Derivatives of Eisenstein series and heights
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In this talk, I describe some recent joint work with **Michael Rapoport** and **Tonghai Yang**. For a long time, I have been interested by the surprising amount of information of an arithmetic geometric nature which seems to appear in the Fourier coefficients of the derivatives of Eisenstein series. The present example has several new features. First, the Eisenstein series in question is a rather familiar classical series. Second, at the critical point $s = \frac{1}{2}$, which is not the center of symmetry, the series has a nonzero value with a geometric meaning. Nonetheless, the next term in the Laurent expansion has a geometric interpretation as well, via Arakelov theory, and specifically, heights.

Eisenstein series.

In 1975, Zagier introduced a striking modular form of weight $\frac{3}{2}$: for $\tau = u + iv \in \mathfrak{H}$, the upper half plane,

$$\mathcal{F}(\tau) = -\frac{1}{12} + \sum_{m=1}^{\infty} H(m)\, q^m + v^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} \frac{1}{16\pi} \int_1^{\infty} e^{-4\pi n^2 v r}\, r^{-\frac{3}{2}}\, dr \cdot q^{-n^2}.$$

where, in the positive coefficients, $H(m)$ is the class number of integral binary quadratic forms of discriminant $-m$. The non-holomorphic part does not have an obvious geometric meaning; a kind of geometric interpretation will be described later.

In fact, $\mathcal{F}(\tau)$ is the special value of an Eisenstein series

$$\mathcal{F}(\tau, s) = \sum_{\Gamma_\infty \backslash \Gamma} (c\tau + d)^{-\frac{3}{2}} |c\tau + d|^{\frac{1}{2} - s} v^{\frac{1}{2}(s - \frac{1}{2})}\, \Phi(\gamma), \qquad \mathrm{Re}(s) > 1,$$

where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, for a certain function $\Phi(\gamma)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\mathcal{F}(\tau) = \mathcal{F}(\tau, \frac{1}{2})$.

The Eisenstein series of interest here are analogous. For a positive square free integer $D \geq 1$, there is an Eisenstein series

$$\mathcal{E}(\tau, s; D) = c(s; D) \sum_{\Gamma_\infty \backslash \Gamma} (c\tau + d)^{-\frac{3}{2}} |c\tau + d|^{\frac{1}{2} - s} v^{\frac{1}{2}(s - \frac{1}{2})}\, \Phi^D(\gamma), \qquad \mathrm{Re}(s) > 1,$$

for a certain function $\Phi^D(\gamma)$, and with a certain normalizing factor $c(s; D)$. This series again has weight $\frac{3}{2}$ and has an analytic continuation in s with functional equation

$$\mathcal{E}(\tau, s; D) = \mathcal{E}(\tau, -s; D).$$

When $D = 1$, $\mathcal{E}(\tau, s; 1)$ is essentially[1] $\mathcal{F}(\tau, s)$.

[1] up to a small modification involving the treatment of 2 in the discriminant...

For now, assume that $D > 1$, and consider the first two terms in the Laurent expansion at the point $s = \frac{1}{2}$. The value at this point has Fourier expansion

$$\mathcal{E}(\tau, \frac{1}{2}; D) = -\frac{1}{12}(-1)^{\mathrm{ord}(D)} \prod_{p|D}(p-1) + \sum_{m \geq 1} \delta(d; D)\, H_0(m; D)\, q^m,$$

where $4m = n^2 d$ for a fundamental discriminant $-d$ with corresponding quadratic character χ_d,

$$\delta(d; D) = \prod_{p|D}(1 - \chi_d(p)),$$

and

$$H_0(m; D) = \sum_{\substack{c|n \\ (c,D)=1}} \frac{h(c^2 d)}{w(c^2 d)},$$

for $h(c^2d)$ (resp. $w(c^2d)$) the class number (resp. number of roots of 1) in the order O_{c^2d} of discriminant $-c^2d$ in the quadratic field $\mathbb{Q}(\sqrt{-m})$. Also, $\mathrm{ord}(D)$ is the number of prime factors of D.

