definition of $A(X)$ is

<u>Def. 1.1.1</u> $A(X) = \Omega B(\coprod_{k\geq 0} BH_k(|GX|)) \cong \mathbb{Z} \times BH(|GX|)^+$.

If X is a basepointed space, $A(X)$ is defined to be $A(\mathrm{Sing}(X))$. Similarly if X is a simplical space, $A(X) \overset{\mathrm{def}}{=} A\,(\mathrm{Sing}|X|)$. If $X \xrightarrow{\simeq} Y$ then $A(X) \xrightarrow{\simeq} A(Y)$.

Note that $\pi_0(H_k(|GX|)) \cong GL_k(\mathbb{Z}[\pi_1 X])$, so $BH_k(|GX|)$ makes sense (as does $BH_k^n(|GX|)$ for all $n \geq 0$). We will use the notation ΣU to denote the reduced suspension of U. If $|X| \simeq \Sigma|Z|$, where Z is a simplicial space connected in each degree, then GX is weakly equivalent to the simplicial James monoid JZ, which in degree q is the free monoid on the pointed space Z_q. In this case we can use JZ in place of the Kan loop group GX in the above constructions. The result is an equivalence $A(\Sigma Z) \simeq \Omega B(\coprod_{k\geq 0} H_k(|JZ|))$.

In studying $A(\Sigma Z)$, we will use constructions from §2 of [W2]. The first, due to Segal, generalizes the bar construction which associates to a monoid it's nerve. Thus, a <u>partial monoid</u> is a basepointed set M together with a partially defined composition law $M \times M \supset M_2 \xrightarrow{\mu} M.M_2$ is required to satisfy i) $M \vee M \subset M_2$, and ii) $(\mu(m_1,m_2),m_3) \in M_2$ iff $(m_1,\mu(m_2,m_3)) \in M_2$. Associated to such a partial monoid is it's nerve: $\{[p] \longmapsto$ composable p-tuples in $M\}$. Face and degeneracy maps are defined in the usual way. One example of a partial monoid is that of Waldhausen's generalized wedge.

Given an inclusion of monoids $A \longrightarrow M$ one defines M_2 to be $M \times A \underset{A\times A}{\cup} A \times M$. The nerve of the resulting partial monoid is denoted by $\{[p] \mapsto \overset{p}{\vee}(M,A)\}$ where $\overset{p}{\vee}(M,A) = \bigcup_{j=1}^{p} A^{j-1} \times M \times A^{p-j}$. Taking $A = \{pt\}$ yields the <u>trivial</u> partial monoid structure on M; the realization of $\{[p] \mapsto \overset{p}{\vee}(M,*)\}$ is weakly equivalent to $\Sigma|M|$. It is often useful to approximate the nerve of a monoid M by generalized wedges. A straightforward arguement (Lemma 2.2.1 of [W2]) yields that if $A \longrightarrow M$ is an $(n-1)$-connected inclusion of monoids, the induced inclusion $\{[p] \mapsto \overset{p}{\vee}(M,A)\} \longrightarrow \{[p] \mapsto \overset{p}{\vee}(M,M)\} = NM$ is $(2n-1)$-connected. As one can easily see, a fixed monoid may admit many different partial monoid structures.

Let M be a monoid, S a set on which M acts. Then one can form the <u>cyclic bar construction</u> of M with "coefficients" in S. It is a simplicial set $N^{cy}(M,S)$ which in degree q is $M^q \times S$. The face and degeneracy maps are given by the following formulae (see [W2], §2):

(1.1.1.5)

$$\begin{aligned}
\partial_0(m_1,\dots,m_q;s) &= (m_2,m_3,\dots,m_q;sm_1)\\
\partial_i(m_1,\dots,m_q;s) &= (m_1,\dots,m_im_{i+1},\dots,m_q;s), \quad 1 \le i \le q-1\\
\partial_q(m_1,\dots,m_q;s) &= (m_1,\dots,m_{q-1};m_qs)\\
s_i\,(m_1,\dots,m_q;s) &= (m_1,\dots,m_i,1,m_{i+1},\dots;s) \quad 0 \le i \le q.
\end{aligned}$$

As noted in [W2], the double bar construction is a special case of the cyclic bar construction where S appears as a cartesian product of a left M-set and a right M-set. When M is a grouplike monoid ($\pi_0 M$ is a group) and $S = M$ with induced M action of the left and right, $N^{cy}(M,M)$ is weakly equivalent to BM^{S^1}. The construction of $N^{cy}(M,S)$ extends in the obvious way to simplicial monoid M acting on a simplicial set S.

It is often case that S itself is a partial monoid which admits a left and right M-action. In this case one wants to know that the cyclic bar construction $N^{cy}(M,S)$ can be done in such a way as to be compatible with the partial monoid structure on S. A <u>left M-module</u> is a partial monoid E together with a basepointed M-action $M \times E \longrightarrow E$ compatible with the partial monoid structure on E. A <u>right</u> M-module is similarly defined, and an <u>M-bimodule</u> is a partial monoid with compatible left and right module structures. Given such an M-bimodule E, the semidirect product $M \ltimes E$ is the partial monoid whose underlying set is $M \times E$ with composition given by $(m,e)(m',e') = (mm', em'+m'e)$ (where the product in M is written multiplicatively, that in E additively). Clearly this construction can be done degreewise when M and E are simplicial. If the partial monoid structure on E has not been specified, we will assume it is the trivial one. Note that in this case it's nerve $\{[p] \longmapsto \overset{p}{\vee}(E,*)\}$ is again a partial monoid with trivial structure, and is a left (resp.

right resp. bi-) module over M if E is. Iteration of this construction yields an M-module structure on a space whose realization is an iterated suspension of $|E|$, and which agrees (up to homotopy) with that induced by the given action of M on E together with the trivial action on the suspension coordinates.

A key result concerning the nerve of a semidirect product is provided by lemma 2.3.1 of [W2]. It states that, under a certain "saturation" condition, there is a map $u : \mathrm{diag}(N^{cy}(M,NE)) \longrightarrow N(M \ltimes E)$, which is a weak equivalence when $\pi_0(M)$ is a group. Here M is a simplicial monoid, E a simplicial M-bimodule, and $N^{cy}(M,NE)$ denote the cyclic bar construction of M acting on the nerve of the partial monoid E. The "diagonal" structure is with respect to the simplicial coordinates coming from $N^{cy}(\_)$ and NE. The saturation condition referred to above, as well as the condition that $\pi_0(M)$ is a group, will always be satisfied in our case. As we will need to know u explicitely later on, we recall that it is given on n-simplices by the formula ([W2], p. 369):

(1.1.2) $u(m_1,\ldots,m_n;e_1,\ldots,e_n) =$

$$= (m_1,(\prod_{i=1}^{n} m_i)e_1m_1;\ m_2,(\prod_{i=2}^{n} m_i)e_2(m_1m_2);\ldots;m_n,m_ne_n(\prod_{i=1}^{n} m_i)) .$$

Let us return to considering JZ and $H_k^n(|JZ|)$ (for connected Z). We will be interested in the case when $Z = X \vee Y$.

Recall first that the James-Milnor splitting yields an equivalence $\Sigma|JZ| \simeq \Sigma(\bigvee_{q\geq 1} |Z|^{[q]})$, the splitting being induced by the word length filtration of $|JZ| \simeq J|Z|$. When $Z = X \vee Y$ one can consider other coarser filtrations. Let $F_r(X,Y) \subset J(X\vee Y)$ denote the subset which in each agree consists of elements of word-length at most r in Y. This is clearly a simplicial subset. Moreover the natural JX-bimodule structure of $J(X\vee Y)$ restricts to a JX-bimodule structure on $F_r(X,Y)$. There is also a natural partial monoid structure on $F_r(X,Y)$, compatible with this action, where two elements are composeable if their product in $J(X\vee Y)$ lies in $F_r(X,Y)$. Under suspension the Hilton-James-Milnor splitting yields an equivalence $\Sigma|F_r(X,Y)| \simeq \Sigma(\bigvee_{q=0}^{r} |F_q(X,Y)/F_{q-1}(X,Y)|)$ of $|JX|$-bimodules, where $F_{-1}(X,Y) = *$, $F_0(X,Y) = J(X) \subset J(X\vee Y)$. In particular, $\Sigma|F_1(X,Y)| \simeq \Sigma(|J(X)|\vee|F_1(X,Y)/F_0(X;Y)|)$; we will denote $F_1(X,Y)/F_0(X,Y)$ by $\bar{F}_1(X,Y)$. The projection maps $F_1(X,Y) \twoheadrightarrow F_1(X,*) \subset JX$ and $F_1(X,Y) \twoheadrightarrow \bar{F}_1(X,Y)$ are JX-bimodule maps, where the partial monoid structure on $\bar{F}_1(X,Y)$ is the trivial one.

If E is a left (resp. right resp. bi-) module over M, then $M_k^n(|E|)$ is a left (resp. right resp. bi-) module over $H_k^n(|M|)$. If M is a <u>partial</u> monoid , then it sometimes makes sense to talk about a partial monoid structure on $H_k^n(|M|)$ even though the latter has not yet been properly defined. For when M is a monoid one has an equivalence

$H_k^n(|M|) \cong \overline{M}_k^n(|M|_+) \subset \mathrm{Map}\,(\overset{k}{\vee} S^n, \overset{k}{\vee} S^n \wedge |M|_+)$, and the latter is defined as a set for all M. In particular when $A \subset M$ is a submonoid and $M_2 = M \times A \underset{A\times A}{\cup} A \times M$, one can define

$(\overline{M}_k^n(|M|_+))_2$ to be

$(\overline{M}_k^n(|A|_+) \times \overline{M}_k^n(|M|_+) \cup \overline{M}_k^n(|M|_+) \times \overline{M}_k^n(|A|_+)) \subset (\overline{M}_k^n(|M|_+))^2$. If $f \in \overline{M}_k^n(|A|_+)$, $g \in \overline{M}_k^n(|M|)_+)$, then $g \circ f = fg$ is the composition

$$\overset{k}{\vee} S^n \xrightarrow{f} \overset{k}{\vee} S^n \wedge |A|_+ \cong \overset{k}{\vee} |A|_+ \wedge S^n \xrightarrow{id\wedge g} \overset{k}{\vee} |A|_+ \wedge S^n \wedge |M|_+ \xrightarrow{\cong}$$
$$\overset{k}{\vee} S^n \wedge |A|_+ \wedge |M|_+ \xrightarrow{id\wedge\mu} \overset{k}{\vee} S^n \wedge |M|_+.$$

fg is similarly defined when $f \in \overline{M}_k^n(|M|_+)$ and $g \in \overline{M}_k^n(|A|_+)$. This applies to the case $M = F_1(X,Y)$, $A = JX$. We summarize these observations as

<u>Lemma 1.1.3</u> The Hilton-James-Milnor splitting of $\Sigma|J(X\vee Y)|$ induces a splitting of $|JX|$-bimodules $\Sigma(|J(X\vee Y)|) \simeq \Sigma(\bigvee_{q\geq 0} |F_q(X,Y)/F_{q-1}(X,Y)|)$, $F_r(X,Y)$ as defined above. $F_r(X,Y)$ admits a partial monoid structure, and the natural projection $F_r(X,Y) \longrightarrow F_r(X,*) = JX$ is a JX-bimodule map. In particular $F_1(X,Y)$ is a generalized wedge, inducing a partial monoid structure on $\overline{M}_k^n(|F_1(X,Y)|_+)$. The projection map $F_1(X,Y) \longrightarrow \overline{F}_1(X,Y)$ induces a map of $H_k^n(|JX|)$-bimodules $\overline{M}_k^n(|F_1(X,Y)|_+) \xrightarrow{p_2} M_k^n(|\overline{F}_1(X,Y)|)$, the latter being endowed with the trivial monoid structure. These maps are compatible with suspension, stabilization and wedge-sum.

□

We also note that the equivalence $H^n_k(|J(X\vee Y)|) \cong \overline{M}^n_k(|J(X\vee Y)|_+)$ is an equivalence of $H^k_n(|J(X\vee Y)|)$-bimodules and hence by restriciton an equivalence of $H^n_k(|JX|)$-bimodules. Now consider the map $\tilde{\mathscr{C}}^i_q : X^{[q-i-q]} \wedge Y \wedge X^{[i]} \longrightarrow X^{[q-1]} \wedge Y$ given by $\tilde{\mathscr{C}}^i_q(x_1,\ldots,x_{q-i},y_1,x'_1,\ldots,x'_i)$ $= (x'_1,x'_2,\ldots,x'_i,x_1,x_2,\ldots,x_{q-i-1},y_1)$. These piece together to yield a "folding" map as in [CCGH]:

$$\Big(\bigvee_{i=1}^{q} \tilde{\mathscr{C}}^i_q\Big) = \tilde{\mathscr{C}}_q : \bigvee_{i=1}^{q} X^{[q-i-1]} \wedge Y \wedge X^{[i]} \longrightarrow X^{[q-1]} \wedge Y, \text{ and}$$

hence

$$\Big(\bigvee_{q\leq 1} \tilde{\mathscr{C}}_q\Big) = \tilde{\mathscr{C}} : \bigvee_{q\geq 1} \Big(\bigvee_{i=1}^{q} X^{[q-i-1]} \wedge Y \wedge X^{[i]}\Big) \longrightarrow \bigvee_{q\geq 1} X^{[q-1]} \wedge Y.$$

Under the James-Milnor splitting $\Sigma|\overline{F}_1(X,Y)|$ can be expanded as $\Sigma\Big(\bigvee_{q\geq 1}\Big(\bigvee_{i=1}^{q} |X^{[q-i-1]} \wedge Y \wedge X^{[i]}|\Big)\Big)$. The $|JX|$-bimodule structure on this wedge is clear, and hence $\tilde{\mathscr{C}}$ induces a map $\mathscr{C} = \Sigma|\tilde{\mathscr{C}}| : \Sigma|\overline{F}_1(X,Y)| \longrightarrow \Sigma\big(\bigvee_{q\geq 1} |X^{[q-1]} \wedge Y|\big)$. $\mathscr{C}$ identifies the left and right $|JX|$-module structures, in the sense that $\mathscr{C}(a \cdot m) = \mathscr{C}(m \cdot a)$ for $a \in \Sigma|\overline{F}_1(X,Y)|$ and $m \in |JX|$.

Finally $H^n_k(|M|)$ is a mapping space (for a simplicial monoid M), for which we will adopt the convention that AB denotes the compositon

$$\bigvee^{k} A^n \wedge M_+ \xrightarrow{A} \bigvee^{k} S^n \wedge M_+ \xrightarrow{B} \bigvee^{k} S^n \wedge M_+,$$

which as a <u>composition</u> product would be written as $B \cdot A$.

§ 1.2 Goodwillie's Calculus

We briefly recall the setup for Goodwillie's calculus of functors, as presented in [G1]. We will only give constructions and results necessary for the following sections, avoiding the somewhat involved definitions describing the connectivity of various families of diagrams - definitions which are needed for the general development given in [G1].

For simplicity we will only consider functors $F : \underline{C} \longrightarrow \underline{D}$ where $\underline{C}$ is either $U, T, U(C)$ or $T(C)$ and $\underline{D}$ is $T, T(C)$ or the category Sp of basepointed spectra. Here U is the category of (Hausdorff) topological spaces weakly equivalent to a C.W. complex, T the category of basepointed spaces in U with basepointed homotopy type of a CW complex. $U(C)$, $T(C)$ denote the corresponding categories of spaces over $C \in \mathrm{obj}(U)$. Note that an object of $T(C)$ is a retractive space Y over C, i.e., $r : Y \longrightarrow C$ admits a right homotopy inverse $i (r \circ i \simeq id)$. Each of these choices of $\underline{C}$ is a closed model category in the sense of Quillen, so one has the usual constructions of homotopy theory. In particular one can consider the restriction of F to $\underline{C}_n$ = the full subcate-gory of n-connected objects in $\underline{C}$. Moreover one has a suitable notion of weak equivalence, and F is called a <u>homotopy functor</u> if F preserves weak equivalences as well as (filtered) homotopy colimits up to weak equivalence ((2.2.1), [G1]). We will only be concerned with homotopy functors.

Let S be a finite set, $C(S)$ the category subsets of S with morphisms corresponding to inclusions. An S-cube in $\underline{\mathcal{C}}$ is a covariant functor $G : C(S) \longrightarrow \underline{C}$. If $S = \{1,2,\dots,n\} = \underline{n}$ G is called an n-cube. Associated to an S-cube is the homotopy-inverse limit $h(G) = \text{holim}(G|_{C_0(S)})$ where $C_0(S)$ denotes the full subcategory of $C(S)$ on all objects except ϕ. The natural coaugmentation map $\lim(G) \longrightarrow \text{holim}(G)$ induces a natural transformation $a(G) : G(\phi) \longrightarrow \text{holim}(G|_{C_0(S)})$. G is <u>h-cartesian</u> if $a(G)$ is a weak equivalence (which is the same as requiring $\text{holim}(G|_{C_0(S)})$ to be weakly equivalent to a point - see remark 1.2.8 of [G1]). We say $F : \underline{\mathcal{C}} \longrightarrow \underline{D}$ (as above) has <u>degree</u> $\underline{n}$ if $F \circ G$ is h-cartesian for every homotopy co-cartesian S-cube $G : S \longrightarrow \underline{C}$ where $|S| = n + 1$. The condition that F has deg.n becomes less restrictive as n increases. That is, $\deg(F) = n \Rightarrow \deg(F) = n+1$ but not conversely (prop. 2.3.2 [G1]; one can think of degree n as meaning "having degree $\leq n$").

Given a homotopy functor F satisfying certain conditions, there is a natural way of producing a functor P_nF of degree n and a natural transformation $F \longrightarrow P_nF$. In fact, P_nF can always be constructed. Starting with $X \in \text{obj}(\underline{C})$ one can define an $(n+1)$-cube $X *_C (\_) : C(\underline{n+1}) \longrightarrow \underline{C} = U(C)$ or $T(C)$ which associates to $T \subset \underline{n+1}$ the space $X *_C T$ = the fibrewise join over C of X with the set T. Now let $(T_nF)(X) = \text{holim}\ (F\circ (X *_C (\_))|_{C_0(\underline{n+1})})$. $a(F\circ (X *_C (\_)))$ defines a transformation $(t_nF)(X) : F(X) \longrightarrow (T_nF)(X)$. One easily checks that $X \longmapsto (T_nF)(X)$ is again a homotopy functor on $\underline{C}$ and that $(t_nF) = (t_nF)(\_)$ defines a natural transformation from F to T_nF. Note that $X^*_{\underline{C}}(\_) : C(\underline{n+1}) \longrightarrow \underline{C}$ is a (strongly) homotopy co-cartesian diagram in $\underline{C}$, so that t_nF is an equi-valence if F is of degree n. Iteration of this construction yields P_nF which is by definition the homotopy colimit of the directed system $\{T^i_nF, t^i_nF\}$.

