ON THE DIMENSION OF SOME LINEAR SPACES OVER $\mathbb{Q}$ (Part 5)

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§0. Foreword.

The section 1 has a preparatory character.

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Acknowledgments. I express my deepest thanks to Professors B.Z.Moroz, I.I. Piatetski-Shapiro, A.G.Aleksandrov, P.Bundshuh and S.G.Gindikin for help and support.

§1. Some simple properties of the binomial coefficients.

For each prime $p \in \mathbb{N}$ let v_p denotes the p-adic valuation on $\mathbb{Q}$.

Lemma 1. *Let $p \in \mathbb{N}+2$ is a prime number, $\{d, r\} \subset \mathbb{N}-1$, $r < p$.*

Then $v_p((dp+r)!/((-p)^d d!r!) - 1) \geq 1$.

Proof. The assertion of the Lemma is the direct corollary of the Wilson's theorem. ■.

Lemma 2. *Let $p \in \mathbb{N}+2$ is a prime number, $\{d, d_1\} \subset \mathbb{N}-1$,*

$$\{r, r_1\} \in [0, p-1] \cap \mathbb{Z},\ d_1 p + r_1 \leq dp + r. \tag{1}$$

Then

$$v_p\left(\binom{dp+r}{d_1 p + r_1}\right) = v_p\left(\binom{d}{d_1}\right), \tag{2}$$

if $r_1 \leq r$,

$$v_p\left(\binom{dp+r}{d_1 p + r_1}\left(\binom{d}{d_1}\binom{r}{r_1}\right)^{-1} - 1\right) \geq 1, \tag{3}$$

if $r_1 \le r$,

$$v_p\left(\binom{dp+r}{d_1p+r_1}\right) = 1 + v_p\left((d-d_1)\binom{d}{d_1}\right), \tag{4}$$

if $r < r_1$,

$$v_p\left(\binom{dp+r}{d_1p+r_1}\binom{r_1}{r}(r_1-r)\left(p\binom{d}{d_1}(d-d_1)\right)^{-1} + (-1)^{r_1-r}\right) \ge 1, \tag{5}$$

if $r < r_1$.

Proof. Clearly, $d_1 \le d$. If $r_1 \le r$, then let $r_2 = r - r_1$, $d_2 = d - d_1$. If $r_1 > r$, then, in view of (1), $d \ge d_1 + 1$; in this case we let

$$r_2 = p + r - r_1,\ d_2 = d - d - 1. \tag{6}$$

Then $d = d_1 + d_2$, $r = r_1 + r_2$,

$$\binom{dp+r}{d_1p+r_1} = ((dp+r)!)/((d_1p+r_1)!(d_2p+r_2)!).$$

Accordindg to the Lemma 1,

$$v_p\left(\binom{dp+r}{d_1p+r_1}(-p)^{-d+d_1+d_2}(d_1)!(r_1)!(d_2)!(r_2)!/(d!r!) - 1\right) \ge 1, \tag{7}$$

$$v_p\left(\binom{dp+r}{d_1p+r_1}\right) = d - d_1 - d_2 + v_p\left(d!r!)/((d_1)!(r_1)!(d_2)!(r_2)!)\right). \tag{8}$$

The equality (2) and the inequality (3) diectly follow from (7) and (8). If the inequality $r < r_1$ holds, then in view of (6) – (8),

$$r_2! \prod_{j=1}^{r_1-r-1}(p + r - r_1 + j) = (p-1)!,\ v_p(r_2!(r_1-r-1)!(-1)^{r_1-r} - 1) \ge 1.$$

Therefore (4) and (5) hold. ■.

Corollary 1. *Let* $p \in \mathbb{N} + 2$ *is a prime number,*

$$\{d,\, d_1,\, d_2,\, r,\, r_1,\, r_2\} \subset \mathbb{N} - 1,\ max(r_1, r_2) < p.$$

Then

$$p^{-d}(dp+r)! \in (-1)^d d!r! + p\mathbb{Z},$$

$$\binom{(d_1+d_2)p + r_1 + r_2}{d_1p + r_1} \in \binom{d_1+d_2}{d_1}\binom{r_1+r_2}{r_1} + p\mathbb{Z}.$$

Proof. This is direct corollary of the Lemmata 1 and 2.
See also Lemma 9 in [32]. ■

Corolary 2. *Let $p \in \mathbb{N}+2$ is a prime number,*

$$\{d,\, r_1\} \subset \mathbb{N},\, r_1 < p,\, d_1 \in \mathbb{N}-1,\, d_1 < d.$$

Then

$$v_p\left(\binom{dp}{d_1p+r_1}\left(d\binom{d-1}{d_1}\binom{p}{r_1}\right)^{-1}+1\right) \geq 1. \tag{9}$$

Proof. Since,

$$d\binom{d-1}{d_1} = (d-d_1)\binom{d}{d_1},\; v_p\left(\binom{p}{r_1}r_1/p-(-1)^{r_1}\right) \geq 1,$$

the equality (9) directly follows from (5). ■

Corolary 3. *Let $p \in \mathbb{N}+2$ is a prime number,*

$$\{d,\, r_1\} \subset \mathbb{N},\, r_1 < p,\, d_1 \in \mathbb{N}-1,\, d_1 < d.$$

Then

$$\binom{dp}{d_1p+r_1} \in d\binom{d-1}{d^{\sim}}\binom{p}{r_1} + p^2\mathbb{Z}.$$

Proof. This is a corollary of the Corrolary 2. See also Lemma 10 in [32] . ■

Lemma 3. *Let $p \in \mathbb{N}+2$ is a prime number, $\alpha \in \mathbb{N}-1$, $\{r_i,\, \rho_i\} \subset [0, p-1] \cap \mathbb{Z}$, where $i = 0, \ldots, \alpha$,*

$$m = \sum_{i=1}^{\alpha} \rho_i p^i, n = \sum_{i=1}^{\alpha} r_i p^i,\, m \leq n. \tag{10}$$

Let $h_i(m,n) = 0$, if $\rho_i \leq r_i$ for some $i \in [0,\alpha] \cap \mathbb{Z}$ (in particular, for $i = \alpha$). But, if $\rho_i > r_i$ for some $i \in [0,\alpha) \cap \mathbb{Z}$, then $\{h \in [1, \alpha - i] \cap \mathbb{Z}: \rho_{i+h} \neq r_{i+h}\} \neq \emptyset$ (in view of (10)), and we put $h_i(m,n) = \inf(\{h \in [1, \alpha - i] \cap \mathbb{Z}: \rho_{i+h} \neq r_{i+h}\})$. Then

$$v_p\left(\binom{n}{m}\right) = \sum_{i=1}^{\alpha} h_i(m,n). \tag{11}$$

Proof. We use induction on α. Clearly, for $\alpha = 0$ the assertion of the Lemma is obvious. Let $\alpha > 0$ and the assertion of the Lemma is true for all the $\alpha' \in [0,\alpha) \cap \mathbb{Z}$. If $h_i(m,n) = 0$ for $i = 0, \ldots, \alpha$, then applying (3) sequentially, we see that

$$v_p\left(\binom{n}{m}\right) = 0 = \sum_{i=1}^{\alpha} h_i(m,n).$$

If $\{i \in \{0, \dots, \alpha\}\colon h_i(m,n) \neq 0\} \neq \emptyset$, then let

$$\beta = \inf(\{i \in \{0, \dots, \alpha\}\colon h_i(m,n) \neq 0\}).$$

In view of (10), $\beta < \alpha$. Let $\alpha^* = \alpha - \beta$,

$$\rho_i^* = \rho_{\beta+i},\ r_i^* = r_{\beta+i},$$

where $i = 0, \dots, \alpha^*$. Let further

$$m^* = \sum_{i=\beta}^{\alpha} \rho_i p^{i-\beta} = \sum_{i=0}^{\alpha^*} \rho_i^* p^i,\ n^* = \sum_{i=\beta}^{\alpha} r_i p^{i-\beta} = \sum_{i=0}^{\alpha^*} r_i^* p^i$$

Since $n^* - m^* = n - m$, it follows from (10) that $n^* \geq m^*$. Applying (3) sequentially, we see that

$$v_p\left(\binom{m}{n}\right) = v_p\left(\binom{m^*}{n^*}\right) = v_p\left(\binom{m^*}{n^*}\right) + \sum_{i=0}^{\beta-1} h_i(m,n). \tag{12}$$

When $i \in [0, \alpha^*] \cap \mathbb{Z}$, then the iequalities $\rho_i^* \leq r_i^*$ and $\rho_{\beta+i} \leq r_{\beta+i}$ are equivalent; in this case $h_i(m^*, n^*) = h_{\beta+i}(m,n) = 0$. When $i \in [0, \alpha^*) \cap \mathbb{Z}$, then $\rho_i^* > r_i^*$ if and only if $\rho_{\beta+i} > r_{\beta+i}$, $h \in [0, \alpha^* - i) \cap \mathbb{Z}$ if and only if $h \in [0, \alpha - \beta - i) \cap \mathbb{Z}$, the inequalities $\rho_{i+h+1}^* \neq r_{i+h+1}$ and $\rho_{\beta+i+h+1}^* \neq r_{\beta+i+h+1}$ are equivalent; therefore, if $\rho_i > r_i$ for some $i \in [0, \alpha^*) \cap \mathbb{Z}$, then $h_i(m^*, n^*) = h_{\beta+i}(m,n)$. We take now on the role of r, r_1, d, d_1 in the Lemma 2 respectively the numbers

$$\rho_\beta,\ r_\beta,\ m' = (m^* - \rho_\beta)/p = \sum_{i=0}^{\alpha'} \rho_i' p^i,\ n' = \sum_{i=0}^{\alpha'} r_i' p^i,$$

where $\alpha' = \alpha^* - 1$, $\rho_i' = \rho_{1+i}^*, r_i' = r_{1+i}, i = 0, \dots, \alpha'$. In view of (5),

$$v_p\left(\binom{n^*}{m^*}\right) = 1 + v_p(n' - m') + v_p\left(\binom{n'}{m'}\right) = h_0(m^*, n^*) + \tag{13}$$

$$v_p\left(\binom{n'}{m'}\right) = h_\beta(m,n) + v_p\left(\binom{n'}{m'}\right) = v_p\left(\binom{n'}{m'}\right) + \sum_{i=0}^{\beta} h_i(m,n).$$

Clearly, $h_i(m', n') = h_{1+i}(m^*, n^*) = h_{1+\beta+i}(m,n)$, where $i = 0, \dots, \alpha'$.

Since $\alpha' = \alpha^* - 1 < \alpha$, it follows that, according to the assumption of induction,

$$v_p\left(\binom{n'}{m'}\right) = \sum_{i=0}^{\alpha'} h_i(m', n') = \sum_{i=1}^{\alpha^*} h_i(m^*, n^*) = \sum_{i=\beta+1}^{\alpha} h_i(m,n). \tag{14}$$

In view of (14), (13) and (12), the equality (11) holds. ■.

§2. Chudnovsky-Hata's multipliers.

Let $\nu \in \mathbb{N}$, $\Delta \in \mathbb{N}$, $\Delta \geq 2$. Let us consider the numbers

$$\binom{\nu\Delta}{k}\binom{\nu\Delta+k}{\nu\Delta-\nu}, \tag{15}$$

where $k = 0, \ldots, \nu\Delta$. How small may be chosen the number $r(\nu) \in (0, +\infty) \cap \mathbb{Q}$ such that $r(\nu)\binom{\nu\Delta}{k}\binom{\nu\Delta+k}{\nu\Delta-\nu} \in \mathbb{Z}$ for all the $k = 0, \ldots, \nu\Delta$? Probably G.V. Chudnovsky was the first man, who discovered, that $r(\nu)$ may be chosen sufficiently small; Hata [16] in details studied this effect. Therefore I name such $r(\nu)$ by Chudnovsky-Hata's multiplier. In this section I follow Hata's paper [16], making use also some results of the section 1. Since $\mathbb{Z}$ contains the numbers (15), it follows that we can seek $r(\nu)$ in the form

$$r(\nu) = r^*(\nu) = 1/d_1^*(\Delta, \nu), \tag{16}$$

where $d_1^*(\Delta, \nu)$, is some common divisor of the numbers (15). According to the Lemma 2, the prime number $p \in \mathbb{N}$ is a divisor of all the numbers (15) if and only if, when

$$0 < \max\left(\{k/p\} - \{\nu\Delta/p\}, \{\nu(\Delta-1)/p\} - \{(k+\nu\Delta)/p\},\right) \tag{17}$$

where $k = 0, \ldots, \nu\Delta$. Let us to prove that the condition (17) is equivalent to the inequality

$$2\{\nu\Delta/p\} < \{\nu(\Delta-1)/p\}. \tag{18}$$

In fact, if, in contrary to (18),

$$\{\nu\Delta/p\} \geq k/p = \max(\{\nu(\Delta-1)/p\} - \{\nu\Delta/p\}, 0), \tag{19}$$

then $\{k/p\} = k/p$, $1 > \max(\{\nu(\Delta-1)/p\}, \{\nu\Delta/p\}) = \{\nu\Delta/p\} + \{k/p\}$ and therefore $\{\nu\Delta/p\} + \{k/p\} = \{(\nu\Delta+k)/p\}$,

$$\{(\nu\Delta+k)/p\} = \max(\{\nu(\Delta-1)/p\}, \{\nu\Delta/p\}) \geq \{\nu(\Delta-1)/p\}$$

with $k \in [0, \nu\Delta] \cap \mathbb{Z}$; but the last inequality, in view of (23), implies the inequality $\{\nu\Delta/p\} < \{k/p\}$, which contradicts to (19). Let the inequality (18) holds and let $\{\nu\Delta/p\} \geq \{k/p\}$ for some $k \in [0, \nu\Delta] \cap \mathbb{Z}$. Then, in view of (18),

$$\{\nu\Delta/p\} + \{k/p\} \leq 2\{\nu\Delta/p\} < \{\nu(\Delta-1)/p\} \leq (p-1)/p < 1,$$

$$\{(\nu\Delta+k)/p\} = \{\nu\Delta/p\} + \{k/p\} < \{\nu(\Delta-1)/p\},$$

and therefore (17) holds. So, if (18) holds, then p is divisor of all the numbers (15). We will study the set

$$X^*_\Delta = \{x \in \mathbb{R}\colon 2\{x\Delta\} < \{x(\Delta-1)\}\,\}. \tag{20}$$

Clearly, $X^*_\Delta \cap \mathbb{Z} = \emptyset$ and $m + X^*_\Delta = X^*_\Delta$, for any $m \in \mathbb{Z}$. Therefore it is suffiient to study the set $X^*_\Delta \cap (0,1)$. Let

$$\xi \in X^*_\Delta \cap (0,1), \tag{21}$$

and let $D^* \in \Delta(\Delta-1)\mathbb{N}$. Let

$$\xi \in [\kappa/D^*, (\kappa+1)/D^*) \tag{22}$$

for some $\kappa \in \{0, \ldots, D^*-1\}$. Then

$$\kappa/(D^*/(\Delta-\epsilon)) \le \xi\Delta < (\kappa+1)/(D^*/(\Delta-\epsilon)),$$

$$[\kappa/(D^*/(\Delta-\epsilon))] \le [\kappa/(D^*/(\Delta-\epsilon))] + \{\kappa/(D^*/\Delta)\} \le \xi\Delta <$$
$$[\kappa/(D^*/(\Delta-\epsilon))] + \{\kappa/(D^*/(\Delta-\epsilon))\} + 1/(D^*/(\Delta-\epsilon)) \le$$
$$[\kappa/(D^*/(\Delta-\epsilon))] + (D^*/(\Delta-\epsilon)-1)/(D^*/\Delta) + 1/(D^*/(\Delta-\epsilon) =$$
$$[\kappa/(D^*/(\Delta-\epsilon))] + 1,$$

where $\epsilon^2 = \epsilon$. Consequently, $[\kappa/(D^*/(\Delta-\epsilon))] = [\xi(\Delta-\epsilon)]$,

$$\xi(\Delta-\epsilon) = \xi(\Delta-\epsilon) - [\xi(\Delta-\epsilon)] = \xi(\Delta-\epsilon) - [\kappa/(D^*/(\Delta-\epsilon))],$$

Therefore, the existence of $\kappa \in [0, D^*-1) \cap \mathbb{Z}$ such that the inclusion (22) and the inequality

$$2\xi\Delta - 2[\kappa/(D^*/\Delta)] < \xi(\Delta-1) - [\kappa/(D^*/(\Delta-1))] \tag{23}$$

hold is equivalent to the condition (21). The inequality (23) is equivalent to the inequality

$$\xi < \kappa/D^* + ((\{\kappa/(D^*/(\Delta-1))\} - 2\{\kappa/D^*/\Delta)]\})(\Delta+1)^{-1}. \tag{24}$$

