THE MAXIMUM PRINCIPLE AND THE GROWTH OF VOLUME MINIMIZING HYPERSURFACES

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hence the lower row of (5.16.1) is exact.

It is clear from the diagram that the composition

$$H^1(K,G) \longrightarrow \oplus H^1(K_v,G) \longrightarrow (M_\Gamma)_{tors}$$

is zero. Now let $\xi_A = \xi_\infty \times \xi_f \in \oplus H^1(K_v,G)$, where $\xi_\infty \in \prod_\infty H^1(K_v,G)$, $\xi_f \in \oplus_{\mathcal{V}_f} H^1(K_v,G)$. Suppose that $\mu(\xi_A) = 0$. Let h_A be the image of ξ_A in $\oplus H^1_{ab}(K_v,G)$. Then the image of h_A in $(M_\Gamma)_{tors}$ is zero, hence h_A is the image of some element $h \in H^1_{ab}(K,G)$. Consider the element
$h \times \xi_\infty \in H^1_{ab}(K,G) \times \prod_\infty H^1(K_v,G)$. It is clear that $h \times \xi_\infty$ is contained in the fiber product over $\prod_\infty H^1_{ab}(K_v,G)$. By Theorem 5.12 $h \times \xi_\infty$ comes from $H^1(K,G)$. The theorem is proved.

The maximum principle and the growth

of volume minimizing hypersurfaces

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Introduction

This paper is devoted to estimating the rate of growth of the volume minimizing hypersurfaces in $\mathbb{R}^n$.

The problem of finding a lower bound on the volume of k–dimensional volume minimizing surfaces X, contained in a domain and passing through an interior point of this domain, often arises in various questions of the calculus of variations, algebraic geometry and complex analysis (see, for example, [7], [8], [9], [14]). Fomenko [8] has stated the following conjecture. Let B be an arbitrary convex domain with piecewise smooth boundary and 0 an interior point for B. Let X be a k–dimensional volume minimizing surface, passing through 0. Then

$$\mathrm{vol}_k(X \cap B) \geq \min \mathrm{vol}_k(\pi^k \cap B),$$

where the minimum is taken over all k–dimensional plane sections of B by planes π^k, passing through 0.

Such an estimate had earlier been found for several special cases. Katsnel'son and Ronkin [11] obtained an analogous inequality for case, when the domain B is a cube in $\mathbb{C}^2$ with centre at 0 and X is a complex analytic set of codimension 1. When B is a ball with centre at 0, the estimate was established by Lelong [13] and Griffiths and King [9] for

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$k = n-2, n-1$ and by Fomenko [8] for any k. Further, Le Hong Van [12] proved the conjecture for two–dimensional volume minimizing surfaces X and for a rectangular parallelepiped B in $\mathbb{R}^n$.

In this paper we consider the problem for volume minimizing hypersurfaces X and an arbitrary convex domain B, symmetric with respect to 0. By using the maximum principle we obtain universal lower bounds on the volumes $\mathrm{vol}_{k-1}(X \cap \partial B)$ and $\mathrm{vol}_k(X \cap B)$ and present several situations, when the bounds obtained are exact. In particular, Fomenko's conjecture, mentioned above, is proved for some new classes of domains B.

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§1 The maximum principle for volume–minimizing hypersurfaces

Let M be a n–dimensional Riemannian manifold. We recall that a twice differentiable surface in M is said to be minimal if its mean curvature is trivial at each point. By applying the maximum principle [1] one can prove, in particular, that two different connected minimal hypersurfaces in M can not touch each other from one side. By the way, a boundary versal of this result is also true (cf. [3]). As well known (see, for example, [2]), the classical minimal surfaces also can be defined as local minima of the volume functional, i.e. its volume does not decrease under infinitely small perturbations with infinitely small support. However, the basic objects of this investigation are globally minimal hypersurfaces, i.e. the hypersurfaces that are minima of the volume with respect to large variations.

A compact k–dimensional surface (possibly with singularities) $S \subset M$ is called volume minimizing if $\mathrm{vol}_K S \leq \mathrm{vol}_k S'$ for any compact k–dimensional surface $S' \subset M$ such that $\partial S' = \partial S$, where $\mathrm{vol}_k S$ denotes the k–dimensional volume of S. A complete noncompact k–dimensional surface (possibly with singularities) $S \subset M$ is called volume minimizing if every its perturbation with compact support does not decrease the k–dimensional volume in an obvious sense.

