MODULI SPACES OF PARABOLIC HIGGS BUNDLES AND PARABOLIC $K(D)$ PAIRS OVER SMOOTH CURVES

Hans U. Boden
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Kôji Yokogawa

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ABSTRACT. This paper is concerned with describing the moduli spaces of rank two parabolic Higgs bundles and parabolic $K(D)$ pairs over a smooth curve. Precisely which parabolic bundles occur in stable $K(D)$ pairs and stable Higgs bundles is determined. Using Morse theory, the moduli space of parabolic Higgs bundles is shown to be connected and simply connected, and a computation of its Poincaré polynomial and Euler characteristic is given.

1. INTRODUCTION

A deep result of Narasimhan and Seshadri states that, over a compact curve C, there is a one-to-one correspondence between irreducible *unitary* representations of $\pi_1(C)$ and stable bundles with $c_1(E) = 0$ [18]. This theorem was later extended to cover arbitrary compact Kähler manifolds [7, 8, 27]. Hitchin then introduced *Higgs* bundles, which are pairs (E, Φ) consisting of a holomorphic bundle E and holomorphic map $\Phi : E \to E \otimes K$ called the Higgs field. The nonabelian Hodge theorem establishes a correspondence between irreducible representations of $\pi_1(X)$ and stable Higgs bundles with $c_i(E) = 0$ for X a compact, Kähler manifold [12, 9, 6, 22, 25].

For noncompact curves C_0 with compactification $C = C_0 \cup \{p_1, \ldots, p_n\}$, Mehta and Seshadri proved a correspondence between stable *parabolic* bundles E_* over C with pardeg $E_* = 0$ and unitary representations of $\pi_1(C_0)$ with fixed holonomy around each p_i, the so-called parabolic points [17]. For regular bundles, the space of Higgs fields $H^0(\mathrm{End}(E) \otimes K)$ is naturally dual to $H^1(\mathrm{End}(E))$, but because parabolic endomorphisms satisfy a vanishing condition at the parabolic points, duality implies that Higgs fields of parabolic bundles can have poles of order one at those points. Allowing the Higgs field to have either *parabolic* or *nilpotent* residues at the p_i, we obtain the two moduli spaces $\mathcal{P}_\alpha$ of parabolic $K(D)$ pairs and $\mathcal{N}_\alpha$ of parabolic Higgs bundles. The subscript α refers to a particular choice of weights. In [28], $\mathcal{P}_\alpha$ is constructed using Geometric Invariant Theory and is proved to be a normal, quasi-projective variety. In [23], stable parabolic $K(D)$ pairs are called filtered regular Higgs bundles and are shown to correspond to filtered regular $\mathcal{D}_X$ modules. In [14], Konno constructs $\mathcal{N}_\alpha$ using gauge theory and shows that stable parabolic Higgs bundles correspond to irreducible parabolic Hermitian-Einstein Higgs bundles. In [21], the nonabelian Hodge theorem for elliptic surfaces is used to show that parabolic Higgs bundles with rational weights correspond to irreducible representations of $\pi_1(C_0)$ with holonomy

around each p_i equal to some root of unity.

The following factorization theorem, due to Simpson [24], states that for X a projective algebraic variety, any $SL(2, \mathbb{C})$ representation of $\pi_1(X)$ with Zariski dense image is either rigid or factors through an algebraic map from X to an orbicurve. We can interpret these representations as stable parabolic Higgs bundles using [21]. In this paper, we study the algebraic and topological properties of the moduli spaces of stable, rank two parabolic Higgs bundles and parabolic $K(D)$ pairs. We first establish necessary and sufficient conditions for a given rank two parabolic bundle E_* to admit a field Φ making (E_*, Φ) a stable parabolic $K(D)$ pair or Higgs bundle. One could theoretically give an algebraic description of the moduli spaces using only these criteria, which we do in one particular case, but the problem appears too complicated for this approach to work in general.

For that reason, we shift gears and study the topological properties of the moduli space of parabolic Higgs bundles, using the approach of Hitchin [12]. There is a circle action on $\mathcal{N}_\alpha$ which preserves its complex and symplectic structure, and the associated moment map is a Morse function. Identifying the critical submanifolds and their indices, we prove that $\mathcal{N}_\alpha$ is a noncompact, connected, simply connected manifold and compute its Betti numbers, which turn out to be independent of the weights α. This behavior is in marked contrast to that exhibited by the moduli space $\mathcal{M}_\alpha$ of parabolic bundles, where the Betti numbers do depend on α [4]. In the sequel, we plan to extend these results to higher rank bundles.

The paper is organized as follows. In §2, one finds the definitions, notation, and preliminary results used throughout the paper. In §2.1, parabolic bundles with auxiliary fields are defined and the three moduli spaces $\mathcal{M}_\alpha$, $\mathcal{N}_\alpha$, and $\mathcal{P}_\alpha$ are introduced. In §2.2, parabolic sheaves are defined and proved to be in one-to-one correspondence with parabolic bundles. The advantage of parabolic sheaves is that one can easily define the natural operations, such as tensor products and duals. These are necessary for the statement of the Serre duality theorem for parabolic bundles, also given in this section. One operation unique to parabolic sheaves, called shifting, is exploited to determine the Picard group of parabolic line bundles.

In §3.1, we establish algebraic criteria for a given parabolic bundle E_* of rank two to admit a field Φ so that (E_*, Φ) is a stable parabolic $K(D)$ pair or a stable parabolic Higgs bundle, and in §3.2, we use these criteria to give a complete description of $\mathcal{P}_\alpha$ and $\mathcal{N}_\alpha$ in the case of rank two bundles over $\mathbb{P}^1$ with three parabolic points.

The topological properties of $\mathcal{N}_\alpha$ are considered in §4. In §4.1, we describe the construction of $\mathcal{N}_\alpha$ as a hyperkähler quotient, following [14]. In §4.2, we extend the arguments of [12] to the case of parabolic Higgs bundles and define a moment map which can be used as a Morse function to calculate the Betti numbers of $\mathcal{N}_\alpha^0$. In §4.3, we use the Morse function to prove that $\mathcal{N}_\alpha^0$ is a noncompact, connected, simply connected manifold whose Betti numbers are independent of α. This last result allows us to determine the Poincaré polynomial and Euler characteristic of $\mathcal{N}_\alpha^0$ as a function of the genus g and the number of parabolic points n, which is done in §4.4.

2. Definitions and Preliminary Results

2.1. Three moduli spaces. Let X be a smooth curve of genus g with n marked points in the reduced divisor $D = p_1 + \cdots + p_n$ and E a holomorphic bundle over X.

Definition 2.1. *A parabolic structure on E consists of weighted flags*

$$E_p = F_1(p) \supset \cdots \supset F_{s_p}(p) \supset 0$$
$$0 \leq \alpha_1(p) < \cdots < \alpha_{s_p}(p) < 1$$

over each $p \in D$. A holomorphic map $\phi : E^1 \longrightarrow E^2$ between parabolic bundles is called parabolic if $\alpha_i^1(p) > \alpha_j^2(p)$ implies $\phi(F_i^1(p)) \subset F_{j+1}^2(p)$ for all $p \in D$. We call ϕ strongly parabolic if $\alpha_i^1(p) \geq \alpha_j^2(p)$ implies $\phi(F_i^1(p)) \subset F_{j+1}^2(p)$ for all $p \in D$.

We use E_* to denote the bundle together with a parabolic structure. Also, we use $\mathrm{ParHom}(E_*^1, E_*^2)$ and $\mathrm{ParHom}(E_*^1, \widehat{E}_*^2)$ to denote the sets of parabolic and strongly parabolic morphisms from E^1 to E^2, respectively. (The decorative notation will become clear in §2.2.) If $\alpha_i^1(p) \neq \alpha_j^2(p)$ for all i, j and $p \in D$, then a parabolic morphism is automatically strongly parabolic. On the other hand, using the notation $\mathrm{ParEnd}(E_*) = \mathrm{ParHom}(E_*, E_*)$ and $\mathrm{ParEnd}^\wedge(E_*) = \mathrm{ParHom}(E_*, \widehat{E}_*)$, then strongly parabolic endomorphisms are nilpotent with respect to the flag data at each $p \in D$.

Let K denote the canonical bundle of X and give $E \otimes K(D)$ the obvious parabolic structure.

Definition 2.2. *A parabolic $K(D)$ pair is a pair (E, Φ) consisting of a parabolic bundle E and a parabolic map $\Phi : E \to E \otimes K(D)$. Such a pair is called a parabolic Higgs bundle if, in addition, Φ is a strongly parabolic morphism.*

Viewing α as a vector-valued function on D, we use it as an index to indicate the parabolic structure on E_*. Let $m_i(p) = \dim(F_i(p)) - \dim(F_{i+1}(p))$, the multiplicity of $\alpha_i(p)$, and $f_p = \frac{1}{2}(r^2 - \sum_{i=1}^{s_p}(m_i(p))^2)$, the dimension of the associated flag variety. Define the parabolic degree and slope of E_* by

$$\mathrm{pardeg}\, E_* = \deg E + \sum_{p \in D} \sum_{i=1}^{s_p} m_i(p)\alpha_i(p),$$
$$\mu(E_*) = \frac{\mathrm{pardeg}\, E_*}{\mathrm{rank}\, E}.$$

If L is a subbundle of E, then L inherits a parabolic structure from E by pullback. We call the bundle E_* *stable (semistable)* if, for every proper subbundle L of E, we have $\mu(L_*) < \mu(E_*)$ (respectively $\mu(L_*) \leq \mu(E_*)$). Likewise, we will call a parabolic $K(D)$ pair (E_*, Φ) *stable* (or *semistable*) if the same inequalities hold on those proper subbundles L of E which are, in addition, Φ-invariant.

Denote by $\mathcal{M}_\alpha$ the moduli space of α-semistable parabolic bundles, by $\mathcal{N}_\alpha$ the moduli space of α-semistable parabolic Higgs bundles, and by $\mathcal{P}_\alpha$ the moduli space

of α-semistable parabolic $K(D)$ pairs. By [17], $\mathcal{M}_\alpha$ is a normal, projective variety of dimension

$$\dim \mathcal{M}_\alpha = (g-1)r^2 + 1 + \sum_{p \in D} \dot{f}_p.$$

(If $g = 0$, this holds only when $\mathcal{M}_\alpha \neq \emptyset$.) Further, in [28, 29], $\mathcal{P}_\alpha$ is shown to be a normal, quasi-projective variety of dimension

$$\dim \mathcal{P}_\alpha = (2g - 2 + n)r^2 + 1$$

which contains $\mathcal{N}_\alpha$ as a closed subvariety of $\mathcal{P}_\alpha$ of dimension

$$\dim \mathcal{N}_\alpha = 2(g-1)r^2 + 2 + 2\sum_{p \in D} f_p.$$

For generic α, a bundle (or pair) is α-semistable $\Leftrightarrow$ it is α-stable. In these cases, the moduli spaces $\mathcal{M}_\alpha, \mathcal{N}_\alpha$ and $\mathcal{P}_\alpha$ are smooth and can be described topologically as certain quotients of the gauge group $\mathcal{G}^{\mathbb{C}} = \mathrm{ParAut}(E_*)$. The same is true for $\mathcal{M}_\alpha^0, \mathcal{N}_\alpha^0$ and $\mathcal{P}_\alpha^0$, the moduli spaces with fixed determinant and trace-free Φ. In this way, it is shown in [14] that $\mathcal{N}_\alpha^0$ is, for generic α, a smooth, hyperkähler manifold of complex dimension

$$\dim \mathcal{N}_\alpha^0 = 2(g-1)(r^2 - 1) + 2\sum_{p \in D} f_p.$$

2.2. Parabolic sheaves and Serre duality.

Suppose now that E is a locally free sheaf on X and $D = p_1 + \cdots + p_n$ is a reduced divisor.