Let us note that for $\kappa = 0$ the system (22) – (24) has no solutions. Let

$$\eta^*(\kappa) = +(\{\kappa/(D^*/(\Delta-1))\} - 2\{\kappa/(D^*/\Delta)]\})(\Delta+1)^{-1},$$

$$\eta^*_0(\kappa) = \max(0, \min(\eta^*(\kappa), 1/D^*)) = \max(0, \eta^*(\kappa)),$$

where $\kappa \in \{0, \ldots, D^*-1\}$. Then the system of conditions (22) – (24) is equivalent to the condition

$$\xi \in [\kappa/D^*, \kappa/D^* + \eta^*_0(\kappa)). \tag{25}$$

So, if $\nu/p = m + \xi$, where

$$m \in [0, (\nu/(2\Delta))^{1/2} - 1] \cap \mathbb{Z},\ \kappa \in \mathbb{Z},\ 1 \le \kappa \le D - 1,$$

$$\xi \in (0, 1),\ \kappa/D^* \le \xi < \kappa/D^* + \eta_0^*(\kappa),$$

i.e. if

$$\nu/(m + \kappa/D^* + \eta_0^*(\kappa)) < p \le \nu/(m + \kappa/D^*), \tag{26}$$

where

$$m \in [0, (\nu/(\Delta+1))^{1/2} - 1] \cap \mathbb{Z},\ \kappa \in \mathbb{Z},\ 1 \le \kappa \le D^* - 1, \tag{27}$$

then the condition $p > (2\nu\Delta)^{1/2}$ and the condition (24) are fulfilled. We take on the role of $d_1^*(\nu)$ the product of all the prime nubers p, which satisfy to the conditions (26) – (27). It is well known that there exists $a > 0$, such that

$$\psi(x) = \sum_{p \le x} = x(1 + O(\exp(-a(log(x))^{1/2})), \tag{28}$$

where $x \ge 2$. Therefore

$$\log(d_1^*(\nu)) = \tag{29}$$

$$\sum_{\kappa=1}^{D^*-1} \sum_{m=0}^{[(\nu/(\Delta+1))^{1/2}-1]} (\psi(\nu/(m + \kappa/D^*)) - \psi(\nu/(m + \kappa/D^* + \eta_0^*(\kappa))).$$

In view of (26) – (28) we have

$$\psi(\nu/(m + \kappa/D^* + \eta_0(\kappa))) = \nu/(m + \kappa/D^* + \eta_0^*(\kappa)) + \tag{30}$$

$$O((\nu/(m+1))(\exp(-(a/\sqrt{2})(log(\nu))^{1/2}),$$

$$\psi(\nu/(m + \kappa/D^*)) = \nu/(m + \kappa/D^*) + \tag{31}$$

$$O((\nu/(m+1))\exp(-(a/\sqrt{2})(log(\nu))^{1/2}).$$

But

$$\sum_{m=0}^{[(\nu/(\Delta+1))^{1/2}-1]} (1/(m + \kappa/D)) - 1/(m + \kappa/D^* + \eta_0^*(\kappa)) =$$

$$(\Gamma'(\kappa/D^* + \eta_0^*(\kappa)))/(\Gamma(\kappa/D^* + \eta_0^*(\kappa))) - (\Gamma'(\kappa/D^*))/(\Gamma(\kappa/D^*))) + O(\nu^{-1/2}),$$

where $\kappa \in [1, D^* - 1] \cap \mathbb{Z}$. Consequently,

$$\log(d_1^*(\nu)) = \nu \sum_{\kappa=1}^{D-1} (\Gamma'(\kappa/D^* + \eta_0^*(\kappa)))/(\Gamma(\kappa/D^* + \eta_0^*(\kappa)))-$$

$$\nu \sum_{\kappa=1}^{D^*-1} (\Gamma'(\kappa/D^*))/(\Gamma(\kappa/D^*)) + O(\nu \log(\nu) \exp(-(a/\sqrt{2})(log(\nu))^{1/2}).$$

Let $\delta_0 = 1/\Delta$, $\eta^\vee(x) = \{(1-\delta_0)x\} - 2\{x\}$, $\eta_0^\vee(x) = \max(0, \eta^\vee(x))$, where $x \in \mathbb{R}$. Let further $\chi^\vee = \chi^\vee(x)$ denotes the characteristic function of the set $\{x \in \mathbb{R}\colon 0 < \eta_0^\vee(x)\}$. If $x \in (0, 1)$, then $\chi^\vee(x) = 0$. Therefore the integral $\int\limits_0^\infty \chi^\vee(x) x^{-2}\, dx$ converges. We wish to finde the value $\Delta \int\limits_0^\infty \chi^\vee(x)\, dx = \int\limits_0^\infty \chi^\vee(\Delta x) x^{-2}\, dx$. Clearly,

$$\int\limits_0^\infty \chi^\vee(\Delta x) x^{-2}\, dx = \sum_{m=0}^\infty \int\limits_m^{m+1} \chi^\vee(\Delta x) x^{-2}\, dx = \sum_{m=0}^\infty \int\limits_0^1 \chi^\vee(\Delta\xi)(m+\xi)^{-2}\, d\xi.$$

In view of (20), $X_\Delta^* = \{x \in \mathbb{R}\colon 0 < \eta_0^\vee(\Delta x)\}$. Clearly, the condition (21) is equivalent to the equality $\chi^\vee(\Delta\xi) = 1$. Therefore, in view of (25),

$$\int\limits_0^1 \chi^\vee(\Delta\xi)(m+\xi)^{-2}\, d\xi =$$

$$\sum_{\kappa=1}^{D^*-1} (1/(m + \kappa/D^*)) - 1/(m + \kappa/D^* + \eta_0^*(\kappa))$$

Consequently, in view of (29) – (31),

$$\log(d_1^*(\nu)) = \nu\Delta \int\limits_0^\infty \chi^\vee(x)\, dx + O(\nu \log(\nu) \exp(-(a/\sqrt{2})(log(\nu))^{1/2}). \tag{32}$$

Let

$$x = m\Delta + \kappa + \xi, \tag{33}$$

where $m \in \mathbb{N} - 1$, $\kappa = 0, \ldots, \Delta - 1$, $0 \le \xi < 1$. Then

$$\int\limits_0^\infty \chi^\vee(\Delta x) x^{-2}\, dx = \sum_{m=0}^\infty \sum_{\kappa=0}^{\Delta-1} \int\limits_0^1 \chi^\vee(m\Delta + \kappa + \xi)(m\Delta + \kappa + \xi)^{-2}\, dx.$$

If $\{x\} \geq \{x/\Delta\}$, then $\{x - x/\Delta\} = \{\{x\} - \{x/\Delta\}\} = \{x\} - \{x/\Delta\}$ and the inequality $\{x-x/\Delta\} > 2\{x\}$ is impossible; on the other hand, clearly, the inequality $\{x\} < \{x/\Delta\}$, implies the equality

$$\{x - x/\Delta\} = \{\{x\} - \{x/\Delta\}\} = \{x\} - \{x/\Delta\} + 1$$

and therefore in this case the inequality $\{x - x/\Delta\} > 2\{x\}$ turns into the the inequality $1 - \{x\} > \{x/\Delta\}$; so, the inequality $\{x - x/\Delta\} > 2\{x\}$ is equivalent to the condition

$$1 - \{x\} > \{x/\Delta\} > \{x\}. \tag{34}$$

In view of (33),

$$\{x\} = \xi,\ \{x/\Delta\} = (\kappa + \xi)/\Delta, \tag{35}$$

and the condition (34) may be rewritten in the form

$$1 - \xi > (\kappa + \xi)/\Delta > \xi,$$

i.e. in the form

$$0 \leq \xi < \min(\kappa/(\Delta - 1),\ (D - \kappa)/(\Delta + 1)),$$

where $\kappa = 0, \dots , \Delta - 1$. Clearly,

$$\min(\kappa/(\Delta - 1),\ (\Delta - \kappa)/(\Delta + 1)) = \kappa/(\Delta - 1),,$$

if $\kappa \leq (\Delta - 1)/2$,

$$\min(\kappa/(\Delta - 1),\ (D - \kappa)/(\Delta + 1)) = (D - \kappa)/(\Delta + 1),$$

if $\kappa \geq (\Delta - 1)/2$. Therefore

$$\int\limits_0^1 \chi^\vee(m\Delta + \kappa + \xi)(m\Delta + \kappa + \xi)^{-2}\, dx = (m\Delta + \kappa)^{-1} -$$

$$(m\Delta + \kappa + \kappa/(\Delta - 1))^{-1} = \Delta^{-1}(m + \kappa/\Delta)^{-1} - (m + \kappa/(\Delta - 1))^{-1},$$

if $\kappa \leq (\Delta - 1)/2$,

$$\int\limits_0^1 \chi^\vee(m\Delta + \kappa + \xi)(m\Delta + \kappa + \xi)^{-2}\, dx = (m\Delta + \kappa)^{-1} -$$

$$(m\Delta + \kappa + (\Delta - \kappa)/(\Delta + 1))^{-1} = \Delta^{-1}(m + \kappa/\Delta)^{-1} - (m + (\kappa + 1)/(\Delta + 1))^{-1},$$

if $\kappa \geq (\Delta-1)/2$. Consequently,

$$\int\limits_0^\infty \chi^\vee(\Delta x)x^{-2}\,dx = \Delta^{-1}\sum_{1\leq\kappa<\Delta/2}\Gamma'(\kappa/(\Delta-1))/\Gamma(\kappa/(\Delta-1))- \tag{36}$$

$$\Delta^{-1}\sum_{1\leq\kappa<\Delta/2}\Gamma'(\kappa/\Delta)/\Gamma(\kappa/\Delta)+$$

$$\Delta^{-1}\sum_{\Delta/2\leq\kappa\leq\Delta-1}\Gamma'((\kappa+1)/(\Delta+1))/\Gamma((\kappa+1)/(\Delta+1))-$$

$$\Delta^{-1}\sum_{\Delta/2\leq\kappa\leq\Delta-1}\Gamma'(\kappa/\Delta)/\Gamma(\kappa/\Delta) = \Delta^{-1}\sum_{1\leq\kappa<\Delta/2}\Gamma'(\kappa/(\Delta-1))/\Gamma(\kappa/(\Delta-1))+$$

$$\Delta^{-1}\sum_{\Delta/2\leq\kappa\leq\Delta-1}\Gamma'((\kappa+1)/(\Delta+1))/\Gamma((\kappa+1)/(\Delta+1))-$$

$$\Delta^{-1}\sum_{1\leq\kappa\leq\Delta-1}\Gamma'(\kappa/\Delta)/\Gamma(\kappa/\Delta) =$$

$$\Delta^{-1}\sum_{1\leq\kappa\leq(\Delta-1)/2}\Gamma'(\kappa/(\Delta-1))/\Gamma(\kappa/(\Delta-1))+$$

$$\Delta^{-1}\sum_{(\Delta-1)/2\leq\kappa\leq\Delta-1}\Gamma'((\kappa+1)/(\Delta+1))/\Gamma((\kappa+1)/(\Delta+1))-$$

$$(1-(-1)^\Delta)(2\Delta)^{-1}\Gamma'(1/2)/\Gamma(1/2) - \Delta^{-1}\sum_{1\leq\kappa\leq\Delta-1}\Gamma'(\kappa/\Delta)/\Gamma(\kappa/\Delta) =$$

$$\Delta^{-1}\sum_{1\leq\kappa\leq(\Delta-1)/2}\Gamma'(\kappa/(\Delta-1))/\Gamma(\kappa/(\Delta-1))+$$

$$\Delta^{-1}\sum_{(\Delta+1)/2\leq\kappa\leq\Delta}\Gamma'(\kappa)/(\Delta+1))/\Gamma(\kappa/(\Delta+1))-$$

$$(1-(-1)^\Delta)(2\Delta)^{-1}\Gamma'(1/2)/\Gamma(1/2) - \Delta^{-1}\sum_{1\leq\kappa\leq\Delta-1}\Gamma'(\kappa/\Delta)/\Gamma(\kappa/\Delta) =$$

$$\Delta^{-1}\sum_{1\leq\kappa\leq(\Delta-1)/2}\Gamma'(\kappa/(\Delta-1))/\Gamma(\kappa/(\Delta-1))+$$

$$\Delta^{-1}\sum_{(\Delta+1)/2\leq\kappa\leq\Delta}\Gamma'(\kappa/(\Delta+1))/\Gamma(\kappa/(\Delta+1))-$$

$$(1-(-1)^\Delta)(2\Delta)^{-1}\Gamma'(1/2)/\Gamma(1/2) - \Delta^{-1}\sum_{1\leq\kappa\leq\Delta/2}\Gamma'(\kappa/\Delta)/\Gamma(\kappa/\Delta)-$$

$$\Delta^{-1}\sum_{\Delta/2\le\kappa\le\Delta-1}\Gamma'(\kappa/\Delta)/\Gamma(\kappa/\Delta)+(1+(-1)^{\Delta})(2\Delta)^{-1}\frac{\Gamma'(1/2)}{\Gamma(1/2)}.$$

Let $p\in\mathbb{N}$, $q\in\mathbb{N}+p$. According to well-known theorem of K.F.Gauß

$$\frac{\Gamma'(p/q)}{\Gamma(p/q)}=-\gamma-log(q)-(\pi/2)\cot(p\pi/q)+ \tag{37}$$

$$\sum_{\kappa=1}^{q-1}(\cos(2kp\pi/q))\log(|1-\exp(-2k\pi/q)|),$$

where γ is the Euler's constant. Let $\sigma(p,q)=\sum\limits_{\kappa=1}^{[q/2]}\cos(kp\kappa\pi/q)$.

Since $\cos(2p(q-k\kappa)\pi/q)=\cos(2k\kappa\pi/q)$ for $k=1,\,\ldots\,,\,[(q-1)/2]$, it follows that $2\sigma(p,q)+1=0$, if $q\in 2\mathbb{N}+1$, and $2\sigma(p,q)+1-(-1)^p=0$, if $q\in 2\mathbb{N}$. So,

$$\sigma(p,q)=-1/2+(-1)^p((-1)^q+1)/4=0,\ \sigma(p,q)=\sum_{[q/2]}^{q-1}\cos(2kp\pi/q). \tag{38}$$

If $q\in 2\mathbb{N}$, then

$$\sum_{\kappa=1}^{q-1}(-1)^k\log(|1-\exp(-2k\pi/q)|)=$$

$$\log\left(\lim_{x\to 1}\left(\left|\frac{\prod\limits_{\kappa=1}^{q}(x-\exp(-2k\pi/q)}{(x-1)\prod\limits_{\kappa=1}^{q/2}(x-\exp(-(2k-1)\pi/(q/2))}\right|\right)\right)-$$

$$\log\left(\left|\prod_{\kappa=1}^{q/2}(x-\exp(-(2k-1)\pi/(q/2))\Big|_{x=1}\right|\right)=$$

$$\log(\lim_{x\to 1}(|((x^{q/2}-1)//(x-1)|))-$$

$$\log(|\left(x^{q/2}+1\right)|_{x=1}|)=\log(q)-2\log(2).$$

If $q\in 2\mathbb{N}+1$, then $k+(q-k)\in 2\mathbb{Z}+1$ and

$$\sum_{\kappa=1}^{q-1}(-1)^k\log(|1-\exp(-2k\pi/q)|)=$$

$$\sum_{\kappa=1}^{(q-1)/2}(-1)^k(\log\left(\frac{|1-\exp(-2k\pi/q)|}{|1-\exp(-2(q-k)\pi/q)|}\right)=0.$$

Therefore

$$\sum_{\kappa=1}^{q-1}(-1)^k\log(|1-\exp(-2k\pi/q)|)=(\log(q)-\log(4))(1+(-1)^q)/2. \tag{39}$$