Theorem 1.1. Let S_1 and S_2 be connected volume minimizing hypersurfaces in M. Assume that either $S_1 \cap S_2$ contains a regular point of both S_1 and S_2, and S_1, S_2 touch each other from one side, or $S_1 \cap S_2$ bounds a domain of S_1 and a domain of S_2. Then S_1 and S_2 are contained in the same volume minimizing hypersurface in M.

Proof. First we consider the case, when S_1 and S_2 lie in one side with respect to each other. Denote by S_i° the set of singularities of S_i $(i = 1,2)$. By virtue of a well–known Federer's result [4,5] codim $S_i^{\circ} \leq 7$. In particular, this fact implies that $S_i' = S_i \setminus S_i^{\circ}$ are connected (smooth) minimal hypersurfaces in M. Moreover, $S_1' \cap S_2' \neq \phi$ by force of the assumption of the theorem. Now, applying the maximum principle to S_1' and S_2' we can conclude that they are extended to the same volume minimizing hypersurface in M. Consequently, so are S_1 and S_2 themselves.

Now we consider the case, when S_1 lies on both sides of S_2. According to the assumption of the theorem there are domains $P_1 \subset S_1$ and $P_2 \subset S_2$ such that $\partial P_1 = \partial P_2 = S_1 \cap S_2$; moreover, P_1 and $S_1 \setminus P_1$ lie on different sides of S_2 and P_2 and $S_2 \setminus P_2$ lie on different sides of S_1. From the global minimality of S_1 and S_2 it follows that $\mathrm{vol}_{n-1} P_1 = \mathrm{vol}_{n-1} P_2$. Set $S_1' = (S_1 \setminus P_1) \cup P_2$. S_1' is a volume minimizing because so is S_1. Obviously, $S_1' \cap S_2 = P_2$ and S_1' and S_2 touch each other from one side.

Clearly, P_2 has regular points. Thus S_1' and S_2 turn out to be in the situation considered above. The proof is completed.

Remark. The maximum principle was formulated in [1] for twice differentiable minimal surfaces. In fact, it, and therefore Theorem 1.1, can be generalized to non–smooth hypersurfaces of a certain type (for details see [3]). Further, the existence of a common regular point of S_1 and S_2 can be replaced by a more weak requirement. Assume that connected volume–minimizing hypersurfaces S_1 and S_2 touch each other from one side nearly an isolated common singular point p, satisfying the following condition

(1.1) S_1 and S_2 have regular tangent cones at p.

Then according to the theorem on perturbation of isolated singularities [10], there exist volume–minimizing hypersurfaces S_1' and S_2', which are regular nearly p and arbitrarily close to S_1 and S_2 respectively. Besides that $S_1' \cap S_2'$ bounds domain of S_1' and S_2'. By force of Theorem 1.1 we have $S_1' \equiv S_2'$. In this way one can conclude that $S_1 \equiv S_2$.

The following simple corollaries of the maximum principle prove useful for our investigation.

Proposition 1.2. Suppose M is a Riemannian manifold, $\sigma : M \longrightarrow M$ is an isometry with a fixed point p. Let X be a connected minimal hypersurface in M, passing through p so that $d\sigma(T_pX) = T_pX$. Then X is invariant under σ, i.e. $\sigma(X) = X$.

Proof. Set $X' = \sigma(X)$. Since σ is an isometry and X is a minimal hypersurface, X' is also minimal. Now, from $T_pX' = d\sigma(T_pX) = T_pX$ it follows that $X' \equiv X$ by virtue of the boundary maximum principle (see, [3]). Thus, the proof is completed.

Corollary 1.3. Let M be a Riemannian manifold, $\sigma : M \longrightarrow M$ an isometry with a fixed point p. Suppose that N is an unique σ–invariant hypersurface, passing through p in a given direction. Then N is an unique minimal hypersurface, passing through p in this direction.

Proof. Suppose X is a minimal hypersurface such that $p \in X$ and $T_pX = T_pN$. Set $X' = \sigma(X)$. We have $T_pX' = \sigma(T_pX) = \sigma(T_pN) = T_pN = T_pX$ because N is invariant with respect to σ. Hence, $X' = X$ by force of Proposition 1.2 and therefore, $X \equiv N$ in accordance with the assumption of the corollary. The proof is completed.

From Corollary 1.3 it immediately follows the following fact

Corollary 1.4. Suppose M is a hypersurface of revolution in $\mathbb{R}^n$. Then all meridians are minimal hypersurfaces. Moreover, every hypersurface, having at a point p the same tangent plane as the meridian, coincides with this meridian.

