RATIONAL CURVES AND RATIONAL SINGULARITIES

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Abstract. We study rational curves on algebraic varieties, especially on normal affine varieties endowed with a $\mathbb{C}^*$-action. For varieties with an isolated singularity, we show that the presence of sufficiently many rational curves outside the singular point strongly affects the character of the singularity. This provides an explanation of classical results due to H. A. Schwartz and G. H. Halphen on polynomial solutions of the generalized Fermat equation.

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Introduction

In 1873 H. A. Schwartz [Schw] (in his studies on the Gauss hypergeometric equation) found polynomial solutions in coprime polynomials $x(s)$, $y(s)$, $z(s)$ of the generalized Fermat equation

$$x^p + y^q + z^r = 0 \tag{1}$$

for every Platonic triple (p,q,r) with $p,q,r \ge 2,\ \ 1/p + 1/q + 1/r > 1$. These solutions are given (up to constant factors, in the notation as in [Schw]) by the following identities.

For a dihedral triple $(p,q,r) = (2,2,d)$:

$$(s^d+1)^2 - (s^d-1)^2 = 4s^d\,.$$

For the tetrahedral triple $(p,q,r) = (2,3,3)$:

$$12\sqrt{3}(s(1+s^4))^2 = (1+2\sqrt{3}s^2 - s^4)^3 - (1-2\sqrt{3}s^2-s^4)^3\,.$$

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For the octahedral triple $(p,q,r)=(2,3,4)$:

$$(1-33s^4-33s^8+s^{12})^2=(1+14s^4+s^8)^3-4\cdot s^3(s(1-s^4))^4\,.$$

For the icosahedral triple $(p,q,r)=(2,3,5)$:

$$[\varphi_{30}(s)]^2=[\varphi_{20}(s)]^3-4^3\cdot 3^3\cdot[\varphi_{12}(s)]^5\,,$$

where

$$\begin{array}{rcc}\varphi_{12}(s) & = & s(1-11s^5-s^{10})\\ \varphi_{20}(s) & = & 1+228s^5+494s^{10}-228s^{15}+s^{20}\\ \varphi_{30}(s) & = & 1-522s^5-10005s^{10}-10005s^{20}+522s^{25}+s^{30}\,.\end{array}$$

Schwartz' solutions are by no means unique, some others having been known to Euler (1756), Hoppe (1859), Liouville (1879), etc. [DaGr]. On the other hand, in 1880 Halphen [Ha] showed that the generalized Fermat equation (1) has no solution in non-constant coprime polynomials when $1/p+1/q+1/r\le 1$. We highly recommend to consult [Kl] for a historical account on the subject, [BaDw, Beu] for a modern one, and [DaGr] for both. (Notice that the above solutions are given there via homogeneous forms in two variables.)

If $(a,b,c)\in\mathbb{C}^3$ is a nonzero constant solution of (1) and $f\in\mathbb{C}[s]$ then clearly $(af^{M/p},\ bf^{M/q},\ cf^{M/r})$ with $M:=\mathrm{lcm}(p,q,r)$ is a solution of (1) which we call *trivial.* The existence of non-trivial solutions is equivalent to the quasirationality of the singularity of the Pham-Brieskorn surface $V_{p,q,r}$ given in $\mathbb{C}^3$ by the equation (1) [KaZa, L. 7]. The latter can be expressed explicitly in terms of the exponents (p,q,r) as in (a) below, whereas the classical results described above admit a geometric interpretation as in (b).

Theorem 0.1.

(a) [BaKa, Ev, KaZa, Cor. of L. 8] *There exists a polynomial curve $C\subseteq V_{p,q,r}$ (i.e. $C=f(\mathbb{C})$, where $f:\mathbb{C}\to V_{p,q,r}$ is a non-constant morphism) that is not contained in an orbit closure of the natural $\mathbb{C}^*$-action on*[1] *$V_{p,q,r}$ if and only if either*
  (i) *one of the exponents p, q, r is coprime with the other two, or*
  (ii) *$gcd(p,q,r)=2$ and $p/2$, $q/2$, $r/2$ are coprime in pairs.*

(b) (Schwartz-Halphen) *There exists a polynomial curve C in $V_{p,q,r}$ not passing through the origin if and only if (p,q,r) is a Platonic triple (or, what is equivalent, iff the singularity of the surface $V_{p,q,r}$ at the origin is a rational double point).*

In sections 1 and 2 we obtain similar results for more general quasihomogeneous affine algebraic varieties, i.e., varieties endowed with an effective $\mathbb{C}^*$-action[2]. The most complete results concerns surfaces with many rational curves. By a *rational curve* we always mean a curve with function field isomorphic to $\mathbb{C}(t)$ or, in other words, the normalization C' of C admits an open embedding into $\mathbb{P}^1$. Our results can be formulated as follows (see Theorems 1.12, 2.7 and 2.13).

[1]This corresponds to a non-trivial polynomial solution of (1).

[2]To avoid possible confusion, notice that algebraic group theorists use this same term in a different sense. Namely, they call a variety *quasihomogeneous* if it possesses an algebraic group action with a Zariski open dense orbit.

Theorem 0.2. *Let V be a normal affine algebraic surface over $\mathbb{C}$ with a good $\mathbb{C}^*$-action*[3] *and with an isolated singularity $p \in V$. Then the following hold.*

(a) *If there exists a closed rational curve C on V which does not pass through the singular point p then (V,p) is a rational singularity.*
(b) *There exists a polynomial curve $f : \mathbb{C} \to V$ on V which does not pass through the point p if and only if (V,p) is a quotient singularity.*
(c) *The surface V admits a nontrivial regular $\mathbb{C}_+$-action if and only if (V,p) is a cyclic quotient singularity.*

The implication "$\Rightarrow$" in (c) is a special case of a more general result of Miyanishi [Mi]. We do not know whether the converse to (a) is also true. Notice that if a surface singularity (V,p) as in the theorem is Gorenstein (for instance, if this is an isolated complete intersection surface singularity) then conditions (a) and (b) above are equivalent (Theorem 2.20).

More generally, (a) is true in any dimension and for not necessarily quasihomogeneous varieties. Namely, we have the following result (see Theorem 1.12).

Theorem 0.3. *If V is a scheme (or a Moishezon variety) with an isolated Cohen-Macaulay singularity $p \in V$ such that*

(*) *a Zariski open subset of V can be covered by closed rational curves C which do not pass through p*

then (V,p) is a rational singularity.

Under the assumption (*) we say that V is *uniruled off p*. Notice that the assumption $p \notin C$ is essential as is easily seen by the example of non-rational quasihomogeneous isolated hypersurface singularities.

As an application, consider the hypersurface V in $\mathbb{C}^n$ given by an equation

$$P := uv - p(x_1, \dots, x_{n-2}) = 0.$$

It admits an effective $\mathbb{C}_+$-action induced by the locally nilpotent derivation ∂ on $\mathbb{C}[x_1, \dots, x_n]$ with

$$\partial(u) := 0\,, \quad \partial(v) := \frac{\partial p}{\partial x_1}\,, \quad \partial(x_1) := u \quad \text{and} \quad \partial(x_i) := 0 \text{ for } i \geq 2.$$

Assume that V has an isolated singularity at 0. The general orbits of the $\mathbb{C}_+$-action constitute a family of polynomial curves on V not passing through 0. Thus we recover the result of Viehweg [Vie] saying that $(V,0)$ is rational.

Throughout the paper we use log-canonical L^2-forms on singular complex spaces as considered in [FlZa$_1$]; we recall their definition and some useful facts in subsection 1.1. As for the classification results 2.6, 2.13 and 2.16 concerning "good" quasihomogeneous surfaces with a $\mathbb{C}_+$-action (as well as for more general ones) see also the forthcoming paper [FlZa$_2$].

1. Rational curves on quasihomogeneous varieties

1.1. **Preliminaries: log-canonical sheaves.** Here we recollect some notions and facts from [FlZa$_1$] that we need in the sequel.

[3] A regular $\mathbb{C}^*$-action on an algebraic variety is called *good* if it possesses a unique attractive fixed point.

1.1. Let X be a normal complex space, and let $\sigma : Y \to X$ be a resolution of singularities such that $E := \sigma^{-1}(\operatorname{Sing} X)_{\mathrm{red}}$ is a divisor with simple normal crossings (SNC divisor in brief). Define

$$\mathcal{L}^{2,m}_{Y,E} := \mathcal{O}_Y(mK_Y + (m-1)E) \qquad \text{and} \qquad \mathcal{L}^{2,m}_X := \sigma_* \mathcal{L}^{2,m}_{Y,E}\,;$$

similarly,

$$\mathcal{L}^{m}_{Y,E} := \mathcal{O}_Y(mK_Y + mE) \qquad \text{and} \qquad \mathcal{L}^{m}_X := \sigma_* \mathcal{L}^{m}_{Y,E}\,.$$

As explained in [Sak], the forms in $H^0(Y, \mathcal{L}^{2,m}_{Y,E})$ are just the meromorphic m-canonical forms on $Y \backslash E$ which are locally L^2 at the points of E. The sheaf $\mathcal{L}^{2,m}_X$ does not depend on the choice of resolution [FlZa$_1$, Sect. 1.1], and its sections over an open subset $U \subseteq X$ are the sections in $H^0(U_{\mathrm{reg}}, \omega_X^{\otimes m})$ which are locally L^2 on U [La$_2$, Bu, KiWa]. In general, $\mathcal{L}^{2,m}_X \subseteq \mathcal{O}_X(mK_X)$. Assuming that K_X is a $\mathbb{Q}$-Cartier divisor, the equality holds if and only if X has at most log-terminal singularities [FlZa$_1$, Prop. 1.17(c)]. Recall that in dimension 2 log-terminal singularities are just quotient singularities (Kawamata [Kaw]; see e.g., [FA, 3.0.1]).

If $D \subseteq X$ is a closed reduced subspace then we may also consider the log-canonical sheaves $\mathcal{L}^{m}_{X,D}$ and $\mathcal{L}^{2,m}_{X,D}$ taking $E := \sigma^{-1}(\operatorname{Sing} X \cup D)_{\mathrm{red}}$ in the above definitions.