Clearly,

$$\sum_{\kappa=1}^{q-1}(-1)^k\log(|1-\exp(-2k\pi/q)|)=\log(q), \tag{40}$$

$$[q/2]+1/2-(1+(-1)^q)/4=q/2. \tag{41}$$

Therefore, in view of (36) – (41),

$$\sum_{1\le\kappa\le(\Delta-1)/2}\Gamma'(\kappa/(\Delta-1)/\Gamma(\kappa/(\Delta-1))= \tag{42}$$

$$-\gamma[(\Delta-1)/2]-[(\Delta-1)/2]\log(\Delta-1))-(\pi/2)\sum_{1\le\kappa\le(\Delta-1)/2}\cot(\pi\kappa/(\Delta-1))+$$

$$\sum_{1\le\kappa\le\Delta-1}\sigma(k,\Delta-1)\log(|(1-\exp(-2ik\pi/(\Delta-1))=$$

$$-\gamma[(\Delta-1)/2]-[(\Delta-1)/2]\log(\Delta-1)-(\pi/2)\sum_{1\le\kappa\le(\Delta-1)/2}\cot(\pi\kappa/(\Delta-1))+$$

$$\sum_{1\le\kappa\le\Delta-1}\left(-1/2+(-1)^k\left((-1)^{\Delta-1}+1\right)/4\right)\log(|(1-\exp(-2ik\pi/(\Delta-1))=$$

$$-\gamma[\Delta-1)/2]-[\Delta-1)/2]\log(\Delta-1)-(\pi/2)\sum_{1\le\kappa\le(\Delta-1)/2}\cot(\pi\kappa/(\Delta-1))+$$

$$(-1/2)\log(\Delta-1))+(\log(\Delta-1)-\log(4))\left(1+(-1)^{\Delta-1}\right)/4=$$

$$-\gamma[\Delta-1)/2]-(\Delta-1)/2)\log(\Delta-1/2)-(\pi/2)\sum_{1\le\kappa\le(\Delta-1)/2}\cot(\pi\kappa/(\Delta-1))+$$

$$(-\log(4))\left(1+(-1)^{\Delta-1}\right)/4,$$

$$\sum_{(\Delta+1)/2\le\kappa\le\Delta}\Gamma'(\kappa/(\Delta+1))/\Gamma(\kappa/(\Delta+1))=$$

$$-\gamma[\Delta+1)/2]-[\Delta+1)/2]\log(\Delta-1))+(\pi/2)\sum_{1\le\kappa\le(\Delta+1)/2}\cot(\pi\kappa/(\Delta+1))+$$

$$\sum_{1\le\kappa\le\Delta-1}\sigma(k,\Delta+1)\log(|(1-\exp(-2ik\pi/(\Delta+1))=$$

$$-\gamma[\Delta+1)/2]-[\Delta+1)/2]\log(\Delta-1))+(\pi/2)\sum_{1\le\kappa\le(\Delta+1)/2}\cot(\pi\kappa/(\Delta+1))+$$

$$\sum_{1\le\kappa\le\Delta-1}\big(-1/2+(-1)^k((-1)^{\Delta+1}+1)/4\big)\log(|(1-\exp(-2ik\pi/(\Delta+1))=$$

$$-\gamma[\Delta+1)/2]-[\Delta+1)/2]\log(\Delta+1))+(\pi/2)\sum_{1\le\kappa\le(\Delta+1)/2}\cot(\pi\kappa/(\Delta+1))+$$

$$(-1/2)\log(\Delta+1)+(\log(\Delta+1)-\log(4))\left((-1)^{\Delta+1}+1\right)/4=$$

$$-\gamma[\Delta+1)/2]-(\Delta+1)/2)\log(\Delta+1))+(\pi/2)\sum_{1\le\kappa\le(\Delta+1)/2}\cot(\pi\kappa/(\Delta+1))+$$

$$(-\log(4))\left((-1)^{\Delta+1}+1\right)/4,$$

$$\sum_{1\le\kappa\le\Delta/2}\Gamma'(\kappa/\Delta)/\Gamma(\kappa/\Delta)\;+\sum_{\Delta/2\le\kappa\le\Delta-1}\Gamma'(\kappa/\Delta)/\Gamma(\kappa/\Delta)= \tag{43}$$

$$-2\gamma[\Delta/2]-2[\Delta/2]\log(\Delta)+$$

$$(-\log(\Delta))+(\log(\Delta)-\log(4))\left((-1)^{\Delta}+1\right)/2=$$

$$-2\gamma[\Delta/2]-\Delta\log(\Delta)+(-\log(4))\left((-1)^{\Delta}+1\right)/2,$$

$$(-\log(4))\left(-(-1)^{\Delta}+1\right)/2-(-\log(4))\left((-1)^{\Delta}+1\right)/2-$$

$$(1-(-1)^{\Delta})\Gamma'(1/2)/(2\Gamma(1/2))+(1+(-1)^{\Delta})\Gamma'(1/2)/(2\Gamma(1/2))=$$

$$(-1)^{\Delta}\log(4)+(-1)^{\Delta}\Gamma'(1/2)/\Gamma(1/2)=$$

$$(-1)^{\Delta}(\log(4)-\gamma-\log(2)-\log(2))=-(-1)^{\Delta}\gamma,$$

$$-\gamma([(\Delta-1)/2]+[(\Delta+1)/2]-2[\Delta/2]+(-1)^{\Delta})=0. \tag{44}$$

Consequently, in view of (36), (42) – (44),

$$\int_0^\infty \chi^\vee(x)x^{-2}\,dx= \tag{45}$$

$$\log(\Delta)-((\Delta-1)/(2\Delta))\log(\Delta-1)-((\Delta+1)/(2\Delta))\log(\Delta+1)-$$

$$(\pi/(2\Delta))\sum_{1\le\kappa\le(\Delta-1)/2}\cot(\pi\kappa/(\Delta-1))+$$

$$(\pi/(2\Delta))\sum_{1\le\kappa\le(\Delta+1)/2}\cot(\pi\kappa/(\Delta+1)).$$

Hata reasons somewhat differently. He uses the equality.

$$\binom{\nu\Delta}{k}\binom{\nu\Delta+k}{\nu\Delta-\nu}=(\nu\Delta)!(\nu\Delta+k)!/(k!(\nu\Delta-k)!(\nu\Delta-\nu)!(\nu+k)!)=$$

$$((\nu\Delta)!)^2/((\nu\Delta+\nu)!(\nu\Delta-\nu)!)\binom{\nu\Delta+\nu}{\nu+k}\binom{\nu\Delta+k}{k},$$

where $k=0,\,\dots\,,\,\nu\Delta$, and making use that

$$\binom{\nu\Delta+\nu}{\nu+k}\binom{\nu\Delta+k}{k}\in\mathbb{N}, \tag{46}$$

for $k=0,\,\dots\,,\,\nu\Delta$, he seeks $r(\nu)$ in the form

$$r(\nu)=r^{**}(\nu)=((\nu\Delta+\nu)!(\nu\Delta-\nu)!)/(((\nu\Delta)!)^2d_1^{**}(\Delta,\nu)), \tag{47}$$

where $d_1^{**}(\Delta,\nu)$, is some common divisor of the numbers (46). According to the Lemma 2, the prime number $p\in\mathbb{N}$ is a divisor of all the numbers (46) if and only if, when

$$0<\max(\{(\nu+k)/p\}-\{\nu(\Delta+1)/p\},\{\nu\Delta/p\}-\{(k+\nu\Delta)/p\}), \tag{48}$$

where $k=0,\,\dots\,,\,\nu\Delta$. Let us to prove that the condition (48) is equivalent to the inequality

$$\{\nu(\Delta-1)/p\}+\{\nu(\Delta+1)/p\}<\{\nu\Delta/p\}. \tag{49}$$

Let the inequality (49) holds and let $\{(\nu+k)/p\}\le\{\nu(\Delta+1)/p\}$ for some $k\in[0,\nu\Delta]\cap\mathbb{Z}$. Then, in view of (49),

$$\{\nu(\Delta-1)/p\}+\{(\nu+k)/p\}\le$$

$$\{\nu(\Delta-1)/p\}+\{\nu(\Delta+1)/p\}<\{\nu\Delta/p\}\le(p-1)/p<1,$$

$$\{(\nu\Delta+k)/p\}=\{\nu(\Delta-1)/p\}+\{(\nu+k)/p\}<\{\nu\Delta/p\},$$

and therefore (48) holds. If (63) holds, then, taking in the inequality (48) $k=0$, we see that

$$\{\nu/p\}>\{\nu(\Delta+1)/p\}=\{\{\nu\Delta/p\}+\{\nu/p\}\},$$

and therefore

$$\{\nu\Delta/p\}+\{\nu/p\}\ge 1,\ \{\nu\Delta/p\}+\{\nu/p\}-1=\{\nu(\Delta+1)/p\}. \tag{50}$$

If

$$\{\nu/p\}>\{\nu\Delta/p\}, \tag{51}$$

then $k = p - p\{\nu/p\} > 0$,, in view of (50),

$$\{\nu\Delta/p\} \geq 1 - \{\nu/p\} = k/p = \{k/p\},$$

and $\{(\nu + k)/p\} = 0 \leq \{\nu(\Delta + 1)/p\}$. Therefore, in view of (48),

$$\{\nu\Delta/p\} > \{(k + \nu\Delta)/p\}) = \{\{k/p\} + \{(\nu\Delta)/p\}\}. \tag{52}$$

But the inequality (52) is equivalent to the inequality

$$1 \leq \{k/p\} + \{(\nu\Delta)/p\} = 1 - \{\nu/p\} + \{(\nu\Delta)/p\},$$

what contradicts to (51). Therefore the inequality

$$\{\nu/p\} \leq \{\nu\Delta/p\}$$

follows from (48). In view of (50),

$$\{\nu\Delta/p\} = \{\nu(\Delta - 1/p\} + \{\nu/p\}\} = \{\nu(\Delta - 1)/p\} + \{\nu/p\} >$$

$$\{\nu(\Delta - 1)/p\} + \{\nu/p\} + \{\nu\Delta/p\} - 1 = \{\nu(\Delta - 1)/p\} + \{\nu(\Delta + 1)/p\}.$$

So, if (49) holds, then p is divisor of all the numbers (46). We will study the set

$$X^{**}_{\Delta} = \{x \in \mathbb{R}\colon \{x(\Delta - 1)\} + \{x(\Delta + 1)\} < \{x\Delta\}\}. \tag{53}$$

Clearly, $X^{*}_{\Delta} \cap \mathbb{Z} = \emptyset$ and $m + X^{**}_{\Delta} = X^{**}_{\Delta}$, for any $m \in \mathbb{Z}$. Therefore it is suffiient to study the set $X^{**}_{\Delta} \cap (0, 1)$. Let

$$\xi \in X^{**}_{\Delta} \cap (0, 1), \tag{54}$$

and let $D^{**} \in \Delta(\Delta^2 - 1)\mathbb{N}$. Let

$$\xi \in [\kappa/D^{**}, (\kappa + 1)/D^{**}) \tag{55}$$

for some $\kappa \in \{0, \ldots, D^{**} - 1\}$. Then

$$\kappa/(D^{**}/(\Delta - \epsilon)) \leq \xi(\Delta - \epsilon) < (\kappa + 1)/(D^{**}/(\Delta - \epsilon)),$$

$$[\kappa/(D^{**}/(\Delta - \epsilon))] \leq [\kappa/(D^{**}/(\Delta - \epsilon))] + \{\kappa/(D^{*}/(\Delta - \epsilon))\} \leq \xi(\Delta - \epsilon) <$$

$$[\kappa/(D^{**}/\Delta)] + \{\kappa/(D^{**}/\Delta)\} + 1/(D^{**}/(\Delta - \epsilon)) \leq$$

$$[\kappa/(D^{**}/\Delta)] + (D^{**}/(\Delta - \epsilon) - 1)/(D^{**}/(\Delta - \epsilon)) + 1/(D^{**}/(\Delta - \epsilon) =$$

$$[\kappa/(D^{**}/(\Delta - \epsilon))] + 1,$$

where $\epsilon^3 = \epsilon$. Consequently,

$$[\kappa/(D^{**}/(\Delta-\epsilon))] = [\xi(\Delta-\epsilon)],$$

$$\xi(\Delta-\epsilon) = \xi(\Delta-\epsilon) - [\xi(\Delta-\epsilon)] = \xi(\Delta-\epsilon) - [\kappa/(D^{**}/(\Delta-\epsilon))],$$

where $\epsilon^3 = \epsilon$. Therefore, the existence of $\kappa \in [0, D^*-1) \cap \mathbb{Z}$, for which the inclusion (55) and the inequality

$$2\xi\Delta - [\kappa(\Delta-1)/D^{**}] - [\kappa(\Delta+1)/D^{**}] < \xi\Delta - [\kappa\Delta/D^{**}], \tag{56}$$

hold, is equivalent to the condition (54). The inequality (56) is equivalent to the inequality

$$\xi < \eta^{**}(\kappa) := \kappa/D^{**} + (\{\kappa\Delta/D^{**}\} - \{\kappa(\Delta-1)/D^{**}\} - \{\kappa(\Delta+1)/D^{**}\})/\Delta. \tag{57}$$

Let us note that for $\kappa = 0$ the system (55) – (57) has no solutions. Let

$$\eta_0^{**}(\kappa) = \max(0, \min(\eta^{**}(\kappa), 1/D^{**})) = \max(0, \eta^{**}(\kappa)),$$

where $\kappa \in \{0, \dots, D^{**}-1\}$. Then the system of conditions (55) – (57) is equivalent to the condition

$$\xi \in [\kappa/D^{**}, \kappa/D^{**} + \eta_0^{**}(\kappa)). \tag{58}$$

So, if $\nu/p = m + \xi$, where

$$m \in [0, (\nu/(2\Delta))^{1/2} - 1] \cap \mathbb{Z},\ \kappa \in \mathbb{Z},\ 1 \le \kappa \le D-1,$$

$$\xi \in (0,1),\ \kappa/D^{**} \le \xi < \kappa/D^{**} + \eta_0^{**}(\kappa),$$

i.e. if

$$\nu/(m + \kappa/D^{**} + \eta_0^{**}(\kappa)) < p \le \nu/(m + \kappa/D^{**}), \tag{59}$$

where

$$m \in [0, (\nu/(\Delta+1))^{1/2} - 1] \cap \mathbb{Z},\ \kappa \in \mathbb{Z},\ 1 \le \kappa \le D^{**} - 1, \tag{60}$$

then the condition $p > (2\nu\Delta)^{1/2}$ and the condition (64) are fulfilled. We take on the role of $\log(d_1^{**}(\nu))$ the product of all the prime nubers p, which satisfy to the conditions (59) – (60). In view of (59) – (60) and (28),

$$\log(d_1^{**}(\Delta,\nu)) = \tag{61}$$

$$\sum_{\kappa=1}^{D^{**}-1} \sum_{m=0}^{[(\nu/(\Delta+1))^{1/2}-1]} (\psi(\nu/(m+\kappa/D^{**})) - \psi(\nu/(m+\kappa/D^{**}+\eta_0^{**}(\kappa))),$$

(62) $$\psi(\nu/(m+\kappa/D^{**}+\eta_0^{**}(\kappa))) = \nu/(m+\kappa/D^{**}+\eta_0^{**}(\kappa))+$$

$$O((\nu/(m+1))(\exp(-(a/\sqrt{2})(log(\nu))^{1/2}),$$

$$\psi(\nu/(m+\kappa/D^{**})) = \nu/(m+\kappa/D^{**}) + O((\nu/(m+1))\exp(-(a/\sqrt{2})(log(\nu))^{1/2}).$$

But

$$\sum_{m=0}^{[(\nu/(\Delta+1))^{1/2}-1]} (1/(m+\kappa/D^{**})) - 1/(m+\kappa/D^{**}+\eta_0^{**}(\kappa)) =$$

$$\Gamma'(\kappa/D^{**}+\eta_0^{**}(\kappa))/\Gamma(\kappa/D^{**}+\eta_0^{**}(\kappa)) - \Gamma'(\kappa/D^{**})/\Gamma(\kappa/D^{**}) + O(\nu^{-1/2}),$$

where $\kappa \in [1, D^{**}-1] \cap \mathbb{Z}$. Consequently,

$$\log(d_1^{**}(\Delta,\nu)) = \nu \sum_{\kappa=1}^{D^{**}-1} \Gamma'(\kappa/D^{**}+\eta_0(\kappa))/\Gamma(\kappa/D^{**}+\eta_0(\kappa)) -$$

$$\nu \sum_{\kappa=1}^{D^{**}-1} \Gamma'(\kappa/D^{**})/\Gamma(\kappa/D^{**}) + O(\nu\log(\nu)\exp(-(a/\sqrt{2})(log(\nu))^{1/2}).$$

Let $\eta^\wedge(x) = \{x\} - \{(1-\delta_0)x\} - \{(1+\delta_0)x\}$, $\eta_0^\wedge(x) = \max(0, \eta^\wedge(x))$, where $x \in \mathbb{R}$. Let further $\chi^\wedge = \chi^\wedge(x)$ denotes the characteristic function of the set $\{x \in \mathbb{R}: 0 < \eta_0^\wedge(\Delta x)\}$. If $x \in (0, 1/(1+\delta_0))$, then $\chi^\wedge(x) = 0$. Therefore the integral $\int\limits_0^\infty \chi^\wedge(x)x^{-2}\,dx$ converges. We wish to finde the value $\Delta\int\limits_0^\infty \chi^\wedge(x)\,dx = \int\limits_0^\infty \chi^\wedge(\Delta x)x^{-2}\,dx$. Clearly,

$$\int\limits_0^\infty \chi^\wedge(\Delta x)x^{-2}\,dx = \sum_{m=0}^\infty \int\limits_m^{m+1} \chi^\wedge(\Delta x)x^{-2}\,dx = \sum_{m=0}^\infty \int\limits_0^1 \chi^\wedge(\Delta\xi)(m+\xi)^{-2}\,d\xi.$$

In view of (53), $X_\Delta^{**} = \{x \in \mathbb{R}: 0 < \eta_0^\wedge(\Delta x)\}$. Clearly, the condition (54) is equivalent to the equality $\chi^\wedge(\Delta\xi) = 1$. Therefore, in view of (58),

$$\int\limits_0^1 \chi^\wedge(\Delta\xi)(m+\xi)^{-2}\,d\xi = \sum_{\kappa=1}^{D^{**}-1} (1/(m+\kappa/D^{**})) - 1/(m+\kappa/D^{**}+\eta_0^{**}(\kappa)).$$

Consequently, in view of (60) – (62),

(63) $$\log(d_1^{**}(\nu)) = \nu\Delta\int\limits_0^\infty \chi^\wedge(x)\,dx + O(\nu\log(\nu)\exp(-(a/\sqrt{2})(log(\nu))^{1/2}).$$

In view of (33),

$$\int_0^\infty \chi^\wedge(\Delta x)x^{-2}\,dx = \sum_{m=0}^\infty \sum_{\kappa=0}^{\Delta-1} \int_0^1 \chi^\wedge(m\Delta+\kappa+\xi)(m\Delta+\kappa+\xi)^{-2}\,dx.$$

If

$$\{x\} > \{x + x/\Delta\} + \{x - x/\Delta\}, \tag{64}$$

then

$$\{x\} > \{x + x/\Delta\} + \{x - x/\Delta\} = \{\{x + x/\Delta\} + \{x - x/\Delta\}\} =$$

$$\{2x\} = \{2\{x\}\},\ 1 \geq 2\{x\},$$

$$\{x\} > \{x + x/\Delta\} = \{\{x\} + \{x/\Delta\}\},\ \{x\} + \{x/\Delta\} \geq 1,\ \{x/\Delta\} > 0,$$

$$\{x\} > \{x - x/\Delta\} = \{\{x\} + \{-x/\Delta\}\},\ \{x\} \geq= 1 - \{1 - \{x/\Delta\}\} = \{x/\Delta\},$$

because $\{x/\Delta\} > 0$. If

$$\{x\} \geq \{x/\Delta\} \geq 1 - \{x\}, \tag{65}$$

then

$$2\{x\} \geq 1,\ \{x + x/\Delta\} + \{x - x/\Delta\} =$$

$$\{\{x\} + \{x/\Delta\} + \{x\} - \{x/\Delta\}\} = \{2\{x\}\} = \{2x\} < \{x\}.$$

So, the conditions (64) and (65) are equivalent.