§2 Universal estimation for boundary of volume minimizing hypersurfaces

Let B be a domain in the Euclidean space $\mathbb{R}^n$ with piecewise smooth boundary ∂B^n, homeomorphic to S^{n-1}, and 0 an interior point for B^n. We suppose that the domain B is symmetric with respect to the point 0. Let X be a volume minimizing hypersurface, passing through 0. In this section we study the problem of finding a lower bound on $\mathrm{vol}_{n-2}(X \cap \partial B)$, which depends only on B.

Theorem 2.1. Let X be a (n–2)–connected volume minimizing hypersurface in B with boundary $\partial X = X \cap \partial B$. Suppose that X contains 0 as an isolated singularity, satisfying the condition (1.1). Then $X \cap \partial B$ contains a (n–3)–dimensional symmetric homeomorphic sphere (i.e. a surface, symmetric with respect to 0 and homeomorphic to S^{n-3}).

Proof. Denote by $\sigma : B \longrightarrow B$ the symmetry through the point 0, i.e. the map, sending each point of B into its opposite point. Obviously, the hypersurface $X' = \sigma(X)$ is also minimizing, has boundary $\partial X' = X' \cap \partial B$ and contains 0 as an isolated singularity of type (1.1). First we prove that $\partial X \cap \partial X' \neq 0$. Really, suppose not. Then $X \cap X'$ is contained in the interior of X and X′. Hence, either X′ lies on one side of X or $X \cap X'$ is a closed (n–2)–dimensional surface, serving as the common boundary of two domains in X and X′ and containing 0 as an isolated singularity of type (1.1) of both X and X′. Consequently, $X \equiv X'$ by force Theorem 1.1 and therefore $\partial X \equiv \partial X'$. The contradiction means that $\partial X \cap \partial X' \neq \phi$. Now we show that if $X \neq X'$, then they intersect each other transversally. Really, if X and X′ have the common tangent plane at some common point, then $X \equiv X'$ by force of the boundary versal of the maximum principle (cf. [3]). From the facts proved above it follows that either $X \equiv X'$ or $X \cap X'$ consists of symmetric (with respect to 0) surfaces, passing through 0 and homeomorphic to the disk D^{n-2} with boundary on ∂B. It is easy to see that in both cases there exists a (n–3)–dimensional symmetric sphere on $\partial X \cap \partial X'$. The proof is completed.

Let H be a symmetric (with respect to 0) integrand over hypersurfaces in ∂B, i.e. $H(Y) = H(\sigma(Y))$ for any hypersurface Y in ∂B.

Theorem 2.2. Let X be a (n–2)–connected volume minimizing hypersurface in B with boundary $\partial X = X \cap \partial B$. Suppose that X contains 0 as an isolated singularity, satisfying the condition (1.1). Then

(2.1) $$H(X \cap \partial B) \geq \inf_{Y} H(Y),$$

<u>where the infimum is taken over all</u> (n–2)–<u>dimensional symmetric (with respect to</u> 0) <u>homeomorphic spheres</u> Y <u>on</u> ∂B.

<u>Proof</u>. Since X is (n–2)–connected and ∂B is homeomorphic to S^{n-1}, $X \cap \partial B$ is homeomorphic to S^{n-2}. According to Theorem 2.1 there is a (n–3)–dimensional symmetric sphere S_0^{n-3} on $X \cap \partial B$. S_0^{n-3} divides $X \cap \partial B$ into two homeomorphic (n–2)–dimensional disks Y_1 and Y_2 such that $\partial Y_1 = \partial Y_2 = S_0^{n-3}$. Since S_0^{n-3} is symmetric, i.e. $\sigma(S_0^{n-3}) = S_0^{n-3}$, the sets

$$Z_1 = Y_1 \cup_{S_0^{n-3}} \sigma(Y_1) \text{ and } Z_2 = Y_2 \cup_{S_0^{n-3}} \sigma(Y_2)$$

are (n–2)–dimensional symmetric homeomorphic spheres on ∂B. Moreover, we have

$$H(Z_1) = 2H(Y_1), \ H(Z_2) = 2H(Y_2).$$

Hence,

$$H(X \cap \partial B) = H(Y_1) + H(Y_2) = \tfrac{1}{2}H(Z_1) + \tfrac{1}{2}H(Z_2) \geq \inf_{Y} H(Y),$$

where the infimum is taken over all (n–2)–dimensional symmetric homeomorphic spheres on ∂B. The proof is completed.

