COSIMPLICIAL OBJECTS IN ALGEBRAIC GEOMETRY

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Part 1

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In these notes we want to point out the importance of cosimplicial schemes in algebraic geometry. Using cosimplicial schemes we constructed the analog of the bundle of fundamental groups in algebraic geometry and we equipped it with an integrable connection. We recover in this way constructions given in [HZ] over complex numbers. However our construction applies to smooth schemes defined over any field of characteristic zero.

The cosimplicial schemes can be used to introduce mixed Hodge structures on homotopy groups as it was done by other methods in [Mo], [H] and [N1]. They seem to be specially suited to treat motives associated to fundamental groups, higher homotopy groups and other topological invariants. At least part of this notes can be viewed as an attempt to obtain results from [D2] about fundamental groupoids in the De Rham setting without any use of the formalism of Tannakian categories. For examples we can define motivic π_1 and also motivic fundamental groupoids for any smooth algebraic variety X, while in [D2] one needs the restriction that $H^1(\overline{X},\mathcal{O}) = 0$ where $\overline{X}$ is a smooth compactification of X.

While trying to emphasize the importance of cosimplicial schemes one can draw some parallels with algebraic topology. Almost at the dawn of algebraic topology simplicial sets played an essential role. Cosimplicial objects were merely a curiosity which nobody seriously occupied. The notable exception, after a long time, was the book of Bousfield and Kan (see [BK]). And then suddenly the cosimplicial objects turned out to be principal tools in solving outstanding problems in algebraic topology.

We do not know what will be the role of cosimplicial objects in future in algebraic geometry. We only want to indicate that they can be useful. More precise description of our results one finds in the section "Review of results".

0. Review of results.

We describe here the topological facts which motivate this paper and we present the outline of our constructions and results.

Let X be an arc–connected and locally arc–connected topological space and let I be a unit interval. Let $p : X^I \longrightarrow X\times X$ be given by $p(\omega) = (\omega(0),\omega(1))$. This is a fibration. Applying the connected component functor π_0 to each fibre of $p : X^I \longrightarrow X\times X$, we get a local system of sets

0.1. $$p : P \longrightarrow X\times X \ .$$

The restriction of this local systems to the diagonal is the local system of fundamental groups on X , while its restriction to $X\times\{x\}$ is the universal covering $\tilde{X} \longrightarrow X$.

These constructions do not generalize straightforward to constructions in algebraic geometry. The constructions in algebraic geometry we present in this paper are based on the following observation.

The standard inclusion $\partial\Delta[1] \hookrightarrow \Delta[1]$ of simplicial sets induces a map

$$p^\bullet : X^{\Delta[1]} \longrightarrow X^{\partial\Delta[1]}$$

of cosimpicial spaces, whose geometric realization is the map

$$p : X^I \longrightarrow X \times X \ .$$

Now we present the main constructions and results of this note.

Let V be a smooth separated scheme of finite type over a field k of characteristic zero. The inclusion $\partial\Delta[1] \hookrightarrow \Delta[1]$ induces a cosimplicial map

0.2. $$p^\bullet : V^{\Delta[1]} \longrightarrow V^{\partial\Delta[1]}$$

of cosimplicial schemes. Let $\Omega_k^\bullet = \Omega^\bullet_{V^{\Delta[1]}/V^{\partial\Delta[1]}}$ be the De Rham complex of smooth, relative $V^{\partial\Delta[1]}$–differentials on $V^{\Delta[1]}$ and let $H^j(tRp_*^\bullet\Omega_k^\bullet)$ be the relative De Rham cohomology groups (R is the component–wise derived functor of $p_*^\bullet$ and t is the functor which associates a total complex to a bicomplex). Following the method of Katz and Oda from [KO] we show the following result.

Theorem A. i) There exists a canonical integrable connection d_k on the relative De Rham cohomology groups $H^j(tRp_*^\bullet\Omega_k^\bullet)$.

ii) If V is a scheme over a field of complex numbers, then the connection $d_{\mathbb{C}}$ extends to the connection

$$d(\mathbb{C}) := (d_{\mathbb{C}})^{an} : H^j(tRp_*^\bullet\Omega_{\mathbb{C}}^\bullet)^{an} \longrightarrow \Omega^1_{V^{an}\times V^{an}} \otimes_{\mathcal{O}_{V^{an}\times V^{an}}} H^j(tRp_*^\bullet\Omega_{\mathbb{C}}^\bullet)^{an}$$

($(\ )^{an}$ denotes the analytic object corresponding to the algebraic one).

iii) The sheaves of $\mathcal{O}_{V\times V}$–modules, $H^j(tRp_*^\bullet\Omega_k^\bullet)$ are locally free.

iv) Let $\mathbb{C}_\bullet$ be the constant sheaf on $(V^{an})^{\Delta[1]}$ equal to $\mathbb{C}$. The horizontal sections of $H^j(tRp_*^\bullet\Omega_{\mathbb{C}}^\bullet)^{an}$ with respect to the connection $d(\mathbb{C})$ are canonically identified with local sections of the locally constant sheaf $H^j(tRp_*^\bullet\mathbb{C}_\bullet)$ on $V^{an}\times V^{an}$.

v) The connection d_k is regular.

It is well known that the category of locally constant sheaves on X is equivalent to the category of $\pi_1(X,x)$–sets. Therefore it is important to identify the fibre of $H^j(tRp_*^\bullet\mathbb{C}_\bullet)$ and the action of $\pi_1(V^{an},x)\times\pi_1(V^{an},x)$.

Let us set $\pi=\pi_1(V^{an},x)$. Let $Z[\pi]$ be a group ring of π and let $I=\ker(Z[\pi]\longrightarrow Z)$ be the augmentation ideal. For any field K of characteristic zero let

$$\mathrm{Alg}(\pi;K):=\varinjlim_{n}\mathrm{Hom}(Z[\pi]/_{I}n;K)\ .$$

($\mathrm{Alg}(\pi;K)$ can be idenitified with the Hopf algebra of regular functions on the Malcev - rationalization of π.) The representation

$$\varphi:\pi\times\pi\longrightarrow(\text{bijections of }\pi)$$

given by $\varphi(\alpha,\beta)(g)=\alpha\cdot g\cdot\beta^{-1}$, induces a representation

$$\Phi:\pi\times\pi\longrightarrow\mathrm{Aut}_{\mathbb{C}\text{–algebra}}(\mathrm{Alg}(\pi;\mathbb{C}))\ .$$

<u>Theorem B</u>. The fibre of $H^0(tRp_*^\bullet\mathbb{C}_\bullet)$ over (v,v) is equal to $\mathrm{Alg}(\pi,\mathbb{C})$ and the representation Φ corresponds to the local system $H^0(tRp_*^\bullet\mathbb{C}_\bullet)$.

Observe that the morphism $p : P \longrightarrow X\times X$ from 0.1 is a groupoid over X. It has a partial composition law $o : P \underset{X}{\times} P \longrightarrow P$, which associates to two paths its sum, an inverse map $\iota : P \longrightarrow P$, which associates to a path α, its inverse α^{-1} and a constant section over the diagonal ΔX of $X\times X$.

We have the following statement.

<u>Theorem $\mathbb{C}$</u>. The morphism $\mathrm{Spec}\, H^0(tRp_*^\bullet\Omega_k^\bullet) \longrightarrow V\times V$ is a Poincaré groupoid equipped with a Poincaré connection.

We shall not give here a precise definition. We only point out the following. Let us set $P := \mathrm{Spec}\, H^0(tRp_*^\bullet\Omega_k^\bullet)$. Then P is equipped with a partial composition law, an inverse map and a constant section over the diagonal. Moreover all these struture maps are compatible with an integrable connection on P.

The fibre of the morphism $\mathrm{Spec}\, H^0(tRp_*^\bullet\Omega_k^\bullet) \longrightarrow V\times V$ over $(x,x) \in V\times V$ we shall denote by $\pi_1^{DR}(V,x)$ and we call it the algebraic De Rham fundamental group of V. Let $\pi_1^{DR}(V,x)(K)$ be a group of K–points of $\pi_1^{DR}(V,x)$. We shall show that for $V = P^1_{\mathbb{Q}}\setminus\{0,1,\infty\}$ the comparison homomorphism

$$b : \pi_1(P^1(\mathbb{C})\setminus\{0,1,\infty\},x) \longrightarrow \pi_1^{DR}(P^1_{\mathbb{Q}}\setminus\{0,1,\infty\},x)(\mathbb{C})$$

involves values of ζ–function.

The group $\pi_1^{DR}(V,x)$ plays an important role in the classification of unipotent differential equations on V.

Definition. Let V be a smooth, geometrically connected, separated scheme of finite type over a field k of characteristic zero. A unipotent differential equation on V is a finite dimensional vector bundle E on V filtered by vector subbundles $\{E_j\}_{j=0}^{n}$ such that $E_0 = \{0\}$ and $E_n = E$. The vector bundle E is equipped with the integrable connection ∇ compatible with the filtration $\{E_j\}$, the associated graded bundle $\mathrm{Gr}E = \bigoplus_{j=0}^{n} \mathrm{Gr}_j E$ is trivial and the connection induced by ∇ on $\mathrm{Gr}E$ is trivial.

Theorem D. Let V be as above and let x be a k–point of V. There is an equivalence of categories:

unipotent differential equations on V

and

algebraic representations of $\pi_1^{DR}(V,x)$ in finite dimensional vector spaces over k.

Let us assume that V^{an} is simply–connected for any embedding of k into $\mathbb{C}$. The fibre of $H^i(tRp_*^{\bullet}\mathbb{Q}_{\bullet})$ at (x,x) is equal to $H^i((V^{an},x)^{\bullet};\mathbb{Q})$. This last group is isomorphic to $H^i(\Omega_x V^{an};\mathbb{Q})$, the i–th cohomology group of the loop space $\Omega_x V^{an}$ at x.

It is well known that

$$H^i(\Omega_x V^{an};\mathbb{Q}) / \sum_{\substack{a+b=i \\ a\,,\,b>0}} H^a(\Omega_x V^{an};\mathbb{Q}) \cdot H^b(\Omega_x V^{an};\mathbb{Q})$$

is equal to the dual vector space of $\pi_{i+1}(V^{an},x) \otimes \mathbb{Q}$.

This suggests a definition of the algebraic De Rham homotopy groups of V.

Definition. Let V be a smooth, separated scheme of finite type over a field k of characteristic zero. Let x be a k–point of V. Assume that $H^0((V,x)^\bullet) = k$. For $i \geq 1$ we set

$$\pi_{i+1}^{DR}(V,x) := \Big(H_{DR}^i((V,x)^\bullet) / \sum_{\substack{a+b=i \\ a,\, b>0}} H_{DR}^a((V,x)^\bullet) \cdot H_{DR}^b((V,x)^\bullet)\Big)^*$$

where $(\ )^*$ is the dual vector space.

If $\sigma : k \hookrightarrow \mathbb{C}$ is an embedding then we denote by $V_\sigma(\mathbb{C})$ the analytic variety corresponding to an algebraic variety $V \underset{\sigma}{\times} \mathbb{C}$. From the standard properties of the groups $\pi_i^{DR}(V,x)$ we get the following result which is usually obtained using étale homotopy.

Theorem E. Let V be a smooth, separated scheme of finite type over a field k of characteristic zero. Assume that $V_\sigma(\mathbb{C})$ is simply–connected for any embedding $\sigma : k \hookrightarrow \mathbb{C}$. Then we have

$$\operatorname{rank}_{\mathbb{Q}}(\pi_i(V_{\sigma_1}(\mathbb{C}))\otimes\mathbb{Q}) = \operatorname{rank}_{\mathbb{Q}}(\pi_i(V_{\sigma_2}(\mathbb{C}))\otimes\mathbb{Q})$$

for any two embeddings $\sigma_1 : k \hookrightarrow \mathbb{C}$ and $\sigma_2 : k \hookrightarrow \mathbb{C}$.

The theory of cosimplicial schemes can be used to define motives associated to fundamental groups. In [D2] this is done only for smooth algebraic varieties X whose smooth compactifications $\overline{X}$ satisfy $H^1(\overline{X},\mathcal{O}) = 0$. The approach through cosimplicial schemes allows to do this without the restriction $H^1(\overline{X},\mathcal{O}) = 0$.

We are very grateful to P. Deligne who pointed out this to us.

In [J] the category of (realizations of) mixed motives is defined in the following way. One takes systems of realizations of $H^i(X)$ for X smooth and quasi–projective. Then motives form the smallest tannakien category generated by such systems of realizations. We do not know if this definition includes mixed motives associated to π_1 and torsors over π_1 (see [D2]).

We propose here a definition which includes such mixed motives. We consider smooth simplicial schemes $X_\bullet$ and smooth cosimplicial schemes $X^\bullet$ such that for each n, X_n and X^n are smooth, quasi–projective. Then one takes systems of realizations of $W^nH^i(X_\bullet)$ and $W^nH^i(X^\bullet)$ for such $X_\bullet$ and $X^\bullet$ which satisfy the following condition:

for every n and i

$$W_nH^i(X_\bullet(\text{or } X^\bullet))/W_{n-1}H^i(X_\bullet(\text{or } X^\bullet))$$

is a pure motive in the sense of Grothendieck (see [M]). One defines the category of mixed motives as the smallest tannakian category generated by such $W^nH^i(X_\bullet)$ and $W^nH^i(X^\bullet)$.

Then mixed motives corresponding to π_1 and torsors over π_1 are included in such category of motives. Also mixed motives corresponding to higher homotopy groups are in this category.

It could be interesting to compare the category of mixed motives defined in [J] and the category proposed here. Are they really different?

Finally we point out three applications of cosimplicial spaces which are not not discussed in this paper.

For $V = P^1_{\mathbb{Q}} \backslash \{0,1,\infty\}$ the horizontal sections of the connection $d_{\mathbb{Q}}$ from Theorem 4.1 are given, among others functions, by classical polylogarithms $\mathrm{Li}_n(z)$ (see [L]). Using this fact one can show the following result.

Theorem. Let $f_1(z),\dots,f_N(z)$ be regular functions from $Y = P^1(\mathbb{C}) \backslash \{a_1,\dots,a_n\}$ to $X = P^1(\mathbb{C}) \backslash \{0,1,\infty\}$ and let $n_1,\dots,n_N$ be integers. Let S be a loop around 0 and T a loop around 1 in X. There is a functional equation

$$\sum_{i=1}^{N} n_i(\mathcal{L}i_n(f_i(z)) - \mathcal{L}i_n(f_i(x))) = 0$$

if and only if there is an equality

$$\sum_{i=1}^{N} n_i (f_i)_* = 0$$

in the group $\mathrm{Hom}(\Gamma^n \pi_1(Y,y)/_{\Gamma^{n+1}\pi_1(Y,y)} ; \Gamma^n \pi_1(X,x)/_{\Gamma^{n+1}\pi_1(X,x)+L})$, where L is a subgroup of $\Gamma^n \pi_1(X,x)$ generated by commutators which contain T at least twice and $\mathcal{L}i_n(z)$ are suitable normalized polylogarithms.