According to (33) and (35), $\{x\} = \xi$, $\{x/\Delta\} = (\kappa+\xi)/\Delta$ and the condition (65) takes form $\xi \geq (\kappa+\xi)/\Delta \geq 1-\xi$, i.e. the form $\xi \geq \max(\kappa/(\Delta-1),\ (\Delta-\kappa)/(\Delta+1))$, where $\kappa = 0, \ldots, \Delta - 1$. Clearly,

$\max(\kappa/(\Delta - 1),\ (D - \kappa)/(\Delta + 1)) = \kappa/(\Delta - 1)$, if $\kappa \geq (\Delta - 1)/2$,

$\max(\kappa/(\Delta - 1),\ (D - \kappa)/(\Delta + 1)) = (D - \kappa)/(\Delta + 1)$, if $\kappa \leq (\Delta - 1)/2$.

Therefore the integral

$$\int_0^1 \chi^\wedge(m\Delta+\kappa+\xi)(m\Delta+\kappa+\xi)^{-2}\,dx$$

is equal to

$$(m\Delta + \kappa + \kappa/(\Delta - 1))^{-1} - (m\Delta + \kappa + 1)^{-1} =$$

$$\Delta^{-1}(m + \kappa/(\Delta - 1))^{-1} - \Delta^{-1}(m + (\kappa + 1)/\Delta)^{-1},$$

if $\kappa \geq (\Delta - 1)/2$, and it is equal to

$$(m\Delta + \kappa + (\Delta - \kappa)/(\Delta + 1))^{-1} - (m\Delta + \kappa)^{-1} =$$

$$\Delta^{-1}(m+(\kappa+1)/(\Delta+1))^{-1}-\Delta^{-1}(m+(\kappa+1)/\Delta)^{-1}),$$

if $\kappa \leq (\Delta-1)/2$. Consequently,

$$\int\limits_0^\infty \chi^\wedge(\Delta x)x^{-2}\,dx = \Delta^{-1}\sum_{0\leq\kappa<\Delta/2}\Gamma'((\kappa+1)/\Delta)/\Gamma((\kappa+1)/\Delta)- \tag{66}$$

$$\Delta^{-1}\sum_{0\leq\kappa<\Delta/2}\Gamma'((\kappa+1)/(\Delta+1))/\Gamma((\kappa+1)/(\Delta+1))+$$

$$\Delta^{-1}\sum_{\Delta/2\leq\kappa\leq\Delta-1}\Gamma'((\kappa+1)/\Delta)/\Gamma((\kappa+1)/\Delta)-$$

$$\Delta^{-1}\sum_{\Delta/2\leq\kappa\leq\Delta-1}\Gamma'(\kappa/(\Delta-1))/\Gamma(\kappa/(\Delta-1))=$$

$$\Delta^{-1}\sum_{0\leq\kappa\leq\Delta-1}\Gamma'((\kappa+1)/\Delta)/\Gamma((\kappa+1)/\Delta)-$$

$$\Delta^{-1}\sum_{0\leq\kappa<\Delta/2}\Gamma'((\kappa+1)/(\Delta+1))/\Gamma((\kappa+1)/(\Delta+1))-$$

$$\Delta^{-1}\sum_{\Delta/2\leq\kappa\leq\Delta-1}\Gamma'(\kappa/(\Delta-1))/\Gamma(\kappa/(\Delta-1))=$$

$$\Delta^{-1}\sum_{0\leq\kappa\leq\Delta-1}\Gamma'((\kappa+1)/\Delta)/\Gamma((\kappa+1)/\Delta)+$$

$$\Delta^{-1}\sum_{1\leq\kappa\leq\Delta-1}\Gamma'(\kappa/\Delta)/\Gamma(\kappa/\Delta)-$$

$$\Delta^{-1}\sum_{0\leq\kappa\leq\Delta-1}\Gamma'((\kappa+1)/(\Delta+1))/\Gamma((\kappa+1)/(\Delta+1))-$$

$$\Delta^{-1}\sum_{1\leq\kappa\leq\Delta-1}\Gamma'(\kappa/(\Delta-1))/\Gamma(\kappa/(\Delta-1))-$$

$$\Delta^{-1}\sum_{1\leq\kappa\leq\Delta-1}\Gamma'(\kappa/\Delta)/\Gamma(\kappa/\Delta)+$$

$$\Delta^{-1}\sum_{\Delta/2\leq\kappa\leq\Delta-1}\Gamma'((\kappa+1)/(\Delta+1))/\Gamma((\kappa+1)/(\Delta+1))+$$

$$\Delta^{-1}\sum_{1\leq\kappa<\Delta/2}\Gamma'(\kappa/(\Delta-1))/\Gamma(\kappa/(\Delta-1)).$$

In view of (43) and (37),

$$\sum_{1\leq\kappa\leq\Delta-1}\Gamma'(\kappa/\Delta)/\Gamma(\kappa/\Delta)=$$

$$-2\gamma[\Delta/2]-\Delta\log(\Delta)+(-\log(4))\left((-1)^{\Delta}+1\right)/2-$$
$$(\Gamma'(1/2)/\Gamma(1/2))(1+(-1)^{\Delta})/2=-2\gamma[\Delta/2]-\Delta\log(\Delta)+$$
$$(-\log(4))\left((-1)^{\Delta}+1\right)/2-(-\gamma-\log(4))\left(1+(-1)^{\Delta}\right)/2=$$
$$-2\gamma[\Delta/2]-\Delta\log(\Delta)+\gamma\left(1+(-1)^{\Delta}\right)/2.$$

Therefore

$$\sum_{0\le\kappa\le\Delta-1}\Gamma'((\kappa+1)/(\Delta+1))/\Gamma((\kappa+1)/(\Delta+1))=$$
$$-2\gamma[(\Delta+1)/2]-\gamma-(\Delta+1)\log(\Delta+1)+\gamma\left(1+(-1)^{(\Delta+1)}\right)/2,$$
$$\sum_{1\le\kappa\le\Delta-1}\Gamma'((\kappa+1)/(\Delta+1))/\Gamma((\kappa+1)/(\Delta+1))=$$
$$-2\gamma[(\Delta-1)/2]-(\Delta-1)\log(\Delta-1)+\gamma\left(1+(-1)^{(\Delta-1)}\right)/2,$$

(67)
$$\Delta^{-1}\sum_{0\le\kappa\le\Delta-1}\Gamma'((\kappa+1)/\Delta)/\Gamma((\kappa+1)/\Delta)+$$
$$\Delta^{-1}\sum_{1\le\kappa\le\Delta-1}\Gamma'(\kappa/\Delta)/\Gamma(\kappa/\Delta)-$$
$$\Delta^{-1}\sum_{0\le\kappa\le\Delta-1}\Gamma'((\kappa+1)/(\Delta+1))/\Gamma((\kappa+1)/(\Delta+1))-$$
$$\Delta^{-1}\sum_{1\le\kappa\le\Delta-1}\Gamma'(\kappa/(\Delta-1))/\Gamma(\kappa/(\Delta-1))=$$
$$-\gamma(4[\Delta/2]-2[(\Delta+1)/2]-2[(\Delta-1)/2]-2(-1)^{\Delta}+$$
$$(\Delta+1)\log(\Delta+1)+(\Delta-1)\log(\Delta-1)-2\Delta\log(\Delta)=$$
$$(\Delta+1)\log(\Delta+1)+(\Delta-1)\log(\Delta-1)-2\Delta\log(\Delta).$$

In view of (36), (66), (67), (45),

(68)
$$\int\limits_0^{\infty}\chi^{\wedge}(\Delta x)x^{-2}\,dx=$$
$$-\log(\Delta)+\frac{\Delta-1}{2\Delta}\log(\Delta-1)+\frac{\Delta+1}{2\Delta}\log(\Delta+1)-$$
$$\frac{\pi}{2\Delta}\sum_{1\le\kappa\le(\Delta-1)/2}\cot(\pi\kappa/(\Delta-1))+$$
$$\frac{\pi}{2\Delta}\sum_{1\le\kappa\le(\Delta+1)/2}\cot(\pi\kappa/(\Delta+1)).$$

The equality (93) was obtained in Hata's paper [16]. Applying the Stirling's formula to $2\log((\nu\Delta)!)-\log((\nu\Delta+\nu)!)-\log((\nu\Delta-\nu)!)$ and making use of (16), (47), (32), (63) and (68), we obtain the equality

$$\log(r^*(\nu))-\log(r^{**}(\nu))=O(\nu\log(\nu)\exp(-(a/\sqrt{2})(\log(\nu))^{1/2}).$$

§3. On Diophantine Approximations of logarithms in cyclotomic fields. Part 1.

Here I use the same method, as in [21] – [40] and apply also the results of the section 2. I work on the Riemann surface $\mathfrak{F}$ of the function $\mathrm{Log}\,(z)$ and identify it with the direct product of the multiplicative group $\mathbb{R}_+^* = \{r \in \mathbb{R}: r > 0\}$ of all the positive real numbers with the operation $\times$, not to be written down explicitly as usual, and the additive group $\mathbb{R}$ of all the real numbers, so that $z_1z_2 = (r_1r_2, \phi_1+\phi_2)$ for any two points $z_1 = (r_1, \phi_1)$ and $z_2 = (r_2, \phi_2)$ on $\mathfrak{F}$. I will illustrate the appearing situations on the half plain (ϕ, r), where $r > 0$. For any $z = (r, \phi) \in \mathfrak{F}$, $\alpha \in \mathbb{R}$ let

$$\theta_0(z) = r \exp i\phi,\ \mathrm{Log}\,(z) = \ln(r) + i\phi,\ \eta_\alpha^*(z) = (r, \phi - \alpha).$$

Then $\theta_0(z) = \exp(\mathrm{Log}\,(z))$, $\mathrm{Log}\,(z_1z_2) = Log(z_1)+\mathrm{Log}\,(z_2)$ for any $\{z,\ z_1,\ z_2\} \subset \mathfrak{F}$. Let further $\rho(z_1, z_2) = |\,\mathrm{Log}\,(z_1) - \mathrm{Log}\,(z_2)|$, where $\{z_1,\ z_2\} \subset \mathfrak{F}$; clearly, $(\mathfrak{F}, \rho)$ is a metric space, $\rho(zz_1, zz_2) = \rho(z_1, z_2)$ for any $\{z,\ z_1,\ z_2\} \subset \mathfrak{F}$. For any $\alpha \in \mathbb{R}$ the map $z \to \eta_\alpha^*(z)$ is the bijection of $\mathfrak{F}$ onto $\mathfrak{F}$ and $\theta_0((\eta_\alpha^*)^m(z)) = \exp(-mi\alpha)\theta_0(z)$ for each $z = (r, \phi) \in \mathfrak{F}$, $\alpha \in \mathbb{R}$ and $m \in \mathbb{Z}$. The group $\mathfrak{F}$ may be considered as $\mathbb{C}$-linear space, if for any $z \in \mathfrak{F}$ and any $s \in \mathbb{C}$ we let $z^s = (|\exp(s\,\mathrm{Log}\,(z))|, \mathrm{Im}\,(s\,\mathrm{Log}\,(z))$. Let D be a domain in $\mathfrak{F}$, and $f(z) = f^\wedge(r, \phi)$ is a complex-valued function $f(z)$ on the domain D. It is well known that $f(z)$ is holomorphic in D if the complex-valued function $f^\wedge(r, \phi)$ of two real variables r and ϕ has continuous partial derivatives in the domain D, and the Cauchy-Riemann conditions

$$(\delta f)(z) := \theta_0(z)(((\partial/\partial z)f)(z)) := \tag{69}$$

$$r(((\partial/\partial r)f^\wedge)(r, \phi)) = -i(((\partial/\partial\phi)f^\wedge)(r, \phi))$$

are satisfied for every point $z = (r, \phi) \in D$. The equality (69) determines a differentiations $\partial/\partial z$ and $\delta = \theta_0(z)\partial/\partial z$ on the ring of all the holomorphic in the domain D functions. Clearly, the function $\mathrm{Log}\,(z)$ is holomorphic on $\mathfrak{F}$ and, moreover, $((\partial/(\partial z)\,\mathrm{Log}\,)(z) = \theta_0(z^{-1})$, $(\delta\,\mathrm{Log}\,)(z) = 1$. I use the functions of C.S.Mejer. Let $\Delta \in \mathbb{N} + 1$, $\delta_0 = 1/\Delta$, $\gamma_1 = (1 - \delta_0)/(1 + \delta_0)$, $\quad d_l = \Delta + (-1)^l$, $\quad l = 1, 2$. I use the auxiliary set $\Omega_0 = \{z \in \mathfrak{F}: |z| \leq 1\}$ to introduce the first of my auxiliary function $f_1(z, \nu)$, and prove that, for each $\nu \in \mathbb{N}$, the function $f_1(z, \nu)$ belongs to the ring $\mathbb{Q}[\theta_0(z)]$; therefore using the principle of analytic continuation we may regard it as being defined in $\mathfrak{F}$. For $\nu \in \mathbb{N}$, let

$$-(-1)^{\nu(\Delta+1)} f_1(z, \nu) = G_{2,2}^{(1,1)}\left(z \middle| \begin{matrix} -\nu d_1, & 1 + \nu d_2 \\ 0, & \nu \end{matrix}\right) = \frac{1}{2\pi i}\int\limits_{L_1} g_{2,2}^{(1,1)}(s)ds, \tag{70}$$

where $g_{2,2}^{(1,1)}(s) = \theta_0(z^s)\Gamma(-s)\Gamma(1 + d_1\nu + s)/(\Gamma(1 - \nu + s)\Gamma(1 + d_2\nu - s))$, and the curve L_1 passes from $+\infty$ to $+\infty$ encircling the set $\mathbb{N} - 1$ in the negative direction, but not including any point of the set $-\mathbb{N}$. So, for the parameters of the Meyer's functions we have

$$p = q = 2,\ m = n = 1,\ a_1 = -\nu d_1,\ a_2 = 1 + \nu d_2,\ b_1 = 0,\ b_2 = \nu,$$

$$\Delta^* = \left(\sum_{k=1}^{q} b_k\right) - \sum_{j=1}^{p} a_j = -\nu - 1 < -1,$$

and, since we take $|z| \leq 1$, convergence conditions of the integral (70) hold. To compute the function $f_1(z, \nu)$, we use the following formula

$$G = (-1)^k \sum_{s \in S_k} \operatorname{Res}(g; s), \tag{71}$$

where $k = 1$, G denotes the integral (70) with $L = L_k$, g denotes the integrand of the integral (70), S_k denotes the set of all the unremovable singularities of g encircled by the curve L_k, and $\operatorname{Res}(g; s)$ denotes the residue of the function g at the point s. Then we obtain the equlity

$$f_1(z, \nu) = (\nu d_1)!/(\nu\Delta)! z^\nu (-1)^{\nu\Delta} \sum_{k=0}^{\nu\Delta} (-\theta_0(z))^k \binom{\nu\Delta}{k} \binom{\nu\Delta + k}{\nu d_1}.$$