Of particular importance is the sheaf $\mathcal{K}_X := \mathcal{L}^{2,1}_X = \sigma_* \omega_{X'} \subseteq \omega_X$ of L^2-canonical forms which was called in [GrRi] the *canonical sheaf* of X. A basic result is the Grauert-Riemenschneider vanishing theorem along with its generalizations due to Kollár and Moriwaki [Moriw] (see also M. Saito [Sai] and Arapura [Ara]), which we formulate for later purposes as follows.

Proposition 1.2. (a) ([GrRi], [Moriw, Thm. 3.2]) *Let $\pi : X' \to X$ be a birational proper morphism of normal complex spaces. Then $\pi_*(\mathcal{K}_{X'}) \cong \mathcal{K}_X$ and $R^i\pi_*(\mathcal{K}_{X'}) = 0$ for $i \geq 1$.*

(b) ([Kol1, Thm. 2.1(i)], [Moriw, Thm. 3.2]) *Let $\pi : X' \to X$ be a morphism of irreducible normal complex spaces which admits a factorization*

$$\pi : X' \xrightarrow{p} Y \xrightarrow{q} X$$

such that

(i) p is projective, i.e. there is a p-ample line bundle $\mathcal{O}_{X'}(1)$;

(ii) q is a Moishezon morphism.[4]

Then all direct image sheaves $R^i\pi_*(\mathcal{K}_{X'})$ $(i \geq 0)$ on X are torsion free.

Proof. (a) In the case that π is projective and X' is smooth, Moriwaki's result [Moriw, Thm. 3.2] gives that the sheaf $R^i\pi_*(\omega_{X'})$, $i \geq 1$, is torsion free on X and so, being concentrated on $\operatorname{Sing} X$, it must vanish. In the general case we can find a desingularization $\sigma : X'' \to X'$ such that $\pi \circ \sigma$ is a projective morphism. As σ is then projective as well, by the first part of the proof the sheaves $R^i\sigma_*(\omega_{X''})$ and $R^i(\pi\circ\sigma)_*(\omega_{X''})$ vanish for $i \geq 1$. By definition, $\sigma_*(\omega_{X''}) = \mathcal{K}_{X'}$ and so $R^i\pi_*(\mathcal{K}_{X'}) \cong R^i(\pi\circ\sigma)_*(\omega_{X''}) = 0$ for $i \geq 1$, proving (a).

(b) If π is a Moishezon morphism and X' is smooth then (b) is shown in [Moriw, 3.2]. The general case is reduced to this as follows: replacing Y by the image of p, we may suppose that p is surjective and Y is irreducible. According to [*loc.cit*, 3.3],

[4] That is, for any point $x \in X$ there exist a neighborhood U of x in X, a compact Moishezon variety V and a closed embedding $\varphi : q^{-1}(U) \hookrightarrow U \times V$ such that $q|q^{-1}(U) = \mathrm{pr}_1 \circ \varphi$, see [Moriw, Sect. 3].

locally with respect to X we can find an irreducible normal complex space $\tilde{Y}$ and a proper bimeromorphic morphism $\mu : \tilde{Y} \to Y$ such that the composition $\tilde{q} := q \circ \mu$ is projective. Let $\tilde{X}' \to \tilde{Y} \times_Y X'$ be a projective morphism that desingularizes the fiber product. Then all maps in the diagram

$$\begin{tikzcd} \tilde{X}' \arrow[r, "\tilde{p}"] \arrow[d, "\mu'"'] & \tilde{Y} \arrow[d, "\mu"'] \arrow[dr, "\tilde{q}"] & \\ X' \arrow[r, "p"] & Y \arrow[r, "q"] & X \end{tikzcd}$$

except possibly q are projective. By *loc.cit.* the direct image sheaves $R^i(\tilde{q} \circ \tilde{p})_*(\omega_{\tilde{X}'})$ are torsion free for all i. By construction, the map μ' is proper and bimeromorphic and so, using (a), $R\mu'_*(\omega_{\tilde{X}'}) \cong \mathcal{K}_{X'}$ in the derived category. Thus

$$R^i(\tilde{q} \circ \tilde{p})_*(\omega_{\tilde{X}'}) \cong R^i\pi_*(\mathcal{K}_{X'})$$

for all i, which gives (b). □

1.3. Let $(X,\, p)$ be an isolated singularity of a normal complex space. Following [KiWa] we define the *m-th L^2-plurigenus* of $(X,\, p)$ to be

$$\delta_m\,(X,\, p) = \dim_{\mathbb{C}} \left[\omega^{[m]}_{X,\,p}/(\mathcal{L}^{2,m}_X)_p\right] ,$$

where $\omega^{[m]}_X$ denotes the reflexive hull of $\omega^{\otimes m}_X$.

We make frequent use of the following results.

Proposition 1.4. (Kempf [KKMS, Prop. on p. 50]) *Let (X,p) be a singularity of a normal complex space. Then (X,p) is a rational singularity if and only if it is Cohen-Macaulay and $\mathcal{K}_{X,p} \cong \omega_{X,p}$. In the case of an isolated normal singularity (X,p) the latter condition is equivalent to $\delta_1\,(X,\, p) = 0$.*

Note that in *loc.cit.* this proposition is shown only for algebraic singularities. However, as remarked in [FlZa$_1$, 1.17(a)] the result holds as well in the complex analytic category.

Proposition 1.5. (Ki. Watanabe [KiWa, Thm. 3.9]) *If (X,p) is an isolated normal surface singularity then $\delta_m\,(X,\, p) = 0\ \forall m \geq 1$ if and only if (X,p) is a quotient singularity.*

1.6. Let V be a smooth variety with a smooth compactification $\bar{V}$ by an SNC-divisor $D = \bar{V}\backslash V$. The *logarithmic plurigenera* $\bar{p}_m(V)$ are defined by

$$\bar{p}_m(V) = \dim\, H^0(\bar{V},\, \mathcal{O}_{\bar{V}}(mK_{\bar{V}} + mD))\,,\ \ m \geq 1,$$

whereas the *logarithmic Kodaira dimension* $\bar{k}\,(V)$ of V is

$$\bar{k}\,(V) = \begin{cases} -\infty & \text{if} \quad \bar{p}_m(V) = 0 \quad \forall m \in \mathbb{N} \\ \min\,\{k \in \mathbb{N} : \limsup\limits_{m\to\infty} \bar{p}_m(V)/m^k < \infty\} & \text{otherwise.} \end{cases}$$

For a possibly singular variety V one defines $\bar{p}_m(V) := \bar{p}_m(V')$ and $\bar{k}\,(V) := \bar{k}\,(V')$, where $V' \to V$ is a resolution of singularities. Let $\bar{V}$ be a compactification of V and $D := \bar{V}\backslash V$. If $\pi : \bar{V}' \to \bar{V}$ is a resolution of singularities such that $D' := \pi^{-1}(D)_{\text{red}}$ is an SNC divisor then by our definitions there are inclusions

$$H^0(\bar{V}',\, \mathcal{O}_{\bar{V}'}(m(K_{\bar{V}'} + D))) \subseteq H^0(\bar{V},\, \mathcal{L}^m_{\bar{V},D}) \subseteq H^0(V,\, \mathcal{L}^m_V)$$

(cf. 1.1). Thus, if the group on the right or in the middle vanishes for $m \geq 1$ then $\bar{p}_m(V) = 0$ and, in particular, $\bar{k}(V) = -\infty$.

In the case of quasihomogeneous singularities we have the following facts.

Theorem 1.7. *Let $A = \bigoplus_{\nu\geq 0} A_\nu$ be a normal $\mathbb{C}$-algebra of finite type and assume that the corresponding affine variety $V = \operatorname{Spec} A$ has at most isolated singularities. Then the following hold.*

(a) ([FlZa$_1$, 2.22 (b)]) *$\delta_m(V,p) = \dim_{\mathbb{C}} (\omega_A^{[m]})_{\leq 0}$, where $\omega_A = H^0(V, \mathcal{O}(K_V))$ is the dualizing module of A. In particular, if V is Cohen-Macaulay then it has at most rational singularities if and only if $(\omega_A)_{\leq 0} = 0$.*

(b) ([FlZa$_1$, 2.22 (a)]) *If $A_0 \neq \mathbb{C}$ (in particular, if V has at least two singular points) then $\delta_m(V,p) = 0$ for all $m \geq 1$ and $p \in \operatorname{Sing} V$.*

(c) ([FlZa$_1$, 2.26 (a), (b)]) *If $A_0 = \mathbb{C}$ then $\bar{p}_m(V\backslash\{p\}) = \dim (\omega_A^{[m]})_0$, where $\{p\} := V(A_+)$ with $A_+ := \bigoplus_{\nu>0} A_\nu$. Consequently, $\bar{k}(V\backslash\{p\}) = -\infty$ if and only if $(\omega_A^{[m]})_0 = 0$ for all $m \geq 1$ or, equivalently, if and only if $\delta_m(V, p) = 0$ for all $m \geq 1$.*

1.8. Let $V = \operatorname{Spec} A$ be a complete intersection of dimension n given in $\mathbb{C}^{n+s}$ by polynomials $p_1, \ldots, p_s \in \mathbb{C}^{n+s}$ that are quasihomogeneous of degrees $d_1, \ldots, d_s$ with respect to weights $w_j > 0$ ($j = 1, \ldots, n+s$) i.e.,

$$p_i(\lambda^{w_1}x_1, \ldots, \lambda^{w_{n+s}}x_{n+s}) = \lambda^{d_i} p_i(x_1, \ldots, x_{n+s}), \qquad i = 1, \ldots, s\,.$$

It is well known that $\omega_A = A[N_A]$ (i.e., $(\omega_A)_\nu = A_{\nu+N_A}$) and $(\omega_A^{[m]})_0 = A_{mN_A}$, where

$$N_A (= N_V) := \sum_{i=1}^{s} d_i - \sum_{j=1}^{n+s} w_j\,,$$

see e.g. [GoWa, 2.2.8 and 2.2.10] or [Fl$_1$, p. 42] for a simple argument. Applying the foregoing results to the case of such complete intersections we get the following corollary.

Corollary 1.9. ([Fl$_1$], [Mora], [FlZa$_1$, 2.23, 2.26(c), 2.28]) *Suppose that V has an isolated singularity at the origin $p := \bar{0} \in \mathbb{C}^{n+s}$. Then*

$$\delta_m(V,p) = \dim_{\mathbb{C}} \sum_{\nu\leq 0} A_{\nu+mN_A}, \qquad \bar{p}_m(V\backslash\{p\}) = \dim_{\mathbb{C}} A_{mN_A}\,,$$

and so

$$\bar{k}(V\backslash\{p\}) = \begin{cases} -\infty & \text{iff} \quad N_A < 0 \\ 0 & \text{iff} \quad N_A = 0 \\ \dim V - 1 & \text{iff} \quad N_A > 0\,. \end{cases}$$

Consequently, (V,p) is a rational singularity if and only if $N_A < 0$.