Definition 2.3. *A parabolic structure on E consists of a weighted filtration of the form*

$$E = E_0 = E_{\alpha_1} \supset \cdots \supset E_{\alpha_l} \supset E_{\alpha_{l+1}} = E(-D),$$
$$0 = \alpha_0 \leq \alpha_1 < \cdots < \alpha_l < \alpha_{l+1} = 1.$$

We can define E_x for $x \in [0,1]$ by setting $E_x = E_{\alpha_i}$ if $\alpha_{i-1} < x \leq \alpha_i$, and then extend to $x \in \mathbb{R}$ by setting $E_{x+1} = E_x(-D)$. We call the resulting filtered sheaf E_* a parabolic sheaf.

We define the coparabolic sheaf $\widehat{E}_*$, by

$$\widehat{E}_x = \begin{cases} E_x & \text{if } x \neq \alpha_i \\ E_{\alpha_{i+1}} & \text{if } x = \alpha_i. \end{cases}$$

A morphism of parabolic sheaves $\phi : E_*^1 \to E_*^2$ is a called parabolic if $\phi(E_x^1) \subseteq E_x^2$ and strongly parabolic if $\phi(E_x^1) \subseteq \widehat{E}_x^2$ for all $x \in \mathbb{R}$.

We shall denote by $\mathfrak{ParHom}(E_*^1, E_*^2)$ and $\mathfrak{ParHom}(E_*^1, \widehat{E}_*^2)$ the sheaves of parabolic and strongly parabolic morphisms, and by $\mathrm{ParHom}(E_*^1, E_*^2)$ and $\mathrm{ParHom}(E_*^1, \widehat{E}_*^2)$ their global sections. We now show that there is an equivalence of the categories of parabolic bundles on X and parabolic sheaves on X.

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Given a parabolic bundle E with flags and weights as in Definition 2.1, we define the filtered sheaf E_* following Simpson [26]. For $p \in D$ and $\alpha_{i-1}(p) < x \le \alpha_i(p)$, set

$$E^p_x = \ker(E \to E_p/F_i(p)),$$
$$E_x = \bigcap_{p \in D} E^p_x.$$

Now extend to all x by $E_{x+1} = E_x(-D)$.

Conversely, given a parabolic sheaf E_*, the quotient E/E_1 is a skyscraper sheaf with support on D and, for each $p \in D$, we get weighted flags in E_p by intersecting with the filtration at p. To be precise, let $\alpha_1(p), \ldots, \alpha_{s_p}(p)$ be the subset of weights such that

$$(1) \qquad \alpha_{i-1}(p) < x \le \alpha_i(p) \Leftrightarrow (E_x/E_1)_p = (E_{\alpha_i(p)}/E_1)_p.$$

Setting $F_i(p) = (E_{\alpha_i(p)}/E_1)_p$, we obtain a parabolic bundle in the sense of Definition 2.1.

Suppose now E^1 and E^2 are parabolic bundles and $\phi \in \mathrm{ParHom}(E^1, E^2)$. We want to show that ϕ induces a morphism of the parabolic sheaves. So, suppose $\alpha^1_{i-1}(p) < x \le \alpha^1_i(p)$ and $\alpha^2_{j-1}(p) < x \le \alpha^2_j(p)$. Since $\alpha^1_i(p) > \alpha^2_{j-1}(p)$, $\phi(F^1_i(p)) \subset F^2_j(p)$ and we see that ϕ maps $\ker(E^1 \to E^1_p/F^1_i(p))$ to $\ker(E^2 \to E^2_p/F^2_i(p))$ for all $p \in D$, from which it follows that ϕ induces a map $\phi : E^1_x \to E^2_x$.

Suppose conversely that E^1_* and E^2_* are parabolic sheaves, $\phi \in \mathrm{ParHom}(E^1_*, E^2_*)$ and $\alpha^1_i(p) > \alpha^2_j(p)$. Set $x = \alpha^1_i(p)$ and $y = \alpha^2_{j+1}(p)$ for notational convenience. Then $\phi(E^1_x) \subset E^2_x$. Since $x > \alpha^2_j(p)$, it follows from (1) that $(E^2_x/E^2_1)_p \subset (E^2_y/E^2_1)_p$ and hence $\phi(F^1_i(p)) \subset F^2_{j+1}(p)$.

It is not hard to see the same correspondence for strongly parabolic morphisms. Thus, we have an equivalence of the categories of parabolic bundles and parabolic sheaves. We use the definitions interchangeably and denote by E_* a parabolic bundle or sheaf, reserving $E = E_0$ for the underlying holomorphic bundle.

For the convenience of readers, we briefly summarize the results in [29] dealing with exact sequences and tensor products of parabolic sheaves. This is necessary for the statement of Serre duality for parabolic bundles, which is a tool we use throughout the paper.
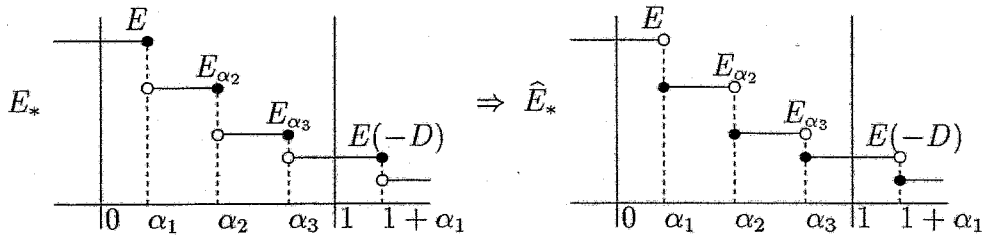

FIGURE 1. The simple relationship between E_* and $\widehat{E}_*$.

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The category of parabolic sheaves $\mathcal{P}$ is not abelian, but is contained in an abelian category $\widetilde{\mathcal{P}}$ as a full subcategory. Objects in $\widetilde{\mathcal{P}}$ are also written by E_* and a morphism $f : E_*^1 \to E_*^2$ is a family of morphisms $f_x : E_x^1 \to E_x^2$. A coparabolic sheaf $\widehat{E}_*$ is realized in $\widetilde{\mathcal{P}}$. The set $\mathrm{ParHom}(E_*^1, \widehat{E}_*^2)$ is just the set of morphisms in $\widetilde{\mathcal{P}}$. In $\widetilde{\mathcal{P}}$, a sequence

$$(2) \qquad\qquad 0 \longrightarrow L_* \longrightarrow E_* \longrightarrow M_* \longrightarrow 0$$

is exact if and only if the induced sequence at x is exact for all $x \in \mathbb{R}$.

Remark. If the sequence (2) is exact, then so is the sequence obtained by tensoring (2) with any parabolic bundle (cf. Proposition 3.3 of [29]) and

$$\mathrm{pardeg}\, E_* = \mathrm{pardeg}\, L_* + \mathrm{pardeg}\, M_*.$$

We can define dual parabolic sheaves $E_*^\vee$, parabolic tensor products $L_* \otimes M_*$, Hom-parabolic sheaves $\mathfrak{ParHom}(L_*, M_*)_*$, and cohomology groups $\mathrm{Ext}^i(L_*, M_*)$. Clearly,

$$\mathrm{pardeg}(L_* \otimes M_*) = \mathrm{rank}(M)\, \mathrm{pardeg}\, L_* + \mathrm{rank}(L)\, \mathrm{pardeg}\, M_*.$$

In addition, we have

$$\begin{aligned}
\mathrm{Ext}^0(L_*, M_*) &= H^0(L_*^\vee \otimes M_*) = H^0(\mathfrak{ParHom}(L_*, M_*)) = \mathrm{ParHom}(L_*, M_*), \\
\mathrm{Ext}^1(L_*, M_*) &= H^1(L_*^\vee \otimes M_*) = H^1(\mathfrak{ParHom}(L_*, M_*)).
\end{aligned}$$

We can identify $\mathrm{Ext}^1(M_*, L_*)$ with the set of equivalence classes of exact sequences of type (2).

The Serre duality theorem is generalized as follows (see Proposition 3.7 of [29]).

Proposition 2.4. *For parabolic sheaves L_* and M_*, there is a natural isomorphism*

$$\theta^i : H^i(L_*^\vee \otimes M_* \otimes K(D)) \xrightarrow{\sim} H^{1-i}(M_*^\vee \otimes \widehat{L}_*)^\vee.$$

Given E_* and $\beta \in \mathbb{R}^n$, define $E_*[\beta]_*$, the parabolic sheaf E_* shifted by β, by

$$E_*[\beta]_x = \bigcap_i E_{x+\beta_i}^{p_i}.$$

Example. The Picard group of parabolic line bundles.
A holomorphic bundle E is regarded as a parabolic bundle with the trivial parabolic structure $E_p \supset 0, \alpha_1(p) = 0$ at each $p \in D$. We call this the special structure on E. Note that every parabolic line bundle L_* is gotten by shifting the special structure on the underlying bundle L, i.e., there is a unique $\beta \in [0,1)^n$ with $L_* = L[\beta]_*$ Viewing $\mathcal{O}_X$ as a parabolic bundle with the special structure, then it is not difficult to verify that

$$(3) \qquad\qquad E_*[\beta]_* = E_* \otimes \mathcal{O}_X[\beta]_*$$

Let e_i denote the standard basis vector in $\mathbb{R}^n$. From (3) we have

$$\begin{aligned}
E_*^1[\beta^1]_* \otimes E_*^2[\beta^2]_* &= E_*^1 \otimes E_*^2[\beta^1 + \beta^2]_*, \\
E_*[\beta]_*^\vee &= E_*^\vee[-\beta]_*, \\
E_*[e_i]_* &= E_* \otimes \mathcal{O}_X(-p_i).
\end{aligned}$$

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These three formulas determine the Picard group of parabolic line bundles on X.

Remark. For any parabolic line bundle L_*, the stability (or semistability) of $E_* \otimes L_*$ is equivalent to that of E_*. Similarly, the stability (or semistability) of $(E_* \otimes L_*, \Phi \otimes 1)$ is equivalent to that of (E_*, Φ).

In particular, apply this to the case of a rank two parabolic bundle E_* with full flags at each p_i and weights $0 \le \alpha_1(p_i) < \alpha_2(p_i) < 1$. Using equation (3) with $\beta_i = \frac{1}{2}(\alpha_1(p_i) + \alpha_2(p_i) - 1)$ notice that $E_*[\beta]_*$ has weights $0 < a_1(p_i) < 1 - a_1(p_i) < 1$ at p_i, where $a_1(p_i) = \frac{1}{2}(\alpha_1(p_i) - \alpha_2(p_i) + 1)$.

3. An Algebraic Description of the Moduli Spaces in Rank Two

3.1. Criteria for the existence of stabilizing fields. In this section, we suppose that E_* is a parabolic bundle of rank two with the weights $\alpha_i \le 1 - \alpha_i$ at p_i and that $n \ge 1$. Consider the following existence questions:

 (I) Does there exist $\Phi : E_* \to E_* \otimes K(D)$ with (E_*, Φ) stable?
 (II) Does there exist $\Phi : E_* \to \widehat{E}_* \otimes K(D)$ with (E_*, Φ) stable?

Such Φ are called stabilizing fields. Of course, if E_* is itself stable, then any Φ (e.g., $\Phi = 0$) gives us an affirmative answer. The other possibilities are if E_* is unstable (meaning not semistable) or if E_* is strictly semistable. In either case, by choosing L_* a line subbundle of maximal parabolic degree, we get a short exact sequence

$$(4) \qquad 0 \longrightarrow L_* \stackrel{i}{\longrightarrow} E_* \stackrel{p}{\longrightarrow} M_* \longrightarrow 0$$

with $\mu(L_*) \ge \mu(E_*)$. Let $\xi \in H^1(M_*^\vee \otimes L_*)$ be the extension class representing (4). If E_* is unstable, then $\mu(L_*) > \mu(E_*)$ and (4) is the Harder-Narasimhan filtration of E_* and is canonical. If E_* is strictly semistable, then $\mu(L_*) = \mu(E_*)$ and (4) is the Jordan-Hölder filtration of E_* and is not, in general, canonical. For example, if E_* is strictly semistable, then the subbundle L_* is canonically determined if and only if the extension ξ is nontrivial.

In the following proposition, the assumption $g \ge 2$ is not essential and after the proof, we treat the case $g \le 1$.