The transformations $\{t^i_nF\}$ induce a natural transformation $p_nF : F \longrightarrow P_nF$. Moverover, choice of a distinguished element $m \in \underline{m+1}$ induces a projection $\underline{m+1} \longrightarrow \underline{m}$ $(T \longmapsto T - T \cap \{m\})$ and hence a natural transformation $C(\underline{m+1}) \longrightarrow C(\underline{m})$. This in turn induces a natural transformation of directed systems $\{T^i_nF, t^i_nF\} \longrightarrow \{T^i_{n-1}, F, t^i_{n-1}F\}$ and hence a natural transformations $P_nF \xrightarrow{q^nF} P_{n-1}F$. Different choices of m yield naturally equivalent choices of q_nF. The <u>Goodwillie Taylor series of F</u> is then by definition the inverse system $\{P_nF, q_nF\}$ which is best viewed as a tower together with the natural transformations p_nF:

(1.2.1)

$$
\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_2F \arrow[d, "q_2F"] \\
 & P_1F \arrow[d, "q_1F"] \\
F \arrow[r, "p_0F"] \arrow[ur, "p_1F"] \arrow[uur, "p_2F"] & P_0F
\end{tikzcd}
$$

The closed diagrams in this tower are homotopy commutative. The $\underline{n^{th}\text{-derivative}}$ of F is by definition the homotopy fibre of q_nF : $D_nF = \operatorname{holim}(P_nF \xrightarrow{q_nF} P_{n-1}F)$. We have not yet explained the conditions necessary for P_nF to have degree n. The precise statement requires some terminology concerning connectivity of diagrams, for which we refer the reader to ([G1], p.9; def. 2.4.5, p. 45). The following will suffice for our purpose. It is a special case of Prop. 2.5.9 of [G1].

Prop. 1.2.2 If there exist integers r and ϵ such that the iterated homotopy fibre of $F \circ G$ is $(r\cdot(n+1)-\epsilon)$-connected for all $(n+1)$-cubes $G: c(\underline{n+1}) \longrightarrow \underline{\underline{C}}_r$ then P_nF has degree n ($\underline{\underline{C}}_r$ = full subcategory of r-connected spaces).

□

In this case D_nF is homogeneous of degree n (it has degree n and $P_iD_nF \simeq$ for $i < n$). We will write $\underline{P^nF}$ for fibre $(F \xrightarrow{p_nF} P_nF)$, and $\underline{P_n^mF}$ for fibre$(P_nF \longrightarrow P_mF)$ when P_kF has degree k for all k (This will always be the case for the functors we are interested in). One also wants to know not just when P_nF is of degree n, but also when the connectivity of $F \xrightarrow{P_nF} P_nF$ tend to ∞ as n tends to ∞. For this Goodwillie introduces the modulus of F, which for our purposes will be the smallest integer $\rho(F)$ such that the above proposition applies with $r = \rho(F) + 1$ and $\epsilon = \epsilon' - n\rho$ for all n, where ϵ' is independent of n (see def. 2.4.4, [G1]). Such an F is said to be <u>analytic of modulus</u> <u>$\rho = \rho(F)$</u>. Goodwillie then proves

<u>Theorem 1.2.3</u> (Th. 2.5.21, [G1]) The connectivity of P_nF tends to ∞ over the category $\underline{\underline{C}}_\rho$, where $\rho = \rho(F)$, $F : \underline{\underline{C}} \longrightarrow \underline{\underline{D}}$

In analogy with functions, $\underline{\underline{C}}_\rho$ is sometimes called the <u>disk of convergance</u> of F. In applying this calculus to F, it is natural to restrict one's attention to the subcategory $\underline{\underline{\mathscr{C}}}_{\rho(F)}$ which in general is the largest subcategory of $\underline{\underline{C}}$ for which the Taylor series of $F|_{\underline{\underline{C}}\rho(F)}$ converges (in the homotopy-theoretical sense). Within this range it provides a powerful machinery for analyzing F, as well as determining the effect of a natural transformation $\eta : F_1 \longrightarrow F_2$ on homotopy groups. It is clear from the above theorem that η will induce a weak equivalence when restricted to $\underline{\underline{C}}_\rho$ $(\rho = \mathrm{Max}(\rho(F_1),\rho(F_2)),$ $F_i : \underline{\underline{C}} \longrightarrow \underline{\underline{D}})$ if η induces an equivalence on derivatives: $D_n(\eta) : D_n(F_1) \longrightarrow D_n(F_2)$, under the condition that $P_0(F_i) \simeq *$. However, there is another way of getting at η. Assume first that $\underline{\underline{C}} = U(C)$ and that $F_i : \underline{\underline{C}} \longrightarrow \underline{\underline{D}}$ have the same modulus $\rho, i = 1,2$. Let $(X,p : X \longrightarrow C)$ be an object in $U(C)$. Then $(X,p : X \longrightarrow C)$ defines a natural transformation $\mathscr{T}_{(X,p)} : U(X) \longrightarrow U(C)$ given on objects by $\mathscr{T}_{(X,p)}(Y,r : Y \longrightarrow X) = (Y,p\circ r : Y \longrightarrow C)$. Analyticity is preserved by the natural transformation

$\mathscr{Y}^{*}_{(X,p)} : F \longrightarrow F \circ \mathscr{Y}_{(X,p)}$. The next result of Goodwillie's concerns only 1^{st} derivatives.

<u>Theorem 1.2.4</u> (Th. 2.7.3, [G1]) If $F_1, F_2 : U(C) \longrightarrow \underline{D}$ are analytic of modulus ρ, and $\eta : F_1 \longrightarrow F_2$ is a natural transformation such that the square

$$\begin{array}{ccc}
P_1(\mathscr{Y}^{*}_{(X,p)}F_1) & \xrightarrow{P_1(\mathscr{Y}^{*}_{(X,p)}\eta)} & P_1(\mathscr{Y}^{*}_{(X,p)}F_2) \\
{\scriptstyle q_1(\mathscr{Y}^{*}_{(X,p)}F_1)}\downarrow & & \downarrow{\scriptstyle q_1(\mathscr{Y}^{*}_{(X,p)}F_2)} \\
P_0(\mathscr{Y}^{*}_{(X,p)}F_1) & \xrightarrow{P_0(\mathscr{Y}^{*}_{(X,p)}\eta)} & P_0(\mathscr{Y}^{*}_{(X,p)}F_2)
\end{array}$$

is homotopy-cartesian for every (X,p) in $U(C)$, then for every $f : Y \longrightarrow X$ in $U(C)_\rho$ the diagram

$$\begin{array}{ccc}
F_1(Y) & \xrightarrow{\eta(Y)} & F_2(Y) \\
{\scriptstyle F_1(f)}\downarrow & & \downarrow{\scriptstyle F_2(f)} \\
F_1(X) & \xrightarrow{\eta(X)} & F_2(X)
\end{array}$$

is homotopy-cartesian.

□

In the case $C = *$ we will denote $\mathrm{fibre}(q_1(\mathscr{Y}^{*}_{(X,p)}F))$ by $(D_1F)_X$; p in this case is unique. The case we are interested in is when $F_2 = A(\Sigma\_) = A\Sigma(\_)$, for which $\rho = 0$ (example 2.4.8, [G1]). Then theorem 1.2.3 yields

<u>Corollary 1.2.5</u> If $\eta : F_1 \longrightarrow A\Sigma(\_)$ is a natural transformation which induces an equivalence

$D_1(\eta)_X : (D_1 D)_X \xrightarrow{\simeq} D(D_1 A\Sigma)_X$ for all connected spaces X,

then η induces an equivalence

$$\eta(f) : \mathrm{fibre}(F_1(Y) \to F_1(X)) \xrightarrow{\simeq} \mathrm{fibre}\,(A(\Sigma Y) \to A(\Sigma X))$$

for all maps f between connected spaces Y and X.

□

The result which makes these techniques applicable to the study of $A(X)$ is the computation, due to Waldhausen at $X = pt$ ([W2], [WM)], and Goodwillie for general X, of the derivatives of $A(X)$: here (Y) denotes the retractive object $(Y \vee X;\ r : Y \vee X \longrightarrow X)$ thought of as an object in $T(X)$.

<u>Theorem 1.2.6</u> [Waldhausen, Goodwillie]

$(D_1 A\Sigma)_X(Y) \simeq \Omega^\infty \Sigma^\infty (\Sigma(\bigvee_{q \geq 1} |X^{[q-1]} \wedge Y|))$ for connected X.

□

We have added the realization functor for consistency of notation, as $A(\_)$ was defined on simplical sets in §1.1. Note that as a homotopy functor $A(\_)$ factors by the realization functor and hence can be viewed as a homotopy functor on spaces, which is necessary in order to apply Goodwillie's calculus as it stands.

Remark 1.2.7 i) Goodwillie's classification theorem for homogeneous functors implies that derivatives are determined by what they do on suspensions. In fact, theorem 1.2.6 can alternatively be written as:

$$(D_1A)_X(Y) \simeq \Omega^\infty\Sigma^\infty\ (\Sigma^{1-q}(\bigvee_{q\geq 1}|X^{[q-1]}\wedge Y|)).$$

ii) There is a slight difference in conventions concerning "stabilization at X" (i.e., passing to $(D_1F)_X$) as defined in [G1], versus the natural generalization of the construction given in [W2]. In [G1] one usually deals with homotopy functors which are reduced; for $(Y,r:Y \longrightarrow X)$ in $U(X)$ $F(Y,r) = \text{fibre}(F(Y) \xrightarrow{F(r)} F(X))$ and evaluatin on the basepoint yields $F(X,id) \simeq *$. On the other hand, $A(X)$ as defined in (1.1.1) is unreduced. Given an unreduced functor F defined in connected spaces, one can extend it to non-connected spaces with finitely many components by defining $F(\coprod_{i=1}^{n} X_i)$ to be $\prod_{i=1}^{n} F(X_i)$. For non-connected $X = (\coprod_{i=1}^{n} X_i)$ $A(X) \simeq \prod_{1}^{n} A(X_i)$ by [W], so this extension is what it should be for $A(X)$. Hence $A(X)$, in the notation of this section, is really $A(X_+, r : X_+ \longrightarrow *)$ for the object $(X_+, r : X_+ \longrightarrow *)$ corresponding in $T(*)$ to X. In what follows we will keep with the convention that $A(X)$ is unreduced $A(X)$, $\bar{A}(X) = \text{fibre}(A(X) \longrightarrow A(*))$ the reduced functor.

iii) It is an interesting question as to what type of constructions in the calculus of several variables (real or complex) have a suitable analogue in Goodwillie's calculus of functors. For example, there seems to be a chain rule that computes the derivatives of a composition $F \circ G$ in terms of the derivatives of F and G. It is easy to show that $D_1(F \circ G)_X$ is $(D_1F)_{GX} \wedge (D_1G)_X$ when $G(X) \simeq *$ and F is reduced. In general, it's formulation seems to require the notion of a generalized spectrum.

We will need the next result in part III. If F is a functor defined on spaces, we will say it is continuous if for each $n \geq 0$ there are natural transformations $\Delta_n \times F(\_) \longrightarrow F(\Delta_n \times \_)$ which induce a natural transformation of realizations $\phi_F : |[k] \longmapsto F(\_)| \longrightarrow F(|[k] \longmapsto (\_)|)$. Here the range of F is either $T, T(C)$ or Sp as before and Δ_n denotes the standard n-simplex.

Lemma 1.2.8 If F is a continuous homotopy functor on $U(C)$ then the natural transformation $\phi_F : |[k] \longmapsto F(\_)| \longrightarrow F(|[k] \longmapsto (\_)|)$ induces a weak equivalence over the category of simplicial objects within the disk of convergence of F.

Pf: Within the disk of convergence of F the Taylor series converges, and the transformation ϕ_F induces a map of Taylor series $\{|[k] \longmapsto P_nF(\_)|(\xrightarrow{\phi_{P_nF}} P_nF(|[k] \longmapsto (\_)|)\}_{n\geq 1}$ and hence a map of derivatives

$|[k] \mapsto D_nF(\_)| \xrightarrow{\phi D_nF} D_nF(|[k] \mapsto (\_)|)$. Goodwillie's classification theorem for derivatives implies that D_nF commutes with realization, that is, ϕD_nF is a weak equivalence for all n. By induction ϕ_{P_nF} is an equivalence for all n. As the Taylor series converges this implies ϕ_F itself is a weak equivalence.

$\square$

There is a slightly more general result one can prove along these lines. Namely, one can consider arbitrary simplicial objects in $U(C)$. Then restricted to such objects there is a weak equivalence $|[k] \mapsto \hat{F}(\_)| \xrightarrow{\phi\hat{F}} \hat{F}(|[k] \mapsto (\_)|)$, where $\hat{F} \stackrel{\text{def}}{=} \underleftarrow{\mathrm{holim}}\,\{P_nF, p_nF\}$ denotes the <u>analytic completion</u> of F (at C). The proof is the same. Now Waldhausen has shown that $A(\_)$ is a continuous homotopy functor ([W]), and by Goodwillie we know that $\bar{A}(\_)$ has modulus 1. Hence we have

<u>Corollary 1.2.9</u> If $X = \{X_k\}$ and X_k is 1-connected for each k, then $\phi_A : |[k] \mapsto \bar{A}(X_k)| \xrightarrow{\simeq} \bar{A}(|[k] \mapsto X_k|)$.

$\square$

Goodwillie's theorem 1.2.4 and it's corollry 1.2.5 can be applied to determine when two natural transformations between homotopy functors are equivalent within the disk of convergence of these functors

<u>Prop. 1.2.10</u> Let F and G be homotopy functors from $\underline{C}$ to $\underline{D}$ where $\underline{C}$ is as before, $\underline{D} = Sp(C)$ or $T(C)$. Let $\eta_1, \eta_2 : F \longrightarrow G$ be two natural transformations of homotopy functors. Then within the disk of convergence (which we define to be the min. of the disks of convergence of F and G) $\eta_1 \simeq \eta_2$ iff $(D_1\eta_1)_X(Y) \simeq (D_1\eta_2)_X(Y)$ within the disk of convergence.

<u>Pf:</u> By Goodwillie ([G2]), one knows that the n^{th} derivative of a homotopy functor admits a description in terms of an n-fold iteration of first derivatives. Thus the condition $(D_1\eta_1)_X(Y) \simeq (D_1\eta_2)_X(Y)$ within the disk of convergence - which is clearly necessary - implies that $(D_n\eta_1) \simeq (D_n\eta_2)$ (at the basepoint $(C, r = id:C \longrightarrow C)$, say) for all n (within the disk of convergence). The hypothesis on $\underline{D}$ allows us to take a C.W. approximation of any element in the image of F or G. The result follows by standard obstruction theory on the skelton of the C.W. approximation, the equivalence $(D_n\eta_1) \simeq (D_n\eta_2)$ and induction on m. This homotopy can be made natural with respect to any diagram which (as a diagram) admits a C.W. approximation. This argument is valid within the disk of convergence.

$\square$

§1.3 Elementary Expansions and Representations in $H_q^n(|JX|)$

As in the previous sections X will denote a basepointed connected simplicial set. Our object in the section will be to construct the maps $\tilde{\rho}_q : \tilde{D}_q(X) \longrightarrow \tilde{A}(\Sigma X)$ of [CCGH] as described in the introduction, to provide some techniques for computing $\tilde{\rho}_q$ on derivatives, and to relate certain restrictions of $\tilde{\rho}_q$ to products of elementary expansions. This will be used in section 2.3 where we compute the trace of $\tilde{\rho}_q$. From the construction of $\tilde{\rho}_q$, it is easy to extend it to a map $\tilde{\rho}_q(JX) : \tilde{D}_q(JX) \longrightarrow \bar{A}(\Sigma X)$. We do this, and prove analogous results for $\tilde{\rho}_q(JX)$ that we will need in part III.

Let $\iota : |X| \longrightarrow |JX|$ denote the standard inclusion. Fixing an indexing of $\overset{q}{\vee} S^n$ and $\overset{q}{\vee} S^n \wedge |JX|$ we let $(S^n)_i$ resp. $(S^n \wedge |JX|_+)_i$ denote the i^{th} term in the appropriate wedge for $1 \leq i \leq q$. Given $(x_1,\ldots,x_q) \in |X|^q$ let $\rho_q(x_1,\ldots,x_q)$ be the map which on $(S^n)_i$ is given by the composition

(1.3.1)

$$(S^n)_i = S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{id \vee f_i} S^n \vee (S^n \wedge |JX|) \xrightarrow{\text{incl}} (S^n \wedge |JX|_+)_i \vee (S^n \wedge |JX|_+)_{i+1}$$

Here coefficients are taken mod q; thus $i+1 = 1$ if $i = q$, $i+1$ otherwise. The basepointed cofibration sequence $S^0 \xrightarrow{i} |JX|_+ \xrightarrow{p} |JX|$ splits up to homotopy after a single suspension. Fixing $j_1 : \Sigma|JX| \longrightarrow \Sigma|JX|_+$ with $\Sigma p \circ j \simeq id$

and letting $j : \Sigma^n|JX| \longrightarrow \Sigma^n|JX|_+$ be $\Sigma^{n-1}(j_1)$, inc. is the map induced by the inclusions $S^n = S^n \wedge S^0 \longrightarrow S^0 \wedge |JX|_+$, $S^n \longrightarrow S^n \wedge |JX| \xrightarrow{i} S^n \wedge |JX|_+$. $f_i(s) = [s, \iota(x_i)] \in S^n \wedge |JX|$ for $s \in S^n$. "pinch" denotes the pinch map associated to the standard embedding $S^{n-1} \longrightarrow S^n$ (of course, any choice of pinch map will do, however we want it to be the same for each i and independent of X). Clearly ρ_q is continuous and defines a map of spaces

$$(1.3.2) \qquad \rho_q : |X|^q \longrightarrow |H_q^n(|JX|)| \cong |\bar{M}_q^n(|JX|_+)|.$$

ρ_q is also equivariant with respect to $\mathbb{Z}/q$, where $\mathbb{Z}/q$ acts on $|X|^q$ by cyclically permuting the coordinates and on $H_q^n(|JX|)$ via the standard embedding $\mathbb{Z}/p \longrightarrow \Sigma_q$ and the usual action of Σ_q on $H_n^q(|JX|)$ by conjugation.

<u>Prop. 1.3.3</u> ρ_q extends to a map
$\bar{\rho}_q : E\mathbb{Z}/q \times_{\mathbb{Z}/p} |X|^q \longrightarrow \Omega\bar{A}(\Sigma X)$, which in turn induces a map
$\tilde{\rho}_q : \Omega^\infty\Sigma^\infty(\Sigma|E\mathbb{Z}/q \wedge_{\mathbb{Z}/q} |X|^{[q]}) \longrightarrow \Omega\bar{A}(\Sigma X)$.