Consider now the projective space $P(\partial B)$, obtained from the sphere ∂B by identifying opposite points. In this doing each (n–2)–dimensional symmetric homeomorphic sphere Y on ∂B corresponds to a (n–2)–dimensional projective homeomorphic subspace $\check{Y}$ in

$P(\partial B)$. By virtue of the symmetry of ∂B the symmetric integrand H induces an integrand $\hat{H}$ over hypersurfaces in $P(\partial B)$ such that

$$H(Y) = 2\hat{H}(\hat{Y}). \tag{2.2}$$

In this way the minimization problem for the integrand H in the class of all (n–2)–dimensional symmetric homeomorphic spheres in ∂B is reduced to the problem of minimizing $\hat{H}$ in the class of all closed hypersurfaces in $P(\partial B)$, realizing the non–trivial element $1 \in H_{n-2}(P(\partial B);\mathbb{Z}_2)$. For an elliptic integrand H the existence of H–minimizing solutions $\hat{Y}$ to the later problem was proved in [4], where $\hat{Y}$ are regarded as integral currents of least mass. Consider the special case $H = \mathrm{vol}_{n-2}$. In this case $\hat{Y}$ can be also regarded as globally minimal compacta in the sense of [6]. As we proved above, there exists a (n–2)–dimensional symmetric homeomorphic sphere (possibly with singularities) Y on ∂B with (n–2)–dimensional volume 2c, where c is the least volume of an arbitrary (n–2)–dimensional disk, spanning an arbitrary (n–3)–dimensional symmetric homeomorphic sphere ∂B. In particular, Y minimizes the volume of (n–2)–dimensional disks, spanning an arbitrary (n–3)–dimensional symmetric homeomorphic sphere on Y. Hence, $Y \setminus Y_0$ is a minimal surface and $\mathrm{codim}\, Y_0 \leq 7$, where Y_0 denotes the set of singularities of Y. Suppose now that X is a (n–2)–connected volume minimizing hypersurface in B, having boundary $\partial X = X \cap \partial B$ and passing through 0 as an isolated singularity of type (1.1). Then from (2.1) we obtain

$$\mathrm{vol}_{n-2}(X \cap \partial B) \geq 2c. \tag{2.3}$$

§3 Universal lower bound on the volume of volume minimizing hypersurfaces

Given n–dimensional compact domains B_r, depending on a real parameter r and expanding with growth of $r : B_r \supset B_{r'}$ for $r > r'$, $B_1 = B$, $0 \in B_0$ and $\dim B_0 < n$. Set $C_r = \partial B_r$. Suppose $p \in C_r$ and let T_pC_r be the tangent plane to C_r at p. Denote by $\vec{n}(p)$ the unit inward normal to C_r at p, directed into the domain B_r. Consider a fixed (n–2)–connected volume minimizing hypersurface X in B, passing through 0 as an isolated singularity of type (1.1). Since X is volume minimizing with boundary ∂X on ∂B, it is not difficult to see that X intersects C_r transversally for almost all $r \in [0,1]$. Fix such a number r. Clearly, $\mathrm{vol}_{n-2}(X \cap C_r) < \infty$. Let $p \in X \cap C_r$. Excluding a set of (n–2)–dimensional volume 0, we may suppose that X and C_r are locally regular about p and intersect transversally. Let $\alpha(p)$ be the angle between hyperplanes T_pX and T_cC_r. We denote by h(p;s) the length of the straightline segment, passing through p along the direction $\vec{n}(p)$ and bounded by C_r and C_s $(s < r)$. Assume that the limit

$$h(p) = \lim_{s \to r} \frac{h(p;s)}{r-s}$$

exists. Clearly, the (n–1)–dimensional volume of X is given by the following formula

$$\mathrm{vol}_{n-1}(X) = \int_0^1 \left[\int_{X \cap C_r} \frac{h(p)\,dS_r}{\sin\alpha(p)} \right] dr + \sum_r \mathrm{vol}_{n-1}(X \cap C_r), \tag{3.1}$$

where dS_r denotes the volume element on $X \cap C_r$. Set

$$G_r(X \cap C_r) = \int_{X \cap C_r} \frac{h(p)\,dS_r}{\sin\alpha(p)},$$

$$H_r(X \cap C_r) = \int_{X \cap C_r} h(p)dS_r .$$

Theorem 3.1. Suppose the domains B_r are symmetric $(0 \leq r \leq 1)$ and let X be a $(n-2)$–connected volume minimizing hypersurface in B with boundary on $C_1 = \partial B$ and contain 0 as an isolated singularity, satisfying the condition (1.1). Then

$$\text{(3.2)} \qquad \mathrm{vol}_{n-1}(X) \geq \int_0^1 \inf_{Y_r} H_r(Y_r)dr ,$$

where Y_r runs all $(n-2)$–dimensional symmetric homeomorphic spheres on C_r. Moreover, if besides that X is orthogonal to C_r at almost all points of X and for almost every $r \in [0,1]$ $X \cap C_r$ minimizes H_r in the class of all $(n-2)$–dimensional symmetric homeomorphic spheres on C_r, then the equality in (3.2) holds.