Therefore, as it has been already remarked, using the principle of analytic continuation we may regard it as being defined in $\mathfrak{F}$. Let $\Omega_1 = \{z \in \mathfrak{F} : |z| \geq 1\}$. Now, let me introduce my second auxiliary function defined for $z \in \Omega_1$. For $\nu \in \mathbb{N}$, let

$$-(-1)^{\nu(\Delta+1)} f_2(z, \nu) = G_{2,2}^{(2,1)}\left(z \middle| \begin{matrix} -\nu d_1, & 1 + \nu d_2 \\ 0, & \nu \end{matrix}\right) = \frac{1}{2\pi i} \int_{L_2} g_{2,2}^{(2,1)}(s) ds, \tag{72}$$

where $g_{2,2}^{(2,1)}(s) = \theta_0((\eta_\pi(z))^s)\Gamma(-s)\Gamma(\nu - s)\Gamma(1 + d_1\nu + s)/\Gamma(1 + d_2\nu - s)$, and the curve L_2 passes from $-\infty$ to $-\infty$ encircling the set $-\mathbb{N}$ in the positive direction, but not including any point of the set $\mathbb{N} - 1$. So, for the parameters of the Meyer's functions we have

$$p = q = m = 2,\ n = 1,\ a_1 = -\nu d_1,\ a_2 = 1 + \nu d_2,\ b_1 = 0,\ b_2 = \nu,$$

$$\Delta^* = \left(\sum_{k=1}^{q} b_k\right) - \sum_{j=1}^{p} a_j = -nu - 1 < -1,$$

and, since we take $|z| \geq 1$, convergence conditions of the integral (72) hold. To compute the function $f_2(z, \nu)$, we use the formula (71), where $k = 2$, G denotes the integral (72) with $L = L_k$, g denotes the integrand of the integral (72), S_k denotes the set of all the unremovable singularities of g encircled by L_k, and $\operatorname{Res}(g; s)$ denotes the residue of the function g at the point s. Then we obtain the equality

$$f_2(z, \nu)(\nu\Delta)!/(\nu d_1)! = (-1)^\nu \sum_{t=\nu+1}^{\infty} R_0(t; \nu)\theta_0(z^{-t+\nu}), \tag{73}$$

where $R_0(t;\nu) = (\nu\Delta)!/(\nu d_1)! \prod_{\kappa=\nu+1}^{\nu\Delta}(t-\kappa)\left(\prod_{\kappa=0}^{\nu\Delta}(t+\kappa)\right)^{-1}$. Let further

$$f_k^*(z,\,\nu) = f_k(z,\,\nu)(\nu\Delta)!/(\nu d_1)!, \tag{74}$$

where $k = 1,\,2$. Expanding the function $R_0(t;\nu)$ into partial fractions, we obtain the equality

$$R_0(t;\nu) = \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}/(t+k)$$

with

$$\alpha^*_{\nu,k} = (-1)^{\nu+\nu\Delta+k}\binom{\nu\Delta}{k}\binom{\nu\Delta+k}{\nu\Delta-\nu}, \tag{75}$$

where $k = 0,\,\dots\,,\,\nu\Delta$. It follows from (73) – (75) that

$$f_2^*(z,\nu) = (-\theta_0(z))^\nu \sum_{t=1+\nu}^{+\infty} (\theta_0(z))^{-t} R_0(t;\nu) =$$

$$(-\theta_0(z))^\nu \sum_{t=1+\nu}^{+\infty} (\theta_0(z))^{-t-k+k} \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}/(t+k) =$$

$$(-\theta_0(z))^\nu \sum_{t=1+\nu}^{+\infty} ((\theta_0(z))^{-t-k}/(t+k)) \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^k =$$

$$(-\theta_0(z))^\nu \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^k \sum_{\tau=1+\nu+k}^{+\infty} ((\theta_0(z))^{-\tau}/\tau)) = \tag{76}$$

$$= \alpha^*(z;\nu)(-\log(1-1/\theta_0(z))) - \phi^*(z;\nu),$$

where $\log(\zeta)$ is a branch of $\mathrm{Log}\,(\zeta)$ with $|\arg(\zeta)| < \pi$,

$$\alpha^*(z;\nu) = (-(\theta_0(z))^\nu \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,\,k}(\theta_0(z))^k = f_1^*(z;\nu), \tag{77}$$

$$\phi^*(z;\nu) = (-\theta_0(z))^\nu \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^k \sum_{\tau=1}^{\nu+k}((\theta_0(z))^{-\tau}/\tau)) = \tag{78}$$

$$(-\theta_0(z))^\nu \sum_{\tau=1}^{\nu}((\theta_0(z))^{-\tau}\alpha^*(z;\nu)/\tau+$$

$$(-\theta_0(z))^\nu \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^k \sum_{\tau=1+\nu}^{\nu+k} ((\theta_0(z))^{-\tau}/\tau)).$$

The change of order of summation by passage to (76) is possible, because the series in the second sum in (76) is convergent, if $|z| \geq 1$ and $\theta_0(z) \neq 1$. We see that in the domain $D_0 = \{z \in \mathfrak{F}\colon |z| > 1\}$ the funcion $f_2^*(z,\nu)$ coincides with the function

$$f_0^*(z,\nu) = \alpha^*(z;\nu)(-\log(1-1/\theta_0(z))) - \phi^*(z;\nu), \tag{79}$$

The form (79) may be used for various applikations. The following case is interesting for us:

$$m \in \mathbb{N}+1,\, k \in \mathbb{Z}, (|k|,\, m) = 1,\, \theta_0(z)) = exp(2ik\pi/m).$$

If $\theta_0(z) = -r\exp(i\psi)$ with $r > 0$ and $-\pi < \psi < \pi$, then $\mathrm{Im}\,(1-1/\theta_0(z)) = 0$ only, when $\mathrm{Re}\,(1-1/\theta_0(z)) > 0$ and we have no problems with $\log(1-1/\theta_0(z))$. So, we take $z = (r,\, \psi-\pi) = (r,\, \theta)$, where $r > 0$, $|\psi| < \pi$ and $-2\pi < \theta = \psi - \pi < 0$; clearly, the function (79) is analytic in the domain

$$D_1 = \{z = (r, \psi-\pi))\colon r > 0,\, -\pi < \psi < \pi\} = \{z = ((r,\theta))\colon r > 0,\, -2\pi < \theta < 0\}.$$

Let

$$D_2(\delta_0) = \{z \in \mathfrak{F}\colon |z| > 1 + \delta_0/2, D_3 = D_2(\delta_0) \cup D_1. \tag{80}$$

Since $D_2(\delta_0) \cap D_1 \neq \emptyset$, it follows that $D_3 = D_2(\delta_0) \cup D_1$ is a domain in $\mathfrak{F}$, and the function (79) is analytic in this domain and coincides with the function $f_2^*(z,\nu)$ in the domain $D_2(\delta_0) \subset D_0$. The conditions, which imply the equality

$$(-1)^{m+p-n}\exp(-i\alpha)\theta_0(z)\times \tag{81}$$

$$\left(\left(\prod_{j=1}^{p}(\delta+1-a_j)\right)(G\circ\eta^*_\alpha)\right)(z) = \left(\left(\prod_{k=1}^{q}(\delta-b_k)\right)(G\circ\eta^*_\alpha)\right)(z)$$

hold for $z \in D_0$ for the Mejer's function

$$G = G^{(m,n)}_{p,q}\left(z\middle|\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q\end{matrix}\right)$$

in our case (see [30], section 1, Corollary of the Lemma 2.1.5, the case C_2.) In fact we have $p = q = 2$, $m = n = 1$, $\alpha = 0$ for the function $f_1(z,\nu)$ and $|z| > 1$ in D_0; therefore the equation (81) holds and takes the form

$$\theta_0(z)((\delta+1+d_1\nu)(\delta-d_2\nu)f_1)(z,\nu) = (\delta(\delta-\nu)f_1)(z,\nu)$$

We have $p = q = m = 2$, $n = 1$, $\alpha = \pi$ for the function $f_2(z, \nu)$ and $|z| > 1$ in D_0; terefore the equation (115) holds and takes the form

$$\theta_0(z)((\delta + 1 + d_1\nu)(\delta - d_2\nu)f_2)(z, \nu) = (\delta(\delta - \nu)f_2)(z, \nu).$$

We see that both the functions $f(z, \nu) = f_k^*(z, \nu)$, where $k = 1, 2$ satisfy to the same differential equation

$$\theta_0(z)(\delta + 1 + d_1\nu)(\delta - d_2\nu)f(z, \nu) = (\delta(\delta - \nu)f)(z, \nu). \tag{82}$$

in the domain D_0. According to the general properties of the Mejer's functions we have the equality

$$\left(\prod_{\kappa=1}^{\Delta-1}(\nu(\Delta - 1) + \kappa)\right)\prod_{\kappa=1}^{d_2}(\delta - d_2\nu - \kappa)f_k^*(z,\, \nu + 1) = \tag{83}$$

$$\left(\prod_{\kappa=1}^{\Delta}(\nu\Delta + \kappa)\right)(\delta - \nu)\prod_{\kappa=1}^{d_1}(\delta + d_1\nu + \kappa)f_k^*(z,\, \nu),$$

where $k = 1, 2$ and $z \in D_0$. Since $f_0^*(z, \nu)$ and polynomial $f_1^*(z, \nu)$ are analytic in the domain $D_0 \cup D_1$, and $f_0^*(z, \nu)$ coincides with $f_2^*(z, \nu)$ in D_0, it follows that the equations (82) and (83) hold in $D_0 \cup D_1$ for $k = 0, 1$. Let

$$D^{\vee}(w, \eta) = (\eta + 1)(\eta + \gamma_1) - 2(1 + \gamma_1)w\eta, \tag{84}$$

$$D^{\wedge}(z, \eta) = D^{\vee}(\theta_0(z), \eta), \tag{85}$$

where

$$w = \theta_0(z) = -r\exp(i\psi),\ r > 0,\ |\psi| < \pi. \tag{86}$$

In view of (86), the polynomials (1) in [36] and (85) coincide.

As in [30], we consider ν^{-1} as an independent variable taking its values in $\mathbb{C}$ including 0. Let F be a bounded closed subset of $\mathfrak{F}$ (in particular, compact F may be an one-point set). Let $\mathfrak{H}_0(F)$ be the ring of all the functions in $\mathbb{Q}(w)$ well defined for every $w \in \theta_0(F)$. For $\varepsilon \in (0, 1)$, let $\mathfrak{H}(F, \varepsilon)$ be the ring of all the functions in $\mathbb{Q}(w, \nu^{-1})$ well defined for every (w, ν^{-1}) with $w \in \theta_0(F)$, $|\nu^{-1}| \leq \varepsilon_0$. Let further $h^{\sim}(\eta) = (\eta - 1)(1 - \delta_0)^{-d_1}(\eta + 1)2^{-2}\eta^{d_1}$.

Lemma 4. *Let F be a closed bounded subset of $D_0 \cup D_1$ (in particular, F may be an one-point set). Let for any $z \in F$ the polynomial (85) has only simple roots and on the set of all the its roots η the map*

$$\eta \to h^{\sim}(\eta) \tag{87}$$

is injective. Then there is $\varepsilon \in (0,1)$ such that, for any $z \in F, \nu \in \mathbb{N} + [1/\varepsilon]$, the functions $f_0^*(z,\nu)$, $f_1^*(z,\nu) = \alpha^*(z;\nu)$ and $\phi^*(z;\nu)$ are solutions of the difference equation

$$x(z,\nu+2) + \sum_{j=0}^{1} q_j^*(z,\nu^{-1})x(z,\nu+j) = 0, \tag{88}$$

moreover, $q_j^*(z,\, \nu^{-1}) \in \mathfrak{H}(F,\, \varepsilon)$ for $j = 0,\, 1,$ and trinomial $w^2 + \sum\limits_{j=0}^{1} q_j^*(z,0)w^j$ coincides with

$$\prod_{k=0}^{1}(w - h(\eta_k)), 89$$

if $\prod\limits_{k=0}^{1}(w - \eta_k)$ coincides with $D^{\vee}(w,\eta)$ from (84).

Proof. Proof may be found in [30]. ∎

This Lemma shows the importance of the properties of the roots of the polynomial (85); all the its roots $\eta_k^{\wedge}(r,\psi,\delta_0)$, where $k = 0,\, 1;\; 0 < r;\,,\psi \in \mathbb{R}$, are presented in the equality (65) of the paper [36].

Let $\delta_0 > 0,\; 2r \geq \delta_0$. It follows from the Lemma 3.1.3 in [35] and its proof that

$$1 \leq |\eta_0^{\wedge}(r,\pi,\delta_0)|\,/|\eta_1^{\wedge}(r,\pi,\delta_0)| \leq |\eta_0^{\wedge}(r,\psi,\delta_0)|\,/|\eta_1^{\wedge}(r,\psi,\delta_0)|, \tag{90}$$

and if $2r \geq \delta_0$, then

$$1 \leq |\eta_0^{\wedge}(r,\pi,\delta_0)+1|\,/|\eta_1^{\wedge}(r,\pi,\delta_0)+1| \leq |\eta_0^{\wedge}(r,\psi,\delta_0)+1|\,/|\eta_1^{\wedge}(r,\psi,\delta_0)+1|, \tag{91}$$

where $\psi \in \mathbb{R}$; moreover, if $2r > 1 + \sqrt{1-\delta_0^2}$, then left inequality in (90) and left inequality in (91) are strong; on the other hand, when $r \geq \delta_0/2$ and $\psi \in [0,\pi)$, then right inequality in (90) and right inequality in (91) are strong. Let $\delta_0 > 0, 2r \geq \delta_0$,

$$\mu_0(r,\delta_0) = \tag{92}$$

$$(|\eta_1^{\wedge}(r,\pi,\delta_0)|\,/|\eta_0^{\wedge}(r,\pi,\delta_0)|)^{(1/\delta_0 - 1)}|\eta_1^{\wedge}(r,\pi,\delta_0)+1|\,/|\eta_0^{\wedge}(r,\pi,\delta_0)+1|,$$

$$c_0(r,\delta_0) = (\mu_0(r,\delta_0))^2 + 1)/(2\mu_0(r,\delta_0)). \tag{93}$$

According to (3.1.43) in [34] and (90) – (92), if $2r \geq \delta_0$, then

$$0 < \mu_0(r,\delta_0) \leq 1, \tag{94}$$

$$1/\mu_0(r,\delta_0)) \leq \tag{95}$$

$$(|\eta_0^\wedge(r,\psi,\delta_0)|\,/|\eta_1^\wedge(r,\psi,\delta_0)|)^{1/\delta_0-1}|\eta_0^\wedge(r,\psi,\delta_0)+1|\,/|\eta_1^\wedge(r,\psi,\delta_0)+1|,$$

and if $2r > 1+\sqrt{1-(\delta_0)^2}$, then right inequality in (94) is strong.

Lemma 5. *Let $\mu > 0$ and all roots of quadratic trinomial $x^2 - px + q$ are positive. Let further $x_{\max}$ and $x_{\min}$ denote respectively the maximal and minimal of these roots. Then the inequality*

$$x_{\min}/x_{\max} > \mu^2 \tag{96}$$

is equivalent to the inequality

$$p/\sqrt{q} < (\mu^2+1)/\mu. \tag{97}$$

Proof. We have

$$x_{\min}/x_{\max} = (x_{\min})^2/q = (x_{\min}/\sqrt{q})^2 = (p/\sqrt{q} - ((p/\sqrt{q})^2-4)^{1/2})^2/4.$$

Therefore (96) is equivalent to the inequality $p/\sqrt{q} - 2\mu > (p/\sqrt{q})^2-4)^{1/2}$, which is equivalent to (97) ■.

Let

$$s = s_0(r,\psi) = |r\exp(i\psi)+1|\,/2 = ((r-1)^2/4 + r\cos^2(\psi/2))^{1/2}, \tag{98}$$

where $\psi \in [-\pi,\pi]$.

Lemma 6. *Let $0 < \delta_0 < 1$ and at least one of the following conditions is fulfilled:*

A) $r \ge 1+\delta_0/2$, $|r-1| \le 2s \le (r+1)$;

B) $\max(2|r-1|,\delta_0) < 4s \le 2(r+1)$, $\delta_0 \le 2r$

C) $2|r-1| < \delta_0 = 4s \le (r+1)$, $\delta_0 \le 2r$,

D) $0 < |r-1|/2 \le s \le (r+1)/2$, $r > (1+\sqrt{1-(\delta_0)^2})/2$,

E)

$$1-\delta_0/2 < r < 1+\delta_0/2,\ |r-1|/2 \le s \le \delta_0/4 \tag{99}$$

and

$$s > \sigma_0(r,\delta_0) = \delta_0((1+\delta_0/2)^2 - r^2)/(4(c_0(r,\delta_0)(1+\delta_0) - \delta_0/2 - r)). \tag{100}$$

Then

(101)
$$|\eta_0^\wedge(r,\psi,\delta_0)|^{1-\delta_0}|(\eta_0^\wedge(r,\psi,\delta_0))^2-1|^{\delta_0} > |\eta_1^\wedge(r,\psi,\delta_0)|^{1-\delta_0}|(\eta_1^\wedge(r,\psi,\delta_0))^2-1|^{\delta_0}.$$

Proof. In the cases A), C) and D), according to the Lemma 4.4 in [36],

$$|\eta_0^\wedge(r,\psi,\delta_0)|^{1-\delta_0}|\eta_0^\wedge(r,\psi,\delta_0)+1|^{\delta_0} > |\eta_1^\wedge(r,\psi,\delta_0)|^{1-\delta_0}|\eta_1^\wedge(r,\psi,\delta_0)+1|^{\delta_0},$$

but, according to the Lemma 4.6 in [36], and in view of (3.1.71) in [35],

$$|\eta_0^\wedge(r,\psi,\delta_0)-1|^{\delta_0}\geq|\eta_1^\wedge(r,\psi,\delta_0)-1|^{\delta_0}>0,$$

So, in the cases A), C) and D) the inequality (100) holds.