Proposition 3.1. *If $g \ge 2$ and E_* is not stable, then*
 (i) *(E_*, Φ) is a stable parabolic $K(D)$ pair for some $\Phi \Leftrightarrow h^1(M_*^\vee \otimes \widehat{L}_*) \ge 1$;*
 (ii) *(E_*, Φ) is a stable parabolic Higgs bundle for some $\Phi \Leftrightarrow h^1(M_*^\vee \otimes L_*) > 1$ or $h^1(M_*^\vee \otimes L_*) = 1$ and $\xi = 0$.*

Proof. Notice first of all that if such a Φ exists, then we can assume it is trace-free. Now consider the short exact sequences of the sheaves of parabolic and strongly parabolic bundle endomorphisms

$$(5) \quad 0 \to E_*^\vee \otimes L_* \otimes K(D) \stackrel{\iota}{\longrightarrow} E_*^\vee \otimes_0 E_* \otimes K(D) \stackrel{\pi}{\longrightarrow} L_*^\vee \otimes M_* \otimes K(D) \to 0,$$

$$(6) \quad 0 \to E_*^\vee \otimes \widehat{L}_* \otimes K(D) \stackrel{\hat{\iota}}{\longrightarrow} E_*^\vee \otimes_0 \widehat{E}_* \otimes K(D) \stackrel{\hat{\pi}}{\longrightarrow} L_*^\vee \otimes \widehat{M}_* \otimes K(D) \to 0,$$

where $\pi, \hat{\pi}$ are the natural surjections, $\iota, \hat{\iota}$ are the natural isomorphisms to the kernels of $\pi, \hat{\pi}$ and

$$E_*^{\vee} \otimes_0 E_* = \mathfrak{ParEnd}_0(E_*)$$

denotes the sheaf of trace-free endomorphisms of E_*. Notice that $H^0(E_*^{\vee} \otimes L_* \otimes K(D))$ and $H^0(E_*^{\vee} \otimes \hat{L}_* \otimes K(D))$ are the relevant subspaces of fields Φ for which L_* is a Φ-invariant subbundle. If (E_*, Φ) is stable, then L_* is not Φ-invariant, and $\pi_*(\Phi) \neq 0$ (similarly for $\hat{\pi}_*(\Phi)$). This proves one implication of the following claim.

Claim 3.2. *Suppose that either E_* is unstable or $\xi \neq 0$, then*
  (i) *for $\Phi \in H^0(E_*^{\vee} \otimes_0 E_* \otimes K(D)), (E_*, \Phi)$ is stable $\Leftrightarrow 0 \neq \pi_*(\Phi)$;*
  (ii) *for $\Phi \in H^0(E_*^{\vee} \otimes_0 \hat{E}_* \otimes K(D)), (E_*, \Phi)$ is stable $\Leftrightarrow 0 \neq \hat{\pi}_*(\Phi)$.*

Proof. To prove ($\Leftarrow$), we just show that L_* is the unique parabolic subbundle of E_* with $\mu(L_*) \geq \mu(E_*)$. Suppose L_*' is another such subbundle. If E_* is unstable, then $\mu(E_*) > \mu(M_*)$ and the projection $L_*' \to M_*$ is the zero map, which shows $L_*' = L_*$. On the other hand, if $L_*' \to M_*$ is not the zero map, then it is an isomorphism and defines a splitting of (4), hence $\xi = 0$. $\square$

Now consider the coboundary maps in the cohomology sequences of (5) and (6)

$$H^0(L_*^{\vee} \otimes M_* \otimes K(D)) \xrightarrow{\delta} H^1(E_*^{\vee} \otimes L_* \otimes K(D)),$$
$$H^0(L_*^{\vee} \otimes \widehat{M}_* \otimes K(D)) \xrightarrow{\hat{\delta}} H^1(E_*^{\vee} \otimes \hat{L}_* \otimes K(D)).$$

Here δ is the zero map since by Serre duality

$$h^1(E_*^{\vee} \otimes L_* \otimes K(D)) = h^0(L_*^{\vee} \otimes \hat{E}_*) = h^0(L_*^{\vee} \otimes \hat{L}_*) + h^0(L_*^{\vee} \otimes \widehat{M}_*) = 0.$$

A diagram chase shows that the dual map of $\hat{\delta}$, $\hat{\delta}^{\vee}: H^0(L_*^{\vee} \otimes E_*) \to H^1(M_*^{\vee} \otimes L_*)$, maps i to ξ. Hence, $\hat{\delta}$ is the zero map if and only if $\xi = 0$. If $\xi \neq 0$, then its image is one dimensional because

$$h^1(E_*^{\vee} \otimes \hat{L}_* \otimes K(D)) = h^0(L_*^{\vee} \otimes E_*) = \begin{cases} 1 & \text{if } L_* \neq M_* \text{ or } \xi \neq 0, \\ 2 & \text{if } L_* = M_* \text{ and } \xi = 0. \end{cases}$$

In the cases covered by the claim, the proposition follows by another application of Serre duality

$$h^0(L_*^{\vee} \otimes M_* \otimes K(D)) = h^1(M_*^{\vee} \otimes \hat{L}_*),$$
$$h^0(L_*^{\vee} \otimes \widehat{M}_* \otimes K(D)) = h^1(M_*^{\vee} \otimes L_*).$$

The remaining cases follow by replacing the claim by the lemma below, which we note is the only step of the argument where we use the assumption $g \geq 2$. $\square$

Lemma 3.3. *If $g \geq 2$ and E_* is not stable, then*
  (i) *(E_*, Φ) is a stable parabolic $K(D)$ pair for some $\Phi \Leftrightarrow \ker \delta \neq 0$;*
  (ii) *(E_*, Φ) is a stable parabolic Higgs bundle for some $\Phi \Leftrightarrow \ker \hat{\delta} \neq 0$.*

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Proof. Since the lemma is a consequence of the claim, when it applies, we can assume that E_* is strictly semistable and $\xi = 0$. Furthermore, we only need to show ($\Leftarrow$).

We introduce some notation. Define the intersection numbers e_i and $\hat{e}_i$ by

$$e_i = \begin{cases} \dim L_{p_i} \cap F_2(p_i) & \text{if } F_2(p_i) \neq 0, \\ 1 & \text{if } F_2(p_i) = 0, \end{cases}$$

$$\hat{e}_i = \dim L_{p_i} \cap F_2(p_i).$$

If $\beta_i = \hat{e}_i + (-1)^{\hat{e}_i} \alpha_i$ and $\gamma_i = 1 - \beta_i$ are the weights of L_{p_i} and M_{p_i}, respectively, then

$$\hat{e}_i = \begin{cases} 0 & \text{if } \beta_i \leq \gamma_i, \\ 1 & \text{if } \beta_i > \gamma_i, \end{cases} \quad \text{and} \quad e_i = \begin{cases} 0 & \text{if } \beta_i < \gamma_i, \\ 1 & \text{if } \beta_i \geq \gamma_i. \end{cases}$$

Set $|e| = \sum e_i$ and $|\hat{e}| = \sum \hat{e}_i$ and notice that $e_i > \beta_i - \gamma_i$ and $\hat{e}_i \geq \beta_i - \gamma_i$, with equality only when $\hat{e}_i = 0$ and $\beta_i = \gamma_i$.

If $\ker \delta \neq 0$ or $\ker \hat{\delta} \neq 0$, then for generic Φ, L_* is not Φ-invariant. Suppose L'_* ($\neq L_*$) is a line subbundle with $\mu(L'_*) \geq \mu(E_*)$. Semistability of E_* implies $\mu(L'_*) = \mu(E_*)$. Then the restriction of p to L'_*, written $p_{L'} : L'_* \longrightarrow M_*$, is an isomorphism since otherwise, $p_{L'} = 0$ and $L'_* = L_*$. Such subbundles are identified with sections of p and are parameterized by $H^0(M_*^\vee \otimes L_*)$. The relevant subspaces of Φ leaving L'_* invariant are $H^0(E_*^\vee \otimes M_* \otimes K(D))$ and $H^0(E_*^\vee \otimes \widehat{M}_* \otimes K(D))$. Thus, (i) will follow once we prove the inequality

$$(7) \qquad h^0(M_*^\vee \otimes L_*) + h^0(E_*^\vee \otimes M_* \otimes K(D)) < h^0(E_*^\vee \otimes_0 E_* \otimes K(D)),$$

which is equivalent to $h^0(M_*^\vee \otimes L_*) < h^0(M_*^\vee \otimes L_* \otimes K(D))$. Likewise, (ii) will follow from

$$(8) \qquad h^0(M_*^\vee \otimes L_*) + h^0(E_*^\vee \otimes \widehat{M}_* \otimes K(D)) < h^0(E_*^\vee \otimes \widehat{E}_* \otimes K(D)),$$

which is equivalent to $h^0(M_*^\vee \otimes L_*) < h^0(M_*^\vee \otimes \widehat{L}_* \otimes K(D))$. Since $\mu(M_*^\vee \otimes L_*) = 0$,

$$h^0(M_*^\vee \otimes L_*) = \begin{cases} 0 & \text{if } M_* \neq L_*, \\ 1 & \text{if } M_* = L_*. \end{cases}$$

On the other hand, because $h^1(M_*^\vee \otimes L_* \otimes K(D)) = h^0(L_*^\vee \otimes \widehat{M}_*) = 0$, it follows that

$$\begin{aligned} h^0(M_*^\vee \otimes L_* \otimes K(D)) &= \deg(M^\vee \otimes L \otimes K(\textstyle\sum_{i=1}^n e_i p_i)) + \chi(X) \\ &= \deg L - \deg M + |e| + g - 1. \end{aligned}$$

Notice that $\deg L - \deg M + |e| > \mu(L_*) - \mu(M_*) = 0$, hence (7) holds provided

$$(9) \qquad \deg L - \deg M + |e| \geq 2 - g \text{ with equality } \Leftrightarrow L_* \neq M_*.$$

This proves part (i) of the lemma when $g \geq 2$. As for part (ii), notice that

$$h^1(M_*^\vee \otimes \widehat{L}_* \otimes K(D)) = h^0(L_*^\vee \otimes M_*) = \begin{cases} 0 & \text{if } M_* \neq L_*, \\ 1 & \text{if } M_* = L_*, \end{cases}$$

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and so (8) follows as long as $\chi(M_*^\vee \otimes \widehat{L}_* \otimes K(D)) > 0$. We have

$$\begin{aligned}
\chi(M_*^\vee \otimes \widehat{L}_* \otimes K(D)) &= \deg(M^\vee \otimes L \otimes K(\textstyle\sum_{i=1}^n \hat{e}_i p_i)) + \chi(X) \\
&= \deg L - \deg M + |\hat{e}| + g - 1.
\end{aligned}$$

Hence (8) holds provided

(10) $$\deg L - \deg M + |\hat{e}| \geq 2 - g.$$

But $\deg L - \deg M + |\hat{e}| \geq \mu(L_*) - \mu(M_*) = 0$ (with equality implying that $\beta_i = \gamma_i$ for all i). This proves part (ii) of the lemma when $g \geq 2$. $\square$

One can deduce the following corollary using Riemann-Roch.

Corollary 3.4. *If $g \geq 3$, then for every semistable E_*, there exists a Higgs field Φ making (E_*, Φ) a stable parabolic Higgs bundle.*

We now explain how to extend these results to lower genus. Clearly, the proposition holds for $g \leq 1$ whenever E_* is unstable or $\xi \neq 0$ by virtue of the claim. So assume that E_* is semistable and $\xi = 0$. The only place where we make essential use of the assumption $g \geq 2$ is in the proof of Lemma 3.3. In particular, we observe from (9) and (10) that the inequalities (7) and (8) fail (respectively) if

(i) $0 < \deg L - \deg M + |e| \leq 2 - g$ with equality $\Leftrightarrow L_* = M_*$,
(ii) $0 \leq \deg L - \deg M + |\hat{e}| \leq 1 - g$.