Proof. From (3.1) and $\sin \alpha(p) \leq 1$ it follows that

$$\text{(3.3)} \qquad \mathrm{vol}_{n-1}(X) \geq \int_0^1 G_r(X \cap C_r)dr \geq \int_0^1 H_r(X \cap C_r)dr.$$

On the other hand, since B_r are symmetric, C_r and $h(p)$ are symmetric too. Therefore H_r are symmetric integrands and we may apply Theorem 2.2 to H_r and B_r. In this way we obtain

$$\text{(3.4)} \qquad H_r(X \cap C_r) \geq \inf_{Y_r} H_r(Y_r) ,$$

where the infimum in the right part is taken over all (n–2)–dimensional symmetric homeomorphic spheres Y_r on C_r. Clearly, (3.2) follows from (3.3) and (3.4). Further, since X is orthogonal to C_r at almost all points of X we have $\sin \alpha(p) = 1$ almost everywhere on X, which implies

$$(3.5) \qquad G_r(X \cap C_r) = H_r(X \cap C_r) ,$$

$$(3.6) \qquad \mathrm{vol}_{n-1}(X \cap C_r) = 0$$

for almost all r, $0 \leq r \leq 1$. Combining (3.1), (3.5) and (3.6) we obtain

$$(3.7) \qquad \mathrm{vol}_{n-1}(X) = \int_0^1 H_r(X \cap C_r)dr.$$

Now, using (3.7) and the last assumption of the theorem $H_r(X \cap C_r) = \inf H_r(Y_r)$ completes the proof.

Consider some special situations. Given a piecewise smooth function f on $\mathbb{R}^n$ such that the set $B_r = \{x \in \mathbb{R}^n : f(x) \leq r\}$ is homeomorphic to n–dimensional ball for any $r \in (0,1]$. Obviously, $C_r = \partial B_r = \{x \in \mathbb{R}^n : f(x) = r\}$ is a piecewise smooth homeomorphic sphere. Taking into account that grad f(x) is orthogonal to C_r at x, one can calculate easily that $h(x) = |\mathrm{grad}\, f(x)|^{-1}$ for any $x \in B_1$. Therefore

$$(3.8) \qquad H_r(X \cap C_r) = \int_{X \cap C_r} \frac{dS_r}{|\mathrm{grad}\, f(x)|} .$$

Theorem 3.2. *Assume that the function* f *is symmetric and positively homogeneous of degree* k ($k < n$). *Let* X *be as in Theorem* 3.1. *Then*

$$\mathrm{vol}_{n-1}(X) \geq \frac{1}{n-k} \inf_{Y_1} H_1(Y_1) , \tag{3.9}$$

where Y_1 *runs all* (n–2)–*dimensional symmetric homeomorphic spheres on* C_1. *Moreover, the equality holds if* grad f *is tangent to* X *almost everywhere on* X *and for almost every* $r \in [0,1]$ *the sphere* $X_r' = (X \cap C_r)/r = \{x/r : x \in X \cap C_r\}$ *minimizes* H_1 *in the class of all* (n–2)–*dimensional symmetric homeomorphic spheres on* C_1.

Proof. Note that the domains $B_r = \{x : f(x) \leq r\}$ are symmetric because f is a symmetric function. Since f is positively homogeneous, i.e. $f(\lambda x) = \lambda^k f(x)$ ($\lambda > 0$), we have

$$\mathrm{grad}\, f(\lambda x) = r^{k-1}\, \mathrm{grad}\, f(x) \tag{3.10}$$

and

$$dS_r = r^{n-2} dS_r' , \tag{3.11}$$

where dS_r and dS_r' denote the volume elements on $X \cap C_r$ and X_r' respectively. Therefore,

$$H_r(X \cap C_r) = \int_{X \cap C_r} h(p) dS_r = \int_{X \cap C_r} \frac{dS_r}{|\mathrm{grad}\ f(x)|}$$

$$\int_{X_r'} \frac{r^{n-2} dS_r'}{r^{k-1} |\mathrm{grad}\, f(x)|} = \int_{X_r'} \frac{r^{n-k-1} dS_r'}{|\mathrm{grad}\, f(x)|} .$$

Hence, we obtain

$$\mathrm{vol}_{n-2}(X) \geq \inf \left(\int_{X_r'} \frac{dS_r'}{|\mathrm{grad}\, f(x)|} \right) \int_0^1 r^{n-k-1}\, dr$$

$$= \frac{1}{n-k} \inf_{Y_1} H_1(Y_1) ,$$

where Y_r runs all (n–2)–dimensional symmetric homeomorphic spheres. The second statement follows immediately from Theorem 3.1 and the fact that grad f is orthogonal to level surfaces C_r. Thus, the proof is complete.