In the case B, in view of the Lemma 4.4 in [36] and (3.1.43) in [35],

$$|\eta_0^\wedge(r,\psi,\delta_0)|^{1-\delta_0}|\eta_0^\wedge(r,\psi,\delta_0)+1|^{\delta_0}\geq|\eta_1^\wedge(r,\psi,\delta_0)|^{1-\delta_0}|\eta_1^\wedge(r,\psi,\delta_0)+1|^{\delta_0}>0,$$

but, according to the Lemma 4.6 in [36], $|\eta_0^\wedge(r,\psi,\delta_0)-1|^{\delta_0}>|\eta_1^\wedge(r,\psi,\delta_0)-1|^{\delta_0}$. So, in the case B) the inequality (100) holds. In the case E) (see the proof of the Lemma 4.9 for the case E) in [36]) the inequality (101) is equivalent to the inequality

$$(|\eta_0^\wedge(r,\psi,\delta_0)|\,/|\eta_1^\wedge(r,\psi,\delta_0)|)^{(1/\delta_0-1)}(|\eta_0^\wedge(r,\psi,\delta_0)+1|\,/|\eta_1^\wedge(r,\psi,\delta_0)+1|)\times$$
$$|\eta_0^\wedge(r,\psi,\delta_0)-1|>|\eta_1^\wedge(r,\psi,\delta_0)-1|,$$

which, in view of (95), will be fulfilled, if the inequality

$$|\eta_0^\wedge(r,\pi,\delta_0)-1|^2>|\eta_1^\wedge(r,\pi,\delta_0)-1|^2(\mu_0(r,\delta_0))^2 \tag{102}$$

will be fulfilled. Since we are found in the case h) of the Lemma 4.6 in [36], it follows that $x_{\max}=|\eta_1^\wedge(r,\pi,\delta_0)-1|^2, x_{\min}=|\eta_0^\wedge(r,\pi,\delta_0)-1|^2$ and (102) will be fulfilled, when (97) will be fulfilled with $p=p_0$, $q=q_0$, where p_0 and q_0 are taken from (70) – (71) in [36]. The equality (3.1.69) in [35] may be rewritten in the form

$$\begin{gathered}|R_0(r,\psi,\delta_0)|^4=|D_0(r,\psi,\delta_0)|^2=r^2+(r^2-(\delta_0/2)^2)^2+ \\ 2r(r^2+(\delta_0/2)^2)t+r^2(\delta_0)^2t^2=r^2(\delta_0)^2((4s^2-r^2-1)/2r)^2+ \\ 2r(r^2+(\delta_0/2)^2)((4s^2-r^2-1)/2r)+r^2+(r^2-(\delta_0/2)^2)^2= \\ 4(\delta_0)^2s^4+(4r^2-(\delta_0)^2(2r^2+1))s^2+(\delta_0/2)^2r^2(r^2-1)+(\delta_0/2)^4= \\ (2rs+(\delta_0/2)^2)^2+(\delta_0)^2\alpha(s),\end{gathered} \tag{103}$$

where

$$\alpha(s)=4s^4-(2r^2+1)s^2+r^2(r^2-1)/4-rs=(2s^2-r^2/2)^2-(s+r/2)^2=$$
$$(2s^2+s+r/2-r^2/2)(2s^2-s-r/2-r^2/2)=$$
$$4(s-(r-1)/2)(s+r/2)^2(s-(r+1)/2))=(2s+r)^2(s-(r-1)/2)(s-(r+1)/2)).$$

Since $\alpha(s)\leq 0$, when $\frac{r-1}{2}\leq s\leq\frac{r+1}{2}$, it follows, in view of (3.1.70) - (3.1.71) in [35] and (103), that

$$\frac{p_0}{2\sqrt{q_0(r,\psi,\delta_0)}}\leq(2(1+\delta_0)s)^{-1}(2(2+\delta_0)s^2-\delta_0(r^2-1)+(\delta_0/2)^2+2rs+(\delta_0/2)^2).$$

Therefore (97) will be fulfilled, if the inequality

$$c_0(r,\delta_0) > (2(1+\delta_0)s)^{-1}(2(2+\delta_0)s^2 - \delta_0(r^2-1) + (\delta_0/2)^2 + 2rs + (\delta_0/2)^2),$$

will be fulfilled; according to (99),

$$2(2+\delta_0)s^2 \le (2+\delta_0)(\delta_0/2)s \le \delta_0 s + (\delta_0/2)^3.$$

Therefore (97) will be fulfilled, if the inequality

$$2(c_0(r,\delta_0)(1+\delta_0) - \delta_0/2 - r)s > \delta_0(1+\delta_0/2)^2 - r^2)/2,$$

will be fulfilled; the last inequality is equivalent to (100). ■

According to the Remark 1 in [36] (see also the section 4 below), if $\varepsilon^3 = \varepsilon$, $k^2 = k$, $|2r-1| \ge \sqrt{1-\delta_0^2}$, then

$$\eta_k^{\wedge}(r,\pi,\delta_0) + \varepsilon = \tag{104}$$

$$(2(r - 1/2 + \varepsilon(1+\delta_0)/2 + (-1)^k \operatorname{sign}(2r-1)(r^2 - r + (\delta_0/2))^2)^{1/2})/(1+\delta_0).$$

and if $\varepsilon^3 = \varepsilon$, $k^2 = k$, $|2r-1| \le \sqrt{1-\delta_0^2}$, then $\eta_k^{\wedge}(r,\pi,\delta_0) + \varepsilon = \overline{\eta_{1-k}^{\wedge}(r,\pi,\delta_0) + \varepsilon}$. If $\varepsilon^2 = \varepsilon$, $k^2 = k$ and $r \ge (1 + (1-\delta_0^2)^{1/2})/2$, then (104) implies the relations

$$\eta_k^{\wedge}(r,\pi,\delta_0) > 0,$$

$$|\eta_k^{\wedge}(r,\pi,\delta_0) + \varepsilon| = \eta_k^{\wedge}(r,\pi,\delta_0) + \varepsilon = \frac{2r - 1 + 2(-1)^k\sqrt{r^2 - r + (\frac{\delta_0}{2})^2}}{1+\delta_0} + \varepsilon. \tag{105}$$

Therefore, if $\varepsilon^2 = \varepsilon, k^2 = k$ and $r \ge \left(1 + \sqrt{1-\delta_0^2}\right)/2$, then

$$(d/dr)|\eta_k^{\wedge}(r,\pi,\delta_0) + \varepsilon| = (d/dr)|\eta_k^{\wedge}(r,\pi,\delta_0)|. \tag{106}$$

In view of (105), if $r > (1-\delta_0^2)^{1/2}/2$, then

$$(d/dr)|\eta_0^{\wedge}(r,\pi,\delta_0)| > 0, \tag{107}$$

and in view of the first of the equalities (3.1.43) in [35], if $r > \frac{1+\sqrt{1-\delta_0^2}}{2}$, then

$$(d/dr)|\eta_1^{\wedge}(r,\pi,\delta_0)| < 0. \tag{108}$$

In view of (106) – (108), if $\varepsilon^2 = \varepsilon, k^2 = k$ and $r \ge (1 + (1-\delta_0^2)^{1/2})/2$, then

$$(-1)^k(d/dr)|\eta_k^{\wedge}(r,\pi,\delta_0) + \varepsilon| > 0. \tag{109}$$

If $r \geq \left(1+\sqrt{1-\delta_0^2}\right)/2$, then, in view of (113) – (114) in [36], and, in view of (92) and (109), $1/\mu_0(r,\delta_0) > 1$, $(d/dr)(1/\mu_0(r,\delta_0)) > 0$. Therefore, according to (93) and (100), if $r > (1+(1-\delta_0^2)^{1/2})/2$, then $(d/dr)c_0(r,\delta_0) > 0$, $(d/dr)\sigma_0(r,\delta_0) < 0$.

Lemma 7. *If* $r=1$, $\Delta \in 2+\mathbb{N}$, *then* $\sigma_0(r,\delta_0) < (1/51)(26/27)$.

Proof. See the Proof of the Corollary of the Lemma 4.9 in [36]. ■

Lemma 8. *Let* $\Delta \in 2+\mathbb{N}$, $\delta_0 = 1/\Delta$. *Then* $((1+(1-\delta_0^2)^{1/2})/2,\, 1)$ *contains the number* $\rho_0(\delta_0)$ *such that* $\sigma_0(r,\delta_0) < 1/51$ *for* $r \in (\rho_0(\delta_0),\, 1+\delta_0/2)$.

Proof. Since $c_0(r,\delta_0) \geq 1$, it follows that the assertion of the Lemma is direct corollary of (100), the Lemma 7 and the Lemma 6. ■

Clearly, for $\rho_0(\delta_0)$ of the Lemma 8 we have the inequality

$$1-\rho_0(\delta_0) < (1-(1-(\delta_0^2))^{1/2})/2 = (\delta_0^2/2)/(1+(1-(\delta_0)^2)^{1/2}) \leq \delta_0/(6+4\sqrt{2}) < \delta_0/2.$$

If $\Delta \in [3,16] \cap \mathbb{N}]$,then $1/51 < \delta_0/4$. Therefore, if $\Delta \in [3,16] \cap \mathbb{N}]$, then

$$D_4(\delta_0) = \left\{\{r,s\} \subset \mathbb{R}\colon \rho_0(\delta_0) < r < 1+\frac{\delta_0}{2}, \max\left(\frac{|r-1|}{2}, \frac{1}{51}\right) < s < \frac{\delta_0}{4}\right\} \neq \emptyset; \tag{110}$$

so the domain $D_4(\delta_0)$ consists of all the points $(r,\psi-\pi) \in \mathfrak{F}$, where, in correspondence with the equality (98), r and $\psi \in (-\pi,\pi)$ satisfy to the condition (110). Let $D_4^*(\delta_0)$ denotes the closure of $D_4(\delta_0)$. According to E) of the Lemma 6, the inequality (101) holds for any $(r,\psi-\pi) \in D_4^*(\delta_0)$. Clearly,

$$D_4(\delta_0) \subset D_5(\delta_0) = \left\{\{r,s\} \subset \mathbb{R}\colon \rho_0(\delta_0) < r < 1+\frac{\delta_0}{2}, \max\left(\frac{|r-1|}{2}, \frac{1}{51}\right) < s < (r+1)/2\right\}.$$

According to B), C) and E) of the Lemma 6, if $\Delta \in [3,16] \cap \mathbb{N}]$, then (101) holds for any $(r,\psi-\pi) \in D_5(\delta_0)$. Let

$$D_6(\delta_0) = \{\{r,s\} \subset \mathbb{R}\colon \delta_0/2 < r,\ \max(2|r-1|,\delta_0) < 4s \leq 2(r+1)\},$$

and let $D_6^*(\delta_0)$ denotes the closure of the domain $D_6(\delta_0)$. According to B) of the Lemma 6, the inequality (101) holds for any $(r,\psi-\pi) \in D_6^*(\delta_0)$. Let further $D_2^*(\delta_0)$ denotes closure of the domain $D_2(\delta_0)$. In view of (80) and according to A) of the Lemma 5, the inequality (101) holds for the arbitrary point $(r,\psi-\pi) \in D_2^*(\delta_0)$. Let $D_7(\delta_0) = D_5(\delta_0) \cup D_6(\delta_0) \cup D_2(\delta_0)$ and $D_7^*(\delta_0)$ denotes the closure of $D_7(\delta_0)$. Since $D_5(\delta_0) \cap D_6(\delta_0) \neq \emptyset$, $D_2(\delta_0) \cap D_6(\delta_0) \neq \emptyset$, it follows that $D_7(\delta_0)$ is a domain in $\mathfrak{F}$, and the inequality (101) holds for any point $(r,\psi-\pi) \in D_7^*(\delta_0)$.

Let D is a domain in $\mathbb{C}$ or $\mathfrak{F}$. and D^* is closure of D. Let

$$a_0^{\sim}(z), \ldots, a_n^{\sim}(z) \tag{111}$$

which are functions continuous on D^* and analytic in D. Let $a_n^{\sim}(z) = 1$ for any $z \in D^*$. Let

$$T(z, \lambda) = \sum_{i=0}^{n} a_i^{\sim}(z)\lambda^k. \tag{112}$$

Let $s \in \mathbb{N}$, $n_i \in \mathbb{N} - 1$, where $i = 1, \ldots, s$ and $\sum_{i=1}^{s} n_i = n$. We say that $T(z, \lambda)$ has $(n_1, \ldots, n_s)$-disjoint system of roots on D^*, if for any $z \in D^*$ the set of all the roots λ of the polynomial $T(z, \lambda)$ splits in s klasses $\mathfrak{K}_1(z), \ldots, \mathfrak{K}_s(z)$ with following properties:

a) for $i = 1, \ldots, s$ the sum of the multiplicities of the roots of the klass $\mathfrak{K}_i(z)$ is equal to n_i;

b) if $i \in [1, s] \cap \mathbb{N}$, $j \in (i, s] \cap \mathbb{N}$, $n_i n_j \neq 0$, then the absolute value of each root of the klass $\mathfrak{K}_i(z)$ is greater than absolute value of the each root of the klass $\mathfrak{K}_j(z)$.

If the polynomial (112) has $(n_1, \ldots, n_s)$-disjoint system of roots on D^*, then for each $i = 1, \ldots, s$ we denote by $\rho^*_{i,0}(z)$ and $\rho^*_{i,1}(z)$ respectively the maximal and minimal absolute value of the roots of the klass $\mathfrak{K}_i(z)$. Let D is bounded domain in $\mathfrak{F}$ such that $D^* \in D_7(\delta_0)$. Let

$$F^{\wedge}(z, \eta) = \prod_{i=1}^{2} (\theta_0(z) - h(\eta_{i-1}(r, \psi, \delta_0))), \tag{113}$$

$$n = s = 2,\ n_1 = n_2 = 1,\ \mathfrak{K}_i(z) = \{h(\eta_{i-1}(r, \psi, \delta_0))\}$$

$$\rho_{i,0} = \rho_{i,1} = |h(\eta_{i-1}(r, \psi, \delta_0))|,$$

where $i = 1, 2$.

Lemma 9. *The polynomial $F^{\wedge}(z, \eta)$ in (113) has* $(1, 1)$*-disjoint system of roots on D^*.*

Proof. According to the previous considerations, the more strong assertion holds: the polynomial (113) has $(1, 1)$-disjoint system of roots on $D_7^*(\delta_0)$. ■

Corollary. *Let D is bounded domain in $\mathfrak{F}$ such that $D^* \subset D_7(\delta_0)$. Then the map (87) is injective for every $z \in D^*$; the functions $f_0^*(z, \nu)$, $\alpha^*(z, \nu)$ and $\phi^*(z, \nu)$ in (79), (77) and (78) satisfy to all the conditions of the Lemma 2 for every $z \in D^*$; therefore for every $z \in D^*$ these functions are solutions of the difference equation of Poincaré type (88), and characteristical polynomial of this equation coincides with the polynomial (89).* ■

Let for each $\nu \in \mathbb{N} - 1$ are given continuous on D^* functions

$$a_0(z; \nu), \ldots, a_n(z, \nu), \tag{114}$$

which are analytic in D. Let $a_n(z : \nu) = 1$ for any $z \in D^*$ and any $\nu \in \mathbb{N} - 1$. Let for any $i = 1, \ldots, n-1$ the sequence of functions $a_i(z; \nu)$ converges to $a_i^{\sim}(z)$ uniformly on D^*, when $\nu \to \infty$. Let us consider now the difference equation

$$a_0(z; \nu)y(\nu + 0) + \ldots + a_n(z; \nu)y(\nu + n) = 0, \tag{115}$$

i.e. we consider a difference equation of the Poincaré type, coefficients (114) of this equation are continuous on D^* and analytic in D, and they uniformly converge to limit functions (111), when $\nu \to \infty$.

Lemma 10. *Let the equation (115) has a solution $y(z, \nu)$, which is continuous on D^* and analytic on D. Let polynomial (112) has $(n_1, \dots, n_s)$-disjoint system of roots on D^*. Let $i \in [1, s] \cap \mathbb{Z}$. Let us consider the set of all the $z \in D$, for which the following inequality holds*

$$\limsup_{\nu \in \mathbb{N}, \nu \to \infty} |y(z, \nu)|^{1/\nu}) < \rho_{i,1}(z); \tag{116}$$

if this set has a limit point in D, then the inequality (116) holds in D^*.

Proof. The proof may be found in [20] (Theorem 1 and its Corollary). ■

Lemma 11. *Let D is bounded domain in $\mathfrak{F}$ such that $D^* \in D_3$. Then*

$$\limsup_{\nu \in \mathbb{N}, \nu \to \infty} |f_0^*(z, \nu)|^{1/\nu}) < \rho_{1,1}(z) = |h^{\sim}(\eta_0(r, \psi, \delta_0))| \tag{117}$$

for any $z \in D^*$.