Let us consider functions of the form $F_{ij}(a_1,\dots,a_n) = \int_{a_i}^{a_j} \frac{dz}{z-a_{i_1}},\dots,\frac{dz}{z-a_{i_k}}$ where $i_s \in \{1,\dots,n\}$. One constructs a morphism of cosimplicial spaces $p^\bullet : Y^\bullet \longrightarrow X^\bullet$ such that the horizontal sections of the canonical Gauss–Manin connection on sheaves $H^i(tRp^\bullet_* \Omega^*_{Y^\bullet/X^\bullet})$ are given by functions F_{ij} .

The third application is somehow different. Let us consider all systems of differential equations which are obtained by succesive extensions of differential equations which are factors of the Gauss–Manin equations associated to smooth morphisms $f : X \longrightarrow S$. It is conjectured that solutions of such equations for $S = \operatorname{Spec} \overline{\mathbb{Q}}(x)$ coincide with G–functions (see [A] page 2). For example classical polylogarithms are solutions of a system of differential equations which is a successive extension of a trivial differential equation by itself. The trivial differential equation $f' = 0$ corresponds to the Guass–Manin equation associated to the projection $p : X \times S \longrightarrow S$.

If we allow morphisms between cosimplicial schemes then to get all G–functions (conjecturely) it is enough to consider all Gauss–Manin equations associated to smooth morphisms between schemes or cosimplicial schemes. One does not need to worry about extensions. They appear automatically if one allows cosimplicial schemes.

We would like to express our gratitude to P. Deligne who told us about problems considered in these notes and who discussed them with us.

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1. Cosimplicial spaces.

1.1. We define a category Δ as follows. The objects Δ_n of Δ are sequences of integers, $\Delta_n = (0,1,\ldots,n)$. The morphisms of Δ are monotonic maps $\mu : \Delta_n \longrightarrow \Delta_m$. Morphisms $\delta^i : \Delta_{n-1} \longrightarrow \Delta_n$, $0 \leq i \leq n$ given by $\delta^i(j) = j$ if $j < i$ and $\delta^i(j) = j+1$ if $j \geq i$ are called coface operators. Morphisms $s^j : \Delta_n \longrightarrow \Delta_{n-1}$, $0 \leq j \leq n-1$ given by $s^j(k) = k$ if $k \leq j$ and $s^j(k) = k-1$ if $k \geq j+1$ are called codegeneracy operators.

A simplicial object in a category $\mathbb{C}$ is a contravariant functor $X_\bullet : \Delta \longrightarrow \mathbb{C}$. For each $n \geq 0$, a simplicial set $\Delta[n]$ is given by the contravariant functor $\Delta[n] : \Delta \longrightarrow$ sets, where

$$\Delta[n](\Delta_m) = \Delta[n]_m = \mathrm{Hom}_\Delta(\Delta_n,\Delta_m)$$

and

$$\Delta[n](\mu)(\lambda) = \lambda \circ \mu \ ,$$

whenever $\lambda \circ \mu$ is defined in the category Δ. We denote by $\partial\Delta[n]$ the simplicial subset of $\Delta[n]$ generated by $\{\Delta[n](\delta^i)(\mathrm{id}_{\Delta_n}) \mid 1 \geq 0\}$.

A cosimplicial object in a category $\mathbb{C}$ is a covariant functor $X^\bullet : \Delta \longrightarrow \mathbb{C}$. The maps $\Delta(\delta^i)$ are called cofaces and $\Delta(s^j)$ are called codegeneracies.

<u>Examples</u>. Let $X_\bullet$ be a simplicial set and let M be an object of a category $\mathbb{C}$. If the category $\mathbb{C}$ has products then $M^{X_\bullet}$ (where in degree n, $M^{X_n} = \prod_{x \in X_n} M$) is a cosimplicial object in $\mathbb{C}$.

The cosimplicial space $\Delta^\bullet$ is defined in the following way. In degree n, Δ^n is the standard n–simplex $\{(t_0,\dots,t_n) \mid \sum_{i=0}^{n} t_i=1 \, , \, t_i \geq 0\}$. The cofaces and codegeneracies are the standard maps.

In the sequel $X^\bullet$ is a cosimplicial space or a cosimplicial scheme.

A sheaf on $X^\bullet$ consists of sheaves F_n on X^n together with maps $F_m \longrightarrow \alpha_* F_n$ (or $\alpha^* F_m \longrightarrow F_n$) for any $\alpha : \Delta_n \longrightarrow \Delta_m$ satisfying obvious compatibility conditions. If $F_\bullet$ is a sheaf on $X^\bullet$ with values in an abelian category $\mathbb{C}$ then the global section functor on $X^\bullet$, $\Gamma_\bullet(F_\bullet;X^\bullet) : n \longrightarrow \Gamma(F_n;X^n)$ is a simplicial object in $\mathbb{C}$. The obvious functor (simplicial objects in $\mathbb{C}$) $\longrightarrow$ (complexes in $\mathbb{C}$) associates to $\Gamma_\bullet(F_\bullet;X^\bullet)$ and hence also to $F_\bullet$, a complex, which we shall also denote by $\Gamma_\bullet(F_\bullet;X^\bullet)$.

If $I_\bullet^*$ is a complex of sheaves of A–modules on a cosimplicial space or a scheme $X^\bullet$ then $\Gamma_\bullet(I_\bullet^*,X^\bullet)$ is a complex of differential, graded A–modules.

Let $K_\bullet^*$ be a bicomplex with commuting differentials $\partial_{ij} : K_j^i \longrightarrow K_j^{i+1}$ and $\delta_{ij} : K_j^i \longrightarrow K_{j-1}^i$. We define the total complex of $K_\bullet^*$ in the following way

$$(\mathrm{Tot}\ K_\bullet^*)_m := \bigoplus_{i-j=m} K_j^i \ ,$$

$$d_m : (\mathrm{Tot}\ K_\bullet^*)_m \longrightarrow (\mathrm{Tot}\ K_\bullet^*)_{m+1} \ \text{ and } \ d_m \mid K_j^i = \partial_{ij} + (-1)^i \delta_{ij} \, .$$

<u>Example</u>. Let $X^\bullet$ be a cosimplicial space with cofaces operators δ^i. After applying the

singular cochains functor to each X^n we get a bicomplex $C^*(X^\bullet)$ with commuting differentials $\partial : C^i(X^j) \longrightarrow C^{i+1}(X^i)$ and $\delta_*^0 - \delta_*^1 + \ldots + (-1)^i\delta_*^j : C^i(X^j) \longrightarrow C^i(X^{j-1})$. The total cochain complex of $X^\bullet$, $\mathrm{Tot}\ C^*(X^\bullet)$ is then defined. We define singular cohomology of $X^\bullet$ in the following way $H^i(X^\bullet) := H^i(\mathrm{Tot}\ C^*(X^\bullet))$.

The total complex $\mathrm{Tot}\ K_\bullet^*$ is equipped with an increasing filtration R, which we call standard given by

$$R_s(\mathrm{Tot}\ K_\bullet^*) := \bigoplus_{j\leq s} K_j^* \ .$$

The first term of the spectral associated with the filtration R is equal to

$$E_1^{p,q} = H^q(K_p^*) \ .$$

If $K_j^* = 0$ for $j < 0$ and $K_j^i = 0$ for $i < 0$ then this spectral sequence converges strongly to $H^{p+q}(\mathrm{Tot}\ K_\bullet^*)$ because $R_a(\mathrm{Tot}\ K_\bullet^*) = 0$ for $a < 0$ and $\bigcup_b R_b(\mathrm{Tot}\ K_\bullet^*) = \mathrm{Tot}\ K_\bullet^*$.

Let $X^\bullet$ be a cosimplicial space or a scheme. A category of sheaves of abelian groups on $X^\bullet$ is an abelian category which we denote by $\mathrm{Ab}(X^\bullet)$. If $F_\bullet$ is a sheaf of abelian groups on $X^\bullet$, one shows that $F_\bullet$ has a right resolution $K_\bullet^*$ in $\mathrm{Ab}(X^\bullet)$ such that $H^r(X^q,K_q^p) = 0$ for $r > 0$. The resolution $K_\bullet^*$, after applying the functor of global sections leads to a bicomplex $\Gamma_\bullet(K_\bullet^*,X^\bullet)$. One defines

$$H^n(X^\bullet,F_\bullet) := H^n(\mathrm{Tot}\ \Gamma_\bullet(K_\bullet^*,X^\bullet)) \ .$$

One verifies that $H^n(X^\bullet, F_\bullet)$ does not depend on the choice of $K_\bullet^*$.

Let $D^+(X^\bullet)$ be the derived category of complexes of sheaves of abelian groups on $X^\bullet$ bounded below. Let $D(S)$ be the derived category of complexes of sheaves on abelian groups on S.

If $u^\bullet : X^\bullet \longrightarrow Y^\bullet$ is a morphism of cosimplicial spaces and $K_\bullet^*$ is a complex of sheaves of abelian groups on $X^\bullet$ then one computes $R\,u_*^\bullet(K_\bullet^*) \in D^+(Y^\bullet)$ in the following way. One takes a complex $L_\bullet^*$ quasi–isomorphic to $K_\bullet^*$ such that the components L_q^p of $L_\bullet^*$ satisfy $(R^i u_*^q)(L_q^p) = 0$ for $i > 0$. One gets then

$$(R\,u_*^\bullet)(K_\bullet^*) = u_*^\bullet(L_\bullet^*) \text{ and } (R^i u_*^\bullet)(K_\bullet^*) = H^i(u_*^\bullet(L_\bullet^*)) .$$

Let S^c be a constant cosimplicial scheme equal to S in each degree. The functor Tot defines trivially the functor $t := R\,\mathrm{Tot} : D^+(S^c) \longrightarrow D(S)$ by the formula $t(K) = \mathrm{Tot}(K)$.

Let $u^\bullet : X^\bullet \longrightarrow S^c$ be a morphism from $X^\bullet$ into a constant cosimplicial space S^c. Let $K_\bullet^*$ be a complex of sheaves of abelian groups on $X^\bullet$. The complex $tR\,u_*^\bullet K_\bullet^*$ equipped with the standard filtration gives a spectral sequence such that

$$E_1^{p,q} = (R^q\, u^p)(K_p^*) \Longrightarrow H^{p+q}(tRu_*^\bullet\, K_\bullet^*) .$$

<u>Example</u>. Let $X^\bullet$ be a smooth, cosimplicial scheme over a field of characteristic zero. Then the De Rham complex of smooth, differential forms on $X^\bullet$, Ω_X^* is a complex of sheaves on $X^\bullet$. In the degree n, it is equal to the De Rham complex $\Omega_{X^n}^*$ on X^n.

Example. Let $X^\bullet$ be a cosimplicial space and let R be a ring. A constant sheaf $R_\bullet$ on $X^\bullet$ is defined in the following way. In a degree n, on X^n it is equal to a constant sheaf, whose fiber is R. The sheaf cohomology $H^i(X^\bullet;R_\bullet)$ are equal to the singular cohomology of $X^\bullet$ with coefficients R.

1.2. We present here without proofs some facts we shall need later.

Let π be a discrete group. Let $I = \ker(Z[\pi] \longrightarrow \pi)$ be an augmentation ideal. Let K be a field of characteristic zero.

We set

$$\mathrm{Alg}(\pi;K) := \varinjlim_{n} \mathrm{Hom}(Z[\pi]/_{I^n};K) .$$

$\mathrm{Alg}(\pi;K)$ is a Hopf algebra. Let $\mathrm{Alg}_n(\pi;K)$ be a subalgebra of $\mathrm{Alg}(\pi;K)$ generated by $\mathrm{Hom}(Z[\pi]/_{I^n};K)$. Then $\mathrm{Alg}_n(\pi;K)$ is also a Hopf algebra. We have

$*_1$ $$\mathrm{Alg}(\pi/_{\Gamma^n\pi};K) = \mathrm{Alg}_n(\pi;K) .$$

We set

$$\pi_K := \mathrm{Spec}\ \mathrm{Alg}(\pi;K)$$

and

$$\pi_K^{(n)} := \operatorname{Spec} \operatorname{Alg}_n(\pi;K) \ .$$

Then π_K and $\pi_K^{(n)}$ are affine group schemes over K. The equality $\operatorname{Alg}(\pi,K) = \varinjlim_n \operatorname{Alg}_n(\pi;K)$ implies that $\pi_K = \varprojlim_n \pi_K^{(n)}$. The isomorphism $*_1$ implies that $(\pi/_{\Gamma^n\pi})_K = \pi_K^{(n)}$. Moreover we have $\pi_K/_{\Gamma^n\pi_K} = \pi_K^{(n)}$.

Therefore π_K is an affine, pro–unipotent group scheme over K. If π is finitely generated then π_K is also pro–algebraic.

If $L : K$ is an extension of fields then $\pi_K \underset{\operatorname{Spec} K}{\times} \operatorname{Spec} L = \pi_L$.

For a group scheme G, let $G(K)$ be a group of K–points of G. Let L be a K–algebra. We shall define a homomorphism

$$r_{L/K} : \pi \longrightarrow \pi_K(L)$$

in the following way. Let $g \in \pi$, then $r_{L/K}(g) : \operatorname{Alg}(\pi;K) \longrightarrow L$ is the evaluation at g homomorphism given by $r_{L/K}(g)(f) = f(g)$. To simplify notations we set $r_K := r_{K/K}$.

For any n the map

$$r_{\mathbb{Q}} : \pi/_{\Gamma^n\pi} \longrightarrow (\pi/_{\Gamma^n\pi})_{\mathbb{Q}}(\mathbb{Q})$$

is the Malcev rational completion of the nilpotent group $\pi/_{\Gamma^n\pi}$. The map

$$r_{\mathbb{Q}} : \pi \longrightarrow \pi_{\mathbb{Q}}(\mathbb{Q})$$

is the Malcev rational completion of π.

1.3. Let $X^\bullet$ and $Y^\bullet$ be two cosimplicial spaces. The space $\hom(X^\bullet,Y^\bullet)$ is a subspace of the product $\prod_{p=0}^{\infty}(Y^p)^{X^p}$ consisting of all sequences $(f_p)_{p=0}^{\infty}$ which commute with cofaces and codegeneracies. We set $\mathrm{Tot}(X^\bullet) := \hom(\Delta^\bullet,X^\bullet)$ (see [BK], [BS]). We shall define a map

$$\alpha^{X^\bullet} : \mathrm{Tot}\; C^*(X^\bullet) \longrightarrow C^*(\mathrm{Tot}\; X^\bullet) \; .$$

Let $f \in C^{n+m}(X^m)$, then $\alpha^{X^\bullet}(f)$ is defined in the following way. Let $\delta \in C_n(\mathrm{Tot}\; X^\bullet)$ be an n–simplex i.e. $\delta : \Delta^n \longrightarrow \mathrm{Tot}\; X^\bullet$. Let $\{\delta_i : \Delta^n \times \Delta^i \longrightarrow X^i\}_{i=0}^{\infty}$ be an adjoint map. The map $\delta_m : \Delta^n \times \Delta^m \longrightarrow X^m$ we consider as an $(n+m)$–chain on X^m. We set $\alpha^{X^\bullet}(f)(\delta) := f(\delta_m)$. One checks that $\alpha^{X^\bullet}$ is a chain map.