1.2. **Canonical forms and rational curves.** It is well known that complete smooth uniruled varieties have no m-canonical forms (see e.g. [Kol, IV, Cor. 1.11]). In the following proposition (b) and (c) we show a similar result for varieties that admit "many" affine rational curves. Notice that the assumption in (a) below is just that the variety is uniruled (cf. [Kol, IV, Def. 1.1]).

Proposition 1.10. *Let $\bar{X}$ be a complete normal algebraic variety over $\mathbb{C}$, let $D \subseteq X$ be a closed reduced subspace and assume that $X := \bar{X}\backslash D$ is smooth.*

(a) *If for a general point x of X there is a non-constant rational curve $f : \mathbb{P}^1 \to X$ passing through x then $H^0(X, \omega_X^{\otimes m}) = 0$ for all $m \geq 1$. In particular, $\bar{k}(X) = -\infty$.*
(b) *If for a general point x of X there is a non-constant curve $f : \mathbb{C} \to X$ passing through x then $H^0(\bar{X}, \mathcal{L}^m_{\bar{X},D}) = 0$ for all $m \geq 1$. In particular, $\bar{k}(X) = -\infty$.*
(c) *If for a general point x of X there is a non-constant curve $f : \mathbb{C}^* \to X$ passing through x then $H^0(\bar{X}, \mathcal{L}^{2,m}_{\bar{X},D}) = 0$ for all $m \geq 1$.*

Proof. The second assertions in (a) and (b) follow from 1.6. For the remaining statements, consider the Hilbert scheme $\mathfrak{H}$ of $\bar{X}$. As $\mathfrak{H}$ has a countable number of irreducible components there is a closed subvariety S' of $\mathfrak{H}$ with the following properties.

(i) The general point of S' is represented by a complete rational curve $\bar{C}$ in $\bar{X}$ such that the normalization of $C := \bar{C} \backslash D$ is isomorphic either to $\mathbb{P}^1$ (or to $\mathbb{C}$ or $\mathbb{C}^*$ in cases (b) and (c), respectively).
(ii) The curves parametrized by S' cover $\bar{X}$.

After passing to a suitable subvariety of S' we may also assume that $\dim S' = \dim X - 1$. Consider the incidence subspace

$$\bar{\mathcal{X}}' := \{(x, [\bar{C}]) \in \bar{X} \times S' \,|\, x \in \bar{C}\}$$

of $\bar{X} \times S'$ which is flat over S'. Passing to the normalizations $\bar{\mathcal{X}}$ of $\bar{\mathcal{X}}'$ and S of S' we get morphisms

$$\begin{array}{ccc} \bar{\mathcal{X}} & \xrightarrow{f} & \bar{X} \\ {\scriptstyle \pi}\downarrow & & \\ S & & \end{array}$$

By construction the fibers of π are all curves and its general fibers are isomorphic to $\mathbb{P}^1$. Moreover, f is dominant and $\dim \bar{\mathcal{X}} = \dim \bar{X}$.

The subspace $\Delta := f^{-1}(D)_{\text{red}}$ of $\bar{\mathcal{X}}$ satisfies $\deg \Delta/S \leq 0$ (resp., ≤ 1 and ≤ 2 in case (b) and (c)). Using the functoriality of m- and $(2, m)$-canonical forms (see [FlZa$_1$, Sect. 1]), with $\mathcal{X} := \bar{\mathcal{X}} \backslash \Delta$ we have inclusions

$$f^*(\omega_X^m) \subseteq \mathcal{L}^m_{\mathcal{X}}\ , \quad f^*(\mathcal{L}^m_{\bar{X},D}) \subseteq \mathcal{L}^m_{\bar{\mathcal{X}},\Delta} \quad \text{and} \quad f^*(\mathcal{L}^{2,m}_{\bar{X},D}) \subseteq \mathcal{L}^{2,m}_{\bar{\mathcal{X}},\Delta}\,.$$

Thus it is sufficient to show that for $m \geq 1$

$$H^0(\mathcal{X}, \mathcal{L}^m_{\mathcal{X}}) = 0\ , \quad H^0(\bar{\mathcal{X}}, \mathcal{L}^m_{\bar{\mathcal{X}},\Delta}) = 0 \quad \text{and} \quad H^0(\bar{\mathcal{X}}, \mathcal{L}^{2,m}_{\bar{\mathcal{X}},\Delta}) = 0$$

in case (a), (b) and (c), respectively. In case (a) this follows easily from the fact that a general fiber $F \simeq \mathbb{P}^1$ of π does not intersect Δ, whence

$$\deg \mathcal{L}^m_{\mathcal{X}}|F = \deg \omega_{\mathcal{X}}^{\otimes m}|F = -2m < 0.$$

Similarly, in case (b) we have

$$\deg \mathcal{L}^m_{\bar{\mathcal{X}},\Delta}|F = m(-2 + \deg \Delta/S) < 0,$$

whereas in the situation of (c)

$$\deg \mathcal{L}^{2,m}_{\bar{\mathcal{X}},\Delta}|F = -2m + (m-1) \deg \Delta/S < 0.$$

$\square$

Remark 1.11. Let $\bar{X}$ and S be algebraic varieties over $\mathbb{C}$ and let $\bar{f}_s : \bar{C}_s \to \bar{X}$ be a family of complete curves in $\bar{X}$ parameterized by S that cover $\bar{X}$ in the sense that there is a proper smooth family $\pi : \bar{\mathcal{X}} \to S$ of curves and a dominant morphism $\bar{f} : \bar{\mathcal{X}} \to \bar{X}$ such that for $s \in S$ we have $\bar{C}_s = \pi^{-1}(s)$ and $f|\bar{C}_s = \bar{f}_s$.

Let $D \subseteq \bar{X}$ be a closed reduced subspace and let $X := \bar{X} \backslash D$. With $\Delta := \bar{f}^{-1}(D)_{\mathrm{red}}$ consider the family of curves

$$f_s : C_s := \bar{C}_s \backslash \Delta \to X$$

with $s \in S$, where $f_s := \bar{f}_s | C_s$. We call $\kappa := \deg \Delta / S$ the *number of punctures* of the family f_s.

If $\bar{X}$ is smooth and D is a divisor then for general $s \in S$ we have estimates

$$(K_{\bar{X}} + D)\bar{f}_{s*}(\bar{C}_s) \leq 2g - 2 + \kappa\,, \tag{2}$$

$$K_{\bar{X}} \bar{f}_{s*}(\bar{C}_s) \leq 2g - 2\,, \tag{3}$$

where g is the genus of $\bar{C}_s$.

Proof. We may assume that S is smooth of dimension $\dim X - 1$. As above $\bar{f}^*(\omega_{\bar{X}}(D)) \subseteq \omega_{\bar{\mathcal{X}}}(\Delta)$. For general $s \in S$ this restricts to a non-zero map $\bar{f}_s^*(\omega_{\bar{X}}(D)) \to \omega_{\bar{C}_s}(\Delta \cap \bar{C}_s)$; note that $\omega_{\bar{\mathcal{X}}}(\Delta)|\bar{C}_s \cong \omega_{\bar{C}_s}(\Delta \cap \bar{C}_s)$ as the normal bundle of $\bar{C}_s$ in $\bar{\mathcal{X}}$ is trivial. Hence

$$(K_{\bar{X}} + D)\bar{f}_{s*}(\bar{C}_s) = \deg \bar{f}_s^*(\omega_{\bar{X}}(D)) \leq \deg \omega_{\bar{C}_s}(\Delta \cap \bar{C}_s) = 2g - 2 + \kappa,$$

proving the first inequality. Applying this to the case $D = \emptyset,\ \Delta = \emptyset$ gives the second inequality. $\square$

1.3. **Rational curves and rational singularities.** In this subsection we prove theorem 0.3 from the introduction.

Theorem 1.12. *Let $p \in X$ be an isolated Cohen-Macaulay singularity of an algebraic scheme (resp., of a compact Moishezon variety) X. If for a general point $x \in X$ there exists a closed rational curve $C \subseteq X$ passing through x but not through p then (X, p) is a rational singularity.*

Proof. In the case of an algebraic scheme we may assume that X and C are projective (first passing to an affine neighbourhood of p in X and then to its normalized projective closure). Thus it is enough to consider the case where X is a compact Moishezon variety. Let $\mathfrak{H}$ be the Hilbert scheme of X. As in the proof of 1.10 there is an irreducible compact subvariety $S' \subseteq \mathfrak{H}$ of dimension $\dim S' = \dim X - 1$ such that the following hold.

(a) A general point of S' represents a rational curve not passing through p;

(b) the curves in S' cover X.

As above let $\mathcal{X}' := \{(x, [D]) \in X \times S' \,\big|\, x \in D\}$ be the incidence subvariety of $X \times S'$ which is flat over S'. Passing to the normalizations $\mathcal{X}$ of $\mathcal{X}'$ and S of S' we get morphisms

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & X \\ {\scriptstyle \pi}\downarrow & & \\ S & & \end{array}$$

By construction the fibers of π are all curves, its general fibers are isomorphic to $\mathbb{P}^1$ and f is finite on every fiber of π. As general fibers of π do not pass through the singular point $p \in X$ and the preimage $A := f^{-1}(p)$ does not contain a whole fiber of π, this analytic subset A has codimension ≥ 2 in $\mathcal{X}$. Moreover $\pi : A \to S$ is a finite morphism and $\mathcal{X} \to X \times S$ is finite.