Proof. In view of (80), expanding the domain D, if necessary, we can suppose that $\{(r, \phi)\colon r \in [2, 3], \phi = 0\} \subset D$. Making use the same arguments, as in [33], Lemma 4.2.1, we see that (117) holds for any $z = (r, \phi) \in \{r \in [2, 3], \phi = 0\}$. According to the Lemma 5, (117) holds for any $z \in D^*$. ■

Let $K/\mathbb{Q}$ be the finite extension of the field $\mathbb{Q}$, $[K : \mathbb{Q}] = d$. Let K has r_1 real places and r_2 complex places. Each such place is the monomorphism of the field K in $\mathbb{R}$, if a place is real, or in $\mathbb{C}$, if a place is not real; we will denote these monomorphisms respectively by $\sigma_1, \dots \sigma_{r_1+r_2}$. Then $d = r_1 + 2r_2$. Let $\mathfrak{B}$ be the fixed integer basis

$$\omega_1, \dots, \omega_d \tag{118}$$

of the field K over $\mathbb{Q}$. Clearly,K is an algebra over $\mathbb{Q}$. With extension of the ground field from $\mathbb{Q}$ to $\mathbb{R}$ appears an isomorphism of the algebra $\mathfrak{K} = K \otimes \mathbb{R}$ onto direct sum $\underbrace{\mathbb{R} \oplus \dots \oplus \mathbb{R}}_{r_1 \text{ times}} \oplus \underbrace{\mathbb{C} \oplus \dots \oplus \mathbb{C}}_{r_2 \text{ times}}$ of r_1 copies of $\mathbb{R}$ and r_2 copies of $\mathbb{C}$. We identify by means of this isomorphism the aIgebra $\mathfrak{K}$ with the specified direct sum. We denote below by π_j, where $j = 1, \dots, r_1 + r_2$, the projection of $\mathfrak{K}$ onto its $j-$th direct summand and also the extension of this projection onto all the kinds of matrices which have all the elements in $\mathfrak{K}$. So, $\pi_j(\mathfrak{K}) = \mathbb{R}$ for $j = 1, \dots, r_1$ and $\pi_j(\mathfrak{K}) = \mathbb{C}$ for $j = r_1 + 1, \dots, r_1 + r_2$. We denote by $\mathfrak{i}_{\mathfrak{K}}$ the embedding of $\mathbb{R}$ in $\mathfrak{K}$ in diagonal way and also the extension of this embedding onto all kinds of the real matrices. So, $\mathbb{R}$ is imbedded by means of $\mathfrak{i}_{\mathfrak{K}}$ in $\mathfrak{K}$ in diagonal way. Each element $Z \in \mathfrak{K}$ has a unique representation in the form:

$$Z = \begin{pmatrix} z_1 \\ \vdots \\ z_{r_1+r_2} \\ \overline{z_{r_1+1}} \\ \vdots \\ \overline{z_{r_1+r_2}} \end{pmatrix}, \tag{119}$$

with $z_j = \pi_j(Z)$ for any $j = 1, \dots, r_1 + r_2$. We denote by $\mathrm{Tr}_{\mathfrak{K}}(Z)$ the sum

$$\sum_{j=1}^{r_1} \pi_j(Z) + \sum_{j=r_1+1}^{r_1+r_2} 2\,\mathrm{Re}\,(\pi_j(Z)),$$

and by $q_\infty^{(\mathfrak{K})}(Z)$ we denote the value $\max(|\pi_1(Z)|, \dots, |\pi_{r_1+r_2}(Z)|)$. Clearly,

$$q_\infty^{(\mathfrak{K})}(Z_1 Z_2) \le q_\infty^{(\mathfrak{K})}(Z_1) q_\infty^{(\mathfrak{K})}(Z_2),$$

$$q_\infty^{(\mathfrak{K})}(Z_1 + Z_2) \le q_\infty^{(\mathfrak{K})}(Z_1) + q_\infty^{(\mathfrak{K})}(Z_2),\ q_\infty^{(\mathfrak{K})}(\mathrm{i}_{\mathfrak{K}}(\lambda) Z) = |\lambda| q_\infty^{(\mathfrak{K})}(Z)$$

for any $Z_1 \in \mathfrak{K}$, $Z_2 \in \mathfrak{K}$, $Z \in \mathfrak{K}$ and $\lambda \in \mathbb{R}$. The natural extension of the norm $q_\infty^{(\mathfrak{K})}$ on the set of all the matrices, which have all the elements in $\mathfrak{K}$ (i.e. the maximum of the norm $q_\infty^{(\mathfrak{K})}$ of all the elements of the matrix) also will be denoted by $q_\infty^{(\mathfrak{K})}$.

If $Z = \begin{pmatrix} z_1 \\ \vdots \\ z_d \end{pmatrix} \in K \subset \mathfrak{K}$, then $z_j = \sigma_j(Z) = \pi_j(Z)$, where $j = 1, \dots, r_1 + r_2$, and $z_{r_1+r_2+j} = \overline{\sigma_{r_1+j}(Z)}$, where $j = 1, \dots, r_2$. In particular,

$$\omega_k = \begin{pmatrix} \sigma_1(\omega_k) \\ \vdots \\ \sigma_{r_1+r_2}(\omega_k) \\ \overline{\sigma_{r_1+1}(\omega_k)} \\ \vdots \\ \overline{\sigma_{r_1+r_2}(\omega_k)} \end{pmatrix},$$

The ring of all the integer elements of the field K will be denoted by $\mathbb{Z}_K$. On $\mathfrak{K}$ there is the involution which turns each element (119) in $\mathfrak{K}$ into $\overline{Z} = \begin{pmatrix} \overline{z_1} \\ \vdots \\ \overline{z_d} \end{pmatrix} \in \mathfrak{K}$.

Every element Z of the algebra $\mathfrak{K}$ has a unique representation in the form

$$Z = \sum_{j=1}^{d} x_j \omega_j \tag{120}$$

with $x_j \in \mathbb{R}$ and ω_j in (118). Let $D \in \mathrm{Mat}_{\,d}(\mathbb{C})$, and ω_j in (118) is its j-th column for $j = 1, \ldots, d$. Let further $X_Z^{(\mathfrak{B})}$ denotes the column $\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$. Then (120) may be rewritten in the form $Z = DX_Z^{(\mathfrak{B})}$. Clearly,

$$K = \left\{ \sum_{j=1}^{d} x_j\omega_j \colon x_j \in \mathbb{Q}, j = 1, \ldots, d \right\}, \mathbb{Z}_K = \left\{ \sum_{j=1}^{d} x_j\omega_j \colon x_j \in \mathbb{Z}, j = 1, \ldots, d \right\}.$$

The ring $\mathbb{Z}_K$ is embedded in the ring $\mathfrak{K}$ as discrete lattice. Moreover, if $Z \in \mathbb{Z}_K \backslash \{0\}$, then $\left(\prod\limits_{i=1}^{r_1} |\sigma_j(Z)| \right) \prod\limits_{i=1}^{r_2} |\sigma_{r_1+i}(Z)|^2 = |\,\mathrm{Nm}_{\,K/\mathbb{Q}}(Z)| \in \mathbb{N}$ and therefore

$$q_\infty^{(\mathfrak{K})}(Z) \geq 1. \tag{121}$$

for any $Z \in \mathbb{Z}_K \backslash \{0\}$. For each $Z \in \mathfrak{K}$ let $\|\mathbb{Z}\|_K = \inf\limits_{W \in \mathbb{Z}_K} \{q_\infty^{(\mathfrak{K})}(Z - W)\}$. The elements of $\mathbb{Z}_K$ we name below by K-integers.

Lemma 12. *If $Z \in \mathfrak{K}$ and $q_\infty^{(\mathfrak{K})}(Z) \leq 1/2$ then $\|Z\|_K = q_\infty^{(\mathfrak{K})}(Z)$.*

Proof. If $q_\infty^{(\mathfrak{K}}(Z - W) < q_\infty^{(\mathfrak{K})}(Z)$ with $W \in \mathbb{Z}_K$, then

$$q_\infty^{(\mathfrak{K})}(W) \leq q_\infty^{(\mathfrak{K})}(Z) + q_\infty^{(\mathfrak{K})}(Z - W) < 2q_\infty^{(\mathfrak{K})}(Z) \leq 1.$$

In view (121), $W = 0$. ■

Let $m \in \mathbb{N}$, $n \in \mathbb{N}$, $a_{i,k} \in \mathfrak{K}$ for $i = 1, \ldots, m,\ k = 1, \ldots, n$, let $\alpha_j^\wedge(\nu) \in \mathbb{Z}_K$, where $j = 1, \ldots, m + n$ and $\nu \in \mathbb{N}$. Let there are $\gamma_0, r_1^\wedge \geq 1, \ldots, r_m^\wedge \geq 1$ such that

$$q_\infty^{(\mathfrak{K})}(\alpha_i(\nu)) < \gamma_0 (r_i^\wedge)^\nu \tag{122}$$

where $i = 1, \ldots, m$ and $\nu \in \mathbb{N}$. Let further $y_k(\nu) = -\alpha_{m+k}^\wedge(\nu) + \sum\limits_{i=1}^{m} a_{i,k}\alpha_i^\wedge(\nu)$ where $k = 1, \ldots, n$ and $\nu \in \mathbb{N}$. If $X = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} \in \mathfrak{K}^n$, then let $\phi_i(X) = \sum\limits_{k=1}^{n} a_{i,k}Z_k$ for $i = 1, \ldots, m$, let $y^\wedge(X) = y^\wedge(X, \nu) = \sum\limits_{k=1}^{n} y_k^\wedge(\nu)Z_k$ for $\nu \in \mathbb{N}$, and let

$$\alpha_0^\wedge(X, \nu) = \sum_{k=1}^{n} \alpha_{m+k}^\wedge(\nu)Z_k \tag{123}$$

for $\nu \in \mathbb{N}$. Clearly, $y^\wedge(X, \nu) = -\alpha_0^\wedge(X, \nu) + \sum\limits_{i=1}^{m} \alpha_i^\wedge(\nu)\phi_i(X)$ for $X \in \mathfrak{K}^n$, $\nu \in \mathbb{N}$; moreover, $\alpha_0^\wedge(X, \nu) \in \mathbb{Z}_K$ for $X \in (\mathbb{Z}_K)^n$ and $\nu \in \mathbb{N}$.

Lemma 13. *Let* $l \in \mathbb{N}$, $\gamma_1 > 0$, *let*

(124) $$\gamma_2 > \frac{1}{2}, R_1 \geq R_2 > 1,\ \alpha_i = (\log(r_i^\wedge R_1/R_2))/\log(R_2),$$

let where $i = 1, \ldots, m$, *let* $\gamma_3 = \gamma_1(R_1)^{(-\log(2\gamma_2 R_2))/\log(R_2)}$,

$$\gamma_4 = \gamma_3 \left(\sum_{i=1}^{m} \gamma_0 (r_i^\wedge)^{(log(2\gamma_2))/\log(R_2)+l} \right)^{-1}$$

(188) $$X \in (\mathbb{Z}_K)^n \backslash \{(0)\},$$

and let for each $\nu \in \mathbb{N} - 1$ *hold the inequalities*

(126) $$q_\infty^{(\mathfrak{K})}(y^\wedge(X, \nu)) \leq \gamma_2 (R_2)^{-\nu} q_\infty^{(\mathfrak{K})}(X),$$

$$\gamma_1 (R_1)^{-\nu} q_\infty^{(\mathfrak{K})}(X) \leq \sup\{q_\infty^{(\mathfrak{K})}(y^\wedge(X, \kappa)) : \kappa = \nu, \ldots, \nu + l - 1\}.$$

Then

(127) $$\sup\{\|\phi_i(X)\|_K (q_\infty^{(\mathfrak{K})}(X))^{\alpha_i} : i = 1, \ldots, m\} \geq \gamma_4.$$

Proof. According to (124), (125) and (121), $2\gamma_2 q_\infty^{(\mathfrak{K})}(X) > 1$. Let

(128) $$\nu_1 = \inf\{\nu \in \mathbb{N} - 1 : \gamma_2 q_\infty^{(\mathfrak{K})}(X)(R_2)^{-\nu} \leq 1/2\} =$$

$$\inf\{\nu \in \mathbb{N} - 1 : \nu \geq \log(2\gamma_2 q_\infty^{(\mathfrak{K})}(X))/\log(R_2).$$

Clearly, $\nu_1 = \log(2\gamma_2 q_\infty^{(\mathfrak{K})}(X))/\log(R_2) + \theta$, where $\theta \in [0, 1)$,

(129) $$\gamma_1 (R_1)^{-\nu_1} q_\infty^{(\mathfrak{K})}(X) =$$

$$\gamma_1 \exp(-(\log(R_1))(log(2\gamma_2 q_\infty^{(\mathfrak{K})}(X))/\log(R_2) + \theta)) q_\infty^{(\mathfrak{K})}(X) =$$

$$\gamma_1 (q_\infty^{(\mathfrak{K})}(X))^{-(\log((R_1)/R_2))/\log(R_)} \exp(-(\log(R_1))((log(2\gamma_2))/\log(R_2) + \theta)) \geq$$

$$\gamma_1 (\exp(-(\log(R_1))(log(2\gamma_2 R_2))/\log(R_2)))(q_\infty^{(\mathfrak{K})}(X))^{-(\log((R_1)/R_2))/\log(R_2)} =$$

$$\gamma_3 (q_\infty^{(\mathfrak{K})}(X))^{-(\log((R_1)/R_2))/\log(R_2)}.$$

In view of (128), $\gamma_2 q_\infty^{(\mathfrak{K})}(X)(R_2)^{-\nu} \leq 1/2$ for $\nu \in \nu_1 + \mathbb{N} - 1$. Let

$$q_\infty^{(\mathfrak{K})}(y^\wedge(X, \nu_2)) = \sup\{q_\infty^{(\mathfrak{K})}(y^\wedge(X, \kappa)) : \kappa = \nu_1, \ldots, \nu + l - 1\}$$

with $\nu_2 \in \{\nu_1, \dots, \nu_1+l-1\}$. Then, in view of (126), (129), (125), (128) and (122),

$$\frac{1}{2} \geq q_\infty^{(\mathfrak{K})}(y^\wedge(X,\nu_2)) \geq \gamma_1(R_1)^{-\nu_1} q_\infty^{(\mathfrak{K})}(X) \geq \gamma_3(q_\infty^{(\mathfrak{K})}(X))^{-(\log((R_1)/(R_2)/\log(R_2)}.$$

Let be fixed $f_1 \in \mathbb{Z}_K, \dots, f_m \in \mathbb{Z}_K$ such that

$$q_\infty^{(\mathfrak{K})}(\phi_i(X) - f_i) = \|\phi_i(X)\|_K. \tag{130}$$

According to (123), (130) and Lemma 12, $g = -\alpha_0^\wedge(X,\nu_2) + \sum\limits_{i=1}^{m} \alpha_i^\wedge(\nu_2) f_i \in \mathbb{Z}_K$ and

$$\gamma_3(q_\infty^{(\mathfrak{K})}(X))^{-(\log((R_1)/R_2))/\log(R_2)} \leq q_\infty^{(\mathfrak{K})}(y^\wedge(X,\nu_2) =$$

$$\|q_\infty^{(\mathfrak{K})}(y^\wedge(X,\nu_2))\|_K \leq q_\infty^{(\mathfrak{K})}(y^\wedge(X,\nu_2) - g) \leq \sum_{i=1}^{m} q_\infty^{(\mathfrak{K})}(\alpha_i^\wedge(\nu_2)) q_\infty^{(\mathfrak{K})}(\phi_i(X) - f_i) =$$

$$\sum_{i=1}^{m} q_\infty^{(\mathfrak{K})}(\alpha_i^\wedge(\nu_2))\|\phi_i(X)\|_K \leq \sum_{i=1}^{m} \|\phi_i(X)\|_K \gamma_0(r_i^\wedge)^{\nu_1+l-1}.$$

But

$$(r_i^\wedge)^{\nu_1} = \exp((\log(r_i^\wedge))((\log(2\gamma_2 q_\infty^{(\mathfrak{K})}(X)))/\log(R_2)+\theta)) =$$
$$(q_\infty^{(\mathfrak{K})}(X))^{(\log(r_i^\wedge))/\log(R_2)}(r_i^\wedge)^{(log(2\gamma_2))/\log(R_2))+\theta))},$$

where $i = 1, \dots, m$. Therefore

$$\gamma_3(q_\infty^{(\mathfrak{K})}(X))^{-(\log((R_1)/(R_2)/\log(R_2)} \leq q_\infty^{(\mathfrak{K})}(y^\wedge(X,\nu_2) \leq \tag{131}$$

$$\sup\{\|\phi_i(X)\|_K(q_\infty^{(\mathfrak{K})}(X))^{(\log(r_i^\wedge))/\log(R_2)} \colon i = 1, \dots, m\} \times$$

$$\sum_{i=1}^{m} \gamma_0(r_i^\wedge)^{(\log(2\gamma_2))/\log(R_2)+l}.$$

¿From (131) follows (127). ■

Corollary. *Let* $a \in \mathfrak{K}$, $\alpha_1^\wedge(\nu) \in \mathbb{Z}_K$, $\alpha_2^\wedge(\nu) \in \mathbb{Z}_K$, $y(\nu) = -\alpha_2^\wedge(\nu) + a\alpha_1^\wedge(\nu)$ *where* $\nu \in \mathbb{N}$.