Let $X^\bullet$ be a smooth cosimplicial manifold or a cosimplicial simplicial complex. Let $\Omega^*(X^\bullet)$ be the De Rham complex of smooth, complex or real valued differential forms on $X^\bullet$ (if $X^\bullet$ is a cosimplicial manifold) or the Sullivan complex of $\mathbb{Q}$–polynomial differential forms on $X^\bullet$ (if $X^\bullet$ is a cosimplicial simplicial complex). Let K be equal to $\mathbb{C}$, R or $\mathbb{Q}$ respectively. Then the chain map

$$\int(X^\bullet) : \mathrm{Tot}\; \Omega^*(X^\bullet) \longrightarrow \mathrm{Tot}\; C^*(X^\bullet,K)$$

given by integration is a quasi–isomorphism. (Of course in the first two cases one integrates only over smooth chains and in the last case over simplicial chains but this is sufficient to have a quasi–isomorphism.) The composition of $\alpha^{X^\bullet}$ with $\int(X^\bullet)$ gives a chain map

$$\int_{X^\bullet} : \mathrm{Tot}(\Omega^*(X^\bullet)) \longrightarrow C^*(\mathrm{Tot}(X^\bullet);K) \ .$$

Now let X be a smooth manifold or a simplicial complex and let $x \in X$. We shall investigate the cosimplicial space $(X,x)^\bullet$. It follows from [C2] that

$$H^0(\mathrm{Tot}\ \Omega^*((X,x)^\bullet) = \mathrm{Alg}(\pi_1(X,x);K)$$

and

$$H^0(C^*((X,x)^\bullet,K) = \mathrm{Alg}(\pi_1(X,x);K) \ .$$

The map

$$\int_{(X,x)^\bullet} : \mathrm{Tot}(\Omega^*((X,x)^\bullet)) \longrightarrow C^*(\mathrm{Tot}((X,x)^\bullet);K)$$

has the following interpretation in terms of iterated integrals. Let $\delta \in \mathbb{C}_*(\mathrm{Tot}((X,x)^\bullet)$ be a zero simplex, which we view as a map $(\delta_i : \Delta^i \longrightarrow X^i)_{i=0}^{\infty}$. We use the following model for Δ^n; $\Delta^n = \{(t_1,\dots,t_n) \mid 0 \le t_1 \le 1, 0 \le t_2 \le t_1,\dots,0 \le t_n \le t_{n-1}\}$. Let us observe that $\delta_n(t_1,\dots,t_n) = (\gamma(t_1),\gamma(t_2),\dots,\gamma(t_n))$ where $\gamma : \Delta[1] \longrightarrow X$ is a loop at x. The form $\omega = \omega_1 \otimes \dots \otimes \omega_n \in \Omega^1(X)^{\otimes n} \longleftrightarrow \Omega^n(X^n)$ one evaluates on δ where δ is a smooth, singular n–simplex or an n–simplex in X. One gets
$\omega(\delta) = \int_{\Delta^n} \delta^*(\omega_1 \otimes \dots \otimes \omega_n) = \int_{\Delta^n} f_1(\gamma(t_1))dt_1 \wedge \dots \wedge f_n(\gamma(t_n))dt_n = \int_\gamma \omega_1,\dots,\omega_n$ – the value of the iterated integral $\int \omega_1,\dots,\omega_n$ on γ.

2. Multiplicative structure.

A (p,q)–shuffle is a permutation π of $\{0,1,2,\dots,p+q-1\}$ which satisfies $\pi(i) < \pi(j)$ if $0 \leq i < j \leq p-1$ or $p \leq i < j \leq p+q-1$. There is a bijection between (p,q)–shuffles and (q,p)–shuffles. If π is a (p,q)–shuffle then the corresponding (q,p)–shuffle π' is given by the formula $\pi'(i) = \pi(p+1)$ for $i = 0,1,\dots,q-1$ and $\pi'(i) = \pi(i-q)$ for $i = q,\dots,p+q-1$. We have that $\operatorname{sign} \pi' = \operatorname{sign} \pi$.

Let $A = \{n \longrightarrow A^{n,\bullet}\}$ be a simplicial object in the category of differential, graded, commutative algebras with face operators δ_α and degeneracy operators s_β. We shall define a product in Tot A ($(\mathrm{Tot}\ A)^n = \bigoplus_{q-p=n} A^{q,p}$) by the following formula.

If $x \in A^{q_1,p_1}$ and $y \in A^{q_2,p_2}$ then we set

$$x*y = \sum_{(p_1,p_2)\text{–shuffles}} (-1)^{p_1 \cdot p_2} \cdot \operatorname{sign}(\pi)(s_{\pi(p_2+p_1-1)} \circ \dots \circ s_{\pi(p_1)}(x)) \cdot (s_{\pi(p_1-1)} \circ \dots \circ s_{\pi(0)}(y)) .$$

The shuffle product is commutative i.e.

$$x*y = (-1)^{(q_1-p_1)(q_2-p_2)} y*x$$

and it commutes with the boundary operator $d : \mathrm{Tot}\ A \longrightarrow \mathrm{Tot}\ A$ i.e.

$$d(x*y) = d(x)*y + (-1)^{\deg x}\, x * d(y)$$

where $\deg x = n$ if $x \in (\mathrm{Tot}\ A)^n$.

For the standard filtration R of the total complex $\mathrm{Tot}(A)$ we have $R_a * R_b \subset R_{a+b}$. Therefore the spectral sequence associated with the filtration R is multiplicative.

Main Example. Let $X^\bullet$ be a cosimplicial, smooth variety and let $\Omega^*(X^\bullet)$ be the algebra of global sections of the De Rham complex of C^∞–complex valued, differencial forms or let $X^\bullet$ be a cosimplicial, smooth, affine scheme over a field of characteristic zero and let $\Omega^*(X^\bullet)$ be the algebra of global sections of the algebraic De Rham complex. It follows from the previous discussion that the group

$$H^*_{DR}(X^\bullet) = H^*(\mathrm{Tot}(\Omega^*(X^\bullet)))$$

is equipped with the commutative product.

If X is affine then the complex $\Omega^*(X)$ of global sections of the algebraic De Rham complex Ω^*_X is suitable to calculate cohomology. Moreover $\Omega^*(X)$ is equipped with a commutative product. This was essential in the example given above. However one can't use $\Omega^*(X)$ if X is an arbitrary quasi–projective scheme. We shall use then the construction of V. Navarro.

Following V. Navarro (see [N] § 4) for any smooth, quasi–projective scheme X over a field k of characteristic zero, there is a sheaf of graded, differential k–algeras on X such that

i) A^*_X is quasi–isomorphic to Ω^*_X;

ii) the quasi–isomorphism is a homomorphism of algebras;

iii) after forgeting the multiplicative structure A_X^* is the canonical cosimplicial resolution of Godement;

(iv) the construction of A_X^* is functorial.

Let $A^*(X)$ be the complex of global sections of A_X^*. Then $A^*(X)$ is the differential, graded, commutative algebra and $H_{DR}^*(X) = H^*(A^*(X))$.

Hence if $X^\bullet$ is a smooth, cosimplicial scheme over k then $H_{DR}^*(X^\bullet)$ is equipped with the commutative product. We have

$$H_{DR}^*(X^\bullet) = H^*(\mathrm{Tot}\ A^*(X))$$

and $H^*(\mathrm{Tot}\ A^*(X^\bullet))$ is equipped with the shuffle product.

3. Hopf algebra structures.

Let $A^* = \{A^p\}_{p\geq 0}$ be a differential, graded k–algebra with a differential ∂ of degree 1 and two augmentations $\epsilon_1, \epsilon_2 : A^* \longrightarrow k$ such that $\epsilon_i(A^p) = 0$ for $p > 0$. Let $(T(A^*), T(\partial)) = \{\bigoplus_{n\geq 0} A^{\otimes n}, \partial^{\otimes n}\}$ be a tensor algebra on $\{A^*, \partial\}$. $(T(A^*), T(\partial))$ is a simplicial object in the category of differential, graded, commutative algebras. The face and degeneracy operators are given by the following formulas

$$\delta_0(w_1\otimes\ldots\otimes w_n) = \epsilon_1(w_1)w_2\otimes\ldots\otimes w_n\,,$$
$$\delta_i(w_1\otimes\ldots\otimes w_n) = w_1\otimes\ldots\otimes w_i\cdot w_{i+1}\otimes\ldots\otimes w_n \quad \text{for } 0 < i < n\,,$$
$$\delta_n(w_1\otimes\ldots\otimes w_n) = \epsilon_2(w_n)w_1\otimes\ldots\otimes w_{n-1}$$

and

$$s_i(w_1\otimes\ldots\otimes w_n) = w_1\otimes\ldots\otimes w_{i-1}\otimes 1\otimes w_i\otimes\ldots\otimes w_n$$

for $0 \leq i \leq n$.

We set $\delta = \delta(\epsilon_1,\epsilon_2) = \sum_{i=0}^{n} (-1)^i \delta_i : A^{*\otimes n} \longrightarrow A^{*\otimes n+1}$. $T(A^*)$ is a bicomplex with commuting differentials $T(\partial)$ and δ. We set

$$B(A^*) := \mathrm{Tot}(T(A^*))$$

with a differential $d = d(\epsilon_1,\epsilon_2)$ given by $d(w_1\otimes\ldots\otimes w_n) := \partial^{\otimes n}(w_1\otimes\ldots\otimes w_n)+(-1)^{|w|}\delta(w_1\otimes\ldots\otimes w_n)$, where $|w| = |w_1|+|w_2|+\ldots+|w_n|$ and $|w_i|$ is the degree of w_i. The complex

$(B(A^*),d(\epsilon_1,\epsilon_2))$ equipped with the shuffle product $*$ is a k–algebra. This algebra is called a bar construction on A^*.

Let $\epsilon_3 : A^* \longrightarrow k$ be also an augmentation such that $\epsilon_3(A^p) = 0$ for $p > 0$. Let us define a comultiplication $\nabla(\epsilon_1,\epsilon_2,\epsilon_3)$

$$\nabla(\epsilon_1,\epsilon_2,\epsilon_3):(B(A^*),d(\epsilon_1,\epsilon_3)) \longrightarrow (B(A^*),d(\epsilon_1,\epsilon_2))\otimes(B(A^*),d(\epsilon_2,\epsilon_3))$$

by the following formula

$$\nabla(w_1\otimes\ldots\otimes w_n) := \sum_{k=0}^{n} (-1)^{k(|w_{k+1}|+|w_{k+2}|+\ldots+|w_n|)}(w_1\otimes\ldots\otimes w_k)\otimes(w_{k+1}\otimes\ldots\otimes w_n)$$

Let us define a product $*'$ in $B(A^*)\otimes B(A^*)$ by the following formula

$$(a\otimes b)*'(a_1\otimes b_1) := (-1)^{\deg(b)\deg(a_1)}(a*a_1)\otimes(b*b_1) \ .$$

<u>Lemma 3.1</u>. The map $\nabla(\epsilon_1,\epsilon_2,\epsilon_3)$ is a chain homomorphism of algebras.

We omit the verification.

<u>Lemma 3.2</u>. The involution $i = i(\epsilon_1,\epsilon_2):(B(A^*),d(\epsilon_1,\epsilon_2)) \longrightarrow (B(A^*),d(\epsilon_2,\epsilon_1))$ defined by the formula

$$i(w_1\otimes w_2\otimes\ldots\otimes w_n) = (-1)^{\frac{(n+1)n}{2}}(-1)^{\sum\limits_{i<j}|w_i||w_j|} w_n\otimes w_{n-1}\otimes\ldots\otimes w_1$$

is a chain homomorphism of algebras.

We omit the verification.

Let $e(\epsilon) : (B(A^*), d(\epsilon,\epsilon)) \longrightarrow (B(k),0)$ be induced by the augmentation $\epsilon : A^* \longrightarrow k$. The map $e(\epsilon)$ is a homomorphism of differential graded algebras.

Let $a(\epsilon_1,\epsilon_2) : (B(k),0) \longrightarrow (B(A^*), d(\epsilon_1,\epsilon_2))$ be a map induced by the map $k \longrightarrow A^*$. The map $a(\epsilon_1,\epsilon_2)$ is also a homomorphism of differential graded algebras.

We shall investigate maps induced by $\nabla(\epsilon_1,\epsilon_2,\epsilon_3)$, $i(\epsilon_1,\epsilon_2)$, $e(\epsilon)$ and $a(\epsilon_1,\epsilon_2)$ on the 0–th cohomology. We shall denote the induced maps by the same letters. Let us notice that $\nabla(\epsilon_1,\epsilon_2,\epsilon_3)$ induces a map

$$\nabla(\epsilon_1,\epsilon_2,\epsilon_3) : H^0(B(A^*)) \longrightarrow H^0(B(A^*))\otimes H^0(B(A^*)) \ .$$

<u>Lemma 3.3</u>. The maps induced on the 0–th cohomology satisfy:

i) $$(\nabla(\epsilon_1,\epsilon_2,\epsilon_3)\otimes \mathrm{id})\circ\nabla(\epsilon_1,\epsilon_2,\epsilon_3) = (\mathrm{id}\otimes\nabla(\epsilon_2,\epsilon_3,\epsilon_4))\circ\nabla(\epsilon_1,\epsilon_2,\epsilon_4) \ ,$$

ii) $$*\circ(\mathrm{id}\otimes i(\epsilon_2,\epsilon_1))\circ\nabla(\epsilon_1,\epsilon_2,\epsilon_1) = a(\epsilon_1,\epsilon_2)\circ e(\epsilon_1)$$

and

$$*\circ(i(\epsilon_1,\epsilon_2)\otimes \mathrm{id})\circ\nabla(\epsilon_1,\epsilon_2,\epsilon_1) = a(\epsilon_2,\epsilon_1)\circ e(\epsilon_1) \ ,$$

iii) $$((a(\epsilon_1,\epsilon_2)\circ e(\epsilon_1))\otimes \mathrm{id})\circ\nabla(\epsilon_1,\epsilon_1,\epsilon_2) = \mathrm{id}$$

and

$$(\mathrm{id}\otimes(a(\epsilon_1,\epsilon_2)\circ e(\epsilon_2)))\circ\nabla(\epsilon_1,\epsilon_2,\epsilon_2) = \mathrm{id}\ .$$

We omit the verification.

The complex $B(A^*)$ is equipped with the standard filtration $\{R_iB(A^*)\}_{i=0}^{\infty}$, which in the case of $B(A^*)$ coincides with the filtration given by length of tensors. On $B(A^*)\otimes B(A^*)$ we consider the tensor product of standard filtrations.

<u>Lemma 3.4</u>. The structure map $\nabla(\epsilon_1,\epsilon_2,\epsilon_3)$, $i(\epsilon_1,\epsilon_2)$, $e(\epsilon)$ and $a(\epsilon_1,\epsilon_2)$ are compatible with standard filtrations.

This is clear from the definitions of these maps.

The filtration $\{R_iB(A^*)\}_{i=0}^{\infty}$ induces a filtration

$$\{\overline{R}_iH^0(B(A^*)) = \mathrm{im}(H^0(R_iB(A^*)) \longrightarrow H^0(B(A^*)))\}_{i=0}^{\infty}$$

of $H^0(B(A^*))$. Let $\mathcal{R}_iH^0(B(A^*))$ be a subalgebra of $H^0(B(A^*))$ generated by $\overline{R}_iH^0(B(A^*))$.