Thus, the only counterexamples to Lemma 3.3 for $g \leq 1$ are given by the semistable, split bundles E_* satisfying (i) and (ii) along with the additional requirements (i') $\ker \delta \neq 0$ and (ii') $\ker \hat{\delta} \neq 0$. First, we list these counterexamples to Lemma 3.3, then we show that the bundles satisfying (i) and (ii) never give rise to any stable parabolic $K(D)$ pairs or stable parabolic Higgs bundles, respectively.

If E_* is semistable and split and satisfies (i) and (i'), i.e., if $h^1(M_*^\vee \otimes \widehat{L}_*) \geq 1$, then there are but two possibilities:

(i–a) $(g, n) \in \{(0, 2), (1, 1)\}$, $E_* = L_* \oplus M_*$ and $L_* = M_*$,
(i–b) $g = 0$, $E_* = L_* \oplus M_*$, $\mu(L_*) = \mu(M_*)$, $\deg L - \deg M + |e| = 1$.

Now if E_* is semistable and split and satisfies (ii) and (ii''), i.e., if $h^1(M_*^\vee \otimes L_*) \geq 1$, then again, we have only two possibilities:

(ii–a) $g = 0$, $E_* = L_* \oplus M_*$, $\mu(L_*) = \mu(M_*)$, and $0 \leq \deg L - \deg M + |\hat{e}| \leq 1$,
(ii–b) $g = 1$, $E_* = L_* \oplus M_*$, $L_* = M_*$.

We now show that if E_* satisfies (i), then (E_*, Φ) is not stable for any $\Phi \in H^0(E_*^\vee \otimes_0 E_* \otimes K(D))$ and if E_* satisfies (ii), then (E_*, Φ) is not stable for any $\Phi \in H^0(E_*^\vee \otimes_0 \widehat{E}_* \otimes K(D))$. For example, suppose that $\deg L - \deg M + |e| = 2 - g$ in (i), so that $L_* = M_*$. Then either $g = 0$ and $n = 2$ or $g = 1 = n$. In either case,

$$H^0(E_*^\vee \otimes_0 E_* \otimes K(D)) = H^0(K(D))^{\oplus 3} = H^0(\mathcal{O}_X)^{\oplus 3}.$$

Thus, any Φ is a constant matrix, one of whose eigenspaces determines a Φ-invariant subbundle violating the condition for stability. Otherwise, if $\deg L - \deg M + |e| = 1 - g$ in (i), then $h^0(M^\vee \otimes L_* \otimes K(D)) = 0$ so that M_* is Φ-invariant for all Φ.

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As for (ii), suppose first of all that $g = 0$ and $\deg L - \deg M + |\hat{e}| \leq 1$. Then $h^0(M_*^\vee \otimes \hat{L}_* \otimes K(D)) = 0$ and M_* is Φ-invariant for all Φ. Now if $g = 1$ and $\deg L - \deg M + |\hat{e}| = 0$, then either $L_* \neq M_*$ and M_* is Φ-invariant for all Φ or $L_* = M_*$ and $H^0(E_*^\vee \otimes_0 \hat{E}_* \otimes K(D)) = H^0(\mathcal{O}_X)^{\oplus 3}$, in which case every Φ is a constant matrix. This proves the following proposition.

Proposition 3.5. *If E_* is not stable and $g \leq 1$, then*

(i) (E_*, Φ) *is a stable parabolic $K(D)$ pair for some $\Phi \Leftrightarrow E_*$ is not one of the bundles occurring in (i–a) or (i–b) and $h^1(M_*^\vee \otimes \hat{L}_*) \geq 1$;*

(ii) (E_*, Φ) *is a stable parabolic Higgs bundle for some $\Phi \Leftrightarrow E_*$ is not one of the bundles occurring in (ii–a) or (ii–b) and either $h^1(M_*^\vee \otimes L_*) > 1$ or $h^1(M_*^\vee \otimes L_*) = 1$ and $\xi = 0$.*

We could ask questions (I) and (II) replacing stability with semistability. Of course, if E_* itself is semistable, then so is (E_*, Φ) for any Φ. So we can assume that E_* is unstable and apply the claim to determine precisely which Φ make (E_*, Φ) stable. One last comment is that if (E_*, Φ) is strictly semistable, then E_* must also be strictly semistable. The converse, however, is false.

3.2. Example: Rank 2 parabolic bundles over $\mathbb{P}^1$ with 3 parabolic points.

In this section, we describe the moduli spaces $\mathcal{M}_\alpha, \mathcal{N}_\alpha$, and $\mathcal{P}_\alpha$ of rank two bundles over $X = \mathbb{P}^1$ with parabolic points in the reduced divisor $D = p_1 + p_2 + p_3$. We suppose that $\mu(E_*) = 0$ and that the weights at p_i are α_i and $1 - \alpha_i$ for some $\alpha \in W = \{(\alpha_1, \alpha_2, \alpha_3) \mid 0 < \alpha_i < \frac{1}{2}\}$. Note that this is equivalent to saying that $\det E_* = \mathcal{O}_X$ (as parabolic bundles) and E_* has full flags at each p_i. For $e = (e_1, e_2, e_3)$, where $e_i \in \{0, 1\}$, we use $\beta(\alpha, e)$ (or simply β) to denote the weights $\beta_i = e_i + (-1)^{e_i} \alpha_i$. Let

$$I = \{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}.$$

Inside W there are four hyperplanes

$$H_e = \{\alpha \mid \beta(\alpha, e) = 1 + \frac{|e|}{2}\}$$

for $e \in I$ whose complement $W \setminus \bigcup_{e \in I} H_e$ consists of five chambers: $C_e = \{\alpha \mid \beta(\alpha, e) > 1 + \frac{|e|}{2}\}$ for $e \in I$ and $C_0 = \{\alpha \mid \beta(\alpha, e) < 1 + \frac{|e|}{2}$ for all $e \in I\}$.

The following is an immediate consequence of the criteria established in the previous section.

Lemma 3.6. *If (E_*, Φ) is a semistable $K(D)$ pair, then the bundle E_* is described as an extension*

$$(11) \qquad\qquad 0 \longrightarrow L_* \longrightarrow E_* \longrightarrow L_*^\vee \longrightarrow 0$$

where L_* satisfies $h^1(L_*^{\otimes 2}) = 1$.

Proof. If E_* is not stable, then by Proposition 3.5, we see that $h^1(L_*^{\otimes 2}) \geq 1$. Since $\mu(L_*) \geq 0$, we see that $\mu(L_*^{\otimes 2}) \geq 0$, and because there are only three weights, this implies $\deg(L_*^{\otimes 2})_0 \geq -2$. Thus $h^1(L_*^{\otimes 2}) = 1$.

If E_* is stable, then by Grothendieck's Theorem, $E = \mathcal{O}(-1) \oplus \mathcal{O}(-2)$. Let L_* be $\mathcal{O}(-1)$ with weights inherited as a subbundle of E_*. Notice that $h^1(L_*^{\otimes 2}) \leq 1$. But by stability of E_*, the extension (11) must be nontrivial, so $h^1(L_*^{\otimes 2}) \geq 1$. $\square$

We now determine all possible line subbundles L_* with $h^1(L_*^{\otimes 2}) = 1$. For fixed $\alpha \in W$, there are four possible line subbundles L_* with $h^1(L_*^{\otimes 2}) = 1$, namely

$$L_*^e = \mathcal{O}_X\left(-1 - \frac{|e|}{2}\right)[-\beta(\alpha, e)]$$

for $e \in I$. We denote by G_*^e the nontrivial extension gotten from (11) with $L_* = L_*^e$. Notice that G_*^e is unique up to isomorphism because $h^1(L_*^{e\otimes 2}) = 1$. Let $F_*^e = L_*^e \oplus L_*^{e\vee}$.

It is not hard to see that G_*^e and $G_*^{e'}$ are isomorphic for $e, e' \in I$. Set $G_* = G_*^e$. This, together with the previous lemma, shows that if (E_*, Φ) is semistable, then E_* is one of the five bundles in the set $\{G_*, F_*^e\}$.

Recall that two bundles E_* and E_*' are called S-equivalent (written $E_* \sim_S E_*'$) if their associated graded bundles are isomorphic, i.e., if $\mathrm{gr}\, E_* \simeq \mathrm{gr}\, E_*'$. We use E_* to denote the isomorphism class of a bundle and $[E_*]$ for its S-equivalence class.

Proposition 3.7. (1) *If $\alpha \in C_0$, then $\mathcal{M}_\alpha = \{G_*\}$.*
(2) *If $\alpha \in C_e, e \in I$, then $\mathcal{M}_\alpha = \emptyset$.*
(3) *If $\alpha \in H_e$, then $\mathcal{M}_\alpha = \{[F_*^e]\}$ and $G_* \sim_S F_*^e$ are the two distinct isomorphism classes of semistable bundles.*

Proof. From the above considerations, if E_* is semistable, then $E_* = G_*$ or F_*^e. But G_* is stable if and only if $\alpha \in C_0$, and F_*^e is never stable. On the other hand, if $\alpha \in H_e$, then G_* and F_*^e are clearly strictly semistable with associated graded bundle $L_*^e \oplus (L_*^e)^\vee$. $\square$

The next lemma shows which auxiliary fields can arise for these five bundles.

Lemma 3.8. *For any $\alpha \in W$, we have*
(i) *G_* is simple, $h^0(G_*^\vee \otimes G_* \otimes K(D)) = 5$, and $h^0(G_*^\vee \otimes \widehat{G}_* \otimes K(D)) = 0$,*
(ii) *$\mathrm{Aut}\, F_*^e = \mathbb{C}^* \times \mathbb{C}^*$, $h^0(F_*^{e\vee} \otimes F_*^e \otimes K(D)) = 5$, and $h^0(F_*^{e\vee} \otimes \widehat{F}_*^e \otimes K(D)) = 1$.*

Proof. For $\alpha \in C_0$, G_* is stable, and therefore simple. But this property is independent of the weights, and it follows that for any $\alpha \in W$,

$$1 = h^0(G_*^\vee \otimes G_*) = h^1(G_*^\vee \otimes \widehat{G}_* \otimes K(D)),$$
$$0 = h^0(G_*^\vee \otimes \widehat{G}_*) = h^1(G_*^\vee \otimes G_* \otimes K(D)).$$

Direct computation shows $\deg(G_*^\vee \otimes G_*)_0 = -3$ and $\deg(G_*^\vee \otimes \widehat{G}_*)_0 = -9$, and part (i) follows using $K(D) = \mathcal{O}(1)$ and Riemann-Roch.

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As for (ii), since $(L_*^{e\otimes 2})_0 = \mathcal{O}(-2)$ and $(L_*^{e\otimes -2})_0 = \mathcal{O}(-1)$, every automorphism of F_*^e is diagonal and $\mathrm{Aut}\, F_*^e = \mathbb{C}^* \times \mathbb{C}^*$. Also, $h^0(L_*^{e\otimes 2} \otimes K(D)) = 0$, so every $\Phi \in H^0(F_*^{e\vee} \otimes F_*^e \otimes K(D))$ has the form

$$\Phi = \begin{pmatrix} a_1 & 0 \\ \phi & a_2 \end{pmatrix}$$

with $\phi \in H^0(L_*^{\otimes -2} \otimes K(D)) = \mathbb{C}$ and $a_i \in H^0(K(D)) = \mathbb{C}^2$. Morevoer, $H^0(F_*^{e\vee} \otimes \widehat{F}_*^e \otimes K(D)) = H^0(L_*^{\otimes -2} \otimes K(D))$, which completes the proof of part (ii) $\square$

We can identify the action of $\mathrm{Aut}(F_*^e)$ on $H^0(F_*^{e\vee} \otimes F_*^e \otimes K(D))$, it is given by conjugation

$$(z_1, z_2) \cdot \Phi = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ \phi & a_2 \end{pmatrix} \begin{pmatrix} z_1^{-1} & 0 \\ 0 & z_2^{-1} \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ z_1^{-1} z_2 \phi & a_2 \end{pmatrix}.$$

Suppose that $\alpha \in C_e$ and set $V = \mathrm{Ext}^1(L_*^{e\vee}, L_*^e) = \mathbb{C}$. Let $\mathcal{E}_*$ be the universal parabolic bundle on $V \times X$ which, when restricted to $\{\xi\} \times X$, is the bundle G_*^ξ in (11) with $L_* = L_*^e$ and extension class ξ. For $\xi \neq 0$, $G_*^\xi \simeq G_*$ and obviously $G_*^0 = F_*^e$.