Consider now the sheaves

$$\mathcal{K}_{\mathcal{X}} := \mathcal{L}^{2,1}_{\mathcal{X}}, \quad \mathcal{K}_X := \mathcal{L}^{2,1}_X \quad \text{and} \quad \mathcal{L} := f^*(\omega_X) \cap \omega_{\mathcal{X}} \subseteq \omega_{\mathcal{X}}\,.$$

Let us show that the quotient $\omega_X/\mathcal{K}_X$ vanishes in a neighborhood of $p \in X$. Indeed, $\mathcal{L}_x = f^*(\omega_X)_x$ if $f(x)$ is a regular point of X. As $\operatorname{codim} A \geq 2$ the sheaf $\mathcal{L}$ coincides with $f^*\omega_X$ in a neighborhood of A. Moreover, by the functoriality of $L^{2,m}$-canonical forms [FlZa$_1$, Sect. 1] $f^*\mathcal{K}_X \subseteq \mathcal{K}_{\mathcal{X}}$. Thus the sheaf $\mathfrak{C}$ in the exact sequence

$$0 \to \mathcal{K}_{\mathcal{X}} \to \mathcal{K}_{\mathcal{X}} + \mathcal{L} \to \mathfrak{C} := (\mathcal{K}_{\mathcal{X}} + \mathcal{L})/\mathcal{K}_{\mathcal{X}} \cong \mathcal{L}/(\mathcal{K}_{\mathcal{X}} \cap \mathcal{L}) \to 0 \tag{4}$$

is a subquotient of $f^*\omega_X/f^*\mathcal{K}_X$. Since $p \in X$ is an isolated singularity, and so $\mathcal{K}_X = \omega_X$ in $X\backslash\{p\}$ near p, it follows that the support of $\mathfrak{C}$ is contained in $A \cup A'$ where $A' \subseteq \mathcal{X}$ is a closed analytic subset disjoint from A. In particular, $\mathfrak{C} \cong \mathfrak{C}|A \oplus \mathfrak{C}|A'$. The sheaf $\pi_*(\mathcal{K}_{\mathcal{X}} + \mathcal{L})$ is contained in $\pi_*(\omega_{\mathcal{X}})$, whence it vanishes as the general fibers of π are isomorphic to $\mathbb{P}^1$. Taking π_* of the sequence (4) we get that $\pi_*(\mathfrak{C})$ is a subsheaf of $R^1\pi_*(\mathcal{K}_{\mathcal{X}})$.

We claim that π can be written as the composition of a projective morphism and a Moishezon morphism. Indeed, as X is Moishezon and $\mathcal{X}'$ is contained in $X \times S'$, the natural projection $\pi' : \mathcal{X}' \to S'$ and then by base change the projection $\mathcal{X}' \times_{S'} S \to S$ are Moishezon as well. As $\mathcal{X} \to \mathcal{X}' \times_{S'} S$ is a finite map it is in particular projective. Thus

$$\pi : \mathcal{X} \longrightarrow \mathcal{X}' \times_{S'} S \longrightarrow S$$

is the desired decomposition.

Applying Proposition 1.2 (b) the sheaf $R^1\pi_*(\mathcal{K}_{\mathcal{X}})$ on S is torsion free and so its subsheaf $\pi_*(\mathfrak{C})$ is torsion free as well. Hence $\mathfrak{C}|A = 0$ since otherwise $\pi_*(\mathfrak{C}|A)$ would be a torsion subsheaf of $\pi_*(\mathfrak{C}) \cong \pi_*(\mathfrak{C}|A) \oplus \pi_*(\mathfrak{C}|A')$. This shows that $f^*(\omega_X) = \mathcal{L} \subseteq \mathcal{K}_{\mathcal{X}}$ in a neighborhood of A. Applying [FlZa$_1$, Prop. 1.9 (b))] we obtain that $\omega_X = \mathcal{K}_X$ in a neighborhood of $p \in X$. Hence by 1.4 the singularity (X, p) is rational, as stated. ☐

Corollary 1.13. *Let $p \in X$ be an isolated Cohen-Macaulay singularity of an affine algebraic variety. If X admits a non-trivial $\mathbb{C}_+$-action then (X, p) is rational. In particular, if X admits an effective action of a non-abelian algebraic group then (X, p) is rational.*

Proof. The first statement is an immediate consequence of the preceding theorem since the general orbits of the C_+-action are closed rational curves in X not passing through p.

If G is a non-abelian algebraic group acting on X then it contains an algebraic subgroup isomorphic to $\mathbb{C}_+$. Indeed, if the solvable radical of G does not contain a subgroup isomorphic to $\mathbb{C}_+$ then this radical is isomorphic to $\mathbb{C}^{*n}$ for some $n \geq 0$ and so G is reductive. It follows then from the structure theory of reductive groups that G contains an algebraic subgroup isomorphic to $\mathbb{C}_+$. Restricing the action of G to such a subgroup the second statement of the corollary follows from the first one. ☐

Remark 1.14. Note that 1.12 is no longer true if the singularity is non-isolated as is seen by the example of a product $X \times \mathbb{P}^1$, where X is a normal variety with a non-rational point.

1.4. **Quasirational singularities of quasihomogeneous varieties.**

Notation 1.15. Let $A = \bigoplus_{\nu \geq 0} A_\nu$ be a graded algebra of finite type over $\mathbb{C}$, and let $V = \operatorname{Spec} A$ be the corresponding affine variety with a good $\mathbb{C}^*$-action. We assume in this subsection that the vertex set $V(A_+)$ (where $A_+ := \bigoplus_{\nu > 0} A_\nu$) consists of one point p which is an isolated normal singularity.

Following Abhyankar [Ab] in the 2-dimensional case we introduce the following notion.

Definition 1.16. Consider the quotient variety $\Gamma := (V \backslash \{p\})/\mathbb{C}^* = \operatorname{Proj} A$. The singularity (V, p) will be called *quasirational* if Γ is uniruled[5].

It is well known (see e.g. 1.10) that in this case the Kodaira dimension of Γ is $-\infty$. It is an open question whether conversely $k(\Gamma) = -\infty$ implies that Γ is uniruled; cf. [Kol, IV 1.12].

Note that Γ has at most cyclic quotient singularities (see [Fl_1] or [FlZa_1, Prop. 2.8]). Hence $H^0(\Gamma, \mathcal{L}_\Gamma^m) \cong H^0(\Gamma, \omega_\Gamma^{[m]})$, where $\omega_\Gamma^{[m]}$ denotes the reflexive hull of $\omega_\Gamma^{\otimes m}$.

If V is a surface then the condition of 1.16 means that Γ is a rational curve or, equivalently, that every irreducible component of the exceptional divisor $\sigma^{-1}(p)$ of the minimal resolution of singularity $\sigma : V' \to V$ is a rational curve (see e.g., [Ab] and also [Or]; alternatively this follows from the fact that the weighted blowing up of V has at most quotient singularities, see [Fl_1]).

The following simple lemma provides a restriction on a singularity to be quasirational.

Lemma 1.17. *If (V, p) is quasirational then $(\omega_A)_0 = 0$. In dimension 2 the converse is also true.*

Proof. By [Fl_2, 8.9] there is an exact sequence

$$0 \to (\Omega_\Gamma^1)^{\vee\vee} \to \tilde{\Omega}_A^1 \xrightarrow{\xi} \mathcal{O}_\Gamma \to 0,$$

where ξ is induced by the Euler derivation and $\tilde{}$ denotes the associated sheaf. Taking determinants it follows that $\omega_\Gamma \cong \tilde{\omega}_A$ and so $(\omega_A)_0 \cong H^0(\Gamma, \omega_\Gamma)$. As mentioned above Γ has at most quotient singularities. Hence, if the Kodaira dimension $k(\Gamma) = -\infty$ then $H^0(\Gamma, \omega_\Gamma)$ and therefore also $(\omega_A)_0$ vanishes. Conversely, if $\dim V = 2$ then Γ is a smooth complete curve, and so the vanishing of $H^0(\Gamma, \omega_\Gamma)$ implies that this curve is rational. □

By an *orbit closure* we mean the closure in V of a one-dimensional orbit of the $\mathbb{C}^*$-action; this is a rational curve passing through the vertex p with a normalization isomorphic to the affine line $\mathbb{C}$. We are interested in the existence of closed rational curves in V different from orbit closures. We have the following proposition.

Proposition 1.18. *For a quasihomogeneous variety V as in 1.15 the following conditions are equivalent:*

[5] i.e., there is a rational dominant map from $\mathbb{P}^1 \times Y \dashrightarrow X$ for some variety Y or, equivalently, for a general point $x \in X$ there is a rational curve $C \subseteq X$ through x; see [Kol, Chapt. IV].

(i) *(V, p) is a quasirational singularity;*
(ii) *for a general point $x \in V$ there exists a closed rational curve $C \subseteq V$ which passes through x, with C being different from an orbit closure.*

If $\dim V = 2$ *these are also equivalent to:*

(iii) *The surface V is rational.*

Proof. Over an open dense subset, say, $U \subseteq \Gamma$ the canonical map $\pi : V\backslash\{p\} \to \Gamma$ is a trivial $\mathbb{C}^*$-fibration so that $\pi^{-1}(U) \cong U \times \mathbb{C}^*$. Assume first that (ii) is satisfied. Projecting curves as in (ii) we see that for a general point x of Γ there is a rational curve in Γ passing through x, whence Γ is uniruled.

Conversely, if Γ is uniruled then for a general point x of V there is a rational curve C in Γ through $\pi(x) \in \Gamma$. For any point $t \in \mathbb{C}^*$ then $(C \cap U) \times \{t\}$ is a rational curve in $U \times \mathbb{C}^* \cong \pi^{-1}(U)$. Thus $\pi^{-1}(U)$ is covered by rational curves that are not contained in orbit closures and (ii) follows. For the equivalence of (iii) with (i), (ii) in the two dimensional case we refer the reader to [KaZa, L. 7]. □

Remark 1.19. (1) According to [Bu] the affine cone V over a smooth cubic 3-fold $\Gamma \subseteq \mathbb{P}^4$ has a rational and quasirational singularity at the origin, whereas $\Gamma = V^*/\mathbb{C}^*$ is not rational [ClGr]. Hence in higher dimensions the equivalence (i)⇔(iii) does not hold, in general.

(2) Clearly, a quasirational singularity of a variety with a good $\mathbb{C}^*$-action is not necessarily rational. For instance, the Pham-Brieskorn surface

$$V_{p,q,r} \subseteq \mathbb{C}^3 \ : \qquad x^p + y^q + z^r = 0$$

has a rational singularity at the origin only for the Platonic surfaces $V_{2,2,m}$, $V_{2,3,3}$, $V_{2,3,4}$, $V_{2,3,5}$ (see 1.9), whereas it is quasirational in many other cases (namely, iff one of the conditions (i), (ii) of 0.1(a) is fulfilled). The simplest example of a quasirational non-rational singularity is $(V_{2,3,7}, \bar{0})$ (see also [Ab, Or] for further examples).