<u>Pf</u>: In [O1] this was proved by constructing a Volodin model for $\Omega\bar{A}(\Sigma X)$ and using it's associated configuration space [FO] to produce $\bar{\rho}_q$ and $\tilde{\rho}_q$. This is probably the easiest way to see why $\bar{\rho}_q$ and $\tilde{\rho}_q$ exist. However it will turn out all we need to know is that $\bar{\rho}_q$ and $\tilde{\rho}_q$ are extensions of ρ_q. To this end, note that ρ_q is compatible with suspension in the n-coordinate. So taking the direct limit under suspension

and stabilization yields a map $|X|^q \longrightarrow |H(|JX|)|$ which we also denote by ρ_q. This map is still $\mathbb{Z}/q$-equivariant, where $\mathbb{Z}/q$ acts on the second space via the embedding $\mathbb{Z}/q \longrightarrow \Sigma_q \longrightarrow \Sigma_\infty$. It suffices to know now that the plus construction $|H(|JX|)| \longrightarrow \Omega A(\Sigma X)$ can be done so as to be equivariant with respect to the action of Σ_∞ and that the action of Σ_∞ on $\Omega A(\Sigma X)$ is trivial up to homotopy. This follows from [FO]. The result is that $\Omega A(\Sigma X) \xrightarrow{i} E\Sigma_\infty \times_{\Sigma_\infty} \Omega A(\Sigma X)$ admits a left homotopy inverse $p : E\Sigma_\infty \times_{\Sigma_\infty} \Omega A(\Sigma X) \longrightarrow \Omega A(\Sigma X)$ $(p \circ i \simeq \mathrm{id})$ and we can take $\bar{\rho}_q$ to be the composition

$$E\mathbb{Z}/q \times_{\mathbb{Z}/q} |X|^q \xrightarrow{(1\times\rho_q)} E\Sigma_\infty \times_{\Sigma_\infty} |H(|JX|)| \to E\Sigma_\infty \times_{\Sigma_\infty} \Omega A(\Sigma X) \xrightarrow{p} \Omega A(\Sigma X).$$

Taking the infinite-loop extension of the adjoint of $\bar{\rho}_q$ yields a map $\Omega^\infty\Sigma^\infty(\Sigma(E\mathbb{Z}/q \times_{\mathbb{Z}/q} |X|^q)) \longrightarrow A(\Sigma X)$. A well-known fact (which we re-prove in section 3.2) is that the projection $E\mathbb{Z}/q \times_{\mathbb{Z}/q} |X|^q \longrightarrow E\mathbb{Z}/q \wedge_{\mathbb{Z}/q} |X|^{[q]} = (E\mathbb{Z}/q)_+ \wedge_{\mathbb{Z}/q} |X|^{[q]}$ admits a stable section s. $\tilde{\rho}_q$ is then the composition

$$\Omega^\infty\Sigma^\infty(\Sigma(E\mathbb{Z}/q \wedge_{\mathbb{Z}/q} |X|^{[q]})) \xrightarrow{s} \Omega^\infty\Sigma^\infty(\Sigma(E\mathbb{Z}/q \times_{\mathbb{Z}/q} |X|^q)) \longrightarrow A(\Sigma X).$$

Finally we note that all of the constructions are natural in X, and hence factor through $\bar{A}(\Sigma X)$.

□

The space $\Omega^{\infty}\Sigma^{\infty}(\Sigma(E\mathbf{Z}/q\lambda_{\mathbf{Z}/q}|X|^{[q]}))$ will be denoted by $\tilde{D}_q(X)$. $\tilde{D}_q(\_)$ can alternatively be thought of as a functor on connected spaces. The following is more or less contained in ([CCGH], §3).

Prop. 1.3.3 i) $(D_1\tilde{D}_q)_X(Y) \simeq \Omega^{\infty}\Sigma^{\infty}(\Sigma|X^{[q-1]} \wedge Y|)$

ii) $(D_1F_q)_X(Y) = \Omega^{\infty}\Sigma^{\infty}(\Sigma(\bigvee_{i=1}^{q}|X^{[i-1]} \wedge Y \wedge X^{[q-1]}|))$, where $F_q(Z) = \Omega^{\infty}\Sigma^{\infty}(\Sigma|Z^{[q]}|)$. The natural transformation $F_q(\_) \longrightarrow \tilde{D}_q(\_)$ induces the fold map on 1^{st} derivatives which is the infinite loop extension of the map

$\bigvee_{i=1}^{q} X^{[q-i-1]} \wedge Y \wedge X^{[i]} \longrightarrow X^{[q-1]} \wedge Y$, $(x_1,\ldots,x_{q-i-1},y,x'_1,\ldots,x'_i)$
$\longmapsto (x'_1,\ldots,x'_i,x_1,\ldots,x_{q-i-1},y)$.

iii) The inclusion
$i_q(X,Y):X^{[q-1]}\wedge Y \longrightarrow (X\vee Y)^{[q]} \longrightarrow E\mathbf{Z}/q\lambda_{\mathbf{Z}/q}(X\vee Y)^{[q]} \longrightarrow$
$E\mathbf{Z}/q\lambda_{\mathbf{Z}/q}(X\vee Y)^{[q]}$ induces an equivalence

$$\begin{aligned}\Omega^{\infty}\Sigma^{\infty}(\Sigma|X^{[q-1]} \wedge Y|) \longrightarrow & \varinjlim_{m} \Omega^{m}\mathrm{fibre}\ (\tilde{D}_q(X\vee\Sigma^{m}Y) \longrightarrow \tilde{D}_q(X)) \\ & = (D_1\tilde{D}_q)_X(Y)\end{aligned}$$

Pf. i) and ii) appear in [CCGH]; the simplest way to see them is to first compute $(D_1F_q)_X(Y)$, which is easy, and then realize that the term $(E\mathbf{Z}/q\lambda_{\mathbf{Z}/q}\_)$ simply has the effect of "dividing by q" (in Goodwillie's words - see [G1]) via the fold map. It should be said that X,Y and Z all denote connected spaces here. Finally iii) follows from i) and ii) since the inclusion $X^{[q-1]} \wedge Y \longrightarrow (X\vee Y)^{[q]}$ induces a map

$\Omega^\infty\Sigma^\infty(\Sigma(|X^{[q-1]} \wedge Y|)) \longrightarrow (D_1F_q)_X(Y)$ which agrees up to homotopy with the infinite loop extension of the inclusion $X^{[q-1]} \wedge Y$ into the last term in the wedge $\bigvee_{i=1}^{q} X^{[i-1]} \wedge Y \wedge X^{[q-i]}$.

$\square$

Recall that for a ring R and $r \in R$ the elementary matrix $e_{ij}(r)$ is the matrix $\mathrm{id} + \bar{e}_{ij}(r)$ where $(\bar{e}_{ij}(r))_{k\ell} = r$ if $(k,\ell) = (i,j)$, o otherwise. One should not try to push the analogy between $H_q^n(|JX|)$ and the group $GL_q(\mathbb{Z}[JX])$ too far, especially for finite n. However one can construct elements of $H_n^q(|JX|)$ which behave enough like elementary matrices to be useful. We call these <u>elementary expansions</u> since they corrospond to the elementary expansion in classical Whitehead simple homotopy theory.

<u>Def. 1.3.5</u> Let X be a connected simplicial set and $\iota : |X| \longrightarrow |JX|$ the standard inclusion. For $x \in |X|$, $e_{ij}(\iota(x)) \in |H_q^n(|JX|)|$ is given on $(S^n)_\ell \subset \bigvee_{k=1}^{q} (S^n)_k$ by

<u>$\ell \neq i$</u> $(S^n)_\ell \xrightarrow{\mathrm{id}} (S^n)_\ell \longrightarrow (S^n \wedge |JX|_+)_\ell$

<u>$\ell = i$</u> $(S^n)_i = S^n \xrightarrow{\text{pinch}} S^n \vee S^n$

$\xrightarrow{\mathrm{id}\ \vee f} S^n \vee (S^n \wedge |JX|) \xrightarrow{\text{inc}} (S^n \wedge |JX|_+)_i \vee (S^n \wedge |JX|_+)_j$

where (as before) we have identified $H_q^n(|JX|)$ with $\bar{M}_q^n(|JX|_+)$. The sequence for $\ell = i$ is exactly as in (1.4.1) with $f(s) = [s,\iota(x)] \in S^n \wedge |JX|$; the only difference is the indexing of the last term. $e_{ij}(-\iota(x))$ is defined the same way, but with $\mathrm{id}\vee f$ replaced by $\mathrm{id}\vee(-f)$ where $-f$ is f composed with a fixed choice of $S^n \xrightarrow{(-1)} S^n$ representing loop inverse. The <u>reduced</u> <u>elementary</u> <u>expansion</u> $\bar{e}_{ij}(\iota(x))$ is given by

<u>$\ell \neq i$</u> $(S^n)_\ell \longrightarrow *$

<u>$\ell = i$</u> $(S^n)_i \xrightarrow{f} (S^n \wedge |JX|) \xrightarrow{\text{inc.}} (S^n \wedge |JX|_+)_i$.

Similarly one can define $\bar{e}_{ij}(-\iota(x))$.

<u>Remark 1.3.6</u> When $i = j$, one could define $e_{ii}(\pm\iota(x))$ to be $\mathrm{id} + \bar{e}_{ii}(\pm\iota(x))$ (loop sum). Also, the definition of $e_{ij}(\pm((x))$ depends on a choice of parameters: choice of pinch map, choice of $j : S^n \wedge |JX| \longrightarrow S^n \wedge |JX|_+$, and choice of $S^n \xrightarrow{(-1)} S^n$ representing -1. These, however, can be fixed so as to be compatible under suspension in the n coordinate and independent of $x \in |X|$ and X. We assume this has been done. All of the manipulations we will do with these elements will be functorial in X and $x \in |X|$.

Often we will want to now that two maps depending on $x \in |X|$ (resp. a diagram depending on $X,Y,\ldots$) are homotopic by a homotopy which is independent of $x \in |X|$ resp. homotopy-cartesian by a homotopy which is independent of the spaces $X,Y,\ldots$) If this can be done, we will say the two maps are <u>canonically</u> homotopic (or that the diagram is <u>canonically</u> h-cartesian).

Prop. 1.3.7 Suppose $f = \ell_{i_1j_1}(\ell(x_1))\cdot\ldots\cdot {}_{i_nj_n}(\iota(x_n))$ for $x_i \in |X|$. Then there is a canonical homotopy $f\cdot f^{-1} \simeq *$, where $f^{-1} = e_{i_nj_n}(-\iota(x_n))\cdot\ldots\cdot e_{i_1j_1}(-\iota(x_1))$.

Pf: There is certainly a homotopy. It can be made canonical by noting that the homotopy can be concentrated in the portion of the sequence which involves finding a null-homotopy of the composition $S^n \xrightarrow{id} S^n \xrightarrow{(-1)} S^n$ for our fixed choice of (-1), and this is independent of $x_i \in |X|$.

Note that we are not making any claims that such a homotopy is unique, even up to homotopy. We will also need

Prop. 1.3.8 For $x_1,\ldots,x_{q-1} \in |X|$, $y \in |Y|$, there is a canonical homotopy between

$e_{12}(-\iota(x_1))\cdot e_{23}(-\iota(x_2))\cdot\ldots\cdot e_{q-1q}(-\iota(x_{q-1}))\bar{e}_{q1}(\iota(y))$ and

$$\bar{e}_{11}((\prod_{i=1}^{q-1} -\iota(x_i))\iota(y))\bar{e}_{21}((\prod_{i=2}^{q-1} -\iota(x_i))\iota(y))\cdot\ldots\cdot\bar{e}_{q1}(\iota(y)).$$

Pf: On the level of matrices this is clear; the product here is taking place in $|J(X\vee Y)|$. Properly speaking, we should write $\prod_{i=j}^{q-1} -\iota(x_i)$ as $(-1)^{q-1-j} \prod_{i=j}^{q-1} \iota(x_i)$ as $|J(X\vee Y)|$ is a monoid without any strict inverses. To realize that the obvious homotopy is canonical, we note that it involves i) reparamerization to pass between the sequence of pinch maps used to evaluate the compositons and ii) reparametrization to reposition the iterated power of (-1) appearing in the expression $(-1)^{q-1} \prod_{i=j}^{q-1} \iota(x_i)$. Both of these can be done independently of the elements $x_1,\ldots,x_{q-1},y$ involved.

□

The next result relates the representaions ρ_q of (1.3.1) to products of elementary expansions. This is needed for the computation of the trace on $\tilde{\rho}_q$ given in §2.3. We define representations $\bar{\rho}_q^{\,1},\bar{\rho}_q^{\,2}$ as follows:

$$
\begin{aligned}
&(1.3.9) \qquad \bar{\rho}_q^1(x_1,\ldots,x_{q-1}) = (\rho_q(x_1,\ldots,x_{q-1},*) \qquad x_i \in X \\
&\qquad\qquad\quad \bar{\rho}_q^2(y) = p_2\rho_q(*,*,\ldots,*,y) \qquad\qquad\qquad y \in Y
\end{aligned}
$$

where $p_2 : H_q^n(|J(X\vee Y)|) \longrightarrow M_q^n(|(|\bar{F}_1(X,Y)|)$ is as in lemma 1.3, X and Y connected.

<u>Proposition 1.3.10</u> As continuous maps $\bar{\rho}_q^1$ and $\bar{\rho}_q^2$ are canoincally homotopic to the following products of elementary expansions:

$$
\begin{gathered}
\bar{\rho}_q^1(x_1,\ldots,x_{q-1}) \xrightarrow{\cong} \\
e_{q-1q}(\iota(x_{q-1}))e_{q-2q-1}(\iota(x_{q-2}))\cdot\ldots\cdot e_{12}(\iota(x_1))
\end{gathered}
$$

$$
\bar{\rho}_q^1(y) \xrightarrow{\cong} \bar{e}_{q1}(y)
$$

<u>Pf:</u> This again only involves reparametrization in the spherical coordinate independent of X and Y, in the case $\bar{\rho}_q^1$. In

the case of $\bar{\rho}_q^2$ we needn't do anything, as the projection map p_2 kills the identity maps along the diagonal and we are left with a single non-zero entry (which in this case we <u>can</u> think of as an entry).

<u>Remark 1.4.11</u> The above canonical homotopies arise from Steinberg identities, which hold in $H_k^n(|GX|)$ up to canonical homotopies. Most types of identities among elementary expansions which hold up to homotopy do <u>not</u> hold up to canonical homotopy. For example, it is not true that the entire representation ρ_q is <u>canonically</u> homotopic to a product of elementary expansions. This type of problem arises whenever one tries to analize such cyclic representations in terms of elementary expansions of matrices.

We have stated the above results using elementary expansions with entries in $\iota(|X|) \subset J|X|$, which is all we will need for chapter 2. However all of the above constructions apply to the more general case where one allows arbitrary entries in $J|X|$ (or even $|GX|$ when $|X|$ is not a suspension). This will be needed in part III. Thus for $y \in J|X| \cong |JX|$, one defines $e_{ij}(y) \in |H_n^q(|JX|)|$ exactly as in definiiton 1.3.5 where $f : S^n \longrightarrow S^n \wedge |JX|$ is the map $f(s) = [s,y] \in S^n \wedge (JX)$. Similarly for the reduced elementary expansion $\bar{e}_{ij}(y)$. Remark 1.3.6 and propositions 1.3.7, 1.3.8 and 1.3.10 apply in this more general context. The version of proposition 1.3.8 we will need is

<u>Proposition 1.3.12</u> For $a_1,\dots,a_{q-1} \in |JX|$, $b \in |\bar{F}_1(X,Y)|$ there is a canonical homotopy between

$$e_{12}(-a_1)\cdot e_{23}(-a_2)\cdot\ldots\cdot e_{q-1}(-a_{q-1})\cdot\bar{e}_{q1}(b) \quad \text{and}$$
$$\bar{e}_{11}((-1)^{q-1}(\prod_{i=1}^{q-1} a_i)b)\cdot\bar{e}_{22}((-1)^{q-2}(\prod_{i=2}^{q-1} a_i)b)\cdot\ldots\cdot$$
$$\bar{e}_{q-1}((-1)a_{q-1}b)\bar{e}_{q-1}(b).$$

□

The representations ρ_q also extend in a natural way to yield a continuous map $\rho_q : |JX|^q \longrightarrow |H_q^n(|JX|)|$, which on a q-triple $(a_1,\dots,a_q) \in |JX|^q$ is given exactly as in (1.3.1) where f_i is now the map $f_i(s) = [s,a_i] \in S^n \wedge |JX|$. Proposition 1.3.3 applies with $|JX|$ in place of $|X|$ for the domain of $\tilde{\rho}_q$; in fact it is easy to see that the map of prop. 1.3.3 factors by this extension. The analogue of proposition 1.3.10 that we will need in part III is

<u>Proposition 1.3.13</u> For $a_1,\dots,a_{q-1} \in |JX|$, $b \in |\bar{F}_1(X,Y)|$, let $\bar{\rho}_q^1(a_1,\dots,a_{q-1}) = \rho_q(a_1,a_2,\dots,a_{q-1},*)$, and $\bar{\rho}_q^2(b) = p_2\rho_q(*,*,\dots,*,b)$ as in (1.4.9). Then as continuous maps $\bar{\rho}_q^1$ and $\bar{\rho}_q^2$ are canonically homotopic to the following product of elementary expansions:

$$\bar{\rho}_q^1(a_1,a_2,\dots,a_{q-1}) \simeq e_{q-1q}(a_{q-1})e_{q-2q-1}(a_{q-2})\cdot\ldots\cdot e_{12}(a_1)$$

$$\bar{\rho}_q^2(b) \simeq \bar{e}_{q1}(b).$$

□

The proofs follow exactly as before.

As a final remark, we should note that in the above propositions involving minus signs, we are <u>not</u> requiring any type of coherence conditions to apply for this minus sign with respect to composition product (which in the limiting case $n \longrightarrow \infty$ will involve the product structure on the generalized ring $\Omega^\infty\Sigma^\infty(|GX|_+))$. We are only stating that certain homotopies can be made canonical. The restriction on the "ring" under consideration that must be made in order for such a coherent (-1) to exist are substantial, as shown by Schwänzl and Vogt in [SV].

CHAPTER II

§2.1 Manipulation in the stable range

We follow closely the arguement of Waldhausen ([W2], Th. 3.1) in proving

Theorem 2.1.1 Let X and Y be pointed simplicial sets, with X connected and Y m-connected. Then the two spaces $NH_k^n(|J(X\vee Y)|)$ and $N^{cy}(H_k^n(|JX|), M_k^n(\Sigma|\overline{F}_1(X,Y)|)$ are q-equivalent, where $q = \min(n-2, 2m+1)$ and $n \geq 1$.

Pf: The notation is that of §1.1. Here the monoid structure on $H_k^n(|J(X\vee Y)|)$ and $H_k^n(|JX|)$ is the usual one, while the partial monoid structure on the $H_k^n(|JX|)$-bimodule $M_k^n(|\overline{F}_1(X,Y)|))$ is trivial. The equivalence follows as in ([W2], Th. 3.1) by the construction of 5 maps, each of which is suitably connected.

The 1^{st} map $H_k^n(|J(X\vee Y)|)$ admits a partial monoid structure where two elements are composeable iff at most one of them lies outside the submonoid $H_k^n(|JX|)$. The nerve of this partial monoid is by definition the generalized wedge $\{[p] \longmapsto \overset{p}{\vee}\ (H_k^n(|J(X\vee Y)|),\ H_k^n(|JX|))\}$. As Y is m-connected, the inclusion $H_k^n(|JX|) \longrightarrow H_k^n(|J(X\vee Y)|)$ is also m-connected. It follows ([W2], Lemma 2.2.1) that the inclusion $\{[p] \longmapsto \overset{p}{\vee}\ (H_k^n(|J(X\vee Y)|),\ H_k^n(|JX|))\} \longrightarrow NH_k^n(|J(X\vee Y)|)$ is (2m+1)-connected.