Let p_X and p_V denote the two projection maps from $V \times X$ and define $\mathcal{L}_*$ to be the pullback bundle $p_X^* L_*^e$. Consider the direct image sheaves of $\mathcal{E}_*^\vee \otimes \mathcal{E}_* \otimes K(D)$ and $\mathcal{L}_*^{\otimes -2} \otimes K(D)$ under p_V, which, by the previous lemma, are locally free sheaves over V whose associated vector bundles, M and N, are trivial with ranks 5 and 1, respectively. Notice that N is canonically isomorphic to $V \times H^0(L_*^{e\otimes -2} \otimes K(D))$. This is key to following construction.

The canonical map $\tilde{\pi} : \mathcal{E}_*^\vee \otimes \mathcal{E}_* \otimes K(D) \to \mathcal{L}_*^{\otimes -2} \otimes K(D)$ of the previous section induces $\tilde{\pi}_* : M \to N$ which is surjective, because the restriction of $\tilde{\pi}_*$ to a fiber above ξ can be identified with $\pi_*^\xi : H^0(G_*^{\xi\vee} \otimes G_*^\xi \otimes K(D)) \to H^0(L_*^{e\otimes -2} \otimes K(D))$, whose cokernel is $H^1(G_*^{\xi\vee} \otimes L_*^e \otimes K(D)) = 0$. Fix some $0 \neq \phi_0 \in H^0(L_*^{e\otimes -2} \otimes K(D))$ and set $Y = \tilde{\pi}_*^{-1}(\{\phi_0\} \times V) \simeq \mathbb{C}^4 \times V$.

Proposition 3.9.
(1) If $\alpha \in C_0$, then $\mathcal{P}_\alpha \simeq H^0(G_*^\vee \otimes G_* \otimes K(D)) = \mathbb{C}^5$ and $\mathcal{P}_\alpha^0 \simeq \mathbb{C}^3$.

(2) If $\alpha \in C_e$, then $\mathcal{P}_\alpha \simeq Y \simeq \mathbb{C}^5$ and $\mathcal{P}_\alpha^0 = \mathbb{C}^3$.

(3) If $\alpha \in H_e$, then $\mathcal{P}_\alpha \simeq H^0(G_*^\vee \otimes G_* \otimes K(D)) = \mathbb{C}^5$ and its strictly semistable part can be identified with a hyperplane.

Remark. In the course of the proof, we will determine the isomorphism classes of semistable parabolic $K(D)$ pairs. This differs from the above only for strictly semistable bundles, because the S-equivalence class of a stable bundle is precisely its isomorphism class. For $\alpha \in H_e$, we will find that there are three distinct components of isomorphism classes of strictly semistable bundles, each is just a copy of $\mathbb{C}^4$.

Proof. Part (1) follows from the fact that $h^0(L_*^{e\otimes 2} \otimes K(D)) = 0$, hence $L_*^{e\vee}$ is a Φ-invariant subbundle of F_*^e for any Φ. Thus, if (E_*, Φ) is stable and $\alpha \in C_0$, then $E_* \simeq G_*$. For part (2), if $(\xi, \Phi) \in Y$, then the associated $K(D)$ pair (G_*^ξ, Φ) is stable by Claim 3.2 since $\pi_*^\xi(\Phi) = \phi_0 \neq 0$. This gives a map $\eta : Y \to \mathcal{P}_\alpha$, which we claim is a bijection. To see this, write $Y = Y' \cup Y''$, where $Y' = Y|_{V \setminus 0}$ and $Y'' = Y|_0$, and

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$\mathcal{P}_\alpha = \mathcal{P}'_\alpha \cup \mathcal{P}''_\alpha$, where $\mathcal{P}'_\alpha$ and $\mathcal{P}''_\alpha$ consist of the $K(D)$ pairs (E_*, Φ) with underlying bundle E_* isomorphic to G_* and F^e_*, respectively. The restriction of M to $V \setminus \{0\}$ is naturally isomorphic to $(V \setminus \{0\}) \times H^0(G^\vee_* \otimes G_* \otimes K(D))$. For $(\xi, \Phi) \in M|_{V \setminus \{0\}}$ and $t \in \mathbb{C}^*$,

$$\pi^{t\xi}_*(\Phi) = t^{-1}\pi^\xi_*(\Phi).$$

It follows from this formula that η induces a bijection between Y' and $\mathcal{P}'_\alpha$

Using the description of the action of $\mathrm{Aut}\, F^e_*$ on $H^0(F^{e\vee}_* \otimes F^e_* \otimes K(D))$ following the proof of the previous lemma, every $(F^e_*, \Phi) \in \mathcal{P}''_\alpha$ is isomorphic to (F^e_*, Φ_0) where

$$\Phi_0 = \begin{pmatrix} a_1 & 0 \\ \phi_0 & a_2 \end{pmatrix}.$$

Hence, η gives a bijection between Y'' and $\mathcal{P}''_\alpha$.

To prove (3), notice that we have a map $\eta : H^0(G^\vee_* \otimes G_* \otimes K(D)) \to \mathcal{P}_\alpha$. Since $\mathcal{P}_\alpha$ is normal, it is enough to show that this map is bijective. Now by Claim 3.2, we see that $\mathcal{P}^s_\alpha \simeq H^0(G^\vee_* \otimes G_* \otimes K(D)) \setminus \mathrm{Ker}\, \pi_*$. The strictly semistable bundles are pairs of the form (F^e_*, Φ) for any Φ, and (G_*, Φ) with $\Phi \in \mathrm{Ker}\, \pi_*$. If $\Phi \in \mathrm{Ker}\, \pi_*$, the subbundle L^e_* is Φ-invariant and we get the extension of parabolic $K(D)$ pairs

$$(12) \qquad 0 \longrightarrow (L^e_*, \phi) \longrightarrow (G_*, \Phi) \longrightarrow (L^{e\vee}_*, \psi) \longrightarrow 0.$$

Thus $\mathrm{gr}(G_*, \Phi) = (L^e_*, \phi) \oplus (L^{e\vee}_*, \psi)$ for $\Phi \in \mathrm{Ker}\, \pi_*$. Consider now the map

$$\lambda : \mathrm{Ker}\, \pi_* \longrightarrow H^0(L^{e\vee}_* \otimes L^e_* \otimes K(D)) \oplus H^0(L^e_* \otimes L^{e\vee}_* \otimes K(D))$$

defined by $\Phi \mapsto (\phi, \psi)$. For $\phi = \psi = 0$, then the extension (12) induces the zero map $L^{e\vee}_* \to L^e_* \otimes K(D)$ (because $H^0(L^{e\otimes 2}_* \otimes K(D)) = 0$) and it follows that $\Phi = 0$. So λ is injective. But the domain and range of λ are both 4-dimensional, and so λ is an isomorphism. Clearly $\mathrm{gr}(F^e_*, \Phi) = (L^e_*, \phi) \oplus (L^{e\vee}_*, \psi)$, and it follows that λ gives a bijection between $\mathrm{Ker}\, \pi_*$ and $\mathcal{P}^{sss}_\alpha$. $\square$

Choosing some $0 \neq \Phi_0 \in H^0(F^{e\vee}_* \otimes \widehat{F^e_*} \otimes K(D)) = \mathbb{C}$ and using the action of $\mathrm{Aut}(F^e_*)$, it is easy to verify that (F^e_*, Φ) is isomorphic to (F^e_*, Φ_0) for all $\Phi \neq 0$. The proof of the last proposition is left as an entertaining exercise in applying the above lemmas.

Proposition 3.10. (1) *If $\alpha \in C_0$, then $\mathcal{N}_\alpha = \{(G_*, 0)\}$.*

(2) *If $\alpha \in C_e, e \in I$, then $\mathcal{N}_\alpha = \{(F^e_*, \Phi_0)\}$.*

(3) *If $\alpha \in H_e$, then $\mathcal{N}_\alpha = \{[F^e_*, 0]\}$ and $(G_*, 0) \sim_S (F^e_*, 0) \sim_S (F^e_*, \Phi_0)$ are the three distinct isomorphism classes of semistable Higgs bundles.*

4. A TOPOLOGICAL DESCRIPTION OF $\mathcal{N}^0_\alpha$ IN RANK TWO

4.1. The function spaces of Biquard and construction of Konno. We begin with a brief overview of the gauge theoretical description of $\mathcal{N}_\alpha$ following [14].

It is convenient to think of the parabolic bundle separate from its holomorphic structure, so we use E_* to denote the underlying topological parabolic bundle (weights α) and $\bar{\partial}_E$ its holomorphic structure. By tensoring with an appropriate line bundle, we can always assume that $\mu(E_*) = 0$. We shall also restrict our attention to

generic weights, i.e., weights α for which α-stability and α-semistability coincide. Let $\mathcal{C}$ denote the affine space of all holomorphic structures on E, and $\mathcal{G}_{\mathbb{C}}$ the group of smooth bundle automorphisms of E preserving the flag structure. Introduce a metric κ adapted to E (κ is unitary and smooth on $E|_{X \setminus D}$, but singular at $p \in D$ in a prescribed way, see Definition 2.3 [3]), and let $\mathcal{A}$ denote the affine space of κ-unitary connections. Define $\mathcal{G}$ to be the subgroup of $\mathcal{G}_{\mathbb{C}}$ consisting of κ-unitary gauge transformations. Letting $\mathcal{C}_{ss}$ and $\mathcal{A}_{flat}$ be the subspaces of α-semistable holomorphic structures and the flat connections, respectively, Biquard proved that

$$\mathcal{M}_\alpha \overset{\text{def}}{=} \mathcal{C}_{ss}/\mathcal{G}_{\mathbb{C}} \cong \mathcal{A}_{flat}/\mathcal{G}$$

by introducing the norms $\| \ \|_{D_k^p}$, defining the weighted Sobolev spaces $\mathcal{C}^p$ and $\mathcal{A}^p$ of D_1^p holomorphic structures and D_1^p κ-unitary connections, and taking quotients by the groups $\mathcal{G}_{\mathbb{C}}^p$ and $\mathcal{G}^p$ of D_2^p gauge transformations for a certain $p > 1$ [3].

The same approach works for parabolic Higgs moduli, at least for generic weights, as was shown by Konno. The arguments in [14] are given for moduli with fixed determinant, but remain equally valid without this condition. We set

$$\mathcal{H} \ = \ \{(\overline{\partial}_E, \Phi) \in \mathcal{C} \times \Omega^{1,0}(\text{End}\,E) \mid \overline{\partial}_E \Phi = 0 \text{ on } X \setminus D \text{ and at each } p \in D,$$
$$\Phi \text{ has a simple pole with nilpotent residue with respect to the flag}\}.$$

Note that $\mathcal{H}$ (this is denoted by $\mathcal{D}$ in [14]) is just the differential geometric definition of the space of parabolic Higgs bundle structures on E_*, for example, the nilpotency condition implies that Φ is strongly parabolic.