Example 1.20. Consider the surface $V = \{F_d(x, y) - z^m\} \subseteq \mathbb{C}^3$, where F_d is a homogeneous polynomial of degree d without multiple factors. The singularity at the origin of the surface V is quasirational if and only if either $d = 2$ or $\gcd(m, d) = 1$ [KaZa]. At the same time, this singularity is rational if and only if $d \leq 2$ or $(d, m) = (3, 2)$, see Corollary 2.23 below.

1.5. **Polynomial curves on quasihomogeneous varieties.** Recall that by a *polynomial curve* on a variety V we mean a rational curve $C \subseteq V$ with one place at infinity, i.e., a curve C with a normalization $C' \cong \mathbb{C}$.

If $V \subseteq \mathbb{C}^n$ is an affine variety then a normalization morphism $\nu : \mathbb{C} \to C \hookrightarrow V$ is given (in the coordinates of the ambient affine space $\mathbb{C}^n$) by a sequence of n polynomials of one variable that are not all constant; this explains our terminology.

As a consequence of 1.7 (b), (c), 1.10(b) and 1.4 we obtain the following statement.

Proposition 1.21. *Let V be a quasihomogeneous normal algebraic variety* [6] *with at most isolated singularities. Suppose that for a point $p \in Sing\, V$ and a general point x of V there exists a polynomial curve $C \subseteq V\backslash\{p\}$ through x. Then $\delta_m(V, p) = 0$ for all $m \geq 1$. In particular, (V, p) is a rational singularity provided that it is Cohen-Macaulay.*

[6] That is, $V = \operatorname{Spec} A$ where $A = \bigoplus_{i \geq 0} A_i$ is a normal graded $\mathbb{C}$-algebra of finite type.

The next corollary gives a geometric application to $\mathbb{C}_+$-actions on quasihomogeneous varieties (cf. Corollary 1.13).

Corollary 1.22. *Let V be a quasihomogeneous normal variety with at most isolated singularities and $p \in \operatorname{Sing} V$. If V admits a non-trivial $\mathbb{C}_+$-action then the conclusions of 1.21 hold. In particular, this holds for any normal affine algebraic variety V with isolated singularities that admits an effective action of a non-abelian non-nilpotent group G.*

Proof. This is an immediate consequence of 1.21. Indeed, in the first case the orbit $\mathbb{C}_+.x$ through a general point $x \in V$ is a polynomial curve in $V\backslash\{p\}$. The second statement follows from the first one since G being non-nilpotent contains an algebraic subgroup isomorphic to $\mathbb{C}^*$, whence V is quasihomogeneous. As G is non-abelian it also has an algebraic subgroup isomorphic to $\mathbb{C}_+$. □

1.23. A similar result holds if V admits a second $\mathbb{C}^*$-action. Of course, one can obtain a second grading of A (and hence a second $\mathbb{C}^*$-action on V) by simply multiplying degrees with a constant k, i.e. $A = \bigoplus_{i\geq 0} A'_i$, where $A'_{kj} = A_j$ and $A'_i = 0$ for $i \not\equiv 0 \mod k$. We say that two gradings (resp., two $\mathbb{C}^*$-actions) are *truly different* if they cannot be obtained by multiplying degrees of a third one with appropriate constants. Equivalently, this means that the associated derivations span a $\mathbb{C}$-vector space of dimension 2 in $\operatorname{Der} A$.

Theorem 1.24. *Let $A = \bigoplus_{i\geq 0} A_i$ be a normal graded $\mathbb{C}$-algebra of finite type defining a quasihomogeneous variety $V = \operatorname{Spec} A$ with at most isolated singularities. If V admits a second truly different $\mathbb{C}^*$-action then $\bar{k}(V\backslash \operatorname{Sing} V) = -\infty$ and $\delta_m(V, p) = 0$ for all $m \geq 1$ and $p \in \operatorname{Sing} V$. In particular, (V, p) is a log-terminal singularity provided that it is $\mathbb{Q}$-Gorenstein.*

Proof. If $A_0 \neq \mathbb{C}$ then the result follows from 1.7 (b) and [FlZa$_1$, 1.26, 2.27]. Suppose now that $A_0 = \mathbb{C}$. Let δ be the derivation associated to the second $\mathbb{C}^*$-action and write $\delta = \sum_{i=k}^{l} \delta_i$, where δ_i is a homogeneous derivation of degree i. We may assume that δ_k, $\delta_l \neq 0$. If $k < 0$ then δ_k is locally nilpotent and so defines (see [Ren]) a $\mathbb{C}_+$-action. Applying 1.22 we are done. Thus we may suppose that $k \geq 0$.

Choose a finite homogeneous generating set, say, E for A as a $\mathbb{C}$-algebra. As $\mathbb{C}^*$ is reductive there is a $\mathbb{C}^*$-invariant subspace $V \subseteq A$ containing E with $\dim_{\mathbb{C}} V < \infty$. For N sufficiently large V is contained in $\bigoplus_{i=0}^{N} A_i$, and so $\delta^n(a) \in \bigoplus_{i=0}^{N} A_i$ for every $a \in E$ and $n \in \mathbb{N}$. If $l > 0$ then (considering the leading term of $\delta^n(a)$) it follows that $\delta_l^n(a) = 0$ for $n \geq N$. Hence δ_l is locally nilpotent, and we can again conclude by 1.22.

Let us finally treat the case where $\delta = \delta_0$ is homogeneous of degree 0. In this case the two $\mathbb{C}^*$-actions commute, and so they define a bigrading on A. In particular there is a grading of A with $A_0 \neq \mathbb{C}$. Applying the first case the desired result follows. □

2. Rational curves on quasihomogeneous surfaces

2.1. **Polynomial curves and quotient singularities.** In the surface case, Proposition 1.21 can be strengthened as follows.

Theorem 2.1. *Let V be an affine normal algebraic surface and let $p \in V$ be a singular point. Assume that for a general point $x \in V$ there exists a polynomial curve $C \subseteq V$ with $x \in C$ but not passing through p. Then (V, p) is a quotient singularity.*

We need the following lemma.

Lemma 2.2. *If a normal surface singularity (V, p) is finitely dominated by a quotient singularity (W, q) then it is also a quotient singularity.*

Proof. By [FlZa$_1$, Cor. 1.27] we have that $\delta_m(V, p) \leq \delta_m(W, q) = 0$ for all $m \geq 1$. Thus by 1.5 (V, p) is a quotient singularity. □

The proof of Theorem 2.1 is based on Proposition 2.3 (b) below. Recall that a complex surface Σ is said to be *ruled* (resp., *affine ruled*) if it is equipped with a proper morphism (resp., a morphism) $\pi : \Sigma \to S$ (called *ruling*) onto a smooth curve S with general fibers isomorphic to $\mathbb{P}^1$ (resp., to $\mathbb{C}$).

Proposition 2.3. *Let $\pi : \Sigma \to S$ be a normal ruled surface over a smooth curve S with generic fiber $F \cong \mathbb{P}^1$. Then the following hold. If there is a Weil divisor H on Σ with $HF = 1$ such that $\Sigma \backslash H$ is affine, then $\Sigma \backslash H$ has at most cyclic quotient singularities.*

This is an immediate consequence of the following result due to Miyanishi [Mi] (see also [Gur]).

Theorem 2.4. (Miyanishi) *Let X be a normal affine algebraic surface, and assume that X contains a cylinder, i.e. a Zariski open dense subset $U \cong \mathbb{C} \times S_0$ where S_0 is a curve. Then X has at most cyclic quotient singularities.*

Proof of Theorem 2.1. Let $\bar{V}$ be a projective compactification of V and denote $D := \bar{V} \backslash V$ the divisor at infinity. As in the proof of Proposition 1.10, there exists a projective normal ruled surface $\pi : \Sigma \to S$ over a smooth complete curve S and a surjective morphism $f : \Sigma \to \bar{V}$ such that $\pi \times f : \Sigma \to S \times \bar{V}$ is finite over a closed subscheme of $S \times \bar{V}$ and such that the divisor $E := f^{-1}(D)_{\mathrm{red}}$ on Σ intersects the general fiber of π in just one point. Consider now $p \in V$ and let $q \in \Sigma$ be a point over p. By construction q is not contained in E. For a fiber F_s over a point s of S different from $\pi(q)$ the scheme $\Sigma \backslash (E \cup F)$ is finite over a closed subscheme of the affine scheme $V \times (S \backslash \{s\})$ and so it is affine as well. Applying 2.3(b) to the divisor $H := E \cup F_s$ we obtain that (Σ, q) is at most a cyclic quotient singularity. As (V, p) is dominated by (Σ, q) under a (analytically) finite map, the assertion follows from Lemma 2.2. □

Note that the assumption that V is affine in Theorem 2.1 is essential. For instance, blowing up a ruled surface repeatedly in one fibre and blowing down any connected curve in the exceptional locus results in a surface with an isolated singularity and many polynomial curves. However, the minimal resolution of a quotient singularity is star shaped (in fact, this is true for any isolated quasihomogeneous surface singularity, see [OW]) whereas by the procedure above we can easily produce singular points, whose minimal resolution is not star shaped.

In view of 2.1 and 1.5 it is natural to ask the following question.

Question 2.5. Let V be an affine normal algebraic variety and let $p \in V$ be an isolated singular point. Assume that for a general point $x \in V$ there exists a polynomial curve $C \subseteq V$ with $x \in C$ but not passing through p. Is then $\delta_m(V, p) = 0$ for all $m \geq 1$?

2.2. **Polynomial curves on quasihomogeneous surfaces.** Recall that a *small finite subgroup* $G \subseteq \mathrm{GL}(2, \mathbb{C})$ is a finite subgroup which does not contain any pseudo-reflection, that is, a linear transformation of finite order $\sigma \in \mathrm{GL}(n, \mathbb{C}) \backslash \{\mathbf{1}_n\}$ which is the identity on a hyperplane. The following fact is well known; in lack of a reference we include a brief argument.