To simplify notation we set $\mathrm{Alg}(\epsilon) := H^0(B(A^*),d(\epsilon,\epsilon))$, $\mathrm{Alg}_n(\epsilon) := \mathcal{R}_i(H^0(B(A^*),d(\epsilon,\epsilon)))$, $\mathrm{Torsor}(\epsilon_1,\epsilon_2) := H^0(B(A^*),d(\epsilon_1,\epsilon_2)$ and $\mathrm{Torsor}_n(\epsilon_1,\epsilon_2) := \mathcal{R}_i(H^0(B(A^*),d(\epsilon_1,\epsilon_2)))$. We have $\mathrm{Alg}_0(\epsilon) = k$,

$\mathrm{Torsor}_0(\epsilon_1,\epsilon_2) = k$, $\bigcup_{i=0}^{\infty} \mathrm{Alg}_i(\epsilon) = \mathrm{Alg}(\epsilon)$ and $\bigcup_{i=0}^{\infty} \mathrm{Torsor}_i(\epsilon_1,\epsilon_2) = \mathrm{Torsor}(\epsilon_1,\epsilon_2)$.

Hence we get

$$\mathrm{Spec}\,\mathrm{Alg}(\epsilon) = \varprojlim_i \mathrm{Spec}\,\mathrm{Alg}_i(\epsilon)$$

and

$$\mathrm{Spec}(\mathrm{Torsor}(\epsilon_1,\epsilon_2)) = \varprojlim_i \mathrm{Spec}(\mathrm{Torsor}_i(\epsilon_1,\epsilon_2)) \ .$$

<u>Theorem 3.5</u>. $\mathrm{Spec}(\mathrm{Alg}(\epsilon))$ is an affine, pro–nilpotent group scheme over k. $\mathrm{Spec}(\mathrm{Alg}_n(\epsilon))$ is an affine, nilpotent group scheme over k. $\mathrm{Spec}((\mathrm{Torsor}(\epsilon_1,\epsilon_2))$ (resp. $\mathrm{Spec}(\mathrm{Torsor}_n(\epsilon_1,\epsilon_2))$ is an affine torsor over $\mathrm{Spec}\,\mathrm{Alg}(\epsilon_i)$ (resp. $\mathrm{Spec}\,\mathrm{Alg}_n(\epsilon_i)$) for $i = 1$ (resp. $i = 2$) on the left side (resp. right side).

Let us assume that $H^i(A^*)$ is a finitely generated k–module for each i. Then $\mathrm{Spec}\,\mathrm{Alg}_n(\epsilon)$ is an algebraic group scheme.

<u>Proof</u>. It follows immediately from Lemma 3.3 that $\mathrm{Spec}\,\mathrm{Alg}(\epsilon)$ is an affine group scheme and that $\mathrm{Spec}\,\mathrm{Torsor}(\epsilon_1,\epsilon_2)$ is an affine torsor over $\mathrm{Spec}\,\mathrm{Alg}(\epsilon_1)$ on the left hand side and over $\mathrm{Spec}\,\mathrm{Alg}(\epsilon_2)$ on the right hand side. Lemma 3.4 and the discussion below Lemma 3.4 imply the corresponding statement for $\mathrm{Spec}\,\mathrm{Alg}_n(\epsilon)$ and $\mathrm{Spec}\,\mathrm{Torsor}_n(\epsilon_1,\epsilon_2)$.

Let I_{n-1} be an ideal in $\mathrm{Alg}_{n-1}(\epsilon)$ generated by $\ker(\mathrm{Alg}_{n-1}(\epsilon) \longrightarrow k)$. Let

$$\tau : \mathrm{Alg}_n(\epsilon) \otimes (\mathrm{Alg}_n(\epsilon)/_{I_{n-1}}) \longrightarrow (\mathrm{Alg}_n(\epsilon)/_{I_{n-1}}) \otimes \mathrm{Alg}_n(\epsilon)$$

be given by $\tau(x\otimes y) = y\otimes x$. Then the structure maps induced by ∇,

$$\nabla' : \mathrm{Alg}_n(\epsilon) \longrightarrow \mathrm{Alg}_n(\epsilon) \otimes (\mathrm{Alg}_n(\epsilon)/_{I_{n-1}})$$

and

$$\nabla'' : \mathrm{Alg}_n(\epsilon) \longrightarrow (\mathrm{Alg}_n(\epsilon)/_{I_{n-1}}) \otimes \mathrm{Alg}_n(\epsilon)$$

satisfy $\tau \circ \nabla' = \nabla''$. This implies that $\mathrm{Spec}\,\mathrm{Alg}_n(\epsilon)$ is a nilpotent group scheme. The isomorphism $\mathrm{Spec}\,\mathrm{Alg}(\epsilon) = \varprojlim_n \mathrm{Spec}\,\mathrm{Alg}_n(\epsilon)$ implies that $\mathrm{Spec}\,\mathrm{Alg}(\epsilon)$ is a pro–nilpotent group scheme.

If $H^i(A^*)$ is a finitely generated k–module for each i then $R_iH^0(B(A^*))$ is a finitely generated k–module. Hence $\mathrm{Alg}_i(\epsilon)$ is a finitely generated k–algebra, consequently $\mathrm{Spec}\,\mathrm{Alg}_i(\epsilon)$ is an algebraic group scheme and $\mathrm{Spec}\,\mathrm{Alg}(\epsilon)$ is a pro–algebraic group scheme.

<u>Example 3.6</u>. Let X be an affine, smooth algebraic variety defined over a field k. Let $\Omega^*(X)$ be the algebra of global sections of the algebraic De Rham complex Ω^*_X. If x,y,z are three k–points then they define augmentations $\epsilon_x,\epsilon_y,\epsilon_z : \Omega^*(X) \longrightarrow k$. Hence the algebra $(B(\Omega^0(X))d(\epsilon_x,\epsilon_z))$ is equipped with structure maps $\nabla(\epsilon_x,\epsilon_y,\epsilon_z)$, $i(\epsilon_x,\epsilon_z)$, $a(\epsilon_x,\epsilon_z)$ and $e(\epsilon_x)$ if $x = z$.

Let us assume now that X is a smooth scheme of finite type over k. Any point $x \in X$ defines an augmentation $\epsilon_k : A^*(X) \longrightarrow k$. Hence the algebra $(B(A^*(X)), d(\epsilon_x,\epsilon_z))$ is equipped with structure maps $\nabla(\epsilon_x, \epsilon_y, \epsilon_z)$, $i(\epsilon_x, \epsilon_z)$, $a(\epsilon_x, \epsilon_z)$ and $e(\epsilon_x)$ if $x = z$.

3.7. Let X be a space or a scheme over a field k . Let $x,y \in X$ be two points of the space X or let x,y be two k–points of the scheme X . Let $(X;x,y)^\bullet$ be the following cosimplicial space or scheme:

$$* \;\begin{matrix}\xrightarrow{d_0}\\ \xrightarrow{d_1}\end{matrix}\; X \;\begin{matrix}\xrightarrow{d_0}\\ \xrightarrow{d_1}\\ \xrightarrow{d_2}\end{matrix}\; X^2 \;\begin{matrix}\xrightarrow{d_0}\\ \vdots\\ \xrightarrow{d_n}\end{matrix}\; X^3 \;\cdots\cdots\; \begin{matrix}\xrightarrow{d_0}\\ \vdots\\ \xrightarrow{d_n}\end{matrix}\; X^n \;\cdots\cdots$$

where $d_0(*) = x$, $d_1(*) = y$;

$$\begin{aligned} d_0(x_1,\dots,x_n) &= (x,x_1,\dots,x_n) \; ; \\ d_i(x_1,\dots,x_n) &= (x_1,\dots,x_i,x_i,\dots,x_n) \; ; \\ d_n(x_1,\dots,x_n) &= (x_1,\dots,x_n,y) \; . \end{aligned}$$

The codegeneracy operators of $(X;x,y)^\bullet$ are projections. If $x = y$ then we set $(X;x)^\bullet := (X;x,y)^\bullet$.

Let us assume that we have one of the following situations:

i) X is a smooth manifold, $\Omega^*(X)$ and $\Omega^*((X;x,y)^\bullet)$ are the De Rham complexes of smooth, differential forms on X and $(X;x,y)^\bullet$.

ii) X is a simplicial complex, $\Omega^*(X)$ and $\Omega^*((X;x,y)^\bullet)$ are the complexes of Sullivan polynomial, differential forms on X and $(X;x,y)^\bullet$.

iii) X is a smooth algebraic variety over k , $\Omega^*(X)$ and $\Omega^*((X;x,y)^\bullet)$ are the De

Rham complexes of smooth, algebraic differential forms on X and $(X;x,y)^{\bullet}$.

iv) X is a smooth, separated scheme of finite type over k, $\Omega^*(X)$ is $A^*(X)$ and $\Omega^*((X;x,y)^{\bullet})$ is $A^*((X;x,y)^{\bullet})$.

Then

$$B(\Omega^*(X),d(\epsilon_x,\epsilon_y)) \longrightarrow \mathrm{Tot}\ \Omega^*((X;x,y)^{\bullet})$$

is a quasi–isomorphism. Therefore $H^0(\mathrm{Tot}\ \Omega^*((X;x,y)^{\bullet})) \approx H^0(B(\Omega^*(X),d(\epsilon_x,\epsilon_y))$.

4. Poincaré groupoids.

We repeat the constructions from section 3 in the relative situation.

Let M be a smooth scheme of finite type over a field k of characteristic zero. The inclusion of simplicial sets $\partial\Delta[1] \hookrightarrow \Delta[1]$ induces a map of cosimplicial schemes

$$p^\bullet : M^{\Delta[1]} \longrightarrow M^{\partial\Delta[1]} .$$

Let $S^1 := \Delta[1]/_{\partial\Delta[1]}$ and let $\Delta[0] \hookrightarrow S^1$ be the image of $\partial\Delta[1]$ in S^1. Then the surjections of simplicial sets

$$\begin{tikzcd} \Delta[1] \arrow[r, two heads] & S^1 \\ \partial\Delta[1] \arrow[u, hook] \arrow[r, two heads] & \Delta[0] \arrow[u, hook] \end{tikzcd}$$

induce inclusions of cosimplicial schemes

$$\begin{tikzcd} M^{S^1} \arrow[r, hook] \arrow[d, "p_1^\bullet"] & M^{\Delta[1]} \arrow[d, "p^\bullet"] \\ M^{\Delta[0]} \arrow[r, hook] & M^{\partial\Delta[1]} \end{tikzcd} .$$

Let $\Omega^\bullet := \Omega^*_{M^{\Delta[1]}/M^{\partial\Delta[1]}}$ be the De Rham complex of smooth, relative $M^{\partial\Delta[1]}$–differentials on $M^{\Delta[1]}$.

Let us assume that M is affine. Let $\pi : M^3 \longrightarrow M^2$ be given by $\pi(a,m,b) = (a,b)$. Let

$\Omega^* := \pi_*\Omega^*_{M^3/M^2}$. Two maps over M^2, $i_0,i_1 : M^2 \longrightarrow M^3$ given by $i_0(x,y) = (x,x,y)$, $i_1(x,y) = (x,y,y)$ define two augmentations $\epsilon_k : \Omega^* \longrightarrow \pi_* i_k^* \Omega^*_{M^2/M^2} = \mathcal{O}_{M^2}$. The complex Ω^* is an $\mathcal{O}_{M^2}$–algebra, hence the bar construction on it, $(B(\Omega^*),d(\epsilon_0,\epsilon_1))$ is also an $\mathcal{O}_{M^2}$–algebra.

Lemma 4.1. There are quasi–isomorphisms of algebras

$$tRp_*^{\bullet}\Omega^*_{M^{\Delta[1]}/M^{\partial\Delta[1]}} \approx tp_*^{\bullet}\Omega^*_{M^{\Delta[1]}/M^{\partial\Delta[1]}} \approx B(\Omega^*) \ .$$

Let $j : M^3 \longrightarrow M^2 \times M^2$ be given by $j(m_1,m,m_2) = ((m_1,m),(m,m_2))$.

We set $B(\Omega^*) \otimes_{\mathcal{O}_M} B(\Omega^*) := j^*(B(\Omega^*) \mathbin{\widehat{\otimes}} B(\Omega^*))$. We shall construct a homomorphism of $\mathcal{O}_{M^2}$–algebras

4.1.1. $$\nabla : B(\Omega^*) \longrightarrow \pi_*(B(\Omega^*) \otimes_{\mathcal{O}_M} B(\Omega^*)) \ .$$

Let U be a Zariski open subset of M^2. Then the group $(\Omega^* \mathcal{O}_{M^2}^{\otimes n})(U) = \Omega^*(M)^{\otimes n} \otimes \mathcal{O}_{M^2}(U)$, while $(\pi_* j^*((\Omega^* \mathcal{O}_{M^2}^{\otimes p}) \otimes_{\mathcal{O}_M} (\Omega^* \mathcal{O}_{M^2}^{\otimes q})))(U) = \Omega^*(M)^{\otimes p} \otimes \mathcal{O}(M) \otimes \Omega^*(M)^{\otimes q} \otimes \mathcal{O}_{M^2}(U)$. We set

$$\nabla_U(w_1 \otimes \ldots \otimes w_n \otimes f) =$$

$$= \sum_{p=0}^{n} (-1)^{p(\deg w_{p+1} + \ldots + \deg w_n)} (w_1 \otimes \ldots \otimes w_p) \otimes 1 \otimes (w_{p+1} \otimes \ldots \otimes w_n) \otimes f \ .$$

The restriction of ∇ to fibers, from the fiber over (x,y) to the fiber over (x,z,y) is the map $\nabla(\epsilon_x, \epsilon_z, \epsilon_y)$.

We shall define an involution

$$i : (B(\Omega^*) \, , d(\epsilon_0, \epsilon_1)) \longrightarrow \tau_*(B(\Omega^*), d(\epsilon_0, \epsilon_1))$$

where $\tau : M^2 \longrightarrow M^2$ $\tau(x,y) = (y,x)$. Let U be a Zariski open subset of $M \times M$. We set

$$i_U : \Omega^*(M)^{\otimes n} \otimes \mathcal{O}(U) \longrightarrow \Omega^*(M)^{\otimes n} \otimes \mathcal{O}(\tau^{-1}(U)) \ ,$$

$$i_U(w_1 \otimes w_2 \otimes \ldots \otimes w_n \otimes f) =$$

$$= (-1)^{\frac{n(n+1)}{2}} (-1)^{\sum_{i<j} |w_i| \cdot |w_j|} w_n \otimes w_{n-1} \otimes \ldots \otimes w_1 \otimes {}^{\tau}f$$

where ${}^{\tau}f = f \circ \tau$.

The restriction of i to the fiber over (x,y) coincides with the map $i(\epsilon_x, \epsilon_y)$.

The map of simplicial sets $S^1 \longrightarrow \Delta[0]$ induces a map of cosimplicial schemes

$e^\bullet : M^{\Delta[0]} \longrightarrow M^{S^1}$. Hence we get a map

$$e : tRp^\bullet_{1*}\Omega^*{}_M S^1/_M\Delta[0] \approx \Delta^* B(\Omega^*) \longrightarrow \mathcal{O}_M$$

where $\Delta : M \longrightarrow M\times M$ is the diagonal.

Finally let $a : B(\mathcal{O}_{M\times M}) \longrightarrow B(\Omega^*)$ be the map induced by $\mathcal{O}_{M\times M} \longrightarrow \Omega^*$. The restriction of e (resp. a) to a fiber over x (resp. (x,y)) is equal to $e(\epsilon_x)$ (resp. $a(\epsilon_x,\epsilon_y)$).