Suppose now that $h(x) = h_r$ is constant on each level surface C_r. Then $H_r(X \cap C_r) = h_r \, \mathrm{vol}_{n-2}(X \cap C_r)$. Therefore, the following result is an immediate corollary of Theorem 3.1.

Corollary 3.3. *Assume that* B_r *and* X *are as in Theorem* 3.1. *Then*

(3.12) $$\mathrm{vol}_{n-1}(X) \geq \int_0^1 h_r \inf_{Y_r} \mathrm{vol}_{n-2}(Y_r) dr,$$

where the infimum is taken over all (n–2)–*dimensional symmetric homeomorphic spheres* on C_r. If besides that X is orthogonal to C_r at almost all points of X and for almost every $r \in [0,1]$ the surface $X \cap C_r$ is volume minimizing among all the (n–2)–dimensional symmetric spheres on C_r, then the equality in (3.12) holds.

§4 Concrete examples

In this section we show several examples, where Theorem 3.2 and Corollary 3.3 enable us to prove Fomenko's conjecture mentioned in the introduction. We shall start by remarking some intuitively obvious facts.

Suppose $N \subset \mathbb{R}^n$ is half of hyperplane with boundary, a (n–2)–dimensional plane $\sum$. Through each point $p \in \mathbb{R}^n$ we draw the two–dimensional plane π_p that is the orthogonal complement to $\sum$. π_p intersects N at an unique point $\Theta(p)$. Denote by $\Theta : \mathbb{R}^n \longrightarrow N$ the map, sending each point p into $\Theta(p)$.

Proposition 4.1. a) The mapping $\Theta : \mathbb{R}^n \longrightarrow N$ is a contracting map, i.e. $\text{dist}(\Theta(p),\Theta(q)) \leq \text{dist}(p,q)$ for any $p,q \in \mathbb{R}^n$.
b) Let C be a hypersurface of revolution with axis $\sum$ in $\mathbb{R}^n$. Then the restriction of Θ to C is a contracting map of C into $C \cap N$.
c) The volume of every figure in C does not increase under Θ.

Proof. Choose orthonormal coordinates $x_1,x_2,\dots,x_{n-2},x_{n-1},x_n$ in $\mathbb{R}^n$ with origin $0 \in \sum$ such that $x_1,x_2,\dots,x_{n-2}$ are orthonormal coordinates of $\sum$. Consider arbitrary points $p = (p_1,p_2,\dots,p_n)$ and $q = (q_1,q_2,\dots,q_n)$ in $\mathbb{R}^n$. Clearly,

$$\Theta(p) = (p_1,p_2,\ldots,p_{n-2}, \sqrt{p_{n-1}^2 + p_n^2},0)$$
$$\Theta(q) = (q_1,q_2,\ldots,q_{n-2}, \sqrt{q_{n-1}^2 + q_n^2},0) .$$

We have

$$\mathrm{dist}(p,q)^2 = \sum_{i=1}^{n} (p_i - q_i)^2 = \sum_{i=1}^{n-2} (p_i-q_i)^2 + (p_{n-1}-q_{n-1})^2 + (p_n-q_n)^2$$

$$\mathrm{dist}(\Theta(p),\Theta(q))^2 = \sum_{i=1}^{n-2} (p_i-q_i)^2 + (\sqrt{p_{n-1}^2+p_n^2} - \sqrt{q_{n-1}^2+q_n^2})^2 .$$

Therefore,

$$\mathrm{dist}(p,q)^2 - \mathrm{dist}(\Theta(p),\Theta(q))^2 = (p_{n-1} - q_{n-1})^2 + (p_n - q_n)^2 -$$

$$(\sqrt{p_{n-1}^2 + p_n^2} - \sqrt{q_{n-1}^2+q_n^2})^2 = p_{n-1}^2 + q_{n-1}^2 - 2p_{n-1}q_{n-1} + p_n^2 + q_n^2 -$$

$$-2p_nq_n - (p_{n-1}^2 + p_n^2) - (q_{n-1}^2 + q_n^2) + 2\sqrt{(p_{n-1}^2+p_n^2)(q_{n-1}^2+q_n^2)}$$