Lemma 2.6. *A surface V with a good $\mathbb{C}^*$-action has an isolated quotient singularity at the vertex $p \in V$ if and only if $V \cong \mathbb{C}^2/G$ where $G \subseteq \mathrm{GL}(2, \mathbb{C})$ is a small finite subgroup.*

Proof. From the assumption that (V, p) is an isolated quotient singularity it follows that the fundamental group $G := \pi_1(V\backslash\{p\})$ (isomorphic to the local fundamental group of the singularity (V, p)) is finite. Thus the universal cover $W' \to V\backslash\{p\}$ extends to a branched covering $\tau : W \to V$ such that W is a smooth surface and $\tau^{-1}(p)$ consists of a single point, say $q \in W$. After replacing the original $\mathbb{C}^*$-action on V by the new one[7] $t^{\operatorname{ord} G}.v$, where $t \in \mathbb{C}^*$, $v \in V$, it can be lifted to a good $\mathbb{C}^*$-action on W with a unique fixed point $q \in W$, so that the morphism τ becomes equivariant. It is well known that in this situation $W \cong \mathbb{C}^2$ and, what is more, $B := H^0(W, \mathcal{O}_W) = \mathbb{C}[X, Y]$ where $X, Y \in B$ are two algebraically independent homogeneous elements of minimal possible degrees, say, d_X resp., d_Y (geometrically, this isomorphism is composed of an embedding $W \hookrightarrow \mathbb{C}^N$ equivariant with respect to a diagonal linear $\mathbb{C}^*$-action on $\mathbb{C}^N$, followed by an equivariant projection onto the tangent plane at the fixed point $q \in W$; see [Za, 8.5]). Thus also the quotient morphism $\tau : W \to V$ is equivariant with respect to the $\mathbb{C}^*$-action $t^d.v$ on V and the diagonal linear $\mathbb{C}^*$-action

$$\lambda : (X, Y) \longmapsto (\lambda^{d_X} \cdot X, \lambda^{d_Y} \cdot Y)$$

on $\mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[X, Y]$. This completes the proof. □

If $G \subseteq \mathrm{GL}(2, \mathbb{C})$ is a small finite subgroup and $\tau : \mathbb{C}^2 \to V := \mathbb{C}^2/G$ is the quotient map then there exists a polynomial curve $\mathbb{C} \to V\backslash\{p\}$, where $p = \tau(0) \in V$. In the next theorem we show that the converse is also true.

Theorem 2.7. *Let V be a normal affine surface with a good $\mathbb{C}^*$-action and an isolated singularity $p \in V$. Then the following conditions are equivalent:*

(i) $\overline{k}(V\backslash\{p\}) = -\infty$;
(ii) $\delta_m(V, p) = 0$ *for all* $m \geq 1$;
(iii) (V, p) *is a quotient singularity;*
(iv) *there exists a polynomial curve* $\mathbb{C} \to V\backslash\{p\}$.

Proof. The equivalences (i)⇔(ii) and (ii)⇔(iii) follow from Theorem 1.7 (c), resp. Watanabe's theorem 1.5. The implication (iii)⇒(iv) has been observed above. For the proof of (iv)⇒(i) note that moving a polynomial curve in $V\backslash\{p\}$ with the $\mathbb{C}^*$-action provides polynomial curves through a general point of $V\backslash\{p\}$. Thus this implication follows from Proposition 1.10 (b). □

Remark 2.8. The implication (i)⇒(iii) provides a strengthening of Proposition 4.7 in[8] [Be_1] which states that a quasihomogeneous normal surface singularity (V, p) with $\overline{k}(V\backslash\{p\}) = -\infty$ is rational. Recall that any quotient singularity is rational (cf. [Bri, Bu, Vie]). Moreover, the quotient V/G of an isolated rational singularity (V, p) by a finite group G acting freely off the singular point $p \in V$ is again rational (cf. Lemma 2.2).

[7]Hereafter we denote by $t.v$ the image of a point $v \in V$ under an element t of a group acting on V.
[8]Cf. also [MaMi].

On the other hand, it is well known that a rational singularity (V, p) of a surface V with a good $\mathbb{C}^*$-action is not necessarily a quotient singularity (and if so then by [Be$_1$, 4.7] or by Theorem 2.7 $\bar{k}\,(V\backslash\{p\}) \geq 0$). To construct concrete examples we use the following proposition.

Proposition 2.9. *Let V be an affine normal surface with a good $\mathbb{C}^*$-action and a quasirational singularity at its vertex, say $p \in V$. Let $\mathbb{Z}_d \cong \langle\zeta\rangle$ be the subgroup of $\mathbb{C}^*$ generated by a primitive d-th root of unity ζ. Then the following hold.*

1. If $d \gg 0$ then the quotient $W := V/\mathbb{Z}_d$ has at most a rational singularity.

2. $\bar{k}\,(W\backslash\{q\}) = \bar{k}\,(V\backslash\{p\})$, where q denotes the vertex of W. In particular, W has a quotient singularity if and only if V has a quotient singularity.

Proof. Let A be the coordinate ring of V with its natural grading $A = \bigoplus_{i\geq 0} A_i$ induced by the $\mathbb{C}^*$-action. The coordinate ring of W is then given by the Veronese subring $B := A^{\mathbb{Z}_d} = \bigoplus_{i\geq 0} A_{id}$. Moreover

$$\omega_B^{[m]} \cong (\omega_A^{[m]})^{\mathbb{Z}_d} = \bigoplus_{i\in\mathbb{Z}} (\omega_A^{[m]})_{id}$$

for all $m \geq 1$. Now $(\omega_A)_{\leq 0}$ is a vector space over $\mathbb{C}$ of finite dimension and $(\omega_A)_0 = 0$, since V has a quasirational singularity (see 1.17). Hence for $d \gg 0$ the homogeneous components $(\omega_A)_{id}$ vanish for all $i \leq 0$ and so $\omega_B = (\omega_B)_{>0}$. Using 1.7(a) it follows that W has a rational singularity at its vertex $q \in W$, proving (1).

To deduce (2), note that by 1.7(c)

$$\bar{p}_m(V\backslash\{p\}) = \dim_{\mathbb{C}}(\omega_A^{[m]})_0 \quad \text{and} \quad \bar{p}_m(W\backslash\{q\}) = \dim_{\mathbb{C}}(\omega_B^{[m]})_0.$$

As $(\omega_B^{[m]})_0 \cong (\omega_A^{[m]})_0$ for $m \geq 1$ this implies that $V\backslash\{p\}$ and $W\backslash\{q\}$ have the same logarithmic Kodaira dimension. The second part of (2) is now a consequence of 2.7. □

Example 2.10. As a concrete example, consider the Pham-Brieskorn surface $V = V_{p,q,r}$ given by the equation $x^p + y^q + z^r = 0$ in $\mathbb{C}^3$. This surface admits a $\mathbb{C}^*$-action via

$$t.(x, y, z) := (t^{qr}x, t^{pr}y, t^{pq}z).$$

Assume that $1/p + 1/q + 1/r < 1$ and that one of the conditions (i), (ii) of Theorem 0.1 (a) is fulfilled so that V has a non-rational quasirational singularity at the origin. If ζ is a d-th root of unity and $d \gg 0$ then by Proposition 2.9 the quotient $W := V/\langle\zeta\rangle$ has a rational singularity which is not a quotient one. Using Corollary 1.9 the logarithmic Kodaira dimension of $V\backslash\{p\}$ (and hence also of $W\backslash\{q\}$) is equal to 1.

Notice that by Theorem 2.7, both $W\backslash\{q\}$ and $V\backslash\{p\}$ do not contain polynomial curves, while by Proposition 1.18 V and W contain closed rational curves different from an orbit closure. Clearly $V \setminus \{p\}$ cannot contain closed rational curves, see 1.12. However, we do not know whether $W \setminus \{q\}$ contains closed rational curves. More generally one is tempted to pose the following question.

Question 2.11. Does an affine normal surface V with a good $\mathbb{C}^*$-action and a rational singularity contain a closed rational curve not passing through its vertex?

2.3. **Surfaces with $\mathbb{C}_+$-actions and cyclic quotient singularities.** Let V be an affine normal surface with a good $\mathbb{C}^*$-action. If V admits a $\mathbb{C}_+$-action then by Theorem 2.7 it has at most a quotient singularity. In this subsection we examine the question as to when a quasihomogeneous surface with a quotient singularity admits a $\mathbb{C}_+$-action.

2.12. Recall that a *cyclic quotient surface singularity* arises locally as the quotient of $\mathbb{C}^2$ by the action of a group $\mathbb{Z}_d \cong \langle\zeta\rangle$ of d-th roots of unity generated by

$$(x,\, y) \longmapsto (\zeta x,\, \zeta^e y), \tag{5}$$

where ζ is a primitive d-th root of unity and $e \in \mathbb{N}$, $\gcd(e, d) = 1$.

By a theorem of Brieskorn [Bri], a normal surface singularity (W, p) is a cyclic quotient singularity if and only if the dual graph Γ of its minimal embedded resolution $\sigma : W' \to W$ is linear (this linear weighted graph Γ represents a *Hirzebruch-Jung string*; see e.g., [La$_1$, BPV, III.2, III.5]). By [KKMS] (see also [Pi$_1$]), (W, p) is a cyclic quotient singularity if and only if it is a toric surface singularity, that is, W is an affine surface with a good effective $\mathbb{C}^* \times \mathbb{C}^*$-action and p is its only fixed point.

We characterize below affine surfaces with a good $\mathbb{C}^*$-action which also admit a regular action of the additive group $\mathbb{C}_+$ (cf. also [DaGi, Be$_2$]).

Theorem 2.13. *Let V be an affine surface with a good $\mathbb{C}^*$-action and an isolated normal singularity $p \in V$. The surface V admits a non-trivial $\mathbb{C}_+$-action if an only if (V, p) is a cyclic quotient singularity: $V \cong \mathbb{C}^2/G$ with $G \cong \mathbb{Z}/d\mathbb{Z}$ being a small finite subgroup of* $\mathrm{GL}\,(2; \mathbb{C})$.

The proof is given in Lemmas 2.14–2.16 below.