The product structure in $B(\Omega^*)$ we shall denote as usual by $*$.

The maps induced by ∇ , i , e and a on the 0–th cohomology we denote by the same letter.

4.2. Let us set $H := H^0(B(\Omega^*)) = H^0(tRp^\bullet_*\Omega^\bullet)$. Then we have the following maps of $\mathcal{O}_{M\times M}$–algebras:

$$\nabla : H \longrightarrow \pi_*(H \otimes_{\mathcal{O}_M} H) \ ,$$
$$i : H \longrightarrow \tau_* H$$
$$e : \bar{H} \longrightarrow \mathcal{O}_M$$
$$a : \mathcal{O}_{M\times M} \longrightarrow H$$

where $H \otimes_{\mathcal{O}_M} H = j^*(H \widehat{\otimes} H)$, $\bar{H} = \Delta^*(H)$ and $\Delta : M \longrightarrow M\times M$, $m \longrightarrow (m,m)$.

Proposition 4.3. Let ∇, i, e and a be as above. Then the following diagram commutes:

i)
$$\begin{tikzcd}
(\mathrm{id}_{M^2} \times_M \pi)^* \circ \pi^* H = (\pi \times_M \mathrm{id}_{M^2})^* \circ \pi^* H \arrow[r, "\nabla"] \arrow[d, "\nabla"'] & (\pi \times_M \mathrm{id}_{M^2})^* (H \otimes_{\mathcal{O}_M} H) \arrow[d, "\nabla \otimes_{\mathcal{O}_M} \mathrm{id}"] \\
(\mathrm{id}_{M^2} \times_M \pi)^* (H \otimes_{\mathcal{O}_M} H) \arrow[r, "\mathrm{id} \otimes_{\mathcal{O}_M} \nabla"] & H \otimes_{\mathcal{O}_M} H \otimes_{\mathcal{O}_M} H
\end{tikzcd}$$

ii)
$$\begin{tikzcd}
\bar{H} \arrow[r, "\nabla"] \arrow[d, "e"'] & \bar{H} \otimes_{\mathcal{O}_M} \bar{H} \arrow[r, "\mathrm{id} \otimes_{\mathcal{O}_M} i", "(\text{or } i \otimes_{\mathcal{O}_M} \mathrm{id})"'] & \bar{H} \otimes_{\mathcal{O}_M} \bar{H} \arrow[d, "*"] \\
\mathcal{O}_M \arrow[rr, "a"] & & \bar{H}
\end{tikzcd}$$

($\bar{H} \otimes_{\mathcal{O}_M} \bar{H}$ is the restriction of $H \otimes_{\mathcal{O}_M} H$ to $M \longrightarrow M^3$ $m \longrightarrow (m,m,m)$.)

iii) The composition

$$\bar{H} \xrightarrow{\nabla} \bar{H} \otimes_{\mathcal{O}_M} \bar{H} \xrightarrow[(\text{or } e \otimes \mathrm{id})]{\mathrm{id} \otimes e} \bar{H}$$

is equal to the identity.

The proposition follows from Lemma 3.3.

Definition–Proposition 4.4. Let M be a smooth scheme of finite type over a field k of characteristic zero. The quasi–coherent sheaf H of $\mathcal{O}_{M \times M}$–algebras on $M \times M$ is called a Poincaré sheaf on M if it is equipped with structure maps ∇, i, e and a as in 4.2 and if it satisfies conditions i), ii) and iii) of 4.3.

Let us set $P := \mathrm{Spec}\, H$. Then P is equipped with the following structure maps derived from $\overline{\nabla}$, i, e and a:

$$o := \mathrm{Spec}\, \overline{\nabla} : P \underset{M}{\times} P \longrightarrow P \text{ over } \pi ,$$

$$(\ )^{-1} := \mathrm{Spec}\, i : P \longrightarrow P \text{ over } \tau ,$$

$$\bar{e} := \mathrm{Spec}\, e : M \longrightarrow \overline{P} := \mathrm{Spec}\, \overline{H} \text{ over } M .$$

These maps have the following property:

a) $$o \circ (\mathrm{id} \times o) = o \circ (o \times \mathrm{id})$$

b) $$o \circ (\mathrm{id} \times \bar{e}) = \mathrm{id} ,\ o \circ (\bar{e} \times \mathrm{id}) = \mathrm{id}$$

c) $$o \circ (\mathrm{id} \times (\ )^{-1}) = \bar{e} ,\ o \circ ((\ )^{-1} \times \mathrm{id}) = \bar{e} .$$

The morphism $P \longrightarrow M{\times}M$ equipped with structure maps o, $(\ )^{-1}$, $\bar{e}$ is called a Poincaré groupoid over M.

If $P \longrightarrow M{\times}M$ is a Poincaré groupoid over M then $\overline{P} \longrightarrow M$ is a group scheme over M and $P_{|M\times\{m\}}$ (resp.: $P_{|\{m\}\times M}$) $\longrightarrow M$ is a principal right (resp. left) Spec $H_{(m,m)}$–bundle over M where $H_{(m,m)}$ is a fiber of H over (m,m).

4.5. Once more the restriction that M is affine is not essential. It follows from [N1] § 4 that for any smooth morphism of constant rank $X \xrightarrow{\pi} S$ between smooth quasi–projective schemes there is a sheaf $A^*_{X/S}$ of graded differential $\mathcal{O}_S$–algebras such that

i) $A^*_{X/S}$ is quasi–isomorphic to $\Omega^*_{X/S}$,

ii) the quasi–isomorphism is a homomorphism of algebras,

iii) the complex $A^*_{X/S}$ is functorial,

iv) $H^i(\pi_* A^*_{X/S}) \approx H^i(R\pi_*\Omega^*_{X/S})$.

Using the complex of sheaves $A^*_{X/S}$ instead of $\Omega^*_{X/S}$ we repeat the construction from this section for any M smooth and quasi–projective over a field k of characteristic zero.

5. Gauss–Manin connection on the bundle of fundamental groups.

Let V be a smooth scheme of finite type over a field k of characteristic zero. $V^{\Delta[1]}$ is a cosimplicial scheme augmented by $V \times V$ and $V^{\partial\Delta[1]}$ is a constant cosimplicial scheme equal to $V \times V$ in each degree. The inclusion of simplicial sets $\partial\Delta[1] \hookrightarrow \Delta[1]$ induces a cosimplicial map

$$p^{\bullet} : V^{\Delta[1]} \longrightarrow V^{\partial\Delta[1]}$$

between cosimplicial schemes. Let $\Omega_k^{\bullet} := \Omega^{*}_{V^{\Delta[1]}/V^{\partial\Delta[1]}}$ be the De Rham complex of smooth, relative $V^{\partial\Delta[1]}$–differentials on $V^{\Delta[1]}$ i.e. in degree n on $V^{\Delta[1]_n}$ we have the complex $\Omega^{*}_{V^{\Delta[1]_n}/V^{\partial\Delta[1]_n}}$. We repeat these constructions and definitions if X is smooth, holomorphic and we get $\Omega^{\bullet}_{hol}$.

<u>Theorem 5.1</u>. Let V be a smooth scheme of finite type over a field k of characteristic zero.

i) There exists a canonical, integrable connection d_k on the relative De Rham cohomology groups $H^i(tRp^{\bullet}_{*}\Omega^{\bullet}_k)$. The connection d_k is compatible with the product structure i.e.

$$d_k(e*e') = d_k(e)*e' + (-1)^q e*d_k(e')$$

if $e \in H^q(tRp^{\bullet}_{*}\Omega^{\bullet}_k)$ and $e' \in H^{q'}(tRp^{\bullet}_{*}\Omega^{\bullet}_k)$.

ii) If $K : k$ is a field extension and $V_K = V \times_k K$ then

$$H^i(tRp_*^\bullet\Omega_K^\bullet) \equiv H^i(tRp_*^\bullet\Omega_k^\bullet) \underset{k}{\otimes} K$$

and

$$d_K = d_k \underset{k}{\otimes} \mathrm{id}_K \ .$$

iii) Let X be a smooth, holomorphic variety. Then there exists a canonical, integrable, compatible with the product structure connection d_{hol} on the relative De Rham cohomology groups $H^i(tRp_*^\bullet\Omega_{hol}^\bullet)$.

iv) Let V be a smooth scheme of finite type over complex numbers $\mathbb{C}$. Then the connection $d_{\mathbb{C}}$ extends to the connection

$$(d_{\mathbb{C}})^{an} : H^i(tRp_*^\bullet\Omega_{\mathbb{C}}^\bullet)^{an} \longrightarrow (\Omega^1_{V\times V})^{an} \underset{(\mathcal{O}_{V\times V})^{an}}{\otimes} (H^i(tRp_*^\bullet\Omega_{\mathbb{C}}^\bullet))^{an}$$

which coincides with the connection d_{hol} for V^{an}. (V^{an} is a holomorphic variety corresponding to V, in general $(\ )^{an}$ is an analytic object: sheaf, variety, morphism ... corresponding to an algebraic object $(\ )$.)

v) The sheaves $H^i(tRp_*^\bullet\Omega_k^\bullet)$ of $\mathcal{O}_{V\times V}$–modules are locally free.

vi) Let X be a smooth, holomorphic variety. Let $\mathbb{C}_\bullet$ be a constant sheaf on $X^{\Delta[1]}$ whose fibers are equal to $\mathbb{C}$. Then the sheaf of horizontal sections of the connection d_{hol} on $H^i(tRp_*^\bullet\Omega_{hol}^\bullet)$ is equal to $H^i(tRp_*^\bullet\mathbb{C}_\bullet)$.

Proof. We shall introduce a connection following the method of Katz and Oda (see [KO]). The algebraic De Rham complex $\Omega^*_{V^{\Delta[1]}}$ is a complex of sheaves on $V^{\Delta[1]}$. For each n we have a map

$$p^n : V^{\Delta[1]_n} \longrightarrow V^{\partial\Delta[1]_n} = V\times V \ .$$

Each complex $\Omega^*_{V^{\Delta[1]_n}}$ admits a canonical filtration

$$F^0_n\Omega^*_{V^{\Delta[1]_n}} \supset F^1_n\Omega^*_{V^{\Delta[1]_n}} \supset F^2_n\Omega^*_{V^{\Delta[1]_n}} \supset \cdots$$

where

$$F^i_n\Omega^*_{V^{\Delta[1]_n}} = \mathrm{image}(\Omega^{*-i}_{V^{\Delta[1]_n}} \underset{\mathcal{O}_{V^{\Delta[1]_n}}}{\otimes} (p^n)^*(\Omega^i_{V\times V}) \longrightarrow \Omega^*_{V^{\Delta[1]_n}}) \ .$$

These filtrations for $n = 0,1,2,\dots$ are compatible with coface operators and codegeneracy operators of $V^{\Delta[1]}$ and therefore they induce a decreasing filtration $\{F^i\}_{i=0,1,2,\dots}$ of $\Omega^*_{V^{\Delta[1]}}$. One calculates that the first term of the hypercohomology spectral sequence associated with the filtration F^i of $\Omega^*_{V^{\Delta[1]}}$ with respect to the functor $tp^\bullet_*$ is equal to

$$E^{p,q}_1(\Omega^*_{V^{\Delta[1]}}) \cong \Omega^p_{V\times V} \underset{\mathcal{O}_{V\times V}}{\otimes} H^q(tRp^\bullet_*\Omega^\bullet_k) \ .$$

The filtration F^i of $\Omega^*_{V^{\Delta[1]}}$ is compatible with the shuffle product $*$ i.e. we have $F^i * F^j \subset F^{i+j}$. This implies that the spectral sequence is multiplicative. The differential d_1 has bidegree $(1,0)$ therefore for every q we get a complex

$$(*_q) \qquad 0 \longrightarrow \mathcal{O}_{V\times V} \underset{\mathcal{O}_{V\times V}}{\otimes} H^q(tRp_*^\bullet\Omega_k^\bullet) \xrightarrow{d_1^{0,q}} \Omega^1_{V\times V} \underset{\mathcal{O}_{V\times V}}{\otimes} H^q(tRp_*^\bullet\Omega_k^\bullet) \longrightarrow$$

$$\xrightarrow{d_1^{1,q}} \Omega^2_{V\times V} \underset{\mathcal{O}_{V\times V}}{\otimes} H^q(tRp_*^\bullet\Omega_k^\bullet) \xrightarrow{d_1^{2,q}} \dots .$$

We have a diagram of cosimplicial schemes

$$\begin{tikzcd} V^{\Delta[1]} \arrow[r, "p^\bullet"] \arrow[d, "p^\bullet"] & V^{\partial\Delta[1]} \arrow[d, "id"] \\ V^{\partial\Delta[1]} \arrow[r, "id"] & V^{\partial\Delta[1]} \end{tikzcd}$$

which induces a map of $\mathcal{O}_{V\times V}$–algebras

$$H^0(tRid_*\Omega^*_{V^{\partial\Delta[1]}/V^{\partial\Delta[1]}}) = \mathcal{O}_{V\times V} \longrightarrow H^0(tRp_*^\bullet\Omega_k^\bullet) .$$

We can repeat our construction of the spectral sequence for a cosimplicial map $V^{\partial\Delta[1]} \xrightarrow{id} V^{\partial\Delta[1]}$. We get a complex $\Omega^*_{V\times V}$ which is a subcomplex of the complex $(*_0)$. From the multiplicative properties of the differentials we get

$$d_1^{0,q}(f*s) = d(f)*s + f*d_1^{0,q}(s)$$

for any $f \in \mathcal{O}_{V\times V} \subset H^0(tRp_*^\bullet\Omega_k^\bullet)$ and $s \in H^q(tRp_*^\bullet\Omega_k^\bullet)$. Hence the map

$$d_1^{0,q} : H^q(tRp_*^\bullet\Omega_k^\bullet) \longrightarrow \Omega^1_{V\times V} \underset{\mathcal{O}_{V\times V}}{\otimes} H^q(tRp_*^\bullet\Omega_k^\bullet)$$

is a connectiom. The differentials $d_1^{p,q}$ satisfy

$$d_1^{p,q}(w*s) = d(w)*s + (-1)^p w*d_1^{0,q}(s)$$

for any $w \in \Omega^p_{V\times V}$ and $s \in H^q(tRp_*^\bullet\Omega_*^\bullet)$. Hence the formula $d_1^{1,q} \circ d_1^{0,q} = 0$ implies that the connection $d_1^{0,q}$ is integrable. We set $d_k := d_1^{0,q}$.

The part ii) follows trivially from the fact that the functor $\underset{k}{\otimes} K$ is exact.

Using holomorphic forms we construct a connection d_{hol} in the same way as we have constructed d_k. Then points iii) and iv) are instantaneous.

To show v) let us notice that the sheaves $H^i(tRp_*^\bullet\Omega_k^\bullet)$ are quasi–coherent sheaves equipped with the connection. This implies that $H^i(tRp_*^\bullet\Omega_k^\bullet)$ are locally free in Zariski topology. (See [Ma] Remark 1.2).