<u>The 2^{nd} map</u> By §1.1, $H^n_k(|J(X\vee Y)|) \xrightarrow{\cong} \bar{M}^n_k(|J(X\vee Y)|_+)$ is an $H^n_k(|JX|)$-bimodule equivalence. The inclusion $F_1(X,Y) \longrightarrow J(X,Y)$ is $(2m+1)$-connected, hence induces a $(2m+1)$-connected map $\bar{M}^n_k(|F_1(X,Y)|_+) \longrightarrow \bar{M}^n_k(|J(X\vee Y)|_+)$ of $H^n_k(|JX|)$-bimodules. This in turn induces an inclusion of generalized wedges:

$$\{[p] \longmapsto \overset{p}{\vee} (\bar{M}^n_k(|F_1(X,Y)|_+),\ \ H^n_k(|JX|))\}$$

$\longrightarrow \{[p] \longmapsto \overset{p}{\vee} (\bar{M}^n_k(|J(X\vee Y)|_+),\ \ H^n_k(|JX|))\}$. This inclusion is $(2m+1)$-connected in each degree by the gluing lemma ([W2], lemma 2.1.2) and induction on p. It follows that the inclusion of simplicial objects is also $(2m+1)$-connected.

<u>The 3^{rd} map</u> $M^n_k(|\bar{F}_1(X,Y)|)$ is an $H^n_k(|JX|)$-bimodule with trivial monoid structure. So the semi-direct product $H^n_k(|JX|) \ltimes M^n_k(|\bar{F}_1(X,Y)|)$ is well-defined. From §1.1, we have projection maps $p_1 : F_1(X,Y) \longrightarrow JX$, $p_2 : F_1(X,Y) \longrightarrow \bar{F}_1(X,Y)$ (which induce the splitting of $F_1(X,Y)$ after suspension). Taken together, p_1 and p_2 induce a map of simplicial partial monoids which on the level of simplicial sets is $\bar{M}^n_k(|F_1(X,Y)|_+) \longrightarrow \bar{M}^n_k(|JX|_+) \times M^n_k(|\bar{F}_1(X,Y)|)$. As in ([W2], p. 374), we consider the restriction to the path components corresponding to H^n_k of the inclusion

$$\begin{aligned} M^n_k(|F_1(X,Y)|_+) &= \mathrm{Map}(\overset{k}{\vee} S^n, \overset{k}{\vee} S^n \wedge |F_1(X,Y)|_+) \\ &\to \mathrm{Map}(\overset{k}{\vee} S^n, \prod^k S^n \wedge |F_1(X,Y)|_+) \\ &\simeq \prod^k \prod^k \Omega^n\Sigma^n(|F_1(X,Y)|_+). \end{aligned}$$

This is an (n-1)-equivalence. Lemma 1, p. 374 of [W2] yields an (n-2)-equivalence
$\Omega^n\Sigma^n(|F_1(X,Y)|_+) \simeq \Omega^n\Sigma^n(|JX \vee \overline{F}_1(X,Y)|_+) \longrightarrow \Omega^n\Sigma^n(|JX|_+) \times \Omega^n\Sigma^n(|\overline{F}_1(X,Y)|)$. The gluing lemma now applies to show that the above map of partial monoids yields a map

$$\{[p] \longmapsto \overset{p}{\vee} (\overline{M}^n_k(|F_1(X,Y)|_+),\ H^n_k(|JX|))\}$$
$$\longrightarrow \{[p] \longmapsto \overset{p}{\vee} (H^n_k(|JX|) \ltimes M^n_k(|\overline{F}_1(X,Y)|), H^n_k\ (|JX|))\}$$

which is (n-2)-connected.

<u>The 4^{th} map</u> Taking the trivial monoid structure on $M^n_k(|F_1(X,Y)|)$ and forming it's nerve, Lemma 2.3 of [W2] provides an equivalence

$$\operatorname{diag}\ (N^{cy}(H^n_k(|JX|),\ \Sigma.M^n_k\ (|\overline{F}_1(X,Y)|)))$$
$$\xrightarrow[\simeq]{u} N(H^n_k(|JX|) \ltimes M^n_k(|\overline{F}_1(X,Y)|)$$
$$= N(\{[p] \longmapsto \overset{p}{\vee}\ (H^n_k(|JX|) \ltimes M^n_k(|\overline{F}_1(X,Y)|),\ H^n_k(|JX|))\}).$$

Here $\Sigma.A$ denotes the simplicial space $\{[p] \longmapsto \overset{p}{\vee}\ (A,*)\}$ which arises on taking the nerve of a trivial partial monoid.

<u>The 5^{th} map</u> Partial geometric realization sends $\Sigma.M^n_k(|\overline{F}_1(X,Y)|)$ to $S^1 \wedge M^n_k(|\overline{F}_1(X,Y)|)$. The pairing map $S^1 \wedge M^n_k(|\overline{F}_1(X,Y)|) \longrightarrow M^n_k(S^1 \wedge |\overline{F}_1(X,Y)|)$ together with partial geometric realization produces a map from the partial geometric realization of

$N^{cy}(H_k^n(|JX|),\ \Sigma.M_k^n(|\overline{F}_1(X,Y)|))$ to $N^{cy}(H_k^n(|JX|),$
$M_k^n(S^1 \wedge |\overline{F}_1(X,Y)|)))$.

By the realization lemma, this map is (2m+1)-connected.

These 5 maps taken together yield the required sequence connecting $N(H_k^n(|J(X\vee Y)|))$ and $N^{cy}(H_k^n(|JX|),$ $M_k^n(\Sigma|\overline{F}_1(X,Y)|))$. Each of the maps is min (n-2,2m+1)-connected and the theorem follows.

$\square$

The maps constructed in the above theorem are compatible with respect to suspension in the n-coordinate as well as pairing under block sum, by which we will always mean the wedge-sum of section 1.1 for the appropriate monoid in question. Taking the limit as n goes to ∞ yields a sequence of maps connecting $\coprod_{k\geq 0} N(H_k(|J(X\vee Y)|))$ and $\coprod_{k\geq 0} N^{cy}(H_k(|JX|),$ $M_k(\Sigma|\overline{F}_1(X,Y)|))$; each of these maps preserves block-sum and is (2m-1)-connected for (m-1)-connected Y. We thus get a sequence of maps between their group completions which is also (m-1)-connected. We will denote $\Omega B(\coprod_{k\geq 0} N^{cy}(H_k(|JX|),$ $M_k(\Sigma|\overline{F}_1(X,Y)|))$ by $C(X,Y)$. $C(X,\_)$ is a homotopy functor on the category of retractive spaces over X : $C(X,\_)(X\vee Y) = C(X,Y)$. Denote $C(\_,\_)$ by C.

<u>Lemma 2.1.2</u> (compare [W2], Lemma 4.2) There is an equivalence of 1^{st} derivatives

$$(D_1A\Sigma)_X(Y) = \lim_{\overrightarrow{n}} \Omega^n(A(\Sigma(X\vee(S^n \wedge Y))) \longrightarrow A(\Sigma X))$$

$$\simeq (D_1C)_X(Y) = \lim_{\overrightarrow{n}} \Omega^n(C(X,S^n \wedge Y) \longrightarrow C(X,*))$$

Pf: This is an immediate consequence of the above theorem; for each n, we have an equivalence $A(\Sigma X) \simeq C(X,*)$ and a $(2n-1)$-equivalence between $A(\Sigma(X\ (S^n \wedge Y)))$ and $C(X,S^n \wedge Y)$. This gives a $(2n-1)$-equivalence between fibre $(A(\Sigma(X_r(S^n\wedge Y))) \longrightarrow A(\Sigma X))$ and fibre $(C(X,S^n\wedge Y_+) \longrightarrow C(X,*))$ which in the above limit yields a weak equivalence.

□

§2.2 The Generalized Waldhausen Trace Map

In this section we construct a trace map, generalizing the construction of Waldhausen in [W2]. The techniques are essentially those of ([W2], §4).

We begin by recalling (Lemma 4.2, [W2]) that for an $(m-1)$-connected space F there are pairing maps

(2.2.1)

$$\mathrm{Map}(\bigvee^k S^n,S^{n+m}) \wedge \mathrm{Map}\,(S^{n+m},S^{n+m}\wedge F) \longrightarrow \mathrm{Map}(\bigvee^k S^n,\bigvee^k S^{n+m} \wedge F)$$

$$\mathrm{Map}(S^{n+m},S^{n+m}\wedge F) \wedge \mathrm{Map}\,(S^n,S^{n+m}) \longrightarrow \mathrm{Map}(S^{n+m},S^{n+2m}\wedge F)$$

which are (3m-1)-connected. The second pairing is induced by first stabilizing: $\mathrm{Map}(S^{n+m},S^{n+m}\wedge F) \wedge \mathrm{Map}(S^n,S^{n+m}) \longrightarrow \mathrm{Map}(S^{n+m},S^{n+m}\wedge F) \wedge \mathrm{Map}(S^{n+m} \wedge F,S^{n+2m} \wedge F)$,
$f \wedge g \longmapsto f \wedge (g \wedge id)$ where $id : S^m \wedge F \longrightarrow S^m \wedge F$.

Let Y be a connected space. By abuse of notation we will denote $|\overline{F}_1(X,S^1 \wedge Y_+)|$ by ΣF - this is not the suspension of $|\overline{F}_1(X,Y_+)|$ as the latter is not properly defined. In what follows we will always have suspended F at least <u>twice</u>. By lemma 1.1.3 there is an equivalence

$$\Sigma^2 F \simeq \Sigma^2 \Big(\bigvee_{q\geq 1} \big(\bigvee_{i=1}^{q} |X^{[q-i-1]} \wedge Y_+ \wedge X^{[i]}| \big)\Big)$$

$$\Sigma \Big(\bigvee_{q\geq 1} \big(\bigvee_{i=1}^{q} |X^{[q-i-1]} \wedge (S^1 \wedge Y_+) \wedge X^{[i]}| \big)\Big) \quad \text{of } JX\text{-bimodules.}$$

As with Waldhausen's construction, the trace map is constructed degreewise on the cyclic bar construction by topologically mimicking (in the appropriate range) the proof of Morita invariance for Hochschild homology ([I1], [B1], [W2]). Under stabilization this agrees with the constructions of Bökstedt, who shows in [B1] that Morita invariance holds for topological Hochschild homology. The point to keep in mind is that in order for this type of technique to work we need a "tensor" (i.e., smash) product decompositon of $\mathrm{Map}\,(\bigvee^k S^n,\bigvee^k S^{n+2m} \wedge F)$ as an $\underline{H^n_k(|JX|)\text{-bimodule}}$.

We consider the following diagram

<u>Diag. 2.2.2</u>

$$\begin{tikzcd}
 & H^k_n(|JX|)^p \times \mathrm{Map}(\overset{k}{\vee} S^n, \overset{k}{\vee} S^{n+2m} \wedge F) \\
H^k_n(|JX|)^p \times \mathrm{Map}(\overset{k}{\vee} S^n,\ S^{n+m} \wedge F) \wedge \mathrm{Map}(S^{n+m},\ \overset{k}{\vee} S^{n+2m} \wedge F)) \arrow[ur, "f^p_1"] \arrow[dd, "g^p_1"'] & \\
 & H^k_n(|JX|)^p \times \mathrm{Map}(\overset{k}{\vee} S^n,\ S^{n+m}) \wedge \mathrm{Map}(S^{n+m}, \overset{k}{\vee} S^{n+2m} \wedge F)) \arrow[ul, "f^p_2"'] \arrow[dl, "g^p"] \\
\mathrm{Map}\ (S^{n+m}, S^{n+3m} \wedge (\bigvee_{q\geq 1} |X^{[q-1]} \wedge Y_+|)) &
\end{tikzcd}$$

The map f^p is induced by the pairing (2.2.1); it factors as $f^p = f^p_1 \circ f^p_2$ where f^p_1, f^p_2 are given explicitly in the proof of theorem 2.2.5 below. Also $g^p = g^p_1 \circ f^p_2$ where g^p_1 is the unique map of quotient spaces induced by the following sequence of maps (compare [W2], p. 380):

(2.2.3)

$$\begin{array}{c}
H^n_k(|JX|)^p \times \mathrm{Map}(\overset{k}{\vee} S^n, S^{n+m} \wedge F) \times \mathrm{Map}(S^{n+m}, \overset{k}{\vee} S^{n+2m} \wedge F) \\
\downarrow h^p_1 \\
\mathrm{Map}(S^{n+m}, \overset{k}{\vee} S^{n+2m} \wedge F) \times H^n_k\ (|JX|^p \times \mathrm{Map}\ (\overset{k}{\vee} S^n, S^{n+2m} \wedge F) \\
\downarrow h^p_2 \\
\mathrm{Map}(S^{n+m}, \overset{k}{\vee} S^{n+2m} \wedge F) \times H^k_{n+2m}(|JX|)^p \times \mathrm{Map}\ (\overset{k}{\vee} S^{n+2m}, S^{n+3m} \wedge F) \\
\downarrow h^p_3 \\
\mathrm{Map}\ (S^{n+m}, S^{n+3m} \wedge (\bigvee_{q\geq 1} |X^{[q-1]} \wedge Y_+|))
\end{array}$$

The map h^p_1 is given by a cyclic switch of factors. $h^p_2(\alpha,\beta,\gamma) = (\kappa, \Sigma^{2m}\beta, \gamma \wedge \mathrm{id})$, $\mathrm{id} : S^m \wedge F \longrightarrow S^m \wedge F$ as above

h_3^p is given explicitly on a $(p+2)$-tuple $(i_1;j_1,\dots,j_p;i_2)$ by the composition

(2.2.4)

$$\begin{aligned}
S^{n+m} &\xrightarrow{i_1} \bigvee^{k} S^{n+2m} \wedge F \\
&\simeq \bigvee^{k} F \wedge S^{n+2m} \\
&\longrightarrow \bigvee^{k} F \wedge (S^{n+2m} \wedge |JX|_+) \quad (\text{induced by } S^0 \longrightarrow |JX|_+) \\
&\xrightarrow{id\wedge j_1} \bigvee^{k} F \wedge (S^{n+2m} \wedge |JX|_+) \xrightarrow{id\wedge i_2} \dots \xrightarrow{id\wedge j_p} \bigvee^{k} F\wedge (S^{n+2m}\wedge |JX|_+) \\
&\simeq \bigvee^{k} S^{n+2m} \wedge F \wedge |JX|_+ \\
&\longrightarrow \bigvee^{k} S^{n+2m} \wedge F \quad (\text{induced by pairing } F \wedge |JX|_+ \longrightarrow F) \\
&\xrightarrow{i_2\wedge id} S^{n+3m} \wedge F \wedge F \\
&\underset{\iota}{\overset{p}{\rightleftarrows}} S^{n+3m} \wedge F_2 \quad (F_2 = \text{composable pairs in } F \times F) \\
&\xrightarrow{id\wedge\mu} S^{n+3m} \wedge F \quad (\mu : F_2 \longrightarrow F) \\
&\xrightarrow{id\wedge\mathscr{C}} S^{n+3m} \wedge \big(\bigvee_{q\geq 1} (X^{[q-1]} \wedge Y_+|)
\end{aligned}$$

The map ι is induced by the inclusion $F_2 \longrightarrow F \times F$. The map p is such that $p \circ \iota = id$; it arises from the splittings of the spaces involved and is explained in more detail in the proof below. We note that the same type of diagram could be used to define g^p directly (and leave it to the reader to verify that g^p, so defined, is homotopic to $g_1^p \circ f_2^p$). However this wouldn't simplify matters - it is g_1^p that we need to use in the construction of the trace map, given in the following theorem.

Theorem 2.2.5 Diagram 2.2.2 determines a map

$$T^k_{n,m} : N^{cy}(H^n_k(|JX|), M^n_k(S^{2m} \wedge F))^{(3m-1)} \longrightarrow \mathrm{Map}(S^{n+m}, S^{n+3m} \wedge (\bigvee_{q\geq 1} |X^{[q-1]} \wedge Y_+|)),$$

functorial in X and Y, where $Z^{(p)}$ denotes the p-skeleton of Z and $m \geq 2$. These maps are compatible with respect to suspension in the n and m coordinates, and take block sum to loop-sum. Thus taken together they induce in the limit a map

$$T : \lim_{\vec{m}} \Omega^{2m}(\Omega B(\coprod_{k\geq 0} N^{cy}(H_k(|JX|), M^k(S^{2n} \wedge \Sigma F))))$$

$$\longrightarrow \Omega^{\infty}\Sigma^{\infty}(\Sigma(\bigvee_{q\geq 1} |X^{[q-1]} \wedge Y_+|)).$$

Pf: Recall first that the composition of elements in $F_1(X,Z)$ is defined iff at most one of them lies outside JX. This induces the trivial monoid structure on $\overline{F}_1(X,Z)$ for connected Z. In the above diagram F will always have been suspended at least twice and by lemma 1.1.3 $\Sigma^2 F$ splits as a wedge sum. The map ι in (2.2.4) under this splitting is homotopic to the inclusion of a wedge summand (i.e., $S^{n+3m} \wedge F \wedge F \simeq S^{n+3m} \wedge (F_2 \vee F'))$ and so admits a left inverse $p : S^{n+3m} \wedge F \wedge F \longrightarrow S^{n+3m} \wedge F_2$ (i.e., $p \circ \iota \simeq \mathrm{id}$). So after fixing a choice of p for $n = 1$, $m = 1$ - call it p' - we can take p to be $\Sigma^{n+3(m-1)}p'$. This operation commutes with suspension and avoids the problem of trying to invert ι up to homotopy. The same situation arises in the definition of the map f_1^p in diag. 2.2.2. It is induced by the pairing which sends $(f \wedge g) \in \mathrm{Map}(\bigvee^k S^n, S^{n+m} \wedge F) \wedge \mathrm{Map}(S^{n+m} \wedge F)$ to the composition

$$\bigvee^k S^n \xrightarrow{f} S^{n+m} \wedge F \xrightarrow{g \wedge id} \bigvee^k S^{n+2m} \wedge F \wedge F \xleftarrow{i} \bigvee^k S^{n+2m} \wedge F_2$$
$$\longrightarrow \bigvee^k S^{n+2m} \wedge F.$$

Again we invert i by choosing a fixed splitting $\Sigma^4 F \wedge F \xrightarrow{p''} \Sigma^4 F_2$ and replacing i by $\bigvee^k \Sigma^{2(m-2)} p''$ going the other way (this is where the condition $m \geq 2$ is used). The splitting of $\Sigma^2 F$ of lemma 1.1.3 is moreover a splitting into a wedge of $|JX|$-bimodules. Thus the projection p'' as well as it's suspensions will be $|JX|$-bimodule maps. It follows that $\bigvee^k \Sigma^{n+2(m-2)} p''$ is a map of $H_k^n(|JX|)$-bimodules which implies the same for the pairing map

$$\mathrm{Map}(\bigvee^k S^n, S^{n+m} \wedge F) \wedge \mathrm{Map}(S^{n+m}, \bigvee^k S^{n+2m} \wedge F) \longrightarrow$$