Writing the Fourier expansion for general s as

$$\mathcal{E}(\tau, s; D) = \sum_m A_m(s, v)\, q^m$$

one has,

$$\mathcal{E}'(\tau, \frac{1}{2}; D) = \sum_m A'_m(\frac{1}{2}, v)\, q^m$$

where the coefficients $A'_m(\frac{1}{2}, v)$ are considerably more complicated. For example, for $m > 0$ for which $\delta(d; D) \neq 0$:

$$\begin{aligned} &A'_m(\frac{1}{2}, v) \\ &\quad = 2\,\delta(d; D)\, H_0(m; D) \Bigg[\, \frac{1}{2}\log(d) + \frac{L'(\frac{1}{2}, \chi_d)}{L(\frac{1}{2}, \chi_d)} - \frac{1}{2}\log(\pi) - \frac{1}{2}\gamma \\ &\qquad\qquad + \frac{1}{2} J(\frac{1}{2}, 4\pi m v) \\ &\qquad\qquad + \sum_{p \nmid D} \Big(\, \log|n|_p - \frac{b'_p(n, 0; D)}{b_p(n, 0; D)} \,\Big) + \sum_{p|D} K_p \log(p) \,\Bigg], \end{aligned}$$

Here, for $k = \mathrm{ord}_p(n)$,

$$\frac{b'_p(n, 0; D)}{b_p(n, 0; D)} = \log(p) \begin{cases} \frac{p^k - 1}{p^k(p-1)} - 2k & \text{if } \chi_d(p) = 1 \\ -\frac{2p(1-(k+1)p^k + kp^{k+1})}{(p-1)(p^{k+1}-1)} & \text{if } \chi_d(p) = 0, \\ -\frac{1+3p-(2k+1)p^k - 3p^{k+1} + 2kp^{k+2}}{(p-1)(p^{k+1}+p^k-2)} & \text{if } \chi_d(p) = -1. \end{cases}$$

and

$$K_p = \begin{cases} -1-k+\frac{p^{k+1}-1}{p-1} & \text{if } \chi_d(p)=0, \\ -k+\frac{(p+1)(p^k-1)}{2(p-1)} & \text{if } \chi_d(p)=-1. \end{cases}$$

Finally,

$$J(\frac{1}{2},t) = \int_0^\infty e^{-tr}\,\big[\,(1+r)^{\frac{1}{2}}-1\,\big]\,r^{-1}\,dr.$$

Similarly explicit formulas can be given in all cases.

Arithmetic Geometry.

The geometric meaning of the value and derivative of $\mathcal{E}(\tau,s;D)$ at $s=\frac{1}{2}$ involves the following arithmetic surface.

Let B be an indefinite quaternion algebra over $\mathbb{Q}$, and let $D(B)$ be the product of the primes at which $B_p = B\otimes\mathbb{Q}_p$ is a division algebra. Let O_B be a maximal order in B. Let $\mathcal{M}$ be the moduli stack over $\mathrm{Spec}(\mathbb{Z})$ for pairs (A,ι), where A is an abelian surface and $\iota: O_B \hookrightarrow \mathrm{End}(A)$ is an action of O_B.

For $D(B)=1$, $B=M_2(\mathbb{Q})$, $O_B=M_2(\mathbb{Z})$ and $\mathcal{M}$ is the moduli stack of elliptic curves over $\mathrm{Spec}(\mathbb{Z})$.

If $D(B)>1$, then $\mathcal{M}$ is proper over $\mathrm{Spec}(\mathbb{Z})$ with good reduction for $p\nmid D(B)$ and with bad fibers, for $p\mid D(B)$, described by Drinfeld-Cherednik uniformization.

The complex points $\mathcal{M}(\mathbb{C})$ of $\mathcal{M}$ have a complex uniformization:

$$\mathcal{M}(\mathbb{C}) \simeq \Gamma\backslash\mathfrak{h}^+\cup\mathfrak{H}^-$$

where $\Gamma = O_B^\times$.

CM–cycles on $\mathcal{M}$ are defined by imposing additional endomorphisms. For (A,ι) given, let

$$V(A,\iota) = \{\, x\in\mathrm{End}(A) \mid \iota(b)x = x\iota(b),\ \text{and } \mathrm{tr}(x)=0\,\}.$$

This $\mathbb{Z}$–lattice has a quadratic form $Q(x)=-x^2$. Let

$$\mathcal{Z}(m) = \Big(\ \text{locus of triples } (A,\iota,x) \text{ with } x\in V(A,\iota),\ Q(x)=m\ \Big) \longrightarrow \mathcal{M}.$$

This defines a divisor on the arithmetic surface $\mathcal{M}$.