Now we shall show the point vi). The sheaf $H^i(tRp_*^\bullet\Omega_{hol}^\bullet)$ is a sheaf of $\mathcal{O}_{X\times X}$–modules equipped with the integrable connection. From the holomorphic Poincaré Lemma it follows that the complex of sheaves

$$0 \longrightarrow \ker d_1^{0,q} \longrightarrow H^q(tRp_*^\bullet\Omega_{hol}^\bullet) \xrightarrow{d_1^{0,q}} \Omega^1_{X\times X} \underset{\mathcal{O}_{X\times X}}{\otimes} H^q(tRp_*^\bullet\Omega_{hol}^\bullet)$$

$$\xrightarrow{d_1^{1,q}} \Omega^2_{X\times X} \underset{\mathcal{O}_{X\times X}}{\otimes} H^q(tRp_*^\bullet\Omega_{hol}^\bullet) \xrightarrow{d_1^{2,q}} \ldots$$

is exact. Therefore the spectral sequence which we used to construct the connection d_{hol} degenerates at E_1–term. This implies that we have an isomorphism

$$H^q(tRp_*^{\bullet}\Omega^*_{X^{\Delta[1]}}) \approx \ker d_1^{0,q} .$$

The complex $\Omega^*_{X^{\Delta[1]}}$ is the resolution of the constant sheaf $\mathbb{C}_{\bullet}$. Hence there is an isomorphism

$$H^q(tRp_*^{\bullet}\Omega^*_{X^{\Delta[1]}}) \approx H^q(tRp_*^{\bullet}\mathbb{C}_{\bullet}) .$$

Therefore we have

$$\ker d_1^{0,q} \approx H^q(tRp_*^{\bullet}\mathbb{C}_{\bullet}) .$$

The important property of connections is regularity. We shall show that the connections constructed by us are regular. However the connections we considered are on locally free $\mathcal{O}_{X\times X}$–modules of infinite dimensions. Therefore first we shall give the following definition.

<u>Definition 5.2</u>. Let $\mathcal{F}$ be a sheaf of locally free $\mathcal{O}_X$–modules possibly infinite dimensional on a smooth, holomorphic variety X equipped with the integrable connection ∇. We say that the connection ∇ is regular if the sheaf $\mathcal{F}$ is a direct limit of locally free, finite dimensional $\mathcal{O}_X$–modules equipped with integrable, regular connections ∇_i compatible with themselves and ∇.

<u>Theorem 5.3</u>. The connection d_{hol} from Theorem 5.2 is regular.

Proof. For any cosimplicial object $X^\bullet$ let $(X^\bullet)_n$ be a part of $X^\bullet$ up to degree n i.e.

$$(X^\bullet)_n := \left\{ X^0 \rightrightarrows X^1 \overset{\rightarrow}{\rightrightarrows} X^2 \dots \overset{\vdots}{\longrightarrow} X^n \right\} .$$

For any cosimplicial map $p^\bullet : X^\bullet \longrightarrow S^\bullet$ let $p^\bullet_n : (X^\bullet)_n \longrightarrow (S^\bullet)_n$ be the restriction of $p^\bullet$ to $(X^\bullet)_n$. For any sheaf $\mathcal{F}_\bullet$ on $X^\bullet$ let $(\mathcal{F}_\bullet)_n$ be the restriction of $\mathcal{F}_\bullet$ to $(X^\bullet)_n$.

Let X be a smooth, holomorphic variety. Then on X×X we have

$$\varinjlim_n H^q(tR(p^\bullet_n)_*(\Omega^\bullet_{hol})_n) = H^q(tRp^\bullet_*\Omega^\bullet_{hol}) .$$

Let us set $\Omega^*_n := \Omega^*_{X^{\Delta[1]_n}/X^{\partial\Delta[1]_n}}$. There is an exact sequence of complexes

$$0 \longrightarrow tR(p^\bullet_n)_*(\Omega^\bullet_{hol})_n \xrightarrow{i} tR(p^\bullet_{n+1})_*(\Omega^\bullet_{hol})_{n+1} \xrightarrow{j} R(p^{n+1})_*\Omega^*_n \longrightarrow 0$$

which induces a finite, long exact sequence of cohomology $\mathcal{O}_{X\times X}$–modules

$$(*) \qquad \dots \longrightarrow H^k(tR(p^\bullet_n)_*(\Omega^\bullet_{hol})_n) \xrightarrow{i_*} H^k(tR(p^\bullet_{n+1})_*(\Omega^\bullet_{hol})_{n+1})$$

$$\xrightarrow{j_*} H^k(R(p^{n+1})_*\Omega^*_n) \xrightarrow{\partial} H^{k+1}(tR(p^\bullet_n)_*(\Omega^\bullet_{hol})_n) \longrightarrow \dots .$$

Let us set $X^\bullet := X^{\Delta[1]}$. The complexes $\Omega^*_{(X^\bullet)_n}$, $\Omega^*_{(X^\bullet)_{n+1}}$ and $\Omega^*_{X^{n+1}}$ are endowed with canonical filtrations. The obvious maps $i_1 : \Omega^*_{(X^\bullet)_n} \longrightarrow \Omega^*_{(X^\bullet)_{n+1}}$ and

$j_1 : \Omega^*_{(X^\bullet)_{n+1}} \longrightarrow \Omega^*_{X^{n+1}}$ are compatible with these filtrations. These filtrations lead to spectral sequences. The maps induced by i_1 and j_1 on E_1–terms are i_* and j_*, hence i_* and j_* commute with connections which are the differentials $d_1^{0,q}$. The complex $R\,p^{n+1}{}_*\Omega^*_{X^{n+1}}$ is filtered quasi–isomorphic to the mapping cone $C(i_1)$ of $i_1 : tRp^\bullet\Omega^*_{(X^\bullet)_n} \longrightarrow tR(p^\bullet_{n+1})_*\Omega^*_{(X^\bullet)_{n+1}}$. The map $C(i_1) \longrightarrow tR(p^\bullet_n)\Omega^*_{(X^\bullet)_n}$ preserves canonical filtrations, hence ϑ, the induced map on E_1–terms commutes with connections. The connections on $H^q(tRp^{n+1}_*\Omega^*_{n+1})$ are regular (see [D1] Théorème 7.9). By the inductive assumption the connections on $H^q(tR(p^\bullet_n)_*(\Omega^\bullet_{hol})_n)$ are regular. Hence it follows from [D1] Proposition 4.6 and the long exact sequence (*) that the connections on $H^q(tR(p^\bullet_{n+1})_*(\Omega^\bullet_{hol})_{n+1})$ are regular. It is clear that these connections are compatible with the connection d_{hol} on $H^q(tRp^\bullet_*\Omega^\bullet_{hol})$.

<u>Theorem 5.4</u>. Let us set $H := H^0(tRp^\bullet_*\Omega_k)$. The structure maps $\nabla : H \longrightarrow \pi_*(H \underset{\mathcal{O}_V}{\otimes} H)$, $i : H \longrightarrow \tau_*H$, $e : \bar{H} \longrightarrow \mathcal{O}_V$ and $a : \mathcal{O}_{V\times V} \longrightarrow H$ are compatible with the connection d_k, connections induced by d_k in tensor products and the trivial connections on $\mathcal{O}_V$ and $\mathcal{O}_{V\times V}$.

<u>Proof</u>. The products $\Delta[1] \times \Delta[1]$ and $\partial\Delta[1] \times \partial\Delta[1]$ we consider as bisimplicial sets. The inclusion $\partial\Delta[1] \times \partial\Delta[1] \longrightarrow \Delta[1] \times \Delta[1]$ induces a bicosimplicial map $p^{\bullet\bullet} : V^{\Delta[1]\times\Delta[1]} \longrightarrow V^{\partial\Delta[1]\times\partial\Delta[1]}$ between bicosimplicial schemes. $V^{\partial\Delta[1]\times\partial\Delta[1]}$ is a constant bicosimplicial scheme equal to $V \times V \times V \times V$ in each bidegree. Let $V^{\partial\Delta[1] \underset{\Delta[0]}{\times} \partial\Delta[1]}$ be a constant bicosimplicial scheme equal to $V \times V \times V$ in each bidegree. Let $i^{\bullet\bullet} : V^{\partial\Delta[1] \underset{\Delta[0]}{\times} \partial\Delta[1]} \longrightarrow V^{\partial\Delta[1]\times\partial\Delta[1]}$ be an inclusion equal to $(m_1,m,m_2) \longrightarrow (m_1,m,m,m_2)$ in each bidegree. Let $p_{\bullet\bullet} : V^{\Delta[1] \underset{\Delta[0]}{\times} \Delta[1]} \longrightarrow V^{\partial\Delta[1] \underset{\Delta[0]}{\times} \partial\Delta[1]}$ be a pull back of $p^{\bullet\bullet}$ by $i^{\bullet\bullet}$.

Let us assume that V is affine. Let $\mathscr{K} := \Omega^*_{V^{\Delta[1]} \times_{\Delta[0]} \Delta[1]}$ be the De Rham complex on $V^{\Delta[1]} \times_{\Delta[0]} \Delta[1]$ and let $\mathscr{F} := \Omega^*_{V^{\Delta[1]}}$. We shall define

$$\tilde{\nabla} : t\, p_*^{\bullet}\, \mathscr{F} \longrightarrow \pi_* t\, p_*^{\bullet\bullet}\, \mathscr{K}$$

in a complete analogy with the map ∇ from 4.1.1. Then the map $\tilde{\nabla}$ is compatible with canonical filtrations and on E_1–terms coincides with the map ∇.

To show that maps i, e and a are compatible with connections one constructs maps of De Rham complexes $\tau : tp_*^{\bullet}\Omega^*_{V^{\Delta[1]}} \longrightarrow \tau_* tp_*^{\bullet}\Omega^*_{V^{\Delta[1]}}$, $\tilde{e} : tRp_1^{\bullet}\Omega^*_{V}S^1 \longrightarrow t\Omega^*_{V^{\Delta[0]}}$ and $\tilde{a} : t\Omega^*_{V^{\partial\Delta[1]}} \longrightarrow tp_*^{\bullet}\Omega^*_{V^{\Delta[1]}}$ compatible with canonical filtrations which induce i, e and a on E_1–terms.

If V is an arbitrary smooth, separated scheme of finite type over k we use complexes A^*_X and $A^*_{X/S}$ from section 2 and 4.5.

Let R be a ring and let $R_{\bullet}$ be a constant sheaf on $(V^{an})^{\Delta[1]}$ whose fiber is R. The sheaf $H^i(tRp_*^{\bullet}\mathbb{C}_{\bullet})$, being a sheaf of horizontal sections of an integrable connection on $V^{an} \times V^{an}$, is a local system on $V^{an} \times V^{an}$. It is well known that the category of local systems on an arc–connected and locally arc–connected topological space X and the category of $\pi_1(X,x)$–sets are equivalent. We shall identify the fiber of $H^0(tRp_*^{\bullet}\mathbb{C}_{\bullet})$ and the corresponding representation of $\pi_1(V^{an} \times V^{an},(v,v))$.

Let $\pi := \pi_1(V^{an},v)$. Let K be a field of characteristic zero. The representation

$$\phi : \pi \times \pi \longrightarrow (\text{bijections of } \pi)$$

given by $\phi(\alpha,\beta)(x) = \alpha \cdot x \cdot \beta^{-1}$ induces representations

$$\Phi : \pi \times \pi \longrightarrow \mathrm{Aut}_{K\text{–algebra}}(\mathrm{Alg}(\pi;k)) \ ,$$

and

$$\Phi' : \pi \times \pi \longrightarrow \mathrm{Aut}(\mathrm{Hom}(Z[\pi],K)) \ .$$

Theorem 5.5. The fiber of the sheaf $H^0(tRp_*^\bullet K_\bullet)$ over $(v,v) \in V^{an} \times V^{an}$ is canonically isomorphic to $\mathrm{Alg}(\pi,K)$. The representation of $\pi \times \pi$ on the fiber of $H^0(tRp_*^\bullet K_\bullet)$ over (v,v) is equal to Φ.

Proof. Let X be a connected and arc–connected topological space. After applying the functor $\mathrm{Tot}(\ )$ to a cosimplicial map

$$p^\bullet : X^{\Delta[1]} \longrightarrow X^{\partial\Delta[1]}$$

we get a path fibration

$$p : X^I \longrightarrow X \times X \ .$$

For any open subset $U \subset X \times X$, which we consider also as a constant cosimplicial space we have $\mathrm{Tot}(p^{\bullet -1}(U)) = p^{-1}(U)$.

Let us set $X = V^{an}$ and $\pi = \pi_1(X,v)$. Let us observe that $p^{\bullet-1}(v,v) = (X,v)^{\bullet}$. It follows from the discussion in section 1.3 that $H^{\bullet}(p^{\bullet-1}(v,v)) = \mathrm{Alg}(\pi;k)$. This shows the first part of the theorem.

We shall define two sheaves on $X \times X$. Let $U \subset X \times X$ be an open set. We set $\mathcal{F}(U) := H^0(p^{-1}(U);K)$ and $G(U) := H^0(p^{\bullet-1}(U);K)$. There is the map of sheaves $\alpha : G \longrightarrow \mathcal{F}$ given by $\alpha(U) := \alpha_{p^{\bullet-1}(U)} : H^0(p^{\bullet-1}(U);K) \longrightarrow H^0(p^{-1}(U);K)$. The sheaf $\mathcal{F}$ corresponds to the local system obtained by applying the functor $H^0(\ ;K)$ to each fiber of $p : X^I \longrightarrow X{\times}X$, while the sheaf G corresponds to the local system $H^0(tRp_*^{\bullet}K_{\bullet})$ on $X \times X$. The group $\pi \times \pi$ acts on $H^0(p^{-1}(v,v);K)$, the fiber over v of the sheaf $\mathcal{F}$, through the representation Φ'. Hence it acts on $H^0(p^{\bullet-1}(v,v);K)$, the fiber over v of the sheaf G, through the representation Φ.

6. Connections.

We shall use the language of A. Grothendieck (see [G] and [BO]).

Let us assume that all schemes and morphisms are over S.

Let $p : E \longrightarrow X$ and $p_1 : E_1 \longrightarrow X$ be morphisms of schemes. For any $f : Y \longrightarrow X$ we denote by $f!(p) : f!E \longrightarrow Y$ the pullback of $p : E \longrightarrow X$ over Y. If $\theta : E_1 \longrightarrow E$ is a morphism such that $p \circ \theta = p_1$ then we denote by $f!\theta : f!E_1 \longrightarrow f!E$ the pullback of θ by f and we have $f!(p) \circ f!\theta = f!(p_1)$.

For each positive integer n, let $X^1(n)$ be the n–th infinitesimal neighbourhood of the diagonal in $X \underset{S}{\times} X$ and $X^2(n)$ the n–th infinitesimal neighbourhood of the diagonal in $X \underset{S}{\times} X \underset{S}{\times} X$.

There is the diagram of canonical projections

$$X \underset{p_2(n)}{\overset{p_1(n)}{\longleftarrow\!\!\longleftarrow}} X^1(n) \overset{p_{31}(n),\; p_{32}(n),\; p_{21}(n)}{\longleftarrow\!\!\longleftarrow\!\!\longleftarrow} X^2(n)$$

6.1. An n–connection on $p : E \longrightarrow X$ is an isomorphism

$$C(n) : p_1(n)!E \xrightarrow{\approx} p_2(n)!E$$

satisfying the cocycle condition

$$p_{31}(n)!(C(n)) = p_{32}(n)!(C(n)) \circ p_{21}(n)!(C(n)) \ .$$

6.2. A connection on $p : E \longrightarrow X$ is a 1–connection on $p : E \longrightarrow X$.

6.3. Let us suppose that S is a scheme over a field of characteristic zero. An integrable connection on $p : E \longrightarrow X$ is a compatible system of n–connections for all $n \in N$.