$\mathrm{Map}(\bigvee^k S^n, \bigvee^k S^{n+2m} \wedge F)$. The map f_2^p is induced by the natural inclusion $S^1 \longrightarrow \Sigma F$ (induced by $S^1 \longrightarrow S^1 \wedge Y_+ \longrightarrow \Sigma F$). It is easy to see that $f^p = f_1^p \circ f_2^p$. $\bigvee^k S^m \wedge F$ is $(m-1)$-connected, so f^p is $(3m-1)$-connected. thus f_1^p is a split surjection on homotopy groups through dimension $(3m-1)$. The splitting is obviously compatible with the maps $(\partial_i)_\#$ on homotopy groups induced by the face maps $\partial_i : N_p^{cy}(\_) \longrightarrow N_{p-1}^{cy}(\_)$ <u>except</u> possibly ∂_p, as the inclusion

$$\mathrm{Map}(\bigvee^k S^n, S^{n+m}) \wedge \mathrm{Map}(S^{n+m}, \bigvee^k S^{n+2m} \wedge F) \longrightarrow \mathrm{Map}(\bigvee^k S^n, S^{n+m} \wedge F) \wedge$$

$\mathrm{Map}\,(S^{n+m}, \overset{k}{\vee}\, S^{n+2m} \wedge F)$ is an inclusion of right $H_k^n(|JX|)$-modules but <u>not</u> $H_k^n(|JX|)$-bimodules. We need to show that the spliting can be done in a way that is also compatible with the last face map, in the appropriate range. The situation is summarized by the following diagram for the case $p = 1$. The same diagram applies for arbitrary p by taking the product with $H_k^n(|JX|)^{p-1}$ everywhere on the left, and crossing with the identity on that factor

(2.2.6)
$$\begin{tikzcd}
H_k^n(|JX|) \times (\mathrm{Map}(\overset{k}{\vee}\, S^n, S^{n+m}) \wedge \mathrm{Map}(S^{n+m}, \overset{k}{\vee}\, S^{n+2m} F)) \arrow[r, "f_2^2"] \arrow[d, "\bar{\partial}_1'"'] & H_k^n(|JX|) \times \mathrm{Map}(\overset{k}{\vee}\, S^n, \overset{k}{\vee}\, S^{n+2m} \wedge F) \arrow[d, "\partial_1'"] \\
\mathrm{Map}(\overset{k}{\vee}\, S^n, S^{n+m} \wedge |JX|_+) \wedge \mathrm{Map}(S^{n+m}, \overset{k}{\vee}\, S^{n+m} \wedge F) \arrow[r, "\bar{f}_1^0"] \arrow[d, dashed, "\bar{\partial}_1''"'] & \mathrm{Map}(\overset{k}{\vee}\, S^n, \overset{k}{\vee}\, S^{n+2m} \wedge F) \arrow[d, "\partial_1''"] \\
\mathrm{Map}(\overset{k}{\vee}\, S^n, S^{n+m}) \wedge \mathrm{Map}(S^{n+m}, \overset{k}{\vee}\, S^{n+2m} \wedge F) \arrow[r, "f_1^0"] & \mathrm{Map}(\overset{k}{\vee}\, S^n, \overset{k}{\vee}\, S^{n+2m} \wedge F)
\end{tikzcd}$$

The vertical composition of maps on the right is $\partial_1 : N_1^{cy} \longrightarrow N_0^{cy}$. It is easy to see that there exist maps $\bar{\partial}_1'$ and $\bar{f}_1^0$ making the top square commute. f_1^0 is $(3m-1)$-connected, and so the desired map $\bar{\partial}_1''$ (indicated by a dotted arrow) can be defined at least through the (3m-1)-skeleton of the middle space on the left, in such a way that the lower square commutes on restriciton to (3m-1)-skeleta. The same arguement applies to yield a map ∂_p'' in degree p on (3m-1)-skeleta. The result is that, upon replacing the spaces appearing in diag. 2.2.2 by their (simplicial (3m-1)-skelta, the collection of simplicial objects

$$\{(H_k^n(|JX|)^p \times (\mathrm{Map}(\overset{k}{\vee} S^n, S^{n+m}) \wedge \mathrm{Map}(S^{n+m}, \overset{k}{\vee} S^{n+2m}) \wedge F)))^{(3m-1)}\}_{p\geq 0}$$

can be given the structure of a bisimplicial object in such a way that $\{f_1^p\}_{p\geq 0}$ and $\{f_2^p\}_{p\geq 0}$ induce bisimplicial maps $f_1^\bullet, f_2^\bullet$ with $f^\bullet = f_1^\bullet \circ f_2^\bullet$ an equivalence. By the realization lemma, we get that $f_1^\bullet$ (which extends to a well-defined map of bi-simplicial objects without restriction to skeleta) is split-surjective on homotopy groups through dimension (3m-1), so that we can construct a right homotopy inverse through the (3m-1)-skeleton of the realization of $N^{cy}(H_k^n(|JX|), M_k^n(S^{2m} \wedge F))$. Taking the disjoint union over n, we see that the basepointed map $\coprod_{n\geq 1} f_i = \coprod_{n\geq 1} f_i(n)$ commutes with block sum. Hence $\coprod_{n\geq 1} f_i(n)$ induces a map of group completions which admits a right homotopy inverse through the (3m-1)-skeleton after group completion. Finally we need to prove:

<u>Claim</u> a) The collection of maps $\{g_1^p\}_{p\geq 0}$ taken together define for each n and k a map

$$g(n,k) = |g_1^{\bullet}| : |N^{cy}(H_k^n(|JX|),\ \mathrm{Map}(\overset{k}{\vee} S^n, S^{n+m} \wedge F) \wedge \mathrm{Map}(S^{n+m}, \overset{k}{\vee} S^{n+2m} \wedge F))|$$

$$\longrightarrow \mathrm{Map}(S^{n+m}, S^{n+3m} \wedge (\bigvee_{q\geq 1} |X^{[q-1]} \wedge Y_+|))$$

b) Under $\coprod_{k\geq 0} g(n,k)$, block-sum maps to loop sum. Hence $\coprod_{k\geq 0} g(n,k)$ extends over group completion, yielding a map

$$\Omega B(\coprod_{k\geq 0} |N^{cy}(H_k^n(|JX|), \mathrm{Map}(\overset{k}{\vee} S^n, S^{n+m} \wedge F) \wedge \mathrm{Map}(S^{n+m}, \overset{k}{\vee} S^{n+2m} \wedge F))|)$$

$$\longrightarrow \mathrm{Map}\ (S^{n+m}, S^{n-3m} \wedge (\bigvee_{q\geq 1} |X^{[q-1]} \wedge Y_+|))$$

The map is compatible with respect to suspension in both the n and m coordinates.

<u>Pf. of Claim</u> The statement a) follows from the fact that the maps g_1^p are compatible with respect to the face maps in the cyclic bar construction. For ∂_i, $0 < i < p$ this is clear. The identity $g_1^{p-1} \circ \partial_0 = g_1^{p-1} \circ \partial_p$ follows by the factorization of g_1^p given in (2.2.3) and (2.2.4) which imply that, up to stabilization via suspension, the difference between the two is simply whether we let $|JX|$ act on the left or right of F and under the folding map $\mathscr{C}$ this difference is eliminated. The point about suspension is that from (2.2.3) one sees there is a difference at which stage certain suspensions

are performed in comparing $g_1^{p-1} \circ \partial_0$ and $g_1^{p-1} \circ \partial_p$. But this difference dissappears by the time we have evaluated h_2^{p-1} in (2.2.3). So this is not a problem. b) is also straightforward, for under the same sequence of maps block sum, which is given by wedge sum, maps to wedge sum in the range of h_3^p and this is exactly how loop-sum is defined. The reader is invited to check this by "adding" two maps in the range of (2.2.4) between i_1 and $i_2 \wedge \mathrm{id}$. It follows immediately that $\coprod_{k\geq 0} g(n,k)$ factors through the group-completion with respect to block-sum. Compatibility with respect to suspension also follows, since this amounts to showing that every diagram we have constructed so far can be simultaneously suspended in the n and m coordinates in a compatible way. This follows by a standard type of argument, completing the proof of the claim.

The proof of the theorem follows by passing to the limit in m. One must take care in making the maps $f_1^{\bullet}$ and $f_2^{\bullet}$ defined on (3m-1)-skeleta compatible as m increases. That this can be done follows by obstruction theory. The result is that T is defined on all subskeleta in the limit as $m \longrightarrow \infty$. Note that in the domain of T we have replaced F by ΣF, which accounts for the extra suspension term on the right-hand side.

□

The map constructed in the above theorem yields a map which we also denote by T:

$$\begin{aligned}&(D_1C)_X(Y_+)\\ &= \varinjlim_m \Omega^m(\mathrm{fibre}(C(X,S^m\wedge Y_+) \longrightarrow C(X,*)))\\ &\xrightarrow{T} \Omega^\infty\Sigma^\infty(\Sigma(\bigvee_{q\geq 1} |X^{[q-1]} \wedge Y_+|)).\end{aligned}$$

Precomposing by the equivalence of lemma 2.1.2 we get

$$\text{(2.2.7)}\quad Tr_X(Y):(D_1A\Sigma)_X(Y_+) \longrightarrow \Omega^\infty\Sigma^\infty(\Sigma(\bigvee_{q\geq 1} |X^{[q-1]}\wedge Y_+|))$$

which we will call the generalized Waldhausen trace map (In the case $X = pt$ we recover the map constructed in [W2]). This map is natural in both X and Y. Taking the fibre with respect to the map $X \longrightarrow pt$ yields (for basepointed Y) the reduced trace map

$$\text{(2.2.8)}\quad \overline{Tr}_X(Y):(D_1(\bar{A}\Sigma)_X(Y) \longrightarrow \Omega^\infty\Sigma^\infty(\Sigma(\bigvee_{q\geq 1} |X^{[q-1]}\wedge Y|))$$

where on the right we have for $q \geq 1$ composed with the (basepointed) projection $Y_+ \longrightarrow Y$). Finally, we can follow by projection to the q^{th} factor $\Omega^\infty\Sigma^\infty(\Sigma|X^{[q-1]}\wedge Y|)$; this yields a map $\overline{Tr}_X(Y)_q : (D_1A\Sigma)_X(Y) \longrightarrow \Omega^\infty\Sigma^\infty(\Sigma|X^{[q-1]} \wedge Y|)$ and for connected X $Tr_X(Y) \simeq \prod_{q\geq 1} Tr_X(Y)_q$.

Remark 2.2.9 It would be very interesting if one could prove directly that $Tr_X(Y)$ is an equivalence for connected X without appealing to theorem 1.2.6. This would entail proving, via K-theoretic techniques, that the mystery homology theory (i.e., the homology theory representing the unknown factor of the 1^{st} derivative of A) vanishes at all spaces X. Note that the difference between (2.2.7) and (2.2.8) is mainly one of notation, as $\overline{Tr}_X(Y_+) = Tr_X(Y)$. For this reason we will usually be interested in the reduced trace map, with results holding for the unreduced trace map by the above equivalence.

§2.3 Computating the trace on $\bar{\rho}_q$

In this section we will complete the proof of theorem A, following the approach used in [CCGH].

In section 1.4 we produced a map
$\tilde{\rho} = \prod_{q\geq 1} \tilde{\rho}_q: \tilde{D}(X) = \prod_{q\geq 1} \tilde{D}_q(X) \longrightarrow \bar{A}(\Sigma X)$ for a connected simplicial set X. This map is natural in X, and is induced (1.4.2) by representations $\rho_q : |X|^q \longrightarrow |H_q^n(|JX|)|$. Replacing X by $X\vee Y$, we can consider the restriction - which we will denote by $\rho_q(X,Y)$ - of ρ_q to $|X|^{q-1} \times |Y| \subset |X\vee Y|^q$. Note that this inclusion induces the inclusion $i_q(X,Y)$ of prop. 1.4.4 iii) after passing to smash products. Thus the composition

$$\tilde{\rho}_q(X,Y): \Omega^\infty\Sigma^\infty(\Sigma|X^{[q-1]}\wedge Y|) \xrightarrow{\Omega^\infty\Sigma^\infty(\Sigma i_q(X,Y))} \Omega^\infty\Sigma^\infty(\Sigma|X\vee Y|^{[q]}) \longrightarrow$$

$$\Omega^\infty\Sigma^\infty(\Sigma(E\mathbf{Z}/q\wedge_{\mathbf{Z}/q}|(X\vee Y)^{[q]}|)) = \tilde{D}_q(X\vee Y) \xrightarrow{\tilde{\rho}_q} \bar{A}(\Sigma(X\vee Y))$$

can alternatively be described as the precomposition of $\Omega^\infty\Sigma^\infty(\Sigma\rho_q(X,Y))$ with the stable section $s : \Omega^\infty\Sigma^\infty(\Sigma|X^{[q-1]} \wedge Y|) \longrightarrow \Omega^\infty\Sigma^\infty(\Sigma|X|^{q-1} \times |Y|)$, followed by the map into $\overline{A}(\Sigma(X\vee Y)$. Proposition 1.4.4 tells us that the map $\Omega^\infty\Sigma^\infty(\Sigma i_q(X,Y))$ induces an equivalence $\Omega^\infty\Sigma^\infty(\Sigma|X^{[q-1]} \wedge Y|) \xrightarrow{\simeq} (D_1\tilde{D}_q)_X(Y)$, and Goodwillie's results tell us that $\tilde{\rho}$ induces an equivalence for connected spaces iff $(D_1\tilde{\rho})_X(Y)$ is an equivalence for all connected X. They also tell us that $(D_1\overline{A}\Sigma)_X(Y)$ and $(D_1\tilde{D})_X(Y)$ are the same for connected X. So we need to show that $\overline{Tr}_X(Y) \circ (D_1\tilde{\rho})_X(Y)$ is an equivalence for all connected X, for this will imply by Goodwillie that $(D_1\tilde{\rho})_X(Y)$ is an equivalence for all connected X.

Theorem 2.3.1 For $p \neq q$, $\overline{Tr}_X(Y)_p \circ \tilde{\rho}_q(X,Y) \simeq *$. When $p = q$, $\overline{Tr}_X(Y)_q \circ \tilde{\rho}_q(X,Y) \simeq (-1)^{q-1}$. These homotopies are canonical in X and Y, and hold for all connected X and $q \geq 1$. Thus $\overline{Tr}_X(Y) \circ (\prod_{q\geq 1} \tilde{\rho}_q(X,Y))$ is an equivalence for connected X, which implies $\overline{Tr}_X(Y) \circ (D_1\tilde{\rho})_X(Y)$ is an equivalence for connected X.

Pf: The last implication follows by proposition 1.4.4. The main object is the evaluation of the trace map $\overline{Tr}_X(Y)$ on $\tilde{\rho}_q(X,Y)$, which we will do in stages. First, we determine what happens to the image of the representation $\rho_q(X,Y)$ under the maps constructed in theorem 2.1.1. This will bring us into the cyclic bar construction. Chasing through the diagrams

(2.2.2)-(2.2.4) will then determine the composition $\overline{\mathrm{Tr}}_X(Y)$. This method applies to a slightly more general class of representations we shall consider in part III. There are points in the proof where particular care is required. We will point them out as they arise.

As in §2.1 we will assume $Z = X \vee Y$ where X is connected and Y is m'-connected. ρ_q (resp. it's restriction $\rho_q(X,Y)$) is induced by a simplicial representation Z^q (resp. $X^{q-1} \times Y) \longrightarrow H_q^n(|JZ|)$ which we will also denote by ρ_q (resp. $\rho_q(X,Y)$). The adjoint of ρ_q (and it's restriction) can be represented simplicially by a map of partial monoids:

$$\{[p] \longmapsto \overset{p}{\vee} (Z^q_{,}*)\} \xrightarrow{\{\overset{p}{\vee} \rho_q\}} NH_q^n(|JZ|) \; .$$

We will construct five diagrams, one for each of the maps in the proof of theorem 2.2.1.

<u>The 1^{st} diagram</u> The first map in th. 2.2.1 was induced by the (2m+1)-conn. inclusion of partial monoids:

$$\{[p] \longmapsto \overset{p}{\vee} (H_q^n(|JZ|), H_q^n(|JX|))\} \xrightarrow{\iota_1} NH_q^n(|JZ|).$$

The generalized wedge on the left contains the image of $\{\overset{p}{\vee} \rho_q\}$ and hence $\{\overset{p}{\vee} \rho_q(X,Y)\}$. Thus $\{\overset{p}{\vee} \rho_q(X,Y)\} = \iota_1 \circ \bar{\rho}_{q,1}$, were $\bar{\rho}_{q,1}$ is a map of generalized wedges, induced in each degree by the representation $\rho_q(X,Y)$, which fits into the commutative diagram:

(2.3.2)

$$\begin{tikzcd}
\{[p] \mapsto \bigvee^{p}(X^{q-1} \times Y,*)\} \arrow[r, "\{\bigvee^{p}\rho_q(X,Y)\}"] \arrow[d, equal] & NH^n_q(|JZ|) \\
\{[p] \mapsto \bigvee^{p}(X^{q-1} \times Y,*)\} \arrow[r, "\bar{\rho}_{q,1}"] & \{[p] \mapsto \bigvee^{p}(H^n_q(|JZ|),H^n_q(|JX|))\} \arrow[u, "\iota_1"']
\end{tikzcd}$$

<u>The 2^{nd} diagram</u> The second map in theorem 2.1.1 is the $(2m' + 1)$-connected map of generalized wedges induced by the $(2m'+1)$-connected inclusion $\bar{M}^n_q(|F_1(X,Y)|_+) \longrightarrow \bar{M}^n_q(|JZ|_+) \cong H^n_q(|JZ|)$. As the image of ρ_q is contained in $\bar{M}^n_q(|F_1(X,Y)|_+)$, we can further factor $\rho_q(X,Y)$ as $\iota_2 \circ \bar{\rho}_{q,2}$. $\bar{\rho}_{q,2}$ is defined exactly as $\bar{\rho}_{q,1}$ - it is the (unique) map of generalized wedges induced by $\rho_q(X,Y)$ which makes the following diagram commute:

(2.3.3)

$$\begin{tikzcd}
\{[p] \mapsto \bigvee^{p}(X^{q-1} \times Y,*)\} \arrow[r, "\bar{\rho}_{q,1}"] \arrow[d, equal] & \{[p] \mapsto \bigvee^{p}(\bar{M}^n_q(|JZ|_+),H^n_q(|JX|))\} \\
\{[p] \mapsto \bigvee^{p}(X^{q-1} \times Y,*)\} \arrow[r, "\bar{\rho}_{q,2}"] & \{[p] \mapsto \bigvee^{p}(\bar{M}^n_q(|F_1(X,Y)|_+),H^n_q(|JX|)|\} \arrow[u, "\iota_2"']
\end{tikzcd}$$

<u>The 3^{rd} diagram</u> The $(n-2)$-connected map $\bar{M}^n_q(|F_1(X,Y)|_+) \xrightarrow{p_1 \times p_2} H^n_q(|JX|) \times M^n_q(|\bar{F}_1(X,Y)|)$ induces the

third map in theorem 2.1.1, where the projections p_1, p_2 are induced by the projections of $F_1(X,Y)$ to JX and $\overline{F}_1(X,Y)$ respectively. Let $\overline{\rho}_q^i = p_i \circ \rho_q(X,Y)$ for $i = 1,2$. Then we have a commuting square

(2.3.4)

$$\begin{tikzcd}
\{[p] \mapsto \overset{p}{V}(X^{q-1}\times Y,*)\} \arrow[r, "\overline{\rho}_{q,2}"] \arrow[d, equal] & \{[p] \mapsto \overset{p}{V}(\overline{M}_q^n(|F_1(X,Y)|_+), H_q^n(|JX|))\} \arrow[d] \\
\{[p] \mapsto \overset{p}{V}(X^{q-1}\times Y,*)\} \arrow[r, "\overline{\rho}_{q,3}"] & \{[p] \mapsto \overset{p}{V}(\overline{M}_q^n(|JX|_+) \ltimes M_q^n(|\overline{F}_1(X,Y)|), H_q^n(|JX|))\}
\end{tikzcd}$$

where $\overline{\rho}_{q,3}$ is induced in each degree by the product $\overline{\rho}_q^1 \times \rho_q^2$.