For $A \in \mathcal{A}$, we use d_A for its covariant derivative, F_A for its curvature, and d_A'' for the $(0,1)$ component of d_A, so $d_A'' \in \mathcal{C}$. Define $\mathcal{E} = \mathcal{A} \times \Omega^{0,1}(\text{End}\,E)$ and $\mathcal{E}^p$ as its completion with respect to the norms $\| \ \|_{D_1^p}$, and set

$$\mathcal{E}_{flat} = \{(d_A, \Phi) \in \mathcal{E}^p \mid d_A'' \Phi = 0, F_A + [\Phi, \Phi^*] = 0\}.$$

(This last space is denoted $\mathcal{D}_{HE}^p$ by Konno.) Using the usual definition of stability on $\mathcal{H}$, Theorem 1.6 of [14] shows that for some $p > 1$,

$$\mathcal{N}_\alpha \overset{\text{def}}{=} \mathcal{H}_{ss}/\mathcal{G}_{\mathbb{C}} \cong \mathcal{E}_{flat}/\mathcal{G}^p.$$

The advantage of the second quotient is that it endows $\mathcal{N}_\alpha$ with a natural hyperkähler structure, namely by viewing it as a hyperkähler quotient of $\mathcal{E}^p$ (in the sense of [13]), whose hyperkähler structure is given by the metric

$$g((\xi, \phi), (\xi, \phi)) = 2i \int_X \text{Tr}(\xi^* \xi + \phi \phi^*),$$

which is Kähler with respect to each of three complex structures

$$I(\xi, \phi) = (i\xi, i\phi), \quad J(\xi, \phi) = (i\phi^*, -i\xi^*), \quad K(\xi, \phi) = (-\phi^*, \xi^*).$$

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4.2. The Morse function for the moduli space of parabolic Higgs bundles.

Assume that E_* is a rank two parabolic bundle with generic weights α_i and $1 - \alpha_i$ at p_i and that $\mu_\alpha(E_*) = 0$. Write $\alpha = (\alpha_1, \ldots, \alpha_n)$. We will always assume $n \geq 1$. We consider the moduli with fixed determinant and trace-free Higgs fields, requiring the following minor modifications in the definitions of the previous section:

  (i) the induced connection d_Λ or holomorphic structure $\overline{\partial}_\Lambda$ on $\Lambda^2 E$ be fixed;
  (ii) the Higgs field be trace-free, i.e. $\Phi \in \Omega^{1,0}(\mathrm{End}_0 E)$.

We denote the corresponding spaces by $\mathcal{A}^0, \mathcal{C}^0, \mathcal{E}^0$, and $\mathcal{H}^0$.

As in [12], we consider the circle action defined on $\mathcal{E}^0$ by $e^{i\theta} \cdot (d_A, \Phi) = (d_A, e^{i\theta}\Phi)$. This action preserves the subspace $\mathcal{E}^0_{flat}$ and commutes with the action of the gauge group $\mathcal{G}^p$, thus it descends to give a circle action ρ on $\mathcal{N}^0_\alpha$. This action commutes with the complex structure defined by I and preserves the symplectic form $\omega_1(X, Y) = g(IX, Y)$, so the associated moment map $\mu_\rho(d_A, \Phi) = \frac{1}{4\pi}\|\Phi\|^2_{D^p_1}$, renormalized for convenience, is a Bott-Morse function and can be used to determine the Betti numbers of $\mathcal{N}^0_\alpha$.

We introduce some notation which will be used throughout the rest of this section. For any line subbundle L_* of E_*, let $e_i(L) = \dim L_{p_i} \cap F_2(p_i) \in \{0, 1\}$. The weight inherited by L_* is then $\beta_i(L) = e_i + (-1)^{e_i}\alpha_i$. We will often suppress the dependence on L and simply write $e = (e_1, \ldots, e_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$. We will also write $\beta(\alpha, e)$ when we want to emphasize the functional dependence of β on α and e. We also use $|e| = \sum_{i=1}^n e_i$.

Theorem 4.1.   (a) *The map $\mu_\rho : \mathcal{N}^0_\alpha \longrightarrow \mathbb{R}$ is a proper Morse function.*

  (b) *Whenever nonempty, $\mathcal{M}^0_\alpha$ is the unique critical submanifold corresponding to the minimum value $\mu_\rho = 0$. The other critical submanifolds are given by $\mathcal{M}_{d,e}$ for an integer d and $e \in \mathbb{Z}^n_2$ satisfying*

$$(13) \qquad -\sum_{i=1}^n \beta_i(\alpha, e) < d \leq g - 1 - |e|/2.$$

   Along $\mathcal{M}_{d,e}$, μ_ρ takes the value $d + \sum_{i=1}^n \beta_i$.

  (c) *The critical submanifold $\mathcal{M}_{d,e}$ is $\widetilde{S}^{h_{d,e}}X$, the 2^{2g} cover of the symmetric product $S^{h_{d,e}}X$ under the map $x \mapsto 2x$ on J_X. Here, $h_{d,e} = 2g - 2 - 2d - |e|$.*

  (d) *The Morse index of $\mathcal{M}_{d,e}$ is given by $\lambda_{d,e} = 2(n + 2d + g - 1 + |e|)$.*

Remark. If $g = 0$, there are always α with $\mathcal{M}^0_\alpha = \emptyset$ (but $\mathcal{N}^0_\alpha \neq \emptyset$). For these α, the minimum value is achieved along some $\mathcal{M}_{d,e}$, which we identify in the next section.

Proof. Properness of μ_ρ follows from the global compactness result for parabolic bundles of Biquard (Theorem 2.14 in [3]). This proves (a). All the other statements rely on the following correspondence between the circle action and the moment map given in [10].

  (1) Critical submanifolds are connected components of the fixed point set of ρ.
  (2) The Morse index of a critical submanifold equals the dimension of the negative weight space of the infinitesimal circle action on its normal bundle.

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Suppose that (d_A, Φ) is a fixed point of the circle action upstairs in $\mathcal{E}_{flat}$. Then $\Phi = 0$ and this shows that one component of the fixed point set in $\mathcal{N}_\alpha^0$ consists of $\mathcal{M}_\alpha^0$, the moduli of stable parabolic bundles with fixed determinant.

The other fixed points arise from when $e^{i\theta} \cdot (d_A, \Phi)$ is gauge equivalent to (d_A, Φ), i.e., when there is a one parameter family $g_\theta \in \mathcal{G}^p$ such that

$$g_\theta^{-1} \Phi g_\theta = e^{i\theta} \Phi,$$
$$g_\theta^{-1} d_A g_\theta = d_A.$$

By the first equation, g_θ is not central, and by the second, we see that d_A is reducible and consequently the holomorphic parabolic bundle splits according to the eigenvalues of g_θ. Write $E_* = L_* \oplus M_*$ as a direct sum of parabolic bundles. We assume (wlog) that $\mu_\alpha(L_*) > 0 > \mu_\alpha(M_*)$. Let $d = \deg L$ and $e = (e_1, \dots, e_n)$ where $e_i = \dim L_{p_i} \cap F_2(p_i)$. Then L inherits the weight $\beta_i = e_i + (-1)^{e_i} \alpha_i$ at p_i as a parabolic subbundle of E_* and

$$(14) \qquad 0 < \mu_\alpha(L_*) = d + \sum_{i=1}^n \beta_i.$$

Since g_θ is diagonal with respect to this decomposition, Φ is either upper or lower diagonal, which means either L or M is Φ-invariant. But α-stability of the pair (E_*, Φ) implies that

$$\Phi = \begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix},$$

where $0 \neq \phi \in \mathrm{ParHom}(L_*, \widehat{M}_* \otimes K(D))$. Thus

$$0 \neq H^0(L_*^\vee \otimes \widehat{M}_* \otimes K(D)) = H^0(L^\vee \otimes M \otimes K(\sum_{i=1}^n (1 - e_i) p_i)).$$

Let $|e| = \sum_{i=1}^n e_i$, then a necessary condition is that

$$(15) \qquad 0 \leq \deg(L^\vee \otimes M \otimes K(\sum_{i=1}^n (1 - e_i) p_i)) = 2(g - 1) - 2d - |e|.$$

Now (13) follows from (14) and (15).

We can use the defining equations for $\mathcal{E}_{flat}^0$ to determine the associated critical values. Take (E_*, Φ) as above, then

$$0 = F_A + [\Phi, \Phi^*] = \begin{pmatrix} F_L - \phi\phi^* & 0 \\ 0 & F_M + \phi^*\phi \end{pmatrix}.$$

Using the Chern-Weil formula for parabolic bundles (Proposition 2.9 of [3]), we get

$$\mu_\rho(d_A, \Phi) = \frac{1}{4\pi} \|\Phi\|^2 = \frac{i}{2\pi} \int_X \mathrm{Tr}(\Phi\Phi^*) = \frac{i}{2\pi} \int_X \phi\phi^* = \frac{i}{2\pi} \int_X F_L = \mathrm{pardeg}(L_*).$$

This completes the proof of (b).

Given $E_* = L_* \oplus M_*$ and Φ as above, then the zero set of ϕ is a nonnegative divisor of degree

$$h_{d,e} = \deg(L^\vee \otimes M \otimes K(\sum_{i=1}^n (1 - e_i) p_i)) = 2g - 2 - 2d - |e|$$

on X, which is just an element of $S^{h_{d,e}}X$. Conversely, given a nonnegative divisor of degree $h_{d,e}$, then we obtain a line bundle U of degree $2d + n$ along with a section of $U^\vee \otimes K(\sum_{i=1}^n (1 - e_i)p_i))$ vanishing on that divisor. There are 2^{2g} choices of L so that $U = L^{\otimes 2} \otimes \Lambda^2 E$, and each choice gives a stable parabolic Higgs bundle (E_*, Φ). The line subbundle L_* is canonically determined from E_*, but Φ is only determined up to multiplication by a nonzero constant. However, it is easy to see that (E_*, Φ) is gauge equivalent to $(E_*, \lambda\Phi)$ for $\lambda \neq 0$, and (c) now follows.

We now calculate the index $\lambda_{d,e}$ of the critical submanifold $\mathcal{M}_{d,e}$, which is given by the negative weight space of the infinitesimal action of ρ, or equivalently, of the gauge transformation g_θ. Letting $H^0(\mathrm{ParEnd}_0(E)) \cdot \Phi$ be the subspace of Higgs fields of the form $[\Psi, \Phi]$ for $\Psi \in H^0(\mathrm{ParEnd}_0(E))$, then the subspace

$$W = H^0(\mathrm{ParEnd}_0^\wedge(E) \otimes K(D))/H^0(\mathrm{ParEnd}_0(E)) \cdot \Phi$$

is Lagrangian with respect to the complex symplectic form

$$\omega((\xi_1, \phi_1), (\xi_2, \phi_2)) = \int_X \mathrm{Tr}(\phi_2\xi_1 - \phi_1\xi_2).$$

So once we determine the weights on W, the weights on the dual space W^* are given by $1 - \nu$ for some weight ν on W (since $\rho(\theta)^*\omega = e^{i\theta}\omega$). With respect to the decomposition $E_* = L_* \oplus M_*$, we have

$$g_\theta = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

with weights $(0, 1, -1)$ on

$$\mathrm{ParEnd}_0^\wedge(E_*) = \mathrm{ParHom}(L_*, \widehat{L}_*) \oplus \mathrm{ParHom}(L_*, \widehat{M}_*) \oplus \mathrm{ParHom}(M_*, \widehat{L}_*).$$

Further, there are no negative weights on $H^0(\mathrm{ParEnd}_0(E)) \cdot \Phi$ and the weights on W^* are $(1, 0, 2)$, so we get

$$\lambda_{d,e} = 2h^0(M_*^\vee \otimes \widehat{L}_* \otimes K(D)) = 2(n + 2d + g - 1 + |e|).$$

This completes the proof of (d). $\square$

4.3. The topology of $\mathcal{N}_\alpha^0$.

Using the results of the previous section, we deduce the following theorem.

Theorem 4.2. (a) *If $g > 0$ or $g = 0$ and $n > 3$, then $\mathcal{N}_\alpha^0$ is noncompact.*
(b) *The Betti numbers of $\mathcal{N}_\alpha^0$ depend only on the quasi-parabolic structure of E_*.*
(c) *If $g > 0$ or $g = 0$ and $n \geq 3$, then $\mathcal{N}_\alpha^0$ is connected and simply connected.*

Proof. Notice that, whenever $\dim \mathcal{N}_\alpha^0 > 0$, then for all (d, e), $\lambda_{d,e} < \dim \mathcal{N}_\alpha^0$. Thus, the Morse function μ_ρ has no maximum value and (a) follows. The only case where $\dim \mathcal{N}_\alpha^0 = 0$ is, of course, $g = 0$ and $n = 3$.