6.4. If $p : E \longrightarrow X$ is a vector bundle then an n–connection on $p : E \longrightarrow X$ is a linear n–connection if $C(n)$ is an isomorphism of vector bundles.

6.5. If $p : E \longrightarrow X$ is a principal G–bundle then an n–connection on $p : E \longrightarrow X$ is an n–connection on a principal G–bundle (G–n–connection) if the following diagram commutes

$$\begin{tikzcd}
(p_1(n)!E) \underset{S}{\times} G \arrow[r, "C(n) \underset{S}{\times} id_G"] \arrow[d, "a_1"] & (p_2(n)!E) \underset{S}{\times} G \arrow[d, "a_2"] \\
p_1(n)!E \arrow[r, "C(n)"] & p_2(n)!E
\end{tikzcd}$$

i.e. if $C(n)$ is a G–morphism (the actions a_1 and a_2 are induced by the action of G on E).

The definition of a linear connection, a G–connection, an integrable linear connection and an integrable G–connection we leave to the reader.

In the analogous way we have a notion of an n–connection, a connection and an integrable connection on any sheaf on X. We shall formulate it only for a sheaf of $\mathcal{O}_X$–modules.

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$–modules.

6.6. A liner n–connection on $\mathcal{F}$ is an isomorphism of $\mathcal{O}_{X^1(n)}$–modules

$$C(n) : p_1(n)^* \mathcal{F} \xrightarrow{\approx} p_2(n)^* \mathcal{F}$$

satisfying the cocycle condition

$$p_{31}(n)^*(C(n)) = p_{32}(n)^*(C(n)) \circ p_{21}(n)^*(C(n)) \ .$$

The definitions of a linear connection and an integrable liner connection on $\mathcal{F}$ we leave to the reader.

If $\mathcal{F}$ is a sheaf of locally free $\mathcal{O}_X$–modules of finite type i.e. $\mathcal{F}$ is a sheaf of sections of a vector bundle $p : E \longrightarrow X$ then the notions of a linear n–connection, a linear connection and an integrable linear connection on $\mathcal{F}$ and on $p : E \longrightarrow X$ coincide.

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$–modules. By the classical definition, a connection on an $\mathcal{O}_X$–module $\mathcal{F}$ is an additive map

$$\nabla : F \longrightarrow F \underset{\mathcal{O}_X}{\otimes} \Omega^1_{X/S}$$

such that $\nabla(af) = a\nabla f + f\otimes da$ if f is a section of $\mathcal{F}$ and a is a section of $\mathcal{O}_X$.

The classical definition of a connection (resp. integrable connection) on an $\mathcal{O}_X$–module and the definition of a linear connection (resp. integrable linear connection) on an $\mathcal{O}_X$–module given in 6.6 coincides (see [BO] Proposition 2.9 and Theorem 2.15).

Definition 6.7. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$–algebras equipped with a connection ∇. We say that the connection ∇ is multiplicative if

$$\nabla(a \cdot b) = \nabla(a) \cdot b + a \cdot \nabla(b)$$

where a and b are sections of $\mathcal{F}$.

Let $\mathcal{P}_X = \mathcal{O}_X \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_X$ and let $\mathcal{P}_X^n = \mathcal{P}_{X}/_{I^{n+1}}$ where $I = \ker(\mathcal{O}_X \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_X \longrightarrow \mathcal{O}_X)$. Let us notice that $\mathcal{O}_{X^1(n)} = \mathcal{P}_X^n$. $\mathcal{P}_X^n$ has two $\mathcal{O}_X$–module structure. For an $\mathcal{O}_X$–module $\mathcal{F}$ we have $p_1(n)^* \mathcal{F} = \mathcal{P}_X^n \underset{\mathcal{O}_X}{\otimes} \mathcal{F}$ and $p_2(n)^* \mathcal{F} = \mathcal{F} \underset{\mathcal{O}_S}{\otimes} \mathcal{P}_X^n$ where for $i = 1$ (resp. 2) we use left (resp. right) $\mathcal{O}_X$–module structure on $\mathcal{P}_X^n$.

Lemma 6.8. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$–algebras. The connection ∇ on $\mathcal{F}$ is multiplicative if and only if the isomorphism

$$C(1) : \mathcal{P}_X^1 \underset{\mathcal{O}_X}{\otimes} \mathcal{F} \longrightarrow \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{P}_X^1$$

corresponding to ∇ is an isomorphism of $\mathcal{P}_X^1$–algebras.

Proof. Let $\theta : \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^1_X$ be given by $\theta(x) = \nabla(x) + x\otimes(1\otimes 1)$. Then we have

$$\theta(x\cdot y) = \nabla(x\cdot y) + x\cdot y\otimes(1\otimes 1) = \nabla(x)\cdot y + x\cdot\nabla(y) + x\cdot y\otimes(1\otimes 1) \ .$$

On the other side

$$\theta(x)\cdot\theta(y) = (\nabla(x) + x\otimes(1\otimes 1))\cdot(\nabla(y) + y\otimes(1\otimes 1)) =$$
$$= \nabla(x)\cdot\nabla(y) + \nabla(x)\cdot(y\otimes(1\otimes 1)) + (x\otimes(1\otimes 1))\cdot\nabla(y) + x\cdot y\otimes(1\otimes 1) \ .$$

The fact that $\nabla(x)\cdot\nabla(y) \in \mathcal{F} \otimes_{\mathcal{O}_X} I^2$ implies that $\theta(x\cdot y) = \theta(x)\cdot\theta(y)$. Therefore the extension of θ to a $\mathcal{P}^1_X$–linear map

$$C(1) : \mathcal{P}^1_X \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{P}^1_X$$

is also an isomorphism of algebras.

Reversing the order of our arguments we show that ∇ is multiplicative.

Lemma 6.9. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$–algebras equipped with the multiplicative connection ∇. Then ∇ induces a connection on $p : \mathrm{Spec}(\mathcal{F}) \longrightarrow X$. If the connection ∇ is integrable then the induced connection is integrable.

Proof. By Lemma 6.8 we have an isomorphism of $\mathcal{P}^1_X$–algebras

$$C(1) : p_1(1)^*(\mathcal{F}) \longrightarrow p_2(1)^*(\mathcal{F})$$

satisfying the cocycle condition.

Applying the functor Spec( ) to the isomorphism $C(1)$ (resp. to all $C(n)$) and to the cocycle condition (resp. to the cocycle conditions for all n) we get a connection (resp. an integrable connection) on $p : \mathrm{Spec}(\mathcal{F}) \longrightarrow X$.

Definition 6.10. Let H be a Poincaré sheaf on M . A Poincaré connection on H is a multiplicative integrable connection D on H such that the structure morphisms ∇ , i , e and a are compatible with the connections D on H , $D \otimes_{\mathcal{O}_X} D$ on $H \otimes_{\mathcal{O}_X} H$ and the trivial connection d on $\mathcal{O}_X$.

Proposition 6.11. Let H be a Poincaré sheaf on M . A Poincaré connection D on H induces an integrable connection ∂ on the bundle $P := \mathrm{Spec}\, H \longrightarrow M{\times}M$ compatible with morphisms $\circ$, $(\ )^{-1}$ and e . Let $G = \mathrm{Spec}\, H_{(m,m)}$. The connection ∂ restricted to the principal G–bundle $P_{|X\times\{m\}} \longrightarrow M{\times}\{m\}$ is a G–connection. The connection ∂ we also call the Poincaré connection.

Proof. It follows from Lemma 6.9 that the bundle $P \longrightarrow M{\times}M$ is equipped with the integrable connection ∂ . The functoriality of Spec( ) implies that the connection ∂ is compatible with structure morphisms $\circ$, $(\ )^{-1}$ and e . The compatibility of ∂ with structure morphisms implies that the connection restricted to $P_{|X\times\{m\}} \longrightarrow M{\times}\{m\}$ is a G–connection.

We shall apply the developed formalism to the sheaf $H^0(tRp_*^{\bullet}\Omega_k^{\bullet})$ and the connection d_k

from Theorem 5.1.

Theorem 6.12. Let V be a smooth scheme of finite type over a field k of characteristic zero. Then the sheaf $H^0(tRp_*^{\bullet}\Omega_k^{\bullet})$ is a Poincaré sheaf on V and the connection d_k is a Poincaré connection on it.

Proof. It follows from Propositions 4.3, 4.2 and 4.5 that $H^0(tRp_*^{\bullet}\Omega_k^{\bullet})$ is a Poincaré sheaf. Theorem 5.4 implies that the connection d_k is a Poincaré connection on $H^0(tRp_*^{\bullet}\Omega_k^{\bullet})$.

7. The algebraic De Rham fundamental group.

Let V be a smooth, separated scheme of a finite type over a field k of characteristic zero and let x be a k–point of V. We know from section 3 that $H^0_{DR}((V,x)^\bullet)$ is a Hopf algebra over k. Therefore Spec $H^0_{DR}((V,x)^\bullet)$ is an affine group scheme over k.

Definition 7.1. We set

$$\pi_1^{DR}(V,x) := \mathrm{Spec}\, H^0_{DR}((V,x)^\bullet)$$

and we call $\pi_1^{DR}(V,x)$ the algebraic De Rham fundamental group of V.

Let us assume that V is defined over the field of complex numbers $\mathbb{C}$. Let $V(\mathbb{C})$ be the set of $\mathbb{C}$–points of V equipped with the complex topology. Then $V(\mathbb{C})$ is a smooth, complex manifold.

Definition 7.2. We set

$$\pi_1^{\mathbb{C}^\infty}(V(\mathbb{C}),x) := \mathrm{Spec}\, H^0(\Omega^*_{\mathbb{C}^\infty}((V(\mathbb{C}),x)^\bullet)$$

where $\Omega^*_{C^\infty}((V(\mathbb{C}),x)^\bullet)$ is the De Rham complex of smooth, complex valued differential forms. The space $V(\mathbb{C})$, being a smooth, complex manifold is also a simplicial complex. Let $S^*((V(\mathbb{C}),x)^\bullet)$ be the Sullivan complex of $\mathbb{Q}$–polynomial, differential forms on $(V(\mathbb{C}),x)^\bullet$.

Definition 7.3. We set

$$\pi_1^B(V(\mathbb{C}),x) := \operatorname{Spec} H^0(S^*((V(\mathbb{C}),x)^\bullet)) .$$

Now we shall compare the groups π_1 defined above.

Let $K:k$ be an extension of fields and let V be defined over k. We set $V_K = V \underset{\operatorname{Spec} k}{\times} \operatorname{Spec} K$.

Theorem 7.4. Let V be a smooth, separated scheme of finite type over k and let x be a k–point of V.

i) If $K:k$ is an extension of fields then

$$*_1 \qquad \pi_1^{DR}(V,x) \underset{k}{\times} K = \pi_1^{DR}(V_K,x) .$$

ii) Let us assume that V is defined over $\mathbb{C}$. Then we have natural isomorphisms

$$*_2 \qquad \pi_1^{DR}(V,x) = \pi_1^{C^\infty}(V(\mathbb{C}),x)$$

and

$$*_3 \qquad \pi_1^B(V(\mathbb{C}),x) \underset{\mathbb{Q}}{\times} \mathbb{C} = \pi_1^{C^\infty}(V(\mathbb{C}),x)$$

iii) $\pi_1^{DR}(V,x)$ (resp. $\pi_1^{C^\infty}(V(\mathbb{C}),x)$, resp. $\pi_1^B(V(\mathbb{C}),x)$) is an affine, pro–unipotent, pro–algebraic group scheme over k (resp. $\mathbb{C}$, resp. $\mathbb{Q}$).

Proof. The point i) follows from the obvious fact that $(\Omega^*_{(V,x)^\bullet})\otimes_k K = \Omega^*_{(V_K,x)^\bullet}$. The first isomorphism from the point ii) is a consequence of the quasi–isomorphism $\Omega^*_{(V,x)^\bullet} \longrightarrow \Omega^*_{\mathbb{C}^\infty,(V8\mathbb{C}),x)^\bullet}$, where $\Omega^*_{\mathbb{C}^\infty,(V(\mathbb{C}),x)^\bullet}$ is the De Rham complex of sheaves of smooth, complex valued differential forms on $(V(\mathbb{C}),x)^\bullet$. The second isomorphism follows from the fact that complexes $S^*((V(\mathbb{C}),x)^\bullet)\otimes\mathbb{C}$ and $\Omega^*_{\mathbb{C}^\infty}((V(\mathbb{C}),x)^\bullet)$ are quasi–isomorphic.

It follows from Section 3, Theorem 3.5 that the considered π_1's are affine pro–algebraic, pro–unipotent group schemes over k (resp. $\mathbb{C}$, resp. $\mathbb{Q}$).

For any group scheme π over k, we denote by $\pi(K)$ the group of K–points of π, where K is a k–algebra.

Let K:k be an extension of fields. The isomorphism $*_1$ induces a homomorphism

$$\alpha_{K/k} = \alpha(V)_{K/k} : \pi_1^{DR}(V,x)(k) \longrightarrow \pi_1^{DR}(V_K,x)(K) \;,$$

and an isomorphism

$$\alpha_K = \alpha(V)_K : \pi_1^{DR}(V,x)(K) \approx \pi_1^{DR}(V_K,x)(K) \;.$$

The isomorphism $*_2$ induces an isomorphism

$$\alpha = \alpha(V) : \pi_1^{C^\infty}(V(\mathbb{C}),x)(\mathbb{C}) \longrightarrow \pi_1^{DR}(V,x)(\mathbb{C}) \;.$$

The isomorphism $*_3$ induces a homomorphism

$$\beta = \beta(V(\mathbb{C})) : \pi_1^B(V(\mathbb{C}),x)(\mathbb{Q}) \longrightarrow \pi_1^{C^\infty}(V(\mathbb{C}),x)(\mathbb{C}) \ .$$

<u>Proposition 7.5</u>. Let us assume that V is defined over k. Let us fix an embedding $\delta : k \hookrightarrow \mathbb{C}$. Let $V_{\mathbb{C}}$ and $V(\mathbb{C})$ be constructed using the embedding δ. We shall denote them also V_δ and $V_\delta(\mathbb{C})$.

In the group $\pi_1^{DR}(V_{\mathbb{C}},x)(\mathbb{C})$ we have two lattices, a k–lattice $\alpha_{\mathbb{C}/k}(\pi_1^{DR}(V,x)(k))$ and a $\mathbb{Q}$–lattice $\alpha(\beta(\pi_1^B(V(\mathbb{C}),x)(\mathbb{Q})))$

$$\alpha_{\mathbb{C}/k} : \pi_1^{DR}(V,x)(k) \longrightarrow \pi_1^{DR}(V_{\mathbb{C}},x)(\mathbb{C}) \longleftarrow \pi_1^B(V(\mathbb{C}),x) : \beta \circ \alpha \ .$$

We shall define two maps

$$b_{\mathbb{Q}} = b(V(\mathbb{C}))_{\mathbb{Q}} : \pi_1(V(\mathbb{C}),x) \longrightarrow \pi_1^B(V(\mathbb{C}),x)(\mathbb{Q})$$

and

$$b_{\mathbb{C}} = b(V(\mathbb{C}))_{\mathbb{C}} : \pi_1(V(\mathbb{C}),x) \longrightarrow \pi_1^{C^\infty}(V(\mathbb{C}),x)(\mathbb{C}) \ .$$

To define both homomorphisms it is enough to evaluate elements of $H := H^\bullet(S^*((V(\mathbb{C}),x)^\bullet))$ and $H' : H^\bullet(\Omega^*_{C^\infty}((V(\mathbb{C}),x)^\bullet))$ on any loop in $\pi_1(V(\mathbb{C}),x)$. Let us observe that $H = H^\bullet(B(S^*(V(\mathbb{C})))$ and $H' = H^\bullet(B(\Omega^*_{\mathbb{C}^\infty}(V(\mathbb{C})))$.