<u>The 4^{th} diagram</u> This is the first place where one encounters complications in computing the trace map on arbitrary representations. From equation (1.1.2) we can see the problem - when M is not a group but only grouplike there may be no simple way to choose f^{-1} for $f \in M$, which one needs to do in order to formally invert the equivalence $u : \mathrm{diag}(N^{cy}(M,NE)) \xrightarrow{\simeq} N(M \ltimes E)$. In our case by first reducing the representation under consideration to $\overline{\rho}_q(X,Y)$ we are able to circumvent this difficulty. For by proposition 1.4.10, $|\overline{\rho}_q^1|$ and $|\overline{\rho}_q^2|$ are canonically homotopic (i.e., can be reparametrized in a way independent of X and Y) to a product of elementary expansions:

(2.3.5)

$$\bar{\rho}^1_q(x_1,\dots,x_{q-1})\cong e_{q-1q}(\iota(x_{q-1}))e_{q-2q-1}(\iota(x_{q-2}))\cdot\dots\cdot e_{12}(\iota(x_1))$$

$\bar{\rho}^2_q(y)\cong \bar{e}_{q1}(\iota(y))$ (the reduced expansion with (q,1) entry $\iota(y)$)

where $\iota(X)$ denotes the image of $x \in |X|$ in $|JX|$ under the natural inclusion $X \longrightarrow JX$, and similarly for Y (for notational simplicity, we have used $|\bar{\rho}^1_q|$ and $|\bar{\rho}^2_q|$ and $|Y|$ respectively. To recover $\bar{\rho}^1_q$ and $\bar{\rho}^2_q$ as above one applies Sing(_) and precomposes with the map $A \longrightarrow \mathrm{Sing}(|A|))$. The notation is explained in section 1.4. For such a product of elementary expansions proposition 1.4.7 yields a canonical homotopy between $f^{-1}f$, $*$ and ff^{-1} where

$f^{-1} = e_{12}(-\iota(x_1))e_{23}(-\iota(x_2))\cdot\dots\cdot e_{q-1q}(-\iota(x_{q-1}))$ for

$f = \bar{\rho}^1_q(x_1,\dots,x_{q-1})$ as above. We can define a map

$|\bar{\rho}^1_{q,4}| : |X|^{q-1} \times |Y| \longrightarrow |N^{cy}_1(H^n_q(|JX|),$

$M^n_q(|\bar{F}_1(X,Y)|))| = |H^n_q(|JX|) \times M^n_q(|\bar{F}_1(X,Y)|))|$ by

$(x_1,\dots,x_{q-1},y) \longmapsto (f,f^{-1}ef^{-1})$ where $f = \bar{\rho}^1_q(x_1,\dots,x_{q-1})$,

$e = \bar{e}_{q1}(y)$ as given in (2.3.5). Extending degreewise and passing to the simplicial setting yields a map $\bar{\rho}_{q,4}$ and a canonically homotopy-commutative diagram

(2.3.6)

$$\begin{array}{ccc}
\{[p]\mapsto \overset{p}{\vee}(X^{q-1}\times Y,*)\} & \xrightarrow{\bar{\rho}_{q,3}} & \{[p]\mapsto \overset{p}{\vee}(H^n_q(|JX|)\ltimes M^n_q(|\bar{F}(X,Y)|),H^n_q(|JX|))\} \\
\| & & \wr\|\uparrow u \\
\{[p]\mapsto \overset{p}{\vee}(X^{q-1}\times Y,*)\} & \xrightarrow{\bar{\rho}_{q,4}} & \mathrm{diag}(N^{cy}(H^n_q(|JX|_+),\Sigma.M^n_q(|\bar{F}_1(X,Y)|)))
\end{array}$$

(recall that $\Sigma.A.$ is just shorthand notation for $\{[p] \mapsto \overset{p}{\vee}(A,*)\})$. The fact that the diagram is canonically homotopy-commutative is important. Note also that on "1-simplices" on the left part of the diagram $\tilde{\rho}_{q,4}$ is given by $\bar{\rho}^{-1}_{q,4}$.

<u>The 5^{th} diagram</u> In theorem 2.1.1 the fifth map is induced by partial geometric realization
$\Sigma.M^n_q((\bar{F}_1(X,Y)|) \mapsto S^1 \wedge M^n_q(|\bar{F}_1(X,Y)|)$ and the pairing
$p : S^1 \wedge M^n_q(|\bar{F}_1(X,Y)|) \longrightarrow M^n_q(S^1 \wedge |\bar{F}_1(X,Y)|)$
$\xrightarrow{\cong} M^n_q(|\bar{F}_1(X,S^1 \wedge Y)|)$. Let $M^n_q(|\bar{F}_1(X,\Sigma.Y)|)$ denote the simplicial object $\{[p] \mapsto M^n_q(|\bar{F}_1(X,\overset{p}{\vee}(Y,*))|)\}$, where the face and degeneracy maps are induced by those of $\Sigma.Y$. There is an obvious map of simplicial objects
$\Sigma.M^n_q(|\bar{F}_1(X,Y)|) \longrightarrow M^n_q(|\bar{F}(X,\Sigma.Y)|)$ which in degree p is given by the inclusion $\overset{p}{\vee} M^q_n(|\bar{F}_1(X,Y)|) \longrightarrow M^q_n(|\bar{F}_1(X,\overset{p}{\vee} Y)|)$.
Partial realization sends $M^n_q(|\bar{F}_1(X,\Sigma.Y)|)$ to
$M^n_q(|\bar{F}_1(X,S^1 \wedge Y)|)$ and the composition
$\Sigma.M^n_q(|\bar{F}_1(X,Y)|) \longrightarrow M^n_q(|\bar{F}_1(X,\Sigma.Y)|) \overset{r}{\longmapsto} M^n_q(|\bar{F}_1(X,S^1 \wedge Y)|)$ is equivalent to the previous composition of partial realization followed by the pairing p. Note that the partial realization map above is $(n-2)$-connected by the same type of argument used in the construction of the third map in theorem 2.1.1. Now the map $\Sigma.M^n_q(|\bar{F}_1(X,Y)|) \xrightarrow{\alpha} M^n_q(|\bar{F}_1(X,\Sigma.Y)|)$ is an $H^n_q(|JX|)$-bimodule map, and so induces a bisimplicial map:

$$N^{cy}(H_q^n(|JX|),\Sigma.M_q^n(|\overline{F}_1(X,Y)|)) \xrightarrow{\beta} N^{cy}(H_q^n(|JX|),M_q^n(|\overline{F}_1(X,\Sigma.Y)|))$$

Let $N_p^{cy}(M,NE)$ denote the simplicial object $\{[k] \mapsto N_{p,k}^{cy}(M,NE) = M^p \times (NE)_k\}$. The representation $\rho_{q,4}^1 : X^{q-1} \times Y \longrightarrow H_q^n(|JX|) \times M_q^n(|\overline{F}_1(X,Y)|) = N_{1,1}^{cy}(H_q^n(|JX|),\Sigma.M_q^n(|\overline{F}_1(X,Y)|)) = N_{1,1}^{cy}(H_q^n(|JX|),M_q^n(|\overline{F}_1(X,\Sigma.Y)|))$ extends uniquely to a map of simplicial objects:

$$\overline{\rho}_{q,5} : X^{q-1} \times \Sigma.Y \longrightarrow (N_1^{cy}(H_q^n(|JX|),M_q^n(|\overline{F}_1(X,\Sigma.Y)|)).$$

It is not true that there is a map $\Sigma.X^{q-1}\times Y \longrightarrow X^{q-1} \times \Sigma.Y$ of simplicial objects which makes the appropriate diagram commute $X^{q-1} \times \Sigma.Y$ is the simplicial object $\{[p] \mapsto X^{q-1} \times (\overset{p}{V}(Y,*))\})$. However there is after passing to smash products. As will be shown in section 3.2, we have stable splitting i_1,i_2 such that $p_1 \circ i_1 \simeq p_2 \circ i_2 \simeq id$ in the square

(2.3.7)

$$\begin{tikzcd}
\Omega^\infty\Sigma^\infty(\Sigma|X^{[q-1]}\wedge Y|) \arrow[r, shift left, "i_1"] \arrow[dd, leftrightarrow, "\int||"'] & \Omega^\infty\Sigma^\infty(\Sigma(|X^{q-1}\times Y|)) \arrow[l, shift left, "p_1"] \arrow[d, "\tilde{\rho}_{q,4}"] & \\
& \Omega^\infty\Sigma^\infty(\mathrm{diag}\, N^{cy}((H_q^n(|JX|),\Sigma.M_q^n(|\overline{F}_1(X,Y)|))| & \\
\Omega^\infty\Sigma^\infty(|X|^{[q-1]} \wedge \Sigma|Y|) \arrow[r, shift left, "i_2"] & \Omega^\infty\Sigma^\infty(|X^{q-1} \times (\Sigma.Y)|) \arrow[l, shift left, "p_2"] \arrow[d, "\tilde{\rho}_{q,5}"] & \arrow[d, "\tilde{\beta}"] \\
& \Omega^\infty\Sigma^\infty|N^{cy}(H_q^n(|JX|),M_q^n(|\overline{F}_1(X,\Sigma.Y|))| &
\end{tikzcd}$$

where $\tilde{\rho}_{q,j} = \Omega^{\infty}\Sigma^{\infty}|\bar{\rho}_{q,j}|$ for $j = 4,5$ and $\bar{\beta}$ is induced by (β). By the construction of $\tilde{\rho}_{q,4}$ and $\tilde{\rho}_{q,5}$ it is straightforward to see that the diagram is canonically h-cartesian. Note that the space appearing in the lower right-hand cover is (n-2) equivalent to $\Omega^{\infty}\Sigma^{\infty}|N^{cy}(H^n_q(|JX|),M^n_q(|\bar{F}_1(X,S^1 \wedge Y)|)|$. This is our fifth diagram.

Before evaluating the trace we make a useful simplication. In order to be consistent with notation, we will assume $Y = \Sigma^{2m-1} \wedge Z_+$ and use $\Sigma^{2m}F$ to denote $\Sigma|\bar{F}_1(X,Y)|$. There is no loss of generality here, because computation of $\overline{Tr}_X(Y)$ involves passing through a direct limit in which Y becomes more and more highly suspended. Now we know that the partial realization map $r : N^{cy}(H^n_q(|JX|),M^n_q(|\bar{F}_1(X,\Sigma.Y)|))$ $\longrightarrow N^{cy}(H^n_q(|JX|),M^n_q(|\bar{F}_1(X,S^1 \wedge Y)|))$ commutes with the simplicial structure in the first coordinate (i.e., the face and degeneracy maps of the cyclic bar construction), and that it maps the simplicial space $N^{cy}_{p\cdot}(H^n_q(|JX|),M^n_q(|\bar{F}_1(X,\Sigma.Y)|))$ to the space $N^{cy}_p(H^n_q(|JX|),M^n_q(F_1(X,S^1 \wedge Y)|))$. And in theorem 2.2.5 we proved that the maps $f^{\cdot}_1$ and $g^{\cdot}_1$ in diag 2.2.2 were well-defined simplicial maps, where the simplicial structure on the range of $g^{\cdot}_1$ - $Map(S^{n+m},S^{n+3m} \wedge (\bigvee_{q\geq 1}|X^{[q-1]} \wedge Y_+|))$ - was trivial. Restricting to the q^{th} component of the reduced trace map $\overline{Tr}_X(Y)$, we have a canonically homotopy-cartesian diagram:

(2.3.8)

$$\begin{tikzcd}
 & & X^{q-1}\times(\Sigma.Y) \arrow[dd, "\bar{\rho}_{q,5}"] \\
M_q^n(|\bar{F}_1(X,\Sigma.Y)|) \arrow[r, phantom, "\cong"] \arrow[dd, "r"'] & N_0^{cy}(H_q^n(|JX|),M_q^n(|\bar{F}_1(X,\Sigma.Y)|)) \arrow[dd, "r"] & \\
 & & N_1^{cy}(H_q^n(|JX|),M_q^n(|\bar{F}_1(X,\Sigma.Y)|)) \arrow[ul, "\partial_0"'] \arrow[dd, "r"] \\
M_q^n(|\bar{F}_1(X,S^1\wedge Y)|) \arrow[r, phantom, "\cong"] & N_0^{cy}(H_q^n(|JX|),M_q^n(|\bar{F}_1(X,S^1\wedge Y)|)) \arrow[dd, dashed, bend right] & \\
 & & N_1^{cy}(H_q^n(|JX|),M_q^n(|\bar{F}_1(X,S^1\wedge Y)|)) \arrow[ul, "\partial_0"'] \arrow[dd, dashed, bend right] \\
 & N_0^{cy}(H_q^n(|JX|),M_{q,1}^n\wedge M_{q,2}^n) \arrow[uu, "f_1^0"'] \arrow[dd, "\bar{\pi}_1\circ g_1^0"'] & \\
 & & N_1^{cy}(H_q^n(|JX|),M_{q,1}^n\wedge M_{q,2}^n) \arrow[ul, "\partial_0"'] \arrow[uu, "f_1^1"'] \arrow[dl, "\bar{\pi}_q\circ g_1^1"] \\
 & \Omega^{n+m}\Sigma^{n+m}(|X^{[q-1]}\wedge Z_+|) &
\end{tikzcd}$$

where $M_{q,1}^n = \mathrm{Map}(\overset{q}{\vee} S^n, S^{n+m}\wedge F)$, $M_{q,2}^n = \mathrm{Map}(S^{n+m}, \overset{q}{\vee} S^{n+2m}\wedge F)$, $\bar{\pi}_q$ is the obvious reduced projection onto the q^{th} component $\Omega^{n+m}\Sigma^{n+3m}|X^{[q-1]}\wedge Z_+|$ and f_i^j, g_i^j are as in (2.2.2). Our object is to show that the composition of the maps on the right is, after realization and up to sign homotopic to the projection $|X|^{q-1}\times\Sigma|Y| \cong |X|^{q-1}\times\Sigma^{2m}|Z|_+ \longrightarrow \Sigma^{2m}|X|^{[q-1]}\wedge|Z|_+$ followed by the standard inclusion $\Sigma^{2m}|X|^{[q-1]}\wedge|Z|_+ \longrightarrow \Omega^{n+m}\Sigma^{n+m}\Sigma^{2m}|X^{[q-1]}\wedge Z_+|$, by a canonical homotopy. The point is that (2.3.8) implies that it suffices to prove this for the sequence of maps which starts off with $\partial_0\circ\tilde{\rho}_{q,5}$ and then runs down the left-hand side. In order to do this, we need to find a map $\rho_{q,6}$ defined on $X^{q-1}\times(\Sigma.Y)$ or it's realization, whose range is $N_0^{cy}(H_q^n(|JX|), M_{q,1}^n \wedge M_{q,2}^n)$ such that $f_1^0 \circ \rho_{q,6}$ is canonically homotopic to $r \circ \partial_0 \circ \bar{\rho}_{q,5}$ (of course, it would suffice to do this $r \circ \bar{\rho}_{q,5}$ directly without using ∂_0, and in fact such a lifting by f_1' can be written down explicitly. However, it is much simpler to do this <u>after</u> mapping first by ∂_0 ; this incidentally makes the computation of $\bar{\pi}_q \circ g_1^0$ easier as well). Now $\bar{\rho}_{q,5}$ is the unique extension to $X^{q-1} \times (\Sigma.Y)$ of the representation $\bar{\rho}_{q,4}^1$ on 1-simplices $X^{q-1} \times Y$ given by

$$(x_1,\ldots,x_{q-1},y) \longmapsto (f,f^{-1}ef^{-1}) \quad f = \bar{\rho}_1^1(x_1,\ldots,x_{q-1}), e = \bar{\rho}_q^2(y)$$

where these are in turn expressed as a product of elementary expansions by (2.3.5). Under ∂_0 this element maps to $(f^{-1}ef^{-1}\cdot f)$ which is canonically homotopic to $(f^{-1}e)$. It follows that we can describe $r \circ \partial \circ \bar{\rho}_{q,5}$ on the realization of $X^{q-1} \times \Sigma.Y$ as the map of <u>spaces</u> given by the representation

$$(2.3.9) \qquad \bar{\rho}_{q,6}^1 : |X^{q-1} \times \Sigma.Y| \longrightarrow |M_q^n(|\bar{F}_1(X,\Sigma.Y)|)|$$
$$(x_1,\ldots,x_{q-1},\tilde{y}) \longmapsto (f^{-1}\tilde{e}), \; f = \bar{\rho}_q^1(x_1,\ldots,x_{q-1}), \quad \tilde{e} = \bar{e}_{q-1}(\iota(\tilde{y}))$$

where f is now considered as a product of elementary expansions yielding a point in $|M_q^n(|\bar{F}_1(X,\Sigma.Y)|)|$. Note that $\tilde{y}$ denotes an element of $|\Sigma Y|$. Writing f^{-1} as $e_{12}(-\iota(x_1))e_{23}(-\iota(x_2))\cdot\ldots\cdot e_{q-1q}(-\iota(x_{q-1}))$ and applying proposition 1.4.8 yields a canonical homotopy between $\bar{\rho}_{q,6}^1$ and the representation

$$(2.3.10)\qquad \bar{\rho}^2_{q,6} : |X^{q-1} \times \Sigma.Y| \longrightarrow |M^n_q(|\bar{F}_1(X,\Sigma.Y)|)|$$
$$(x_1,..,x_{q-1},\tilde{y}) \longmapsto \bar{e}_{11}(z_1)\bar{e}_{21}(z_2)\cdot\ldots\cdot e_{q1}(z_q)$$

where $z_i = (\prod_{j=i}^{q-1} - \iota(x_i))\iota(\tilde{y}) \in \pm\ (|\bar{F}_1(X,\Sigma.Y)| = \pm\Sigma^{2m}F$ (were "-"denotes the inverse under loop sum). We can write $\tilde{y} \in \Sigma|Y| \cong S^m \wedge |Z| \wedge S^m$ as $\tilde{y} = (s,z,s_2)$. Define $\bar{\rho}^3_{q,6}$ by