We first recall Theorem 3.1 of [4]. Let $W = \{\alpha \mid 0 < \alpha_i < \frac{1}{2}\}$ be the weight space and for any (d, e), define the hyperplane $H_{d,e} = \{\alpha \mid d + \beta(\alpha, e) = 0\}$. The set $W \setminus \cup_{d,e} H_{d,e}$ consists of the generic weights, i.e., those for which stability and

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semistability coincide. Suppose $\delta \in H_{d,e}$, then stratifying $\mathcal{M}_\delta^0$ by the Jordan-Hölder type of the underlying parabolic bundle, we see that

$$\mathcal{M}_\delta^0 = (\mathcal{M}_\delta^0 \setminus \Sigma_\delta) \cup \Sigma_\delta,$$

where Σ_δ consists of strictly semistable bundles, i.e., semistable bundles E_* with $\operatorname{gr} E_* = L_* \oplus M_*$ for two parabolic line bundles of parabolic degree zero. Suppose that α and α' are generic weights on either side of $H_{d,e}$ and that $\operatorname{pardeg}_\alpha(L_*) < 0$. If both $\mathcal{M}_\alpha^0$ and $\mathcal{M}_{\alpha'}^0$ are nonempty, then Theorem 3.1 of [4] states that there are canonical, projective maps

$$\begin{array}{ccc} \mathcal{M}_\alpha^0 & & \mathcal{M}_{\alpha'}^0 \\ \phi \searrow & & \swarrow \phi' \\ & \mathcal{M}_\delta^0 & \end{array}$$

which are isomorphisms on $\mathcal{M}_\delta^0 \setminus \Sigma_\delta$ and are $\mathbb{P}^a$ and $\mathbb{P}^{a'}$ bundles along Σ_δ, where $a = h^1(M_*^\vee \otimes L_*) - 1$ and $a' = h^1(L_*^\vee \otimes M_*) - 1$. In particular, since $\Sigma_\delta = J_X$, Corollary 3.2 of [4] gives

$$P_t(\mathcal{M}_\alpha^0) - P_t(\mathcal{M}_{\alpha'}^0) = (P_t(\mathbb{P}^a) - P_t(\mathbb{P}^{a'}))P_t(J_X).$$

To prove (b), we must show that $P_t(\mathcal{N}_\alpha^0) = P_t(\mathcal{N}_{\alpha'}^0)$ for weights on either side of a hyperplane $H_{d,e}$. Note that $d = \deg L$ and $e = e(L)$, and set $\hat{d} = -n - d$ and $\hat{e}_i = 1 - e_i$. Since

$$d + \beta(\alpha, e) = \operatorname{pardeg}_\alpha(L) < 0 < \operatorname{pardeg}_{\alpha'}(L) = d + \beta(\alpha', e),$$

and $\hat{d} + \beta(\alpha', \hat{e}) < 0 < \hat{d} + \beta(\alpha, \hat{e})$, it follows that the indexing sets of (d, e) satisfying (13) for $\mathcal{N}_\alpha^0$ and $\mathcal{N}_{\alpha'}^0$ are identical except for (d, e) and $(\hat{d}, \hat{e})$ listed above; the pair (d, e) satisfies (13) for α but not for α' and vice versa for $(\hat{d}, \hat{e})$. Thus, we claim

$$0 = P_t(\mathcal{M}_\alpha^0) - P_t(\mathcal{M}_{\alpha'}^0) + t^{\lambda_{d,e}} P_t(\mathcal{M}_{d,e}) - t^{\lambda_{\hat{d},\hat{e}}} P_t(\mathcal{M}_{\hat{d},\hat{e}}),$$

which, setting $\Delta = t^{\lambda_{\hat{d},\hat{e}}} P_t(\mathcal{M}_{\hat{d},\hat{e}}) - t^{\lambda_{d,e}} P_t(\mathcal{M}_{d,e})$ is equivalent to

$$(16) \qquad \Delta = \frac{(t^{2a'+2} - t^{2a+2})(1+t)^{2g}}{1 - t^2}.$$

First, we compute

$$h_{d,e} = 2g - 2 - 2d - |e|, \qquad \lambda_{d,e} = 2(n + 2d + g - 1 + |e|),$$

$$h_{\hat{d},\hat{e}} = 2g - 2 + n + 2d + |e|, \qquad \lambda_{\hat{d},\hat{e}} = 2(g - 1 - 2d - |e|).$$

Next, notice that if $h > 2g - 2$, then $P_t(\tilde{S}^h(X)) = P_t(S^h(X))$ (see p. 98 of [12]). But both $h_{d,e}$ and $h_{\hat{d},\hat{e}}$ are greater than $2g-2$, which we see as follows. Since $\frac{e_i}{2} \leq \beta_i(\alpha, e) \leq \frac{1+e_i}{2}$, we have $\frac{|e|}{2} \leq \sum_{i=1}^n \beta_i(\alpha, e) \leq \frac{n+|e|}{2}$. It now follows that $2d+|e| < 2d+2\beta(\alpha, e) < 0$ and $2d + n + |e| > 2d + 2\sum_{i=1}^n \beta_i(\alpha', e) > 0$.

Now use the result of [15] to interpret $P_t(S^h X)$ as the coefficient of x^h in

$$\frac{(1 + xt)^{2g}}{(1 - x)(1 - xt^2)},$$

and compute in terms of residues to see

$$\begin{aligned}
\Delta &= t^{\lambda_{\tilde{d},\tilde{e}}} P_t(S^{h_{\tilde{d},\tilde{e}}} X) - t^{\lambda_{d,e}} P_t(S^{h_{d,e}} X) \\
&= \operatorname*{Res}_{x=0} \left(\frac{t^{\lambda_{\tilde{d},\tilde{e}}}}{x^{h_{\tilde{d},\tilde{e}}+1}} - \frac{t^{\lambda_{d,e}}}{x^{h_{d,e}+1}} \right) \left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \right).
\end{aligned}$$

This last function is analytic at $x = \infty$ and has a removable singularity at $x = 1/t^2$, thus

$$\begin{aligned}
\Delta &= - \operatorname*{Res}_{x=1} \left(\frac{t^{\lambda_{\tilde{d},\tilde{e}}}}{x^{h_{\tilde{d},\tilde{e}}+1}} - \frac{t^{\lambda_{d,e}}}{x^{h_{d,e}+1}} \right) \left(\frac{(1+xt)^{2g}}{(1-x)(1-xt^2)} \right) \\
&= \frac{(t^{\lambda_{\tilde{d},\tilde{e}}} - t^{\lambda_{d,e}})(1+t)^{2g}}{1-t^2}.
\end{aligned}$$

But we can compute directly that $2a' + 2 = \lambda_{\tilde{d},\tilde{e}}$ and that $2a + 2 = \lambda_{d,e}$ and (16) follows. This proves (b) in case both $\mathcal{M}_\alpha^0$ and $\mathcal{M}_{\alpha'}^0$ are nonempty. In case one of the moduli is empty, we use the following lemma (see the remark).

To prove (c), we use the fact that $\mathcal{M}_\alpha^0$ is connected and simply-connected, which follows for $g = 0$ from [2] and for $g \geq 1$ from [5]. Since $\lambda_{d,e}$ is always even, (c) will follow if $\lambda_{d,e} > 0$ for all (d,e). This is true if $\mathcal{M}_\alpha^0 \neq \emptyset$. However, if $g = 0$ we must be careful since there are weights α with $\mathcal{M}_\alpha = \emptyset$. In that case, we must show that there is a unique pair (d,e) with $\lambda_{d,e} = 0$, and also that $\mathcal{M}_{d,e}$ is connected and simply connected. This is the content of the following lemma. $\square$

Lemma 4.3. (i) *If $g \geq 1$, then $\lambda_{d,e} > 0$ for every (d,e) satisfying (13).*
(ii) *If $g = 0$ and $n \geq 3$, then there is at most one pair (d,e) satisfying (13) with $\lambda_{d,e} = 0$. Such a pair (d,e) exists if and only if $\mathcal{M}_\alpha = \emptyset$, and in that case, $\mathcal{M}_{d,e} = \mathbb{P}^{n-3}$. Here, $\mathcal{M} = \mathcal{M}^0$ since $g = 0$.*

Remark. We now explain why this lemma proves part (b) of the Proposition when one of the moduli is empty. Suppose $\mathcal{M}_\alpha = \emptyset$, then it follows that the moment map μ_ρ is positive with minimum value $d + \sum_{i=1}^n \beta(\alpha, e)$ for the pair (d,e) identified in part (ii) of the lemma. Since (d,e) does not satisfy (13) for α', $H_{d,e}$ is the relevant hyperplane. This identifies the birth and death strata as $\mathcal{M}_{\alpha'}$ and $\mathcal{M}_{d,e}$, and thus all the other strata for α and α' are identical. The rest follows from the fact that $\mathcal{M}_{\alpha'} = \mathbb{P}^{n-3}$, first proved by Bauer [2].

Proof. Suppose that $\lambda_{d,e} = 0$ for a pair (d,e) satisfying (13). We first show that $g = 0$. Recall that $\beta_i(\alpha, e) = e_i + (-1)^{e_i} \alpha_i$. Using the fact that $0 = \lambda_{d,e} = n + 2d + g + |e| - 1$, the condition (13) and the inequality $\beta_i(\alpha, e) < \frac{e_i+1}{2}$, we see that

$$(17) \qquad \frac{n + |e| + g - 1}{2} < \sum_{i=1}^n \beta_i(\alpha, e) < \frac{n + |e|}{2}.$$

This is only possible if $g = 0$, which we now assume.

Setting $\gamma_i = 1 - \beta_i = (1 - e_i)(1 - \alpha_i) + e_i\alpha_i$, then equation (17) is equivalent to

$$\frac{n - |e|}{2} < \sum_{i=1}^{n} \gamma_i < \frac{n - |e| + 1}{2}.$$

Writing $\gamma_i = \frac{1-e_i}{2} + (1 - e_i)(\frac{1}{2} - \alpha_i) + e_i\alpha_i$, we get immediately

$$(18) \qquad 0 < \sum_{i=1}^{n}(1 - e_i)(\frac{1}{2} - \alpha_i) + e_i\alpha_i < \frac{1}{2}.$$

The advantage of the (18) is that each summand is positive.

We now prove uniqueness of the pair (d, e). If $\lambda_{d',e'} = 0$ for $(d', e') \neq (d, e)$, then it follows that $|e| - |e'| = 2(d' - d)$ is even, which implies that $e_i \neq e_i'$ for at least two i, which we assume (wlog) to include $i = 1, 2$. Now (α, e) and (α, e') both satisfy the inequality (18). Add them together and notice that since $e_1 \neq e_1'$ and $e_2 \neq e_2'$, the sum of the left hand sides is at least $\alpha_1 + (1/2 - \alpha_1) + \alpha_2 + (1/2 - \alpha_2) = 1$, which violates the (summed) inequality and therefore gives a contradiction.

It follows from $\lambda_{d,e} = 0$ and $g = 0$ that $n + |e| - 1$ is even and $h_{d,e} = n - 3$. Thus $\mathcal{M}_{d,e} = S^h X = S^h \mathbb{P}^1 = \mathbb{P}^{n-3}$. The rest of the lemma follows from the the inequality (17), together with the following proposition, which we have chosen to state as it is of independent interest. $\square$

Proposition 4.4. *If $g = 0$, then the moduli space $\mathcal{M}_\alpha \neq \emptyset \Leftrightarrow$*

$$(19) \qquad \sum_{i=1}^{n} e_i + (-1)^{e_i}\alpha_i < \frac{n + |e| - 1}{2}.$$

for every $e = (e_1, \ldots, e_n)$, $e_i \in \{0, 1\}$, with $n - |e| + 1$ even.