Lemma 2.14. *Let $V = \mathbb{C}^2/G$ be a quotient of $\mathbb{C}^2$ with respect to a small subgroup $G \subseteq \mathrm{GL}\,(2; \mathbb{C})$. If G is a cyclic group then the surface V admits a nontrivial $\mathbb{C}_+$-action.*

Proof. We may assume that the action of the group $G \cong \mathbb{Z}_d$ on $\mathbb{C}^2$ is generated by the linear transformation

$$g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^e \end{pmatrix},$$

where ζ is a primitive d-th root of unity. As G is small, e and d have to be coprime. The locally nilpotent derivation

$$\partial : \mathbb{C}[X, Y] \longrightarrow \mathbb{C}[X, Y]\,,$$

$$\partial(X) := 0, \qquad \partial(Y) := X^e$$

commutes with the action of G i.e., $\partial(g.f) = g.\partial(f)$ for every $f \in \mathbb{C}[X, Y]$ and every $g \in G$. Hence ∂ descends to a non-zero locally nilpotent derivation say, ∂' on the subalgebra of invariants $A := \mathbb{C}[X, Y]^G$ and so $\mathbb{C}_+$ acts effectively on A (as well as on the surface $V = \operatorname{Spec} A$) by $t \longmapsto \exp\,(t\partial')$ (see e.g., [Ren]). □

The converse is also true (see Lemma 2.16 below). But first we need the next simple lemma. Recall that a derivation, say, ∂ of a domain, say, A (over $\mathbb{C}$) naturally extends to the fraction field $\operatorname{Frac} A$ leaving invariant the integral closure $\bar{A}$ of A in $\operatorname{Frac} A$ [Sei]. We have the following statements.

Lemma 2.15. *Let $A \hookrightarrow B$ be an integral extension of finitely generated domains over $\mathbb{C}$, and let $\partial \in \operatorname{Der} B$ be a derivation such that $\partial(A) \subseteq A$. Then the following hold.*

(a) [Vas] *If the restriction $\partial\,|\,A$ is locally nilpotent then so is ∂.*

(b) *Suppose moreover that B is graded and that A is a graded subring. If the restriction $\partial\,|\,A$ is homogeneous then ∂ is homogeneous.*

Proof. (a) We provide an alternative geometric proof supposing that A and B are defined over $\mathbb{C}$. Let $\pi : Y := \operatorname{Spec} B \to X := \operatorname{Spec} A$ be the map induced by $A \hookrightarrow B$. Every orbit $\mathrm{O}_x := \mathbb{C}_+.x \subseteq X$, where $x \in X$ is a closed point, of the associated $\mathbb{C}_+$-action on X is isomorphic to $\mathbb{C}_+$ or reduced to a point. As X is affine, O_x is closed in X. By assumption the infinitesimal generator of the $\mathbb{C}_+$-action lifts to a vector field on Y and so $\pi^{-1}(\mathrm{O}_x) \to \mathrm{O}_x$ is unramified. Thus the preimage $\pi^{-1}(\mathrm{O}_x)$ is a disjoint union of copies of $\mathbb{C}_+$ or of points, and the $\mathbb{C}_+$-action on O_x lifts to an (algebraic) action on $\pi^{-1}(\mathrm{O}_x)$. In particular, the derivation ∂ induces a locally nilpotent derivation on the affine coordinate ring of $\pi^{-1}(\mathrm{O}_x)$.

For an algebraic function $f \in B$ consider the subset A_n of all closed points $x \in X$ such that the restriction of $\partial^n(f)$ to $\pi^{-1}(\mathrm{O}_x)$ vanishes identically. By the above reasoning, $\bigcup_{n \geq 0} A_n$ is the set of all closed points of X. Thus for some n the Zariski closure of A_n in X must be equal to X, and then $\partial^n(f)$ vanishes identically on Y.

(b) Let $f \in B_k$ be a homogeneous element and consider an equation of integral dependence $P(f) = 0$, where

$$P = X^n + a_1 X^{n-1} + \cdots + a_n$$

with $a_i \in A$. We may assume that a_ν is homogeneous of degree $k\nu$. Applying ∂ gives

$$0 = \partial(P(f)) = P'(f) \cdot \partial(f) + P_\partial(f) \qquad \text{with} \qquad P_\partial := \sum_{\nu=0}^{n} \partial(a_{n-\nu}) X^\nu .$$

As $P_\partial(f)$ and $P'(f)$ are both homogeneous and B is a domain, it follows that $\partial(f)$ is also homogeneous. ☐

Lemma 2.16. *Let V and $p \in V$ be as in 2.13. If V admits a non-trivial $\mathbb{C}_+$-action then $V \cong \mathbb{C}^2/G$ for a small cyclic subgroup $G \subseteq \mathrm{GL}\,(2; \mathbb{C})$.*

Proof. Since V admits a non-trivial $\mathbb{C}_+$-action with a fixed point $p \in V$, we have $\overline{k}\,(V \backslash \{p\}) = -\infty$, and hence by Theorem 2.7, (V, p) is a quotient singularity. Thus, Lemma 2.6 can be applied. We keep the notation as in the proof of this lemma; in particular, we have $V = W/G$, where $W \cong \mathbb{C}^2$ and $G \subseteq \mathrm{GL}\,(2, \mathbb{C})$ is a small finite subgroup, with quotient morphism $\tau : W \to V$. Furthermore, $B := H^0(W, \mathcal{O}_W) \cong \mathbb{C}[X, Y]$ with $X, Y \in B$ being homogeneous of degree d_X resp., d_Y, and τ is equivariant with respect to the corresponding (linear diagonal) $\mathbb{C}^*$-action on $\mathbb{C}^2 = \operatorname{Spec} \mathbb{C}[X, Y]$.

The derivation $\partial = \partial_\varphi \in \operatorname{Der} A$ corresponding to the $\mathbb{C}_+$-action, say $\varphi : \mathbb{C}_+ \times V \to V$ is known to be locally nilpotent (see e.g., [Ren]) and non zero. Since the leading term ∂_ℓ of ∂ is again a locally nilpotent derivation (see e.g., [Ren, ML]) we can replace ∂ by ∂_ℓ and the action $\varphi(t, -) = e^{t\partial}$ by the new action $e^{t\partial_\ell}$. Thus we may assume that ∂ is homogeneous.

The derivation ∂ is induced by a homomorphism, say, $h : \Omega^1_A \to A$ and so it gives a homomorphism $h \otimes 1 : \Omega^1_A \otimes_A B \to B$. As $A \to B$ is unramified in codimension 1, the reflexive hull of $\Omega^1_A \otimes_A B$ is just Ω^1_B, and so we get an induced homomorphism

$\tilde{h} : \Omega^1_B \to B$. Composing this map with the total differential $d : B \to \Omega^1_B$ we obtain a derivation $\tilde{\partial}$ that lifts ∂ and is locally nilpotent by 2.15(a). Taking the exponential of $\bar{\partial}$ the $\mathbb{C}_+$-action $\mathbb{C}_+ \times V \to V$ lifts to an action $\mathbb{C}_+ \times W \to W$ such that the diagram

$$\begin{tikzcd} \mathbb{C}_+ \times W \arrow[r, "\tilde{\varphi}"] \arrow[d, "\mathrm{id}\times\tau"'] & W \arrow[d, "\tau"] \\ \mathbb{C}_+ \times V \arrow[r, "\varphi"] & V \end{tikzcd}$$

is commutative. Let us show that this action commutes with the action of G on W. Indeed, it preserves the G-orbits and so, given $z \in \mathbb{C}^2$ and $g \in G$, for any $\lambda \in \mathbb{C}_+$ there exists an element $\tilde{g}(\lambda) \in G$ such that $\lambda.(g.z) = \tilde{g}(\lambda).(\lambda.z)$. Since $\tilde{g}(\lambda)$ is a continuous function on the connected variety $\mathbb{C}_+$ with values in G, it must be constant: $\tilde{g}(\lambda) = \tilde{g}(0) = g$, and so $g\lambda = \lambda g$ for all $g \in G,\ \lambda \in \mathbb{C}_+$, as stated.

In what follows we distinguish two cases:

Case (1): $d_X < d_Y$. As the action of G on the algebra $B \cong \mathbb{C}[X, Y]$ is homogeneous, it follows that

$$g.X = \alpha(g) \cdot X \qquad \text{and} \qquad g.Y = \beta(g) \cdot Y$$

for two characters $\alpha,\ \beta : G \to S^1$. For an element $g \in G$, let $\operatorname{ord} \alpha(g) = kp_1$ and $\operatorname{ord} \beta(g) = kp_2$ with $\gcd(p_1, p_2) = 1$. If $p_1 \neq 1$ then $\alpha(g^{kp_1}) = 1$ and $\beta(g^{kp_1}) \neq 1$, so that the element g^{kp_1} acts on $W = \mathbb{C}^2$ as a pseudo-reflection. This leads to a contradiction since the covering $\tau : W \to V$ is unramified outside the origin $q = \bar{0} \in \mathbb{C}^2 = W$.

Hence it follows that $p_1 = 1$ and (by symmetry) $p_2 = 1$ so that the orders of $\alpha(g)$ and $\beta(g)$ are equal for any $g \in G$. Thus $G \cong \alpha(G) \subseteq S^1$ is a cyclic group, as desired.

Case (2): $d_X = d_Y$. Let $\widetilde{\partial} = \widetilde{\partial}_{\widetilde{\varphi}}$ be the infinitesimal generator of the $\mathbb{C}_+$-action $\widetilde{\varphi}$ on B so that $\widetilde{\partial}$ is a locally nilpotent derivation of the polynomial algebra $B = \mathbb{C}[X, Y]$. By our assumption, ∂ is homogeneous and so, by Lemma 2.15, $\widetilde{\partial}$ is as well homogeneous of degree, say d. As $\widetilde{\partial}$ is locally nilpotent, we can find a homogeneous polynomial $P(X, Y) \in B$ of positive degree with $\widetilde{\partial}(P) = 0$. The kernel $\ker \widetilde{\partial}$ of $\widetilde{\partial}$ being factorially closed (see e.g., [ML]), every factor in the decomposition $P = \prod_{i=1}^m (\alpha_i X + \beta_i Y)$ vanishes under $\widetilde{\partial}$. Hence after a linear change of coordinates (which is homogeneous since $d_X = d_Y$) we may assume that $\widetilde{\partial}(X) = 0$. Write now

$$\widetilde{\partial}(Y) = \sum_{i=0}^{s} Y^i F_i(X) \in B_{d+d_Y}$$

with $F_s \not\equiv 0$. If $s \geq 1$ then an easy induction shows that

$$\widetilde{\partial}^n(Y) = c_n \cdot Y^{ns-n+1} F_s(X)^n + (\text{lower order terms in } Y) \neq 0$$

with $c_n \in \mathbb{N},\ c_n \neq 0$, which contradicts the assumption that $\widetilde{\partial}$ is locally nilpotent. Hence $s = 0$ and so

$$\widetilde{\partial}(Y) = c \cdot X^{d+d_Y}$$

for some $c \in \mathbb{C}^*$. Replacing Y by Y/c we can achieve that $c = 1$.