Let $\gamma \in \pi_1(V(\mathbb{C}),x)$ and let $\omega = \sum_{k=1}^{n} \omega_{i_1} \otimes \ldots \otimes \omega_{i_k}$ be a representative of an element in H or H′ . We can assume that all ω_{i_α} are one–forms. We set

$$*_5 \qquad \omega(\gamma) := \sum_{k=1}^{n} \int_\gamma \omega_{i_1}, \ldots, \omega_{i_k} \ .$$

The formula $*_5$ defines two homomorphisms

$$b_{\mathbb{Q}} = b(V(\mathbb{C}))_{\mathbb{Q}} : \pi_1(V(\mathbb{C}),x) \longrightarrow \pi_1^B(V(\mathbb{C}),x)(\mathbb{Q})$$

and

$$b_{\mathbb{C}} = b(V(\mathbb{C}))_{\mathbb{C}} : \pi_1(V(\mathbb{C}),x) \longrightarrow \pi_1^{C^\infty}(V(C),x)(\mathbb{C})$$

which satisfy $\beta \circ b_{\mathbb{Q}} = b_{\mathbb{C}}$.

Proposition 7.6. The homomorphism

$$b_{\mathbb{Q}} : \pi_1(V(\mathbb{C}),x) \longrightarrow \pi_1^B(V(\mathbb{C}),x)(\mathbb{Q})$$

is the Malcev rational completion of $\pi_1(V(\mathbb{C}),x)$.

Proof. This follows from the discussion in 1.3.

Corollary 7.7. The homomorphism

$$b_{\mathbb{C}} : \pi_1(V(\mathbb{C}),x) \longrightarrow \pi_1^{C^\infty}(V(\mathbb{C}),x)(\mathbb{C})$$

is the Malcev $\mathbb{C}$–completion of $\pi_1(V(\mathbb{C}),x)$.

Let us observe the following analogy. For a smooth algebraic variety V over a number field k ($\delta : k \hookrightarrow \mathbb{C}$ is an embedding) the fundamental group of $V_\delta(\mathbb{C})$, $\pi_1(V_\delta(\mathbb{C}),x)$ cannot be defined in algebraic way. However its finite completion $\pi_1(V_\delta(\mathbb{C}),x)^\wedge$ is constructed in purely algebraic way. It follows from Theorem 7.4 and Corollary 7.7 that the Malcev $\mathbb{C}$–completion of $\pi_1(V_\delta(\mathbb{C}),x)$ can be constructed in purely algebraic way.

8. The Betti lattice in $\pi_1^{DR}(P^1_{\mathbb{C}}\setminus\{0,1,\infty\})(\mathbb{C})$.

Let V be a smooth algebraic variety over a field k and let $x \in V$ be a k–point. Let $\sigma : k \longrightarrow \mathbb{C}$ be an embedding. In the group $\pi_1^{DR}(V_{\mathbb{C}},x)(\mathbb{C})$ we have two lattices

$$\alpha_{\mathbb{C}/k} : \pi_1^{DR}(V,x)(k) \longrightarrow \pi_1^{DR}(V_{\mathbb{C}},x)(\mathbb{C}) \longleftarrow \pi_1(V(\mathbb{C}),x) : b_{\mathbb{C}}\circ\alpha \ .$$

The aim of this section is to calculate both lattices in $\pi_1^{DR}(V_{\mathbb{C}},x)(\mathbb{C})/\Gamma^5\pi_1^{DR}(V_{\mathbb{C}},x)(\mathbb{C})$ for $V = P^1_{\mathbb{Q}}\setminus\{0,1,\infty\}$.

Let us set $V = P^1_k\setminus\{a_1,\dots,a_{n+1}\}$. For simplicity we shall assume that $a_{n+1} = \infty$. The De Rham algebra $\Omega^*(V)$ is quasi–isomorphic to the differential, graded algebra

$$\Lambda^*(V) : 0 \longrightarrow k \xrightarrow{\ 0\ } \Lambda^1(V) := k\cdot w_1+\dots+k\cdot w_n \longrightarrow 0 \ ,$$

where $w_i = \frac{dz}{z-a_i}$ $i = 1,\dots,n$. The obvious inclusion

$$\Lambda^*(V) \longrightarrow \Omega^*(V)$$

is the required quasi–isomorphism. Hence it follows that $H^0_{DR}((V,x)^\bullet) \approx H^0(B(\Lambda^*(V))$.

Let $T(\Lambda^1(X))$ be a tensor algebra on $\Lambda^1(X)$. $T(\Lambda^1(X))$ has a natural filtration $\{L_n(T(\Lambda^1(X))\}_{n=0}^{\infty}$ by length of tensors. Observe that $T(\Lambda^1(X)) = \bigoplus_{n=0}^{\infty} L_{n+1}(T(\Lambda^1(X))/_{L_n}(T(\Lambda^1(X))$ and therefore $T(\Lambda^1(X))$ is a graded abelian group.

Lemma 8.1. The abelian group $H^0_{DR}((V,x)^\bullet)$ is equal to $T(\Lambda^1(V))$. The isomorphism $H^0_{DR}((V,x)^\bullet) \approx T(\Lambda^1(V))$ is compatible with filtrations of $H^0_{DR}((V,x)^\bullet)$ and $T(\Lambda^1(V))$.

Proof. The chain complex $B(\Lambda^*(V))$ in the degree zero is equal to $T(\Lambda^1(V))$. Notice that all differentials in $B(\Lambda^*(V))$ are equal to zero. Hence it follows that $H^0(B(\Lambda^*(V)) \approx T(\Lambda^1(V))$. The isomorphism is of course compatible with filtrations, hence it is an isomorphism of filtered abelian groups.

Notice that the Hopf algebra structure on $H^0(B(\Lambda^*(V))$ induces the Hopf algebra structure on $T(\Lambda^1(V))$.

Let $L = L(x_1,...,x_n)$ be a free Lie algebra over k on n–elements $x_1,...,x_n$. For each n we form the quotient Lie algebra $L/_{\Gamma^n L}$. The Lie algebra $L/_{\Gamma^n L}$ we equipped with a multiplication given by the Baker–Hausdorff formula. The group $L/_{\Gamma^n L}$ is the group of k–points in an affine, unipotent, algebraic group G_n defined over k. Let us set $G := \varprojlim_n G_n$.

Let us choose a base $\{e_i\}_{i\in I}$ of L given by basic Lie elements. The first n elements of this base are $x_1,...,x_n$. Let $\{e_i^*\}_{i\in I}$ be the dual base of L^*.

Let $S((L/_{\Gamma^i L})^*)$ be a symmetric algebra on $(L/_{\Gamma^i L})^*$, the dual space of $L/_{\Gamma^i L}$. Let $S(L^*) = \varinjlim_n (S((L/_{\Gamma^n L})^*))$. The product in $L/_{\Gamma^n L}$ induces a Hopf algebra structure on $S(L^*)$. The algebra $S(L^*)$ is a Hopf algebra of regular functions on G. We shall define a gradation $\{S(L^*)_k\}_{k=0}^{\infty}$ of $S(L^*)$ in the following recursive way.

$S(L^*)_0$ consists of constant functions,

$S(L^*)_p$ is a k–vector space generated by functionals dual to elements of the base

$\{e_i\}_{i\in I}$ which have length p and by products $f_1 \cdot \ldots \cdot f_\ell$ where $f_i \in S(L^*)_{n_i}$ and $\sum_{i=1}^{\ell} n_i = p$. It is clear that $S(L^*) = \bigoplus_{i=0}^{\infty} S(L^*)_i$.

Lemma 8.2. There is a unique homomorphism of Hopf algebras

$$\varphi^* : S(L^*) \longrightarrow T(\Lambda^1(V))$$

such that

i) φ^* preserves gradations,
ii) $\varphi^*(x_i^*) = -w_i$; $i=1,\ldots,n$.

Proof. Let e_k belong to the base of L given by basic Lie elements and let e_k^* be the dual functional. Assume that e_k has length p. Assume also that $\varphi^*(e_i^*)$ is already defined for all e_i such that the length of e_i is strictly smaller than p. We set $\varphi^*(e_k^*) = \sum_t a_t w_{t_1} \otimes \ldots \otimes w_{t_p}$. The condition that φ^* is a morphism of Hopf algebras determines coefficients a_t uniqely.

Let us set $\pi := \pi_1^{DR}(V,x)$.

Corollary 8.3. The morphism φ^* induces isomorphisms $\varphi : \pi \longrightarrow G$ and $\varphi_n : \pi/\Gamma^n\pi \longrightarrow G/\Gamma^n G$.

Proof. The morphism of Hopf algebras β^* induces a homomorphism of affine groups $\varphi : \pi \longrightarrow G$. The homomorphism $\varphi_2 : \pi/\Gamma^2\pi \longrightarrow G/\Gamma^2 G$ is an isomorphism, hence for any n, $\varphi_n : \pi/\Gamma^n_\pi \longrightarrow G/\Gamma^n G$ is an isomorphism. We have $\pi = \varprojlim_n \pi/\Gamma^n_\pi$ and $G = \varprojlim_n G/\Gamma^n G$. This implies that φ is an isomorphism.

Let $V = P^1_{\mathbb{Q}} \setminus \{0,1,\infty\}$. We want to calculate the image of a homomorphism

$$\chi : \pi_1(V(\mathbb{C}),x) \xrightarrow{b} \pi_1^{C^\infty}(V(\mathbb{C}),x)(\mathbb{C}) \xrightarrow{\alpha} \pi_1^{DR}(V_{\mathbb{C}},x)(\mathbb{C}) \xrightarrow{\alpha_{\mathbb{C}}} \pi_1^{DR}(V,x)(\mathbb{C}) \xrightarrow{\varphi_5} (G/\Gamma^5 G)(\mathbb{C}) \ .$$

Let $e_1 = x$, $e_2 = y$, $e_3 = [x,y]$, $e_4 = [x[x,y]]$, $e_5 = [y[x,y]]$, $e_6 = [x[x[x,y]]]$, $e_7 = [x[y[x,y]]]$ and $e_8 = [y[y[x,y]]]$ be the base of $L/\Gamma^5 L$. Let $\varphi^*(e_1^*) = -\frac{dz}{z}$ and $\varphi^*(e_2^*) = \frac{-dz}{z-1}$. Let us set $\psi = b \circ \alpha \circ \alpha_{\mathbb{C}}$.

The image of $\gamma \in \pi_1(V(\mathbb{C}),x)$ on $(G/\Gamma^5 G)(\mathbb{C})$ is described completely by values $e_i^*(\chi(\gamma)) = e_i^*(\varphi_5 \circ \psi(\gamma)) = \varphi^*(e_i^*)(\psi(\gamma)) = \varphi^*(e_i^*)(\gamma)$ for $i = 1,2,\ldots,8$. The value of $\varphi^*(e_i^*)$ on γ is the value of the iterated integral corresponding to $\varphi^*(e_i^*)$ on γ. We give here the first few elements $\varphi^*(e_i^*)$. Let us set $w_1 = \frac{dz}{z}$ and $w_2 = \frac{dz}{z-1}$. Then we have

$$\varphi^*(e_3^*) = \frac{1}{2}(w_1 \otimes w_2 - w_2 \otimes w_1) \ ,$$

$$\varphi^*(e_4^*) = \frac{1}{3} w_1 \otimes w_2 \otimes w_1 - \frac{1}{6} w_2 \otimes w_1 \otimes w_1 - \frac{1}{6} w_1 \otimes w_1 \otimes w_2 \ ,$$

$$\varphi^*(e_5^*) = -\frac{1}{3} w_2 \otimes w_1 \otimes w_2 + \frac{1}{6} w_1 \otimes w_2 \otimes w_2 + \frac{1}{6} w_2 \otimes w_2 \otimes w_1 \ \cdots \ .$$

After the calculations we get the following result.

<u>Theorem 8.4</u>. Let $S \in \pi_1(V(\mathbb{C}),x)$ be a loop around 0 and let $T \in \pi_1(V(\mathbb{C}),x)$ be a loop around 1. Then we have

$$\chi(S) = (-2\pi i,\ 0,\ -2\pi i \log(1-x),\ -2\pi i\ L_2(z) + \pi i \log z \log(1-z)\ ,$$

$$\pi i \log^2(1-z),\ 2\pi i\ L_3^1(z) + \pi i \log z\ L_2(z) - \tfrac{1}{3}\ \pi i \log^2 z \log(1-z)\ ,$$

$$2\pi i\ L_3^2(z) - \tfrac{2}{3}\ \pi i \log z \log^2(1-z) + \pi i \log(1-z) L_2(z),\ -\tfrac{1}{3}\ \pi i \log^3(1-z))$$

and

$$\chi(T) = (0,\ -2\pi i,\ 2\pi i \log z,\ -\pi i \log^2 z,\ -2\pi i\ L_2(z) - \pi i \log z \log(1-z) + 2\pi i\,\xi(2)\ ,$$

$$\tfrac{1}{3}\pi i \log^3 z, 2\pi i L_3^1(z) + \pi i \log z\ Li_2(z) + \tfrac{2}{3}\pi i \log(1-z)\log^2 z$$

$$-2\pi i \log z\ \ \xi(2) - 2\pi i\ \ \xi(3),\ 2\pi i\ L_3^2(z) +$$

$$\pi i L_2(z)\log(1-z) + \tfrac{1}{3}\ \pi i \log z \log^2(1-z) - 2\pi i \log(1-z)\,\xi(2) - 2\pi i\,\xi(3))$$

where the functions $L_2(z)$, $L_3^1(z)$ and $L_3^2(z)$ are given by the following formulas:

$$L_2(z) = \int_0^z -\frac{1}{2}\left[\frac{\log z}{1-z} + \frac{\log(1-z)}{z}\right] dz\ ,$$

$$L_3^1(z) = \int_0^z \left\{\left[\frac{1}{12}\log z \log(1-z) + \frac{1}{2}\ L_2(z)\right]\frac{1}{z} + \frac{1}{12}\log^2 z\ \frac{1}{1-z}\right\} dz\ ,$$

$$L_3^2(z) = \int_0^z \left\{\left[-\frac{1}{2}\ L_2(z) + \frac{1}{12}\log z \log(1-z)\right]\frac{1}{1-z} + \frac{1}{12}\log^2(1-z)\ \frac{1}{z}\right\} dz\ .$$

Let us observe that the image of χ can be interpreted as a position of $\pi_1(V(\mathbb{C}),x)$ in $\pi_1^{DR}(V,x)(\mathbb{C})/\Gamma^5$ in the coordinates given by φ_5.

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