(2.3.11)
$$\bar{\rho}^3_{q,6} : |X^{q-1}| \times (S^m \wedge |Z|_+ \wedge S^m) \longrightarrow |M^n_{q,1} \wedge M^n_{q,2}|$$
$$(x_1,..,x_{q-1},s,z,s_2) \longmapsto (\bar{e}_{11}(z_1')\cdot\ldots\cdot\bar{e}_{q1}(z_q')) \wedge \iota^1(s_2)$$

where $M^q_{n,i}$ is as in (2.3.8), ι^1 is the inclusion

$$S^m \xrightarrow{\iota_1} \Omega^{n+m}S^{n+2m} \xrightarrow{\iota_2} |\mathrm{Map}(S^{n+m},\overset{q}{\vee}\ S^{n+2m} \wedge F)|,$$

where ι_1 is the standard inclusion and ι_2 is induced by the map $S^{n+2m} \xrightarrow{\bar{\iota}_2} \overset{q}{\vee}\ S^{n+2m} \wedge S^0 \longrightarrow \overset{q}{\vee}\ S^{n+2m} \wedge F$ where $\bar{\iota}_2$ maps to the <u>first</u> factor in the wedge. $z_i' = (\prod_{j=i}^{q-1} - \iota(x_i))\iota(s_1,z) \in \pm\ \Sigma^m F$, and the product of reduced elementary expansions in (2.3.11) is viewed as an element of $\mathrm{Map}(\overset{q}{\vee}\ S^n,S^{n+m} \wedge F)$. It is straightforward to verify that the diagram

$$(2.3.12)\qquad \begin{array}{ccc} |X^{q-1}|\times(S^m \wedge |Z|_+\wedge S^m) & \xrightarrow{\bar{\rho}^3_{q,6}} & |M^n_{q,1} \wedge M^n_{q,2}| \\ \wr\!\!\mid \big\downarrow & & \big\downarrow |f^0_1| \\ |X^{q-1} \times \Sigma Y| & \xrightarrow{\quad\bar{\rho}^2_{q,6}\quad} & |M^n_q(|\bar{F}_1(X,\Sigma.Y)|)| \end{array}$$

is canonically homotopy-commutative. So $\bar{\rho}^3_{q,6}$ provides the necessary lift in order to evaluate $\overline{\mathrm{Tr}}(Y)$. This evaluation is achieved, according to (2.2.4) and theorem 2.2.5, by switching the terms in (2.3.11) and composing. Since ι^1 is just the standard inclusion to the first factor in the wedge, we get $(\bar{e}_{11}(z_1')\cdot\ldots\cdot\bar{e}_{q1}(z_q'))\circ\iota^1(s_2) = \bar{e}_{11}(z_1')\circ\iota^1(s_2)$ which implies that $\bar{\pi}_p\circ g_1^0\circ\bar{\rho}^3_{q,6}$ is canonically null-homotopic for $p\neq q$, and that $\bar{\pi}_q\circ g_1^0\circ\bar{\rho}^3_{q,6}$ is the map $|X|^{q-1}\wedge S^m\wedge |Z|_+\wedge S^m\longrightarrow \Omega^{n+m}\Sigma^{n+3m}(|X|^{[q-1]}\wedge|Z|_+)$ given by $(x_1,\ldots,x_{q-1},s_1,z,s_2)\longmapsto(\prod_{j=1}^{q-1}-\iota(x_j),s_1,\iota(z),s_2)$ which up to reparametrization independent of X and Z is $(-1)^{q-1}$. The standard inclusion of $\Sigma^{2m}|X|^{[q-1]}\wedge|Z|_+$ composed with the projection $|X|^{q-1}\times(\Sigma^{2m}|Z|_+)\longrightarrow|X|^{[q-1]}\wedge\Sigma^{2m}|Z|_+$. This implies that the composition of the maps on the right is as required, completing the proof of the theorem.

□

The equivalence $\tilde{D}(X)\xrightarrow{\rho}\bar{A}(\Sigma X)$ is natural with respect to X, so that if $f : X\longrightarrow Y$ is a map of connected simplicial sets there is a homotopy-commutative diagram

(2.3.13)
$$\begin{array}{ccc}\tilde{D}(X) & \xrightarrow{\rho_X} & \bar{A}(\Sigma X)\\ {\scriptstyle\tilde{D}(f)}\downarrow & & \downarrow{\scriptstyle\bar{A}(\Sigma f)}\\ \tilde{D}(Y) & \xrightarrow{\rho_Y} & \bar{A}(\Sigma Y)\end{array} .$$

It also follows that ρ restricts to yield equivalences

$$\prod_{q=m+1}^{n} \tilde{D}_q(X) = P_n^m\tilde{D}(X) \xrightarrow{P_n^m\rho} P_n^m\bar{A}(\Sigma X) \tag{2.3.14}$$

natural in X for all $0 \leq m < n \leq \infty$, because ρ is a natural transformation of homotopy functors and hence commutes with Goodwillie Calculus. However, it is not true that ρ or $P_n^m\rho$ are natural with respect to maps $\Sigma X \xrightarrow{g} \Sigma Y$ which do not desuspend up to homotopy. This point is important to keep in mind if the ultimate aim is to understand $\bar{A}(X)$ for 1-connected simplicial sets X which are not homotopy equivalent to suspensions. We will return to this point in part III.

CHAPTER III

§3.1 Splittings of Homotopy Functors and Weight Filtrations

We will establish a simple criterion for splitting a homotopy functor F (cf. §1.2) as a product of it's derivatives on the subcategory $U(C)_{\rho(F)}$ of $U(C)$, where $\rho(F)$ is the modulus of F. For simplicity we will take $C = pt$ and assume that F is reduced (i.e. that we have passed to the fibre of $F(x) \longrightarrow F(*)$ for all $(X \longrightarrow *)$ in $obj(U(*))$). All of the results however apply with an arbitrary base space C in place of $*$. We leave it to the reader to make the necessary translation. The n^{th} derivative of F at $*$ will simply be written as D_nF.

Def. 3.1.1 A weight filtration of a reduced homotopy functor F is a direct system of reduced homotopy functors $\{\omega_rF\}_{r\geq 0}$ satisfying:

i) There are compatible natural transformations $\eta_r : \omega_rF \longrightarrow F$ inducing a weak equivalence of reduced functors

$$\underset{\overrightarrow{r}}{\text{holim}} \{\omega_rF\} \xrightarrow{\simeq} F.$$

ii) η_r induces an equivalence of approximations

$$p_i(\eta_r) : P_i(\omega_rF) \xrightarrow{\simeq} P_i(F) \quad i \leq r \quad \text{for all} \quad r \geq 0.$$

iii) $\rho(\omega_r F) \geq \rho(F)$ for all $r \geq 0$.

$\{\omega_r F\}_{r\geq 0}$ is <u>minimal</u> if

iv) $\omega_0 F \simeq *$ and fibre $(\omega_{r-1}F \longrightarrow \omega_r F)$ is homogeneous of degree r for all $r \geq 1$. We note that inductively this is the same as requiring

iv′) $\omega_r F \longrightarrow P_r(\omega_r F)$ is an equivalence for all $r \geq 0$ within the disk of convergence of $\omega_r F$.

The following is implicit in Goodwillie's short proof of the Snaith splitting of $\Omega^\infty\Sigma^\infty(JX)$ (p.p. 66-68, [G1]).

<u>Lemma 3.1.2</u> If F as above admits a minimal weight filtration $\{\omega_r F\}$, then $F \simeq \prod_{n\geq 1} D_n F$ within the disk of convergence of F.

<u>Pf:</u> As on p. 68 [G1], consider the diagram

(3.1.3)

$$\begin{tikzcd}
\omega_{r-1}F \arrow[r, "\iota"] \arrow[d, "p_{r-1}(\omega_{r-1}F)"] & \omega_r F \arrow[r] \arrow[d, "p_{r-1}(\omega_r F)"] & \Omega^{-1}\text{fibre }(\omega_{r-1}F \longrightarrow \omega_r F) \arrow[d] \\
P_{r-1}(\omega_{r-1}F) \arrow[r, "P_{r-1}(\iota)"] & P_{r-1}(\omega_r F) \arrow[r] & P_{r-1}(\Omega^{-1}\text{fibre}(\omega_{r-1}F \longrightarrow \omega_r F)).
\end{tikzcd}$$

By ([G1], chap. III), fibre $(\omega_{r-1} \longrightarrow \omega_r F)$ is homogeneous and hence canonically deloopable, $p_r(\omega_{r-1}F)$ is a weak equivalence and $P_{r-1}\ (\Omega^{-1}$ fibre $(\omega_{r-1}F \longrightarrow \omega_r F))$ by iv) or iv′)

and induction on r. Hence $P_{r-1}(\iota)$ is an equivalence. $(\omega_{r-1}F) \simeq \prod_{n\geq 1} D_n(\omega_{r-1}F) \simeq \prod_{n=1}^{r-1} D_n(F)$ by induction on r, iv'), ii), and iii), within the disk of convergence of F. Now the splitting $\omega_r F \simeq \omega_{r-1}F \times G_r$ via the equivalence $P_{r-1}(\iota)$ yields $G_r = D_r(\omega_r F)$ and hence $\omega_r F \simeq \prod_{n\geq 1} D_n(\omega_r F) \simeq \prod_{n=1}^{r} D_r(F)$ within the disk of convergence of F.

□

We will want to apply this lemma to simplicial functors. Recall (e.g., [W3], prop. 6.3) that if $X. \longrightarrow Y. \longrightarrow Z.$ is a sequence of simplicial spaces which is a fibration in each degree and Z_n is connected for each n then $|X.| \longrightarrow |Y.| \longrightarrow |Z.|$ is a fibration up to homotopy. Although one can do better, this implies (by induction on n) that an n-dimensional cube of simplicial spaces is homotopy cartesian (upon realization) if, in each degree, it is homotopy-cartesian and all of the spaces are (n-1)-connected. We can remove the condition on connectivity if we start with a diagram of simplicial spaces which can be sufficiently delooped in a way compatible with the simplicial structure, for then by delooping we can make the diagrams sufficiently connected and proceed as above. Thus we have

Lemma 3.1.3. Suppose $F. = \{F_r\}_{r\geq 0}$ is a simplicial object in the category of reduced homotopy functors from (spaces) to (∞-loop spaces) = the category of basepointed infinite loop spaces. Suppose each F_r admits a minimal weight filtration and that the face and degeneracy maps of $F.$ are weight-preserving. Finally assume that $\rho(|F.|) \geq \rho(F_r)$ for each r . Then the Goodwillie Taylor series of $|F.|$ splits as a product of it's derivatives within the disk of convergence of $|F.|$. Moreover $D_n(|F.|) \simeq |[r] \longmapsto D_n(F_r)|$ for each $n \geq 0$.

Pf: The condition on $\rho(F_r)$ ensures that the Taylor series for F_r converges on the disk of convergence of $|F.|$ for each r . Since $F.$ is a simplicial ∞-loop space functor, the delooping arguement above shows that $T_n^k(|F.|)$ and $|T_n^k(F.)|$ are weakly equivalent for each r and k, where $T_n^k(F.)$ is the simplicial functor $\{r \longmapsto T_n^k(F_r)\}_{r\geq 0}$. Passing to the limit as $k \longrightarrow \infty$ yields a weak equivalence $P_n(|F.|) \simeq |P_n(F.)|$ for each r . Since the weight filtrations on F_r are compatible with the simplicial structure, lemma 1.2 yields equivalences

$$P_n(|F.|) \simeq |P_n(F.)| \simeq |\prod_{j=1}^{n} D_j(F.)| \simeq \prod_{j=1}^{n} |D_j(F.)| \simeq \prod_{j=1}^{n} D_j(|F.|)$$

for each n . Note that we are using the equivalence $D_j(|F.|) \simeq |D_j(F.)| = |[r] \longmapsto D_j(F_r)|$ which follows from Goodwillie's classification theorem for homogeneous functors.

$\square$

Remark 3.1.4 The conditon that $F.$ be a simplicial functor to (∞-loop spaces) rather than (spaces) is not really a restriction in the presence of a minimal weight filtration, since the resulting splitting of F_r for each r as a pro-duct of derivatives makes F_r an ∞-loop space functor within the disk of convergence of F_r.

§3.2 Some Applications

Let X be a basepointed space, $G \subset \Sigma_n$. We can consider the functor $F_G(X) = \Omega^\infty\Sigma^\infty(EG \times_G X^n)$, which is a homotopy functor on the category of basepointed spaces. G fixes the basepoint $(*,*,*,\ldots,*)$ of X^n, so the fibration $EG \times_G X^n \longrightarrow BG$ admits a section $s : BG \longrightarrow EG \times_G X^n$ determined by this base point. Let $EG \times_G X^n/BG$ denote the cofiber of s, and let $\overline{F}_G(X) = \Omega^\infty\Sigma^\infty(EG \times_G X^n/BG)$. $\overline{F}_G(X)$ is then a reduced homotopy functor and $F_G(X) \simeq \overline{F}_G(X) \times \Omega^\infty\Sigma^\infty(BG)$.

Proposition 3.2.1 Over the category of basepointed connected spaces $\overline{F}_G$ has degree n and splits as a product of it's derivatives. In particular, $(D_n\overline{F}_G)(X) = \Omega^\infty\Sigma^\infty(EG\wedge_G X^{[n]})$ naturally splits off of $\overline{F}_G(X)$. This splitting of $\overline{F}_G$ yields a splitting of the n^{th} delooping B^nF_G for all $n \geq 1$.

Pf: The proof is easy, and typical of the way in which the methods of the previous section apply. Let $\iota_j : X^j \longrightarrow X^n$ denote the embedding $\iota_j(x_1,\ldots,x_j) = (x_1,\ldots,x_j,*,*,\ldots,*)$. Let $\mathcal{F}_j(X^n)$ denote the orbit of $\iota_j(x_j)$ in X^n under the usual action of Σ_n which permutes entries. Let

$(\omega_j\overline{F}_G)(X) = \Omega^\infty\Sigma^\infty(\overline{\mathcal{F}}_j(X^n)) \subset \overline{F}_G(X)$ where $\overline{\mathcal{F}}_j(X^n) = EG \wedge_G(\mathcal{F}_j(X^n)_+)$. Certainly $\omega_0(F_G)(X) \simeq *$. Assume by induction that $\omega_r F_G(X)$ is of degree r and splits as a product of it's derivatives (at $*$).

Ω^{-1} fiber $(\omega_r \overline{F}_G(X) \longrightarrow \omega_{r+1}\overline{F}_G(X) \simeq \Omega^\infty\Sigma^\infty(EG\wedge_G\mathcal{F}_{r+1}(X^n)/\mathcal{F}_r(X^n))$.

$\mathcal{F}_{r+1}(X^n)/\mathcal{F}_r(X^n) \simeq \overset{k}{\vee}\, X^{[r+1]}$ where $k = \binom{n}{r+1}$. This is a homogeneous functor of degree $(r + 1)$ by ([G]). By induction the weight filtration is minimal, and after splitting delopable. The proposition follows.

$\square$

Of course, this splitting is known. We have included it as an example, as we have referred to it previous sections.

As another example, let $C_m = \{C_m(n)\}_{n\geq 0}$ denote the little n-cubes operad of Boardman-Vogt. $C_m(n)$ is a topological space via the standard function-space (compact-open) topology. Precomposition with an element $\sigma \in \Sigma_n$ yields a well-defined action of Σ_n on $C_m(n)$ given by $f \longmapsto f\circ\sigma$. Then n ordered inclusions $i^j : \underline{n-1} \longrightarrow \underline{n}$ = ordered set of n elements induce restriction maps $i_j : C_m(n) \longrightarrow C_m(n-1)$. The n ordered projection maps $h_j : \underline{n}_+ \longrightarrow \underline{n-1}_+$ $(\underline{m}_+ = \underline{m} \coprod \mathrm{pt})$ given by $h_j(k)$ = i) k if $k < j$, ii) $*$ if $k = j$, iii) $k-1$ if $k > j$ yield maps

$h^j : X^{n-1} = \mathrm{Map}_*(\underline{n-1}_+,X) \longrightarrow X^n = \mathrm{Map}_*(\underline{n}_+,X)$ given by $h^j(g) = g \circ h^j$. One can form the configuration space

$C(\mathbb{R}^m,X) = (\coprod_{n\geq 0} C_m(n) \times X^n/\sim$ with $\sim$ generated by the two compatible types of identifications : $(f,g \circ \sigma) \sim (f \circ \sigma, g)$, $(f,h^j(\bar{g})) \sim (i_j(f),\bar{g})$ for $f \in C_m(n)$, $g \in X^n$, $\bar{g} \in X^{n-1}$.

The approximation theorem yields a map $C(\mathbb{R}^m,X) \longrightarrow \Omega^m\Sigma^m$ which is an m-fold loop map and a weak equivalence (as before X is a basepointed connected space). $C(\mathbb{R}^m,X)$ is filtered by $\mathcal{F}^r_m(X) = (\coprod_{n\geq 0}^{r} C_m(n) \times X^n)/\sim \subset C(\mathbb{R}^m,X)$. The inclusion $\mathcal{F}^{r-1}_m(X) \xrightarrow{\iota^r_m} \mathcal{F}^r_m(X)$ is a closed cofibration with cofibre $\iota^r_m \simeq C_m(n) \wedge_{\Sigma_n} X^{[n]}$. Under the associative pairing $C(\mathbb{R}^m,X) \times C(\mathbb{R}^m,X) \xrightarrow{\theta} C(\mathbb{R}^m,X)$ induced by the action of the operad C_m, $\mathcal{F}^r_m(X) \times \mathcal{F}^s_m(X)$ maps to $\mathcal{F}^{r+s}_m(X)$. Using the monoid $C(\mathbb{R}^m,X)$ is place of $\Omega^m\Sigma^m X$ we have

<u>Lemma 3.2.2</u> Let X be a connected basepointed space. The filtration $\{\mathcal{F}^r_m(X)\}$ of $C(\mathbb{R}^m,X)$ induces minimal weight filtrations of the functors $X \longmapsto \Omega^\infty\Sigma^\infty(\Omega^m\Sigma^m X)$, $X \longmapsto \Omega^\infty\Sigma^\infty((\Omega^{m-1}\Sigma^m X)^{S^1})$ and $X \longmapsto \Omega^\infty\Sigma^\infty(ES^1 \times (\Omega^{m-1}\Sigma^m X)^{S^1}/BS^1)$.