$$= 2[\sqrt{(p_{n-1}^2+p_n^2)(q_{n-1}^2+q_n^2)} - (p_{n-1}q_{n-1}+p_nq_n)].$$

On the other hand,

$$(p_{n-1}^2+p_n^2)(q_{n-1}^2+q_n^2)-(p_{n-1}q_{n-1}+p_nq_n)^2=p_{n-1}^2q_{n-1}^2+$$

$$p_{n-1}^2q_n^2+p_n^2q_{n-1}^2+p_n^2q_n^2-p_{n-1}^2q_{n-1}^2-p_n^2q_n^2-2p_{n-1}q_{n-1}p_nq_n$$

$$=p_{n-1}^2q_n^2+p_n^2q_{n-1}^2-2p_{n-1}q_{n-1}p_nq_n = (p_{n-1}q_n-p_nq_{n-1})^2 \geq 0\,.$$

Consequently, $\mathrm{dist}(p,q) \geq \mathrm{dist}(\Theta(p),\Theta(q))$, proving the statement a) of the proposition. Consider now an arbitrary curve γ in C. Since C is a hypersurface of revolution, it is clear that the image $\Theta(\gamma)$ of γ under Θ is contained in $C\cap N$. Combining the statement a) with the fact that γ and $\Theta(\gamma)$ can be approximated by broken lines with length, arbitrarily close to the length of γ and $\Theta(\gamma)$ respectively, we can prove the statement b). Finally, the statement c) follows from b). Thus, the proof is completed.

From Proposition 4.1 it immediately follows

Corollary 4.2. Every meridian on a hypersurface of revolution is volume minimizing.

Let B is a convex domain in $\mathbb{R}^n$. Consider the "concave map" $\sigma : \mathbb{R}^n\setminus B \longrightarrow \partial B$, carrying each point $p \in \mathbb{R}^n\setminus B$ into the base of the perpendicular, drawn from the point p onto ∂B. It is easy to see that the map σ is defined uniquely, even for case when ∂B is just piecewise smooth.

Proposition 4.3 a) For any points $p,q \in \mathbb{R}^n\setminus B$ the following inequality holds

$$\mathrm{dist}(p,q) \geq \mathrm{dist}(\sigma(p),\sigma(q))$$

b) The volume of every figure in $\mathbb{R}^n\setminus B$ does not increase under σ.

Proof. First of all we note a simple property of the "concave map" σ, namely,

$$\text{dist}_{\mathbb{R}^n}(p,q) \geq \text{dist}_{\mathbb{R}^n}(\sigma(p),\sigma(q)) \tag{4.1}$$

for any points $p,q \in \mathbb{R}^n\setminus B$. Consider now the image $\sigma([p,q])$ of the line segment $[p,q]$, joining p and q, and an arbitrary broken line γ, approximating $\sigma([p,q])$. The curve γ consists of line segments $[\sigma(p_i),\sigma(p_{i+1})]$, $0 \leq i \leq m-1$, where $p_0,p_1,\ldots,p_m \in [p,q]$, $p_0=0$, $p_m=q$. According to (4.1) we have

$$\text{length}(\gamma) = \sum_{i=0}^{m-1} \text{dist}(\sigma(p_i),\sigma(p_{i+1})) \leq \sum_{i=0}^{m-1} \text{dist}(p_i,p_{i+1}) = \text{dist}(p,q).$$

Hence, we obtain: length $(\sigma([p,q])) \leq \text{dist}(p,q)$. Consequently,

$$\text{dist}_{\partial B}(\sigma(p),\sigma(q)) \leq \text{length}(\sigma([p,q])) \leq \text{dist}(p,q).$$

In this way the statement a) has been proved. The statement b) immediately follows from a). The proof is completed.

Corollary 4.4. Suppose B is a convex domain in $\mathbb{R}^n$, π half a hyperplane with boundary Σ. Let the map Θ be defined as in Proposition 4.1 and assume that $\Theta(\partial B) \subset \pi\setminus B$. Then $\pi\cap\partial B$ is volume minimizing in ∂B.

Proof. Assume that X is an arbitrary hypersurface in ∂B with boundary $\partial X \subset \pi \cap \partial B$. Denote by $\tilde{\pi}$ the hyperplane, containing π. Since B is convex, the intersection $\tilde{\pi} \cap B$ is also convex. Let X' denote the image (with multiplicity) of X under Θ. Clearly, $\partial X' = \Theta(\partial X) = \partial X$. Hence, $\mathrm{vol}_{n-2}(X') \leq \mathrm{vol}_{n-2}(X)$ by force of Proposition 4.1. Consider "concave map" $\sigma: \tilde{\pi} \setminus B \longrightarrow \tilde{\pi} \cap \partial B$ and denote by π' the part of $\pi \subset \partial B$, bounded by ∂X, and by X" the image (with multiplicity) of X' under σ. Clearly, X" covers the whole π_1. Now, applying Proposition 4.3 we have

$$\mathrm{vol}_{n-2}(\pi') \leq \mathrm{vol}_{n-2}(X'') \leq \mathrm{vol}_{n-2}(X').$$

Therefore, $\mathrm{vol}_{n-2}(\pi_1) \leq \mathrm{vol}_{n-2}(X)$, completing the proof.