As $\widetilde{\partial}$ commutes with the action of the group G on $\mathbb{C}^2$ we have $\widetilde{\partial}(g.X) = g.\widetilde{\partial}(X) = 0$ for every $g \in G$. It follows that $g.X = \alpha(g) \cdot X$ for a character $\alpha : G \to S^1$. Furthermore, if $g.Y = \xi X + \eta Y$ then the equalities

$$\alpha(g)^{d+d_Y} \cdot X^{d+d_Y} = g.X^{d+d_Y} = g.\widetilde{\partial}(Y) = \widetilde{\partial}(g.Y) = \eta \widetilde{\partial}(Y) = \eta \cdot X^{d+d_Y}$$

show that $\eta = \alpha(g)^{d+d_Y}$. If $\alpha(g) = 1$ then the matrix of g has the form

$$g = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}.$$

Since g has a finite order, ξ must be zero. In other words, the homomorphism $\alpha : G \to S^1$ is injective and so $G \cong \alpha(G)$ is a cyclic group. This completes the proof of the lemma and of Theorem 2.13. □

The next example shows that the assumption of normality in Theorem 2.13 is essential. Indeed, the existence of a $\mathbb{C}_+$-action on an isolated surface singularity does not imply that this singularity is normal.

Example 2.17. Consider the polynomial algebra $\mathbb{C}[X,\, Y]$ with the derivation $\partial := X \cdot \frac{\partial}{\partial Y}$. Apparently, ∂ is a locally nilpotent, homogeneous derivation of degree 0 with respect to the standard grading on $\mathbb{C}[X,\, Y]$. Hence ∂ induces a homogeneous locally nilpotent derivation on the subring

$$A := \mathbb{C}[X^2,\, XY,\, Y^2,\, X^3,\, Y^3] = \bigoplus_{\nu \neq 1} \mathbb{C}[X,\, Y]_\nu\,.$$

Note that the normalization of A is $\mathbb{C}[X,\, Y]$ and that A is not a quotient singularity.

2.4. **Rational curves on surfaces with a Gorenstein singularity.** Recall the following notion.

Definition 2.18. A singularity $(V,\, p)$ is called *Gorenstein* if it is Cohen-Macaulay and $\omega_{V,p} \cong \mathcal{O}_{V,p}$.

Note that a normal surface singularity is always Cohen-Macaulay, and so it is Gorenstein if and only if there exists a nowhere vanishing holomorphic 2-form ω on $V \backslash \{p\}$.

We recollect below some useful facts on Gorenstein singularities.

Theorem 2.19. (a) (See e.g., [FlVo, A.2.5]) *Any complete intersection singularity is Gorenstein.*

(b) [KeWa] *A quotient singularity $(V, p) \cong (\mathbb{C}^n/G,\, \bar{0})$ with a small finite subgroup G of* $\mathrm{GL}\,(n;\, \mathbb{C})$ *is Gorenstein if and only if $G \subseteq \mathrm{SL}\,(n;\, \mathbb{C})$.*

(c) *In particular, a cyclic quotient surface singularity $(V,\, p) \cong (\mathbb{C}^2/G,\, \bar{0})$ is Gorenstein if and only if $G \subseteq \mathrm{SL}\,(2,\, \mathbb{C})$ is a cyclic subgroup generated by*

$$(x, y) \longmapsto (\zeta x,\, \zeta^{-1} y)$$

where $\zeta \in \mathbb{C}$ is a primitive root of unity of degree, say, d. In the latter case, $(V,\, p) \cong (V_{2,2,d},\, \bar{0})$ where $V_{2,2,d} \subseteq \mathbb{C}^3$ is the dihedral surface with the equation $x^2 + y^2 + z^d = 0$.

(d) [Wah, 2.5, 4.6] *A rational surface singularity $(V,\, p)$ is Gorenstein if and only if it is a rational double point (i.e., $\mathrm{mult}_p V = 2$). In the latter case, $(V,\, p)$ is a quotient singularity $(\mathbb{C}^2/G,\, 0)$ with a small finite subgroup $G \subseteq \mathrm{SL}\,(2;\, \mathbb{C})$ (e.g., see* [Pi$_2$]*).*

(e) [Wah, 4.5, A.6] *Every rational surface singularity is $\mathbb{Q}$-Gorenstein, and so a cyclic quotient (in a canonical way) of a Gorenstein singularity (called* the canonical cover*). The canonical cover is rational if and only if the original singularity is a quotient singularity.*

One can consult e.g., [TSH] and 2.10 above for concrete examples of affine surfaces with a good $\mathbb{C}^*$-action and an isolated rational singularity which is not Gorenstein. Note that $(V_{p,q,r}, \bar{0})$ in 2.10 is just the canonical Gorenstein cyclic cover of (W, q).

For Gorenstein surface singularities we have the following result.

Theorem 2.20. *Let V be an affine surface with a good $\mathbb{C}^*$-action and an isolated Gorenstein singularity $p \in V$. The following conditions are equivalent:*

(i) $\overline{k}\,(V\backslash\{p\}) = -\infty$;

(ii) $(V,\,p)$ *is a quotient singularity;*

(iii) $(V,\,p)$ *is a rational singularity;*

(iv) *there exists a closed rational curve* $C \subseteq V\backslash\{p\}$;

(v) *there exists a polynomial curve* $f : \mathbb{C} \to V\backslash\{p\}$.

Proof. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (v) have been established in Theorem 2.7 above. The implication (v) $\Rightarrow$ (iv) is evident; (iv) $\Rightarrow$ (iii) follows from Theorem 1.12. For Gorenstein singularities, (ii) and (iii) are equivalent; see [Bri] for (ii) $\Rightarrow$ (iii) and Theorem 2.19(d) above for (iii) $\Rightarrow$ (ii). This proves the theorem. □

Example 2.21. The Pham-Brieskorn hypersurface

$$V_{\bar{p}} := \left\{ \sum_{i=1}^{n} x_i^{p_i} = 0 \right\} \subseteq \mathbb{C}^n \qquad (\text{with} \quad p_i \geq 2 \quad \forall i)$$

is quasihomogeneous with weights $w_i = 1/p_i$, $i = 1, \ldots, n$, and the defining equation has degree 1. According to 1.9 $(V, 0)$ is a rational singularity if and only if

$$\sum_{i=1}^{n} \frac{1}{p_i} > 1\,.$$

On the other hand, it is known [Ste, BrMa, DaGr, Prop.5.2] that for

$$\sum_{i=1}^{n} \frac{1}{p_i} \leq \frac{1}{n-2}$$

the Diophantine equation $\sum_{i=1}^{n} x_i^{p_i} = 0$ does not admit non-constant solutions in polynomials coprime in pairs, that is any polynomial curve $\mathbb{C} \to V_{\bar{p}}$ meets a codimension 2 coordinate subspace of $\mathbb{C}^n$.

In particular, for $n = 3$ (in virtue of Theorem 2.20) we have the equivalences

$$\overline{k}\,(V_{\bar{p}}\backslash\{\bar{0}\}) = -\infty \qquad \Leftrightarrow \qquad \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} > 1$$

$\Leftrightarrow$ the surface V_{p_1,p_2,p_3} has a rational (or equivalently, a quotient) singularity $\Leftrightarrow$ it is one of the Platonic surfaces $V_{2,2,n}$, $V_{2,3,3}$, $V_{2,3,4}$, $V_{2,3,5}$. It has a cyclic quotient ($A_{d-1}-$)singularity if and only if it is a dihedral surface $V_{2,2,d}$ (cf. [Be$_1$, 4.6] and [MaMi]), and only in this case it admits an effective $\mathbb{C}_+$-action [KaZa, L. 4].

2.22. Question. The affine Fermat cubic 3-fold

$$V := \{x_1^3 + \ldots + x_4^3 = 0\}$$

in $\mathbb{C}^4$ has a rational singularity at the origin. Does it admit a non-trivial regular $\mathbb{C}_+$-action?

Other examples are provided by the following result.

Corollary 2.23. *Let V be a surface in $\mathbb{C}^3$ defined by the equation $F_d(x,y)=z^m$, where $F_d\in\mathbb{C}[x,y]$ is a homogeneous form of degree d without multiple factors. If $d\ge 3$ and $(d,m)\neq(3,2)$ then any closed rational curve $C\subseteq V$ passes through the singular point $\bar 0\in V$. Consequently, the Diophantine equation $F_d(x,y)=z^m$ has a non-constant polynomial solution in coprime polynomials if and only if $d\le 2$ or $(d,m)=(3,2)$.*

Proof. Indeed, in that case the surface V has an isolated singularity at the origin, and the defining equation is weighted homogeneous of degree $d_1=md$ with respect to the weights $w_1=w_2=m$, $w_3=d$, so that by 1.9 $\bar k(V\backslash\{\bar 0\})=-\infty$ if and only if

$$N_V=d_1-\sum_{j=1}^{3}w_j=(d-2)m-d<0 \quad \text{if and only if} \quad d\le 2 \quad \text{or} \quad (d,m)=(3,2)\,.$$

Thus the first part follows from Theorem 2.20 above.

If $d=2$ resp., $(d,m)=(3,2)$ then in appropriate coordinates the equation of the surface V can be written as $x^2+y^2+z^m=0$ resp., $x^3+y^3+z^2=0$, i.e., V is linearly isomorphic to the dihedral surface $V_{2,2,m}\cong\mathbb{C}^2/\mathbb{Z}_m$ resp., to the tetrahedral surface[9] $V_{2,3,3}=\mathbb{C}^2/G$ with the tetrahedral group $G\subseteq\mathrm{GL}(2,\mathbb{C})$ (isomorphic to the alternating group A_4). Thus, there exist polynomial curves $\mathbb{C}\to V\backslash\{\bar 0\}$ (for concrete examples of such curves see the Introduction). ☐

Remark 2.24. Corollary 2.23 for $(d,m)\neq(3,3),(4,3)$ or $(d,2)$ with $d\le 16$ also follows from Theorem 1 in [Sch] (cf. [KaZa, Schmidt's Lemma]).

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