Let there are γ_0, $r_1^\wedge \geq 1$ *such that* $q_\infty^{(\mathfrak{K})}(\alpha_1(\nu)) < \gamma_0(r_1^\wedge)^\nu$, *where* $\nu \in \mathbb{N}$. *Let*

$$l \in \mathbb{N},\ \gamma_1 > 0,\ \gamma_2 > \frac{1}{2}, R_1 \geq R_2 > 1, \alpha_1 = (\log(r_1^\wedge R_1/R_2))/\log(R_2),$$

$$\gamma_3 = \gamma_1(R_1)^{(-\log(2\gamma_2 R_2))/\log(R_2)}, \gamma_4 = \gamma_3\left(\gamma_0(r_1^\wedge)^{(log(2\gamma_2))/\log(R_2)+l}\right)^{-1}.$$

Let $X \in \mathbb{Z}_K$ *and let for each* $\nu \in \mathbb{N} - 1$ *hold the inequalities*

$$\gamma_1(R_1)^{-\nu} q_\infty^{(\mathfrak{K})}(X) \leq \sup\{q_\infty^{(\mathfrak{K})}(y_1(\kappa)X) \colon \kappa = \nu, \dots, \nu+l-1))\},$$

$$q_\infty^{(\mathfrak{K})}(y(\nu)X) \leq \gamma_2(R_2)^{-\nu} q_\infty^{(\mathfrak{K})}(X).$$

Then $\|aX\|_K (q_\infty^{(\mathfrak{K})}(X))^\alpha \geq \gamma_4$.

Proof. This Corrolary is the Lemma 13 for $m = n = 1$. ■

Let $l \in \mathbb{N}$, let p be prime in $(2, +\infty)$, let K be a finite extension of $\mathbb{Q}$ let $\mathfrak{p}$ be a prime ideal in $\mathbb{Z}_K$ and $p \in \mathfrak{p}$, let f be the degree of $\mathfrak{p}$, let $(p) = \mathfrak{p}^e\mathfrak{q}$, where $\mathfrak{q}$ is a entire ideal, not contained in $\mathfrak{p}$, let $v_\mathfrak{p}$ be additive $\mathfrak{p}$-valuation, which prolongs v_p; so, if π is a $\mathfrak{p}$-prime number, then $v_\mathfrak{p}(\pi) = 1/e$. If f is the degree of the ideal $\mathfrak{p}$ then

$$v_\mathfrak{p}\left(w^{p^\beta} - w\right) \geq 1, \tag{132}$$

where $\beta \in \mathbb{N}f$, $w \in K$ and $v_\mathfrak{p}(w) \geq 0$. In viw of (132), (77), and (75),

$$v_\mathfrak{p}(\alpha^*(z;\, p^\beta l) - \alpha^*(z;\, l)) > 1/e,$$

if $\beta \in \mathbb{N}f$, $\theta_0(z) \in K$ and $v_\mathfrak{p}(\theta_0(z)) \geq 0$. In view of (78),

$$\phi^*(z;\nu) = \tag{133}$$

$$(-\theta_0(z))^\nu \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^k \sum_{\tau=1}^{\nu+k}((\theta_0(z))^{-\tau}/\tau)) =$$

$$(-\theta_0(z))^\nu \sum_{\tau=1}^{\nu}((\theta_0(z))^{-\tau}\alpha^*(z;\nu)/\tau +$$

$$(-\theta_0(z))^\nu \sum_{k=0}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^k \sum_{\tau=1+\nu}^{\nu+k}((\theta_0(z))^{-\tau}/\tau)) =$$

$$(-1)^\nu \sum_{\tau=1}^{\nu(\Delta+1)} \frac{1}{\tau} \sum_{k=\max(0,\,\tau-\nu)}^{\nu\Delta} \alpha^*_{\nu,k}(\theta_0(z))^{\nu-\tau+k};$$

therefore, if $\nu = p^\beta l$, $f = 1$, $\beta \in \mathbb{N}f$, $p > l(\Delta+1)$, $\theta_0(z) \in K$ and $v_\mathfrak{p}(\theta_0(z)) \geq 0$, then, according to the Lemma 2,

$$1 - \beta \leq \tag{134}$$

$$v_\mathfrak{p}\left(\phi^*(z;\nu) - \sum_{\substack{\eta\in[1,\,\Delta+1]\cap\mathbb{Z} \\ k\in[p^\beta(\eta-l),\,p^\beta l\Delta]\cap\mathbb{Z} \\ k\geq 0,\, v_\mathfrak{p}(k)>0}} \frac{(-1)^{pl}}{p^\beta\eta}(\theta_0(z))^{p^\beta(l-\eta)+k}\alpha^*_{\nu,k}\right),$$

$$1/e - \beta \leq \tag{135}$$

$$v_{\mathfrak{p}}\left(\phi^*(z;\nu)-\sum_{\substack{\eta\in[1,\,\Delta+1]\cap\mathbb{Z}\\ k\in[p^{\beta-1}(\eta-l),\,p^{\beta-1}l\Delta]\cap\mathbb{Z}\\ k\geq 0}}\frac{(-1)^{pl}p}{p^\beta\eta}(\theta_0(z))^{p^{\beta-1}(l-\eta)+k}\alpha^*_{\nu/p,k}\right).$$

We make the pass (134) $\to$ (135) β times and obtain the inequality

$$1/e-\beta\leq v_{\mathfrak{p}}\left(\phi^*(z;\,p^\beta l)-p^{-\beta}\phi^*(z;\,l)\right),\tag{136}$$

where $\beta\in\mathbb{N}$, $p>l(\Delta+1)$, $p\in\mathfrak{p}$ and $\mathfrak{p}$ is ideal of the first degree.

Lemma 14. *If* $m\in\mathbb{N}+1$, $K=\mathbb{Q}[\exp(2pi/m)]$, $\alpha^*(z;\,l_1)\phi^*(z;\,l_2))\neq 0$ *for some* $z\in K\diagdown\{0\}$, $l_1\in\mathbb{N}$, $l_2\in\mathbb{N}$, *then for any* $\mu\in\mathbb{N}$ *the sequenses*

$$\alpha^*(z;\,\nu),\,\phi^*(z;\,\nu),\tag{137}$$

where $\nu\in\mu+\mathbb{N}$ *form a linear independent system over* K.

Proof. There exists $d^*\in\mathbb{N}$ such that

$$d^*z\in\mathbb{Z}_K,\,d^*z\alpha^*(z;\,l_1)\in\mathbb{Z}_K,\,d^*z\phi^*(z;\,l_2)\in\mathbb{Z}_K.$$

Let a prime $p\in\mathbb{N}m+1$ satisfies to the inequality

$$p>|\operatorname{Nm}_{K/\mathbb{Q}}(d^*z\alpha^*(z;\,\nu))|+|\operatorname{Nm}_{K/\mathbb{Q}}(d^*z\phi^*(z;\,\nu))+$$

$$|\operatorname{Nm}_{K/\mathbb{Q}}(d^*z)|+|\operatorname{Nm}_{K/\mathbb{Q}}(d^*)+(\Delta+1)(l_1+l_2).$$

Let $\mathfrak{p}$ is a prime ideal containing p. Then $v_{\mathfrak{p}}\left(\alpha^*(z;\,l_1)\right)=v_{\mathfrak{p}}\left(\phi^*(z;\,l_2)\right)=0$, and, in view of (136), $v_{\mathfrak{p}}\left(\phi^*(z;\,p^\beta l_2)\right)=-\beta$, but $v_{\mathfrak{p}}\left(\alpha^*(z;\,p^\beta l_1)\right)=0$ with $\beta\in\mathbb{N}$. ■

We choose for each $m=2,\,\ldots\,,37$ the corresponding $\Delta=\Delta(m)$ and check for it and $z=(1,2\pi/m)$ the filfilment of the condition of the Lemma 14. For $m\in[2,4]$ let $\Delta=\Delta(m)=7$, $l_1=l_2=1$.

If $m=2$, then $\theta_0(z)=-1$, and in view of (75), (77), (78),

$$\alpha^*(z;\nu)=\sum_{k=0}^{7}\binom{7}{k}\binom{7+k}{6}>0.$$

If $m\in\mathbb{N}$, $\Delta=\Delta(m)=7$ then, in view of (75), (77), (79),

$$(1/7)\alpha^*(z;1)=(1/7)(-\theta_0(z))\sum_{k=0}^{7}\binom{7}{k}\binom{7+k}{6}(-\theta_0(z))^k\equiv$$

$$(-(\theta_0(z))(1+(\theta_0(z))^6-2(\theta_0(z))^7)\,(\mod 7).$$

If $m=3$, then $(1/7)\alpha^*(z;\nu)/(-\theta_0(z))\equiv 2(1-\theta_0(z))\not\equiv 0\,(\mod 7)$.

If $m = 4$, then $(1/7)\alpha^*(z;\nu)/(-\theta_0(z)) \equiv -2i \not\equiv 0 (\mod 7)$. Let $\mathfrak{q}$ be a prime ideal containing the number 2, $\mathfrak{q} = (2)$, if $m = 2, 3$, $\mathfrak{q} = (1+i)$, $(2) = \mathfrak{q}^2$, if $m = 4$. Then in view of (75), $v_{\mathfrak{q}}(\alpha^*_{1,7}) = v_{\mathfrak{q}}\left(-\binom{7}{7}\binom{14}{6}\right) = 0$ and in view of (133),

$$v_{\mathfrak{q}}\left(\phi^*(z,1) + (1/8)\alpha^*_{1,7}\right) \geq -2.$$

If $m \in [5, 12] \cap \mathbb{N}$, we take $\Delta = \Delta(m) = 5$, $l_1 = l_2 = 1$. If $m \in \mathbb{N}$, $\Delta = \Delta(m) = 5$ then in view of (75), (77), (78),

$$(1/5)\alpha^*(z;1) = (-(\theta_0(z))\sum_{k=0}^{5}\binom{5}{k}\binom{5+k}{4}(-\theta_0(z))^k \equiv$$

$$(-(\theta_0(z))(1 + (\theta_0(z))^4 - 2(\theta_0(z))^5)(\mod 5).$$

If $m = 5$, then $(1/5)\alpha^*(z;1)/(-(\theta_0(z)) \equiv -1 + \exp(-2\pi i/5) \not\equiv 0(\mod 5)$.

If $m = 6$, then

$$(1/5)\alpha^*(z;1)/(-(\theta_0(z)) \equiv$$

$$1 + \exp(4\pi i/3) - 2\exp(-\pi i/3) = \exp(2\pi i/3) \not\equiv 0 (\mod 5).$$

If $m \in \{7, 9, 11\}$, then the numbers $\exp(2\pi k/m)$, where $k = 0, \ldots, \phi(m)$ compose an integer basis of the field $\mathbb{Q}[\exp(2\pi/m)]$, $\alpha^*_{1,\Delta} \neq 0$, $\phi(m) \geq 6$ and

$$\alpha^*(z;1) = -\exp(2\pi/m)\sum_{k=0}^{\Delta}\alpha^*_{1,k}\exp(2k\pi/m) \neq 0.$$

If $m = 8$, then $\frac{1}{5}\alpha^*(z;1) \equiv -\exp(\pi/4)2\exp(\pi/4) (\mod 5)$. If $m = 10$, then

$$\prod_{k=1}^{4}(2 - \exp(2k\pi/5) = 31 \tag{138}$$

$$(1/5)\alpha^*(z;1) \equiv (-(\theta_0(z))(1 + (\theta_0(z))^4 - 2(\theta_0(z))^5) \equiv$$

$$-\exp(\pi/5)(-2 + \exp(4\pi/5)) \not\equiv 0(\mod 5).$$

If $m = 12$, then $\mathrm{Nm}_{K/\mathbb{Q}}(-2-i) = \mathrm{Nm}_{K/\mathbb{Q}}(-2+i) = 25$, the ideal (5) is equal to the product of ideals $(-2-i)$, $(-2+i)$, and therefore

$$(1/5)\alpha^*(z;1) \equiv (-(\theta_0(z))(1 + (\theta_0(z))^4 - 2(\theta_0(z))^5) \equiv$$

$$-\exp(i\pi/6)(1 + \exp(2i\pi/3)) - 2\exp(5i\pi/6)) =$$

$$-\exp(i\pi/6)(-\exp(8i\pi/6)) - 2\exp(5i\pi/6)) = -2 - i \not\equiv 0(\mod 5).$$

Let $\mathfrak{q}$ be a prime ideal containing the number 5. Then, in view of (133),

$$v_{\mathfrak{q}}\left(\phi^*(z,1) - \frac{-1}{5}(\alpha^*_{1,4} + \alpha^*_{1,5}\theta_0(z)\right) \geq 1,$$

$$v_{\mathfrak{q}}\left(\phi^*(z,1)-(1-2\theta_0(z)\right)\geq 1.$$

If $m=5$, then in view of (138),

$$v_{\mathfrak{q}}(1-2\exp(2i\pi/5))=v_{\mathfrak{q}}(-\exp(2i\pi/5)(2-\exp(8i\pi/5))=0.$$

If $m=6$, then $(2-\exp(10i\pi/6)(2-\exp(2i\pi/6)=3$, and

$$v_{\mathfrak{q}}(1-2\exp(2i\pi/6))=v_{\mathfrak{q}}(-\exp(2i\pi/6)(2-\exp(10i\pi/5))=0.$$

If $m\in\{7,\ 9,\ 11\}$, then the numbers $\exp(2\pi k/m)$, where $k=0,\ \dots,\ \phi(m)$ compose an integer basis of the field $\mathbb{Q}[\exp(2i\pi/m)]$, $\alpha^*_{1,\,5}\neq 0$, $\phi(m)\geq 6$, in view of (78),

$$\phi^*(z;\nu)=(-1)^{\nu}\sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}\sum_{\tau=1}^{\nu+k}((\theta_0(z))^{\nu+k-\tau}/\tau))= \tag{139}$$

$$(-1)^{\nu}\sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}\sum_{s=0}^{\nu+k-1}((\theta_0(z))^{s}/(\nu+k-s)=$$

$$(-1)^{\nu}\sum_{k=0}^{\nu\Delta-1}\alpha^*_{\nu,k}\sum_{s=0}^{\nu+k-1}((\theta_0(z))^{s}/(\nu+k-s)+$$

$$(-1)^{\nu}\alpha^*_{\nu,\nu\Delta}\sum_{s=0}^{\nu(\Delta+1)-1}((\theta_0(z))^{s}/(\nu(\Delta+1)-s),$$

and, since $\Delta=5$ now, it follows that

$$\phi^*(z;1)=-\left(\sum_{k=0}^{4}\alpha^*_{1,k}\sum_{s=0}^{k}\frac{\exp}{(}2is\pi/m)1+k-s\right)-$$

$$\alpha^*_{1,5}\sum_{s=0}^{5}\frac{\exp}{(}2is\pi/m)6-s\neq 0.$$

If $m=8$, then $\prod\limits_{k=1}4(2-\exp(2i(2k-1)\pi/8))=17$, and

$$v_{\mathfrak{q}}(1-2\exp(2i\pi/8))=v_{\mathfrak{q}}(-\exp(2i\pi/8)(2-\exp(14i\pi/8))=0.$$

If $m=10$, then $\prod\limits_{k=1}2(2-\exp(2i(2k-1)\pi/10))(2-\exp(-2i(2k-1)\pi/12))=11$, and

$$v_{\mathfrak{q}}(1-2\exp(2i\pi/10))=v_{\mathfrak{q}}(-\exp(2i\pi/10)(2-\exp(18i\pi/12))=0.$$

If $m=12$, then $\prod\limits_{k=1}2(2-\exp(2i(4k-3)\pi/12))(2-\exp(-2i(4k-3)\pi/12))=13$, and

$$v_{\mathfrak{q}}(1-2\exp(2i\pi/12))=v_{\mathfrak{q}}(-\exp(2i\pi/12)(2-\exp(22i\pi/12))=0.$$

If $m \in [13, 37] \cap \mathbb{N}$, we take $\Delta = \Delta(m) = 3$, $l_1 = l_2 = 1$. Then $\phi(m) \geq 6$, $\alpha^*_{1,3} \neq 0$, and

$$\alpha^*(z; 1) = -\exp(2\pi/m) \sum_{k=0}^{3} \alpha^*_{1,\,k} \exp(2k\pi/m) \neq 0,$$

moreover, in view of (139), $\phi^*(z; 1) \neq 0$ in this case. So, the condition of the Lemma 14 holds for all the $m = 2, \ldots, 37$. We have proved the following

Lemma 15. *For each $\mu \in \mathbb{N}$ the two sequences (137) are linearly independent over $\mathbb{C}$.* ■

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