<u>Pf:</u> The filtration of $\Omega^\infty\Sigma^\infty(\Omega^m\Sigma^m)$ by $\Omega^\infty\Sigma^\infty(\mathcal{F}^r_m(X))$ is minimal by the same type of argument as in the previous proposition and yields the Snaith splitting. Recall ([W2], §2) that for a grouplike monoid M, $|N^{cy}_\cdot(M)| \simeq (BM)^{S^1}.C(\mathbb{R}^n,X)$ is grouplike and so $|N^{cy}(C(\mathbb{R}^n,X))| \simeq (\Omega^{m-1}\Sigma^m X)^{S^1}.\Omega^\infty\Sigma^\infty(\_)$ commutes with geometric realization and so it suffices to show that $F_\cdot(X) = \{[p] \longmapsto \Omega^\infty\Sigma^\infty(N^{cy}_p(C(\mathbb{R}^m,X)))\}$ admits a minimal

weight filtration in each degree compatible with the simplicial structure. Let

(3.2.3)

$$\omega t_r(N_p^{cy}(C(\mathbb{R}^m,X))) = \bigcup_{\substack{i,\dots,i_{p+1}\\ \Sigma i_j=r}} \mathcal{F}_m^{i_1}(X)\times\dots\times\mathcal{F}_m^{i_{p+1}}(X) \subset N_p^{cy}(C(\mathbb{R}^m,X))$$

$\theta(\mathcal{F}_m^r(X) \times \mathcal{F}_m^s(X)) \subset \mathcal{F}_m^{r+s}(X)$ and so $\{[p] \longmapsto \omega t_r N_p^{cy}(C(\mathbb{R}^m,X))\}$ defines a well-defined simplical subset $\omega t_r N_{\cdot}^{cy}(C(\mathbb{R}^m,X))$ of $N_{\cdot}^{cy}(C(\mathbb{R}^m,X))$. Cofibre $(\omega t_{r-1} N_p^{cy}(C(\mathbb{R}^m,X)) \longrightarrow \omega t_r N_p^{cy}(C(\mathbb{R}^m,X)))$ is of the form $\bigvee_{G\subset\Sigma_r} E\Sigma_r \lambda_G X^{[r]}$ where the wedge is over all G of the form $\Sigma_{i_1}\oplus\dots\oplus\Sigma_{i_{p+1}} \subset \Sigma_r$ $(\Sigma i_j = r)$, and we have seen that $X \longmapsto \Omega^\infty\Sigma^\infty(E\Sigma_r\lambda_G X^{[r]})$ is homogeneous of degree r. Hence the weight filtration is minimal in each degree and lemma 3.1.3 applies to split $\Omega^\infty\Sigma^\infty((\Omega^{m-1}\Sigma^m X)^{S^1})$ as a product of it's derivatives. For the last functor, we use the result of Dunn and Fiedorwicz [DF] which provides a configuration-space model for $[ES^1 \times_{S^1} X^{S^1})/BS^1$ for connected X. To state their result, let $E\mathbb{Z}/*+1$ denote the cyclic space $\{[p] \longmapsto E(\mathbb{Z}/p+1)\}$ ([DF], Example 1) with cyclic simplicial structure induced by the cyclic simplicial structure on the crossed simplicial group $\mathbb{Z}/*+1 = \{[p] \longmapsto \mathbb{Z}/p+1\}$ (in the sense of Fiederowicz and Loday) whose standard realization is $\cong S^1$. Any cyclic space can be viewed as a cocyclic space by precomposition with the duality isomorphism $D : \Lambda(C_*) \xrightarrow{\cong} \Lambda(C_*)^{op}$ of Connes ([C1], where $\Lambda(C_*)$ is denoted Λ). Via this identifcation, one can form the tensor product (over $\Lambda(C_*)$) of $E\mathbb{Z}/*+1$ and a cyclic simplicial space S_*, resulting in a space

$$E\mathbb{Z}/*+1 \otimes_{\Lambda(C_*)} S_* = \coprod_{n\geq 0} E\mathbb{Z}/n+1 \times S_n/\sim, \quad (\lambda^*(x),y) \sim (x,(D\lambda)^*y)$$

for $x \in E\mathbb{Z}(n+1)$, $y \in S_m$ and $\lambda : [m] \longrightarrow [n]$ a morphism in $\Lambda(C_*)$. Dunn and Fiedorowicz prove

<u>Theorem 3.2.4</u> ([DF], p.8) Let S_* be a cyclic space. Then $E\mathbb{Z}/*+ \otimes_{\Lambda(C_*)} S_*$ is equivalent to the pushout of the diagram

$$\begin{array}{ccc} BS^1 \times \mathrm{Fix}(|S_*|) & \longrightarrow & ES^1 \times_{S^1} |S_*| \\ \downarrow & & \\ \mathrm{Fix}\ (|S_*|) & & \end{array}$$

where $\mathrm{Fix}(|S_*|)$ is the S^1 fixed-point set of $|S_*|$ (with S^1-action induced by the cyclic structure on S_*, as in [BF]).

□

In particular taking S_* to be $N_\cdot^{cy}(C(\mathbb{R}^m,X))$ with the usual cyclic structure we get

$ES^1 \times_{S^1} (\Omega^{m-1}\Sigma^m X)^{S^1}/BS^1 \simeq ES^1 \times_{S^1} |(N_\cdot^{cy}(C/\mathbb{R}^m,X))|/BS^1 \simeq$ $E\mathbb{Z}/*+1 \otimes_{\Lambda(C_*)} N_*^{cy}(C(\mathbb{R}^m,X))$. Now $E\mathbb{Z}/*+1 \otimes_{\Lambda(C_*)} S_*$ is filtered by $\{g^r(S_*) = \coprod_{n=0}^{r} E\mathbb{Z}/n+1 \times S_n/\sim\}$. Denote $N_\cdot^{cy}(C(\mathbb{R}^m,X))$ temporarily by S_*. The above filtration (3.2.3) is invariant under the cyclic structure on S_*, and hence induces a cyclic simplicial filtration $\{wt_r S_*\}$ of S_* (functiorial in X). In order to show that $\{\Omega^\infty\Sigma^\infty(E\mathbb{Z}/*+1 \otimes_{\Lambda(C_*)} wt_r S_*)\}$ is a minimal weight filtration we note it suffices to prove the homogeneity of the fibre of

$(\Omega^\infty\Sigma^\infty(g^n(wt_{r-1}S_*)/g^{n-1}(wt_{r-1}S_*)) \to \Omega^\infty\Sigma^\infty(g^n(wt_r S_*)/g^{n-1}(wt_r S_*)$.

This latter space can be written as

$$\Omega^\infty\Sigma^\infty\Big(\bigvee_{\substack{i_1,\dots,i_{n+1}\\ \Sigma i_j=r,\, i_j\geq 1}} E\mathbb{Z}/n+1 \,\lambda_{\mathbb{Z}/n+1}(C_m(i_1)\lambda_{\Sigma_{i_1}} X^{[i_1]})\wedge\dots$$
$$\wedge(C_m(i_{n+1})\lambda_{\Sigma_{i_{n+1}}} X^{[i_1]})\Big)$$

Each term in the wedge sum is of the form $E\Sigma_r\lambda_G(A_G \wedge X^{[r]})$ for some $G \subset \Sigma_r$ and A_G a G-space where we take the diagonal G-action on $A_G \wedge X^{[r]}$. Again, the functors $X \longmapsto \Omega^\infty\Sigma^\infty(E\Sigma_r\lambda_G(A_G \wedge X^{[r]}))$ are homogeneous by [G1]. This completes the proof.

□

<u>Remarks 3.2.6</u> i) The above theorem applies more generally, by the same arguments, with $\Omega^n\Sigma^{n+1}(X)$ replaced by a functor $F(X)$ satisfying the property that $\Omega F(X)$ admits a filtration $\{\mathcal{F}_r\Omega F(X)\}$ functorial in X such that $\mathcal{F}_r F(X)/\mathcal{F}_{r-1}F(X) \simeq A_r\lambda_{\Sigma_r} X^{[r]}$ for some Σ_r-space A_r.

ii) For $n = 1$ the splitting of $\Omega^{\infty}\Sigma^{\infty}(ES^1 \times_{S^1} (\Sigma X)^{S^1}/BS^1$ as a product of derivatives was one the main results of [CC]. Also, many of these splittings have been obtained by C.F. Bödingheimer. The techniques described here can be used to recover his results.

iii) These techniques can be "equivariantized" to yield equivariant splitting theorems for the functors described in lemma 3.2. In the simplest case one recovers the equivariant Snaith splitting of $\Omega^V\Sigma^V{}_G(\Omega^n\Sigma^n X)$ proved by Lewis, May and Steinberger for a compact Lie group G and basepointed G-space X.

It is often the case that there exists a number of different weight filtrations on a given homotopy functor, which yield different splittings. This occurs in particular when F can be written as an iterated composition of homotopy functors $G_1 \circ \ldots \circ G_m$ where each G_i has a natural weight filtration. The result in this case is a sequence of filtrations of F, each of which is a refinement of the one previous.

As an example, one could consider the homotopy functor $F(X) = \Omega^{\infty}\Sigma^{\infty}(\Sigma^r E\Sigma_q \lambda_{G_q} [(\Omega^m\Sigma^m) \circ \ldots \circ (\Omega^m\Sigma^m)(X)]^{[q]}$, where $G_q \subset \Sigma_q$ acts on the q-fold smash product on the right by the usual action of Σ_q. $F(X)$ is really then a homogeneous functor of degree q evaluated on $(\Omega^m\Sigma^m)^{(s)}(X)$. Considered as a functor in X not it is no longer homogeneous. There are s different weight filtrations that we could construct on F, the i^{th} one arising from using the lexicographic ordering on the pro-duct of the filtration of $\Omega^m\Sigma^m(\_)$ described above for the first i terms in the composition. In constructing such a filtration one needs to start with a product filtration of $((\Omega^m\Sigma^m)^{(s)}(X))^{[q]}$ and then group by orbits under the action of G_q to get a G_q-equivariant filtration of $((\Omega^m\Sigma^m)^{(s)}(X))^{[q]}$. Starting with the finest filtration of $(\Omega^m\Sigma^m)^{(s)}(X)$ which uses all s copies of $\Omega^m\Sigma^m$ and essentially just measures word-length in X, the result is a weight filtration of $F(X)$ which is minimal - the argument is exactly as in lemma 3.2.2 above. One uses the filtration of $[(\Omega^m\Sigma^m)^{(s)}(X)]^{[q]}$ made G_q-equivariant to yield a description of $E\Sigma_q \lambda_{G_q}((\Omega^m\Sigma^m)^{(s)}(X))^{[q]}$ up to homotopy as an iterated co-fibration sequence where the n^{th} subquotient is of the form $E\Sigma_n \lambda_{\Sigma_n} A_n \wedge X^{[n]}$ for some Σ_n-space A_n. So we conclude

<u>Corollary 3.2.6</u> $F(X) = \Omega^\infty\Sigma^\infty(\Sigma^r E\Sigma_q \lambda_{G_q}((\Omega^m\Sigma^m)^{(s)}(X))^{[q]})$ splits as a product of it's derivatives (at a point) for all $r,m,s,q \geq 1$ and connected X. The splitting is, moreover, functorial in X.

□

We will want an explicit description of this splitting in the case $r = 1$, $G_q = \mathbb{Z}/q$, $m = s = 1$. For fixed n and q, let S_n^q = set of the equivalence classes of q-tuples $[i_1,\ldots,i_q]$, where $i_j \geq 1$ and $\Sigma i_j = n$, with the equivalence relation $[i_1,\ldots,i_q] = [i_1',\ldots,i_q']$ if there exist $\tau \in \mathbb{Z}/q$ such that $i_j' = i_{\tau(j)}$ for all $1 \leq j \leq q$. $\mathbb{Z}/q$ acts on the indices $\{1,\ldots,q\}$ by cyclic permutation: if m_τ is the integer corresponding to τ under the usual identification of $\mathbb{Z}/q$ with $\{0,1,2,\ldots,q-1\}$ as a set, then

$\tau(j) = (j+m_\tau) - [\frac{j+m_\tau-1}{q}]q$, $[r]$ = the largest integer $\leq r$.

Under this action of $\mathbb{Z}/q$, the set of q-tuples $\tilde{S}_n^q = \{(i_1,\ldots,i_q) \mid i_j \geq 1, \ \Sigma i_j = n\}$ break up into orbit types: the orbited a q-tuple $(i_1,\ldots,i_q)$ will contain q'elements if and only if $\frac{q}{q'}$ is an integer dividing n and $(i_1,\ldots,i_q)$ is in the equivalence class of

$(i_1,\ldots,i_{q'})^{q/q'} = (i_1,i_2,\ldots,i_{q'},i_1,\ldots,i_{q'},\ldots,i_1,i_2,\ldots,i_{q'})$.

For such q' let $\tilde{S}_n^q(q')$ denote the subset of such q-tuples and $S_n^q(q')$ the equivalence class of such q-tuples. Note that the projection map $\tilde{S}_n^q(q') \xrightarrow{\pi_n^q} \tilde{S}_n^q(q')/\mathbb{Z}/q = S_n^q(q')$ is a principle $\mathbb{Z}/q'$-fibration of sets. Now the word-length filtration of $J(X) \simeq \Omega\Sigma X$ induces a product filtration on $(JX)^{[q]}$, which when made $\mathbb{Z}/q$-equivariant and extended to a filtration of $E\mathbb{Z}/q\lambda_{\mathbb{Z}/q}(JX)^{[q]}$ describes $E\mathbb{Z}/q\lambda_{\mathbb{Z}/q}(JX)^{[q]}$ as an iterated cofibration whose n^{th} subquotient is of the form:

$$(3.2.7)\qquad \mathcal{F}_n(E\mathbb{Z}/q\lambda_{\mathbb{Z}/q}(JX)^{[q]})/\mathcal{F}_{n-1}(E\mathbb{Z}/q\lambda_{\mathbb{Z}/q}(JX)^{[q]})$$

$$\simeq \bigvee_{I\in S_n^q} E\mathbb{Z}/q\ \lambda_{\mathbb{Z}/q}X(I) = \bigvee_{\substack{q' \\ q'|q \text{ and } \frac{q}{q'}|n}} \ \bigvee_{I\in S_n^q(q')} E\mathbb{Z}/q\ \lambda_{\mathbb{Z}/q}X(I)$$

where, for an equivalence class $I \in S_n^q(q')$, $X(I)$ is the wedge $\bigvee_{(i_1,\dots,i_q)\in(\pi_n^q)^{-1}(I)} X^{[i_1]} \wedge X^{[i_2]} \wedge \dots \wedge X^{[i_q]}$.

Note that as <u>spaces</u> all of the terms in the wedge describing $X(I)$ are the same. However the $\mathbb{Z}/q$ action or $(JX)^{[q]}$ induces a free basepointed $\mathbb{Z}/q'$-action on this wedge (for $I \in S_n^q(q')$) which cyclically permutes the terms.

Application of $\Omega^\infty\Sigma^\infty(\Sigma\_)$ to the subquotient in (3.2.7) yields a homogeneous functor of degree n in X. So in this case the splitting of Corollary 3.2.6 is given explicitly by

$$(3.2.8]\qquad \Omega^\infty\Sigma^\infty(\Sigma E\mathbb{Z}/q\ \lambda_{\mathbb{Z}/q}(JX)^{[q]})$$

$$\simeq \Omega^\infty\Sigma^\infty(\Sigma(\bigvee_{n\geq 1}(\bigvee_{\substack{q' \\ q'(q \text{ and } \frac{q}{q'}|n}}(\bigvee_{I\in S_n^q(q')} E\mathbb{Z}/q\lambda_{\mathbb{Z}/q}X(I))))).$$

Of course, up to weak equivalence natural in X this can also be written as a product

$$(3.2.9) \qquad \prod_{n\geq 1} \Big(\prod_{\substack{q' \\ q'|q \text{ and } \frac{q}{q'}|n}} \Big(\prod_{I\in S_n^q(q')} F_I(X)\Big)\Big)$$

where $F_I(X) \overset{\text{def}}{=} \Omega^\infty\Sigma^\infty(\Sigma E\mathbb{Z}/q\wedge_{\mathbb{Z}/q}X(I))$. We will, finally, want to know about the 1^{st} derivative of F_I at an arbitrary space. As before $(D_1F)_X(Y)$ denotes $(D_1F)_X(X \vee Y, r : X \vee Y \longrightarrow X)$.

Proposition 3.2.10 Let $m = \frac{nq'}{q} = \frac{n}{(\frac{q}{q'})}$ with n,q and q' as above. Then there is a natural inclusion

$$F_I(X;Y) \overset{\text{def}}{=} \Omega^\infty\Sigma^\infty(\Sigma(\bigvee_{i=1}^{m} X^{[i-1]} \wedge Y \wedge X^{[m-i]}) \wedge (X^{[m]})^{[\frac{q}{q'}-1]})$$

$\longrightarrow F_I(X \vee Y)$ for $I \in S_n^q(q')$ such that the induced map

$$F_I(X,Y) \longrightarrow F_I(X \vee Y) \to \text{fibre}(F_I(X \vee Y) \longrightarrow F_I(X)) \longrightarrow (D_1F_I)_X(Y)$$

is an equivalence.

Pf: This is another example of how $E\mathbb{Z}/q\wedge_{\mathbb{Z}/q}(\_)$ "divides by q" upon passage to 1^{st} derivatives. One simply has to look at how $\mathbb{Z}/q$ acts on $X(I)$. First, there are q' terms in the wedge describing (X), and $\mathbb{Z}/q$ cylically permutes these terms via the epimorphism $\pi : \mathbb{Z}/q \twoheadrightarrow \mathbb{Z}/q'$. This has the effect of "dividing by q'" which can be realized on the level of spaces before passing to 1^{st} derivatives by the equivalence

(3.2.11)

$$E\mathbb{Z}/q\lambda_{\mathbb{Z}/q}\Big(\bigvee_{(i_1,\ldots,i_q)\in(\pi_n^q)^{-1}(I)} X^{[i_1]}\wedge\ldots\wedge X^{[i_q]}\Big) \cong E\mathbb{Z}/p\lambda_{\mathbb{Z}/p}(X^{[i_1]}\wedge\ldots\wedge X^{[i_{q'}]})^{[p]}$$

where $p = \frac{q}{q'}$, and $\mathbb{Z}/p \cong \ker(\pi:\mathbb{Z}/q \twoheadrightarrow \mathbb{Z}/q')$ acts on $(X^{[i_1]} \wedge\ldots\wedge X^{[i_{q'}})^{[p]}$ by cyclically permuting the copies of $(X^{(i_1]}\wedge\ldots\wedge X^{[i_{q'}]})$. Writing $X^{[i_1]}\wedge\ldots\wedge X^{[i_{q'}]})$, as $X^{[m]}$, it follows as in proposition 1.4.4 iii) that the inclusion

$$(3.2.12)\quad \Big(\bigvee_{i=1}^{m} X^{[i-1]} \wedge Y \wedge X^{[m-1]}\Big) \wedge (X^{[m]})^{[p-1]} \longrightarrow$$

$$\longrightarrow ((X \vee Y)^{[m]})^{[p]} \longrightarrow E\mathbb{Z}/p\lambda_{\mathbb{Z}/p}((X \vee Y)^{[m]})^{[p]}$$

induces the equivalence described in the statement of the propostion after applying $\Omega^\infty\Sigma^\infty(\Sigma\_)$ and passing to $(D_1F_I)_X(Y)$ on the right.

□

This corollary will be used in part III to extend the computation of $\overline{\mathrm{Tr}}_X(Y)$, in much the same way proposition 1.4.4 iii) was used in §2.3 when we computed the trace on the image of $\tilde{\rho}_q$ after passing to first derivatives.

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