To define the corresponding element

$$\hat{\mathcal{Z}}(m,v) = (\mathcal{Z}(m),\Xi(m,v)) \in \widehat{\mathrm{CH}}^1(\mathcal{M})$$

in the arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{M})$, we add a Green's function depending on a positive parameter $v > 0$. Let

$$V = \{x \in B \mid \mathrm{tr}(x) = 0\ \},$$

with quadratic form $Q(x) = -x^2$. Let

$$L = V \cap O_B.$$

Then, for $z \in \mathfrak{H}^+ \cup \mathfrak{H}^-$,

$$\Xi(m, v)(z) = \sum_{\substack{x \in L \\ Q(x)=m}} \int_1^\infty e^{-2\pi v R(x,z)}\, r^{-1}\, dr,$$

where $R(x, z)$ is a quadratic expression which vanishes precisely at the points in $\mathcal{Z}(m)(\mathbb{C})$. This function is a Green's function for $\mathcal{Z}(m)(\mathbb{C})$ on $\mathcal{M}(\mathbb{C})$.

For $m < 0$, the function $\Xi(m, v)$ is still well defined and is now a smooth function on $\mathcal{M}(\mathbb{C})$. In this case, set

$$\hat{\mathcal{Z}}(m, v) = (0, \Xi(m, v)) \in \widehat{\mathrm{CH}}^1(\mathcal{M}).$$

Finally, let $\mathcal{A} \to \mathcal{M}$ be the universal family over $\mathcal{M}$ with zero section ϵ, and let

$$\omega = \epsilon^* \Omega^2_{\mathcal{A}/\mathcal{M}}.$$

This line bundle on $\mathcal{M}$ has a natural metric defined by

$$||\alpha_z||^2 = e^{-2C} \left| \left(\frac{i}{2\pi}\right)^2 \int_{\mathcal{A}_z(\mathbb{C})} \alpha \wedge \bar{\alpha} \right|,$$

where

$$2C = \log(4\pi) + \gamma$$

for γ Euler's constant. The resulting metrized line bundle

$$\hat{\omega} = (\omega, ||\ ||) \in \widehat{\mathrm{Pic}}(\mathcal{M})$$

can be paired with the classes in $\widehat{\mathrm{CH}}^1(\mathcal{M})$ via the Gillet–Soulé height pairing

$$\langle\ ,\ \rangle : \widehat{\mathrm{CH}}^1(\mathcal{M}) \times \widehat{\mathrm{CH}}^1(\mathcal{M}) \longrightarrow \mathbb{R}.$$

Also, let

$$\hat{\mathcal{Z}}(0, v) = -\Big(\hat{\omega} + (0, \log(v))\Big) \in \widehat{\mathrm{CH}}^1(\mathcal{M}) \simeq \widehat{\mathrm{Pic}}(\mathcal{M}).$$

Theorem. *Assume that $D = D(B) > 1$. Then*
(i)

$$\mathcal{E}(\tau, \frac{1}{2}; D) = \sum_m \deg(\hat{\mathcal{Z}}(m, v))\, q^m =: \phi_{\deg}(\tau).$$

(ii)

$$\mathcal{E}'(\tau, \frac{1}{2}; D) = \sum_m \langle\ \hat{\mathcal{Z}}(m, v), \hat{\omega}\ \rangle\, q^m + \mathbf{c}$$

for some constant $\mathbf{c}$.

The series on teh RHS of (ii) is the analogue for the arithmetic surface $\mathcal{M}$ of the Hirzebruch–Zagier generating function for volumes of curves of the Hilbert modular surface.

The ambiguity $\mathbf{c}$ occurs in the constant term, since we do not know the exact value of $\langle\hat{\omega}, \hat{\omega}\rangle$, the arithmetic volume of $\mathcal{M}$.

An analogous result will hold for the case $D(B) = 1$, i.e., for the modular curve. In this case, the Chow groups defined by Bost and Kühn must be used to accomodate the singularities of the singular Petersson metric on $\hat{\omega}$. The degree map is then defined by

$$\deg : \widehat{\mathrm{CH}}^1(\mathcal{M}) \longrightarrow \mathbb{R}, \qquad (Z, g_Z) \mapsto \int_{\mathcal{M}(\mathbb{C})} \omega_Z,$$

where ω_Z is the $(1,1)$–form on the right side of the Green's equation

$$dd^c g_Z + \delta_Z = \omega_Z.$$

The result of the Maryland thesis of Jens Funke then proves that:

Proposition.

$$\phi_{\deg}(\tau) = \mathcal{F}(\tau)$$

i.e., the negative Fourier coefficients of Zagier's function are Arakelov degrees of the cycles $\hat{\mathcal{Z}}(-n^2, v)$ *on the modular curve.*