Remark. For $n = 3$, $\mathcal{M}_\alpha$ is either empty or a point. In this case, the proposition can be verified directly by comparing the inequalities (19) to the well-known fusion rules (or the quantum Clebsch-Gordan conditions):

$$\mathcal{M}_\alpha \neq \emptyset \Leftrightarrow |\alpha_1 - \alpha_2| \leq \alpha_3 \leq \min(\alpha_1 + \alpha_2, 1 - \alpha_1 - \alpha_2).$$

Proof. Like the proof of part (b) of the theorem, we shall use the techniques of [4]. Recall the weight space $W = \{\alpha \mid 0 \leq \alpha_i \leq 1/2\}$ and the hyperplanes $H_{d,e} = \{\alpha \mid d + \beta(\alpha, e) = 0\}$ defined earlier. We call connected components of $W \setminus \cup_{d,e} H_{d,e}$ *chambers*. A chamber C is called *null* if the associated moduli space $\mathcal{M}_\alpha$ is empty in genus 0 for every $\alpha \in C$. The proposition follows once we show that every null chamber is given by $C_{d,e} = \{\alpha \mid d + \beta(\alpha, e) > 0\}$, where $2d = 1 - n - |e|$.

Associated to the configuration of hyperplanes in W is a graph with one vertex for each chamber and an edge between two vertices whenever the two chambers are separated by a hyperplane. We shall see that in terms of this graph, null chambers have valency one. The (unique) hyperplane separating a null chamber from the rest of W is called a *vanishing wall*. If $\delta \in H_{d,e}$, a vanishing wall, and α, α' are nearby weights on either side of $H_{d,e}$, then the proof of Proposition 5.1 of [4] shows that $\mathcal{M}_\delta = \Sigma_\delta$ and, assuming that $\mathcal{M}_{\alpha'} = \emptyset$, the map ϕ is a fibration with fiber $\mathbb{P}^a$, where

$a = h^1(M_*^{\vee} \otimes L_*) - 1$. Moreover, $h^1(L_*^{\vee} \otimes M_*) = 0$ and this last equation in fact characterizes vanishing walls.

We claim now that every vanishing hyperplane is given by $H_{d,e}$ for $2d = 1 - n - |e|$. Now if $d = \deg L$ and $e = e(L)$, then direct computation shows that $h^1(L_*^{\vee} \otimes M_*) = 2d + n + |e| - 1$. On the other hand, if $n + |e| - 1$ is even and $d = \frac{1-n-|e|}{2}$, then $H_{d,e}$ is a vanishing hyperplane.

Along $H_{d,e}$, the relevant line bundles of parabolic degree 0 are given by $L_* = \mathcal{O}_X(\frac{-n-|e|+1}{2})[-\beta]_*$ and $M_* = \mathcal{O}_X(\frac{-n+|e|-1}{2})[-\gamma]_*$, where $\delta \in H_{d,e}, \beta = \beta(\delta, e)$ and $\gamma_i = 1 - \beta_i$. Since $h^1(L_*^{\vee} \otimes M_*) = 0$ and $h^1(M_*^{\vee} \otimes L_*) = n - 2$, it follows that the null chamber is defined by $C_{d,e} = \{\alpha \mid \beta(\alpha, e) > \frac{n+|e|-1}{2}\}$. To verify this is indeed a chamber, we prove that no other hyperplane cuts through $C_{d,e}$. This will also show that null chambers have valency one in the graph associated to the configuration of hyperplanes.

So suppose to the contrary that $\alpha \in H_{d',e'} \cap C_{d,e}$. Then we have $\sum (-1)^{e_i}\alpha_i > \frac{n-|e|-1}{2}$ and $\sum (-1)^{e'_i}\alpha_i = -|e'| - d' = k \in \mathbb{Z}$. If $e_i = e'_i = 0$, then $((-1)^{e_i} + (-1)^{e'_i})\alpha_i < 1$ and in all other cases, $((-1)^{e_i} + (-1)^{e'_i})\alpha_i \leq 0$. Using a similar property for $e'' = 1 - e'$, we see

$$\frac{n - |e| - 1}{2} + k < \sum_{i=1}^{n}((-1)^{e_i} + (-1)^{e'_i})\alpha_i < \sum_{e_i = e'_i = 0} 1,$$

$$\frac{n - |e| - 1}{2} - k < \sum_{i=1}^{n}((-1)^{e_i} + (-1)^{e''_i})\alpha_i < \sum_{e_i = e''_i = 0} 1.$$

These are strict inequalities of integers, so after adding one to the left hand sides and summing the two inequalities (which are no longer strict), we see $n - |e| + 1 \leq \sum_{e_i = 0} 1 = n - |e|$, a contradiction. $\square$

4.4. The Betti numbers of the moduli space of parabolic Higgs bundles.

The results of the previous section show that the Betti numbers of $\mathcal{N}_\alpha^0$ depend only on the genus g and number n of parabolic points. In this section, we give a formula for the Poincaré polynomial of $\mathcal{N}_\alpha^0$. Such a general calculation is not possible for $P_t(\mathcal{M}_\alpha^0)$ without first specifying α, so take $\alpha = (\frac{1}{3}, \ldots, \frac{1}{3^n})$. Using Proposition 4.4 (taking $e = (0, 1, \ldots, 1)$) it is clear the α lies in a null chamber. We could calculate $P_t(\mathcal{M}_\alpha^0)$ using the Atiyah-Bott procedure for parabolic bundles as in [5], but there is an easier method which exploits the fact that α lies in a null chamber. First of all, using the results of §6.4 in [5], we get

$$P_t(\mathcal{M}_\alpha^0) = \frac{(1+t^2)^{n-1}(1+t^3)^{2g}}{(1-t^2)^2} - \frac{(1+t)^{2g}}{(1-t^2)} \sum_{\lambda,e} t^{2d_{\lambda,e}}.$$

Note that $d_{\lambda,e}$ depends on g ($d_{\lambda,e} = d_{\lambda,e}(g = 0) + g$), but the indexing set $\{\lambda, e\}$ is independent of g. Since $\mathcal{M}_\alpha^0(g = 0) = \emptyset$, this determines the sum and we see that

$$P_t(\mathcal{M}_\alpha^0) = (1+t^2)^{n-1}\left(\frac{(1+t^3)^{2g} - t^{2g}(1+t)^{2g}}{(1-t^2)^2}\right).$$

It follows from Theorem 4.1 that

$$P_t(\mathcal{N}_\alpha^0) = P_t(\mathcal{M}_\alpha^0) + \sum_{d,e} t^{\lambda_{d,e}} P_t(\mathcal{M}_{d,e}),$$

where the sum is taken over (d,e) satisfying (13), which, for our choice of α, is simply $e_1 - |e| \leq d \leq [g - 1 - \frac{|e|}{2}]$, where $[x]$ is the greatest integer less than x. Setting $j = 2d + n + |e| - 1$, then j satisfies:

$$n + 2e_1 - |e| - 1 \leq j \leq 2g + n - 3 \quad \text{and} \quad j - n - |e| + 1 \text{ is even.}$$

Also $\lambda_{d,e} = 2(g + j)$ and $h_{d,e} = 2g + n - j - 3$.

Fixing e_1 and $|e|$, for each d, there are $\binom{n-1}{|e|-e_1}$ strata given by the choice of e. Thus, for each j, there are $q_j = \sum_{i=0}^{j} \binom{n-1}{i}$ strata (note that $q_j = 2^{n-1}$ for $j \geq n - 1$) and we see

$$\sum_{d,e} t^{\lambda_{d,e}} P_t(\mathcal{M}_{d,e}) = \sum_{|e|=0}^{n} \binom{n-1}{|e|-e_1} \sum_{d=e_1-|e|}^{[g-1-|e|/2]} t^{\lambda_{d,e}} P_t(\widetilde{S}^{h_{d,e}} X)$$

$$= \sum_{j=0}^{2g+n-3} q_j t^{2(g+j)} P_t(\widetilde{S}^{2g+n-j-3} X)$$

$$= \sum_{j=0}^{n-2} q_j t^{2(g+j)} P_t(\widetilde{S}^{2g+n-j-3} X) + \sum_{j=0}^{2g-2} 2^{n-1} t^{2(g+n+j-1)} P_t(\widetilde{S}^{2g-j-2} X).$$

We refer to the last two sums by $\widetilde{S}_1$ and $\widetilde{S}_2$. Using the Binomial Theorem and the general formula (p. 98 of [12]) $P_t(\widetilde{S}^h X) = (2^{2g} - 1)\binom{2g-2}{h} t^h + P_t(S^h X)$, we see that

$$\widetilde{S}_1 = \sum_{j=0}^{n-2} q_j t^{2(g+j)} P_t(S^{2g+n-j-3} X) = S_1,$$

$$\widetilde{S}_2 = \sum_{j=0}^{2g-2} 2^{n-1} t^{2(g+n+j-1)} P_t(S^{2g-j-2} X) + \sum_{j=0}^{2g-2} 2^{n-1}(2^{2g} - 1)\binom{2g-2}{j} t^{4g+2n+j-4}$$

$$= S_2 + 2^{n-1}(2^{2g} - 1) t^{2(2g+n-2)}(1 + t)^{2g-2},$$

where S_1 and S_2 are the sums obtained by removing the tildes from the summands of $\widetilde{S}_1$ and $\widetilde{S}_2$. According to a result of [15], $P_t(S^h X)$ is the coefficient of x^h in

$$\frac{(1 + xt)^{2g}}{(1 - x)(1 - xt^2)}.$$

This allows us to evaluate S_i as follows:

$$S_1 = \operatorname*{Res}_{x=0} \left(\sum_{j=0}^{n-2} \frac{q_j t^{2(g+j)} (1 + xt)^{2g}}{x^{2g+n-j-2}(1 - x)(1 - xt^2)} \right),$$

$$S_2 = \operatorname*{Res}_{x=0} \left(\frac{2^{n-1} t^{2(g+n-1)} (1 + xt)^{2g}}{x^{2g-1}(1 - x)(1 - xt^2)^2} \right).$$

But each of these rational functions is analytic at $x = \infty$, so we can use the Cauchy Residue Formula to evaluate instead at the poles $x = 1$ and $x = 1/t^2$. Letting $Q_n(t) = \sum_{k=0}^{n-2} q_k t^{2k}$ and noticing that $Q_n(1) = \sum_{k=0}^{n-2} q_k = 2^{n-2}(n-1)$, we get

$$S_1 = \left(Q_n(t)t^{2g} - 2^{n-2}(n-1)t^{2(2g+n-2)}\right)\frac{(1+t)^{2g}}{(1-t^2)},$$

$$S_2 = 2^{n-1}\left(t^{2(g+n-1)} + t^{4g+2n-3}((2g-1)t - 2g)\right)\frac{(1+t)^{2g}}{(1-t^2)^2}.$$

But since $Q_n(t)(1-t^2) + 2^{n-1}t^{2(n-1)} = (1+t^2)^{n-1}$, it follows that

$$
\begin{aligned}
P_t(\mathcal{N}_\alpha^0) &= P_t(\mathcal{M}_\alpha^0) + \tilde{S}_1 + \tilde{S}_2 \\
&= P_t(\mathcal{M}_\alpha^0) + S_1 + S_2 + 2^{n-1}(2^{2g}-1)t^{2(2g+n-2)}(1+t)^{2g-2} \\
&= \frac{(1+t^3)^{2g}(1+t^2)^{n-1} + 2^{n-1}t^{2n+4g-3}(1+t)^{2g}[(2g-1)t-2g]}{(1-t^2)^2} \\
&\quad - \frac{2^{n-2}(n-1)t^{2n+4g-4}(1+t)^{2g}}{1-t^2} + 2^{n-1}(2^{2g}-1)t^{4g+2n-4}(1+t)^{2g-2}.
\end{aligned}
$$

Evaluating this at $t = 1$ shows that the Euler characteristic of $\mathcal{N}_\alpha^0$ is

$$\chi(\mathcal{N}_\alpha^0) = 2^{2g+n-3}\left(2g^2 + (2n-1)g + \frac{(n-1)(n-2)}{2} + 2^{2g} - 1\right).$$

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MAX-PLANCK-INSTITUT FÜR MATHEMATIK, GOTTFRIED-CLAREN-STRASSE 26, 53225 BONN, GERMANY

E-mail address: hboden@mpim-bonn.mpg.de, yokogawa@mpim-bonn.mpg.de