Example 1. Consider a n–dimensional rectangular parallelepiped B with centre 0 and edges $a_1 \leq a_2 \leq \ldots \leq a_n$. Denote by F and F' the faces of B that are perpendicular to the longest edge a_n. Let M be the central hyperplane, parallel to the faces F and F'. Set $\beta = M \cap \partial B$. Obviously, β is the boundary of a (n–1)–dimensional rectangular parallelepiped with centre 0 and edges $a_1 \leq a_2 \leq \ldots \leq a_{n-1}$. Suppose γ is a (n–2)–dimensional symmetric homeomorphic sphere in ∂B. Since γ and β are both symmetric spheres, their intersection $\gamma \cap \beta$ contains a (n–3)–dimensional symmetric sphere A, dividing each of γ and β into two symmetric parts: $\gamma = \gamma_1 \cup \gamma_2$, $\beta = \beta_1 \cup \beta_2$, $\partial\gamma_1 = \partial\gamma_2 = \partial\beta_1 = \partial\beta_2 = A$, $\mathrm{vol}_{n-2}(\gamma_1) = \mathrm{vol}_{n-2}(\gamma_2)$, $\mathrm{vol}_{n-2}(\beta_1) = \mathrm{vol}_{n-2}(\beta_2)$. Clearly, β_1 and β_2 are volume minimizing with respect to the boundary A. Hence, $\mathrm{vol}_{n-2}(\gamma_1) \geq \mathrm{vol}_{n-2}(\beta_1)$, $\mathrm{vol}_{n-2}(\gamma_2) \geq \mathrm{vol}_{n-2}(\beta_2)$. Consequently, $\mathrm{vol}_{n-2}(\gamma) \geq \mathrm{vol}_{n-2}(\beta)$. Thus, we have

(4.2) $$\mathrm{vol}_{n-2}(\beta) = \inf_{\gamma} \mathrm{vol}_{n-2}(\gamma),$$

where the infimum is taken over all (n–2)–dimensional symmetric homeomorphic spheres on $C_1 = \partial B$.

At a vertex P of the parallelepiped we consider the n–dimensional cube with edge $1/2a_1$, having a vertex at P and faces, parallel to the faces of B. Denote by PQ its main diagonal, passing through P. We have $|PQ| = \frac{a_1}{2}\sqrt{n}$. Let $S \in PQ$, $|QS| = \frac{sa_1}{2}\sqrt{n}$ $(0 \le s \le 1)$. Denote by B_s the rectangular parallelepiped with centre 0, that has a vertex at S and faces, parallel to the faces of B.

Clearly, the domains B_s $(0 \le s \le 1)$ satisfy the condition of Corollary 3.3. Moreover, $B_1 = B$, B_0 is a (n–1)–dimensional rectangular parallelepiped, that has centre 0 and edges $a_2-a_1,\dots,a_n-a_1$ and lies on the central hyperplane, perpendicular to the shortest edge a_1. Set $\beta_s = M \cap \partial B_s$ $(0 \le s \le 1)$. Similarly to (4.2) one can get the equality

(4.3) $$\mathrm{vol}_{n-2}(\beta_2) = \inf_{\gamma} \mathrm{vol}_{n-2}(\gamma)\ ,$$

where γ runs all (n–2)–dimensional symmetric homeomorphic spheres on $C_s = \partial B_s$. On the other hand, by a simple calculation we have $h(x) = a_{1/2}$ for any $x \in B$. Now from (3.12) and (4.3) it follows that

(4.4) $$\mathrm{vol}_{n-1}(X \cap B) \ge \int_0^1 \frac{a_1}{2} \mathrm{vol}_{n-2}(\beta_s) ds$$
$$= \mathrm{vol}_{n-1}(M \cap B)\ ,$$

where X is an arbitrary (n–2)–connected volume minimizing hypersurface in B with boundary on ∂B, which contains 0 as an isolated singularity, satisfying the condition (1.1). Thus, in this case Fomenko's conjecture is true.

Example 2. Consider two–dimensional ellipsoids B_r in $\mathbb{R}^3$, given by the equations

$$f = f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - r^2 = 0 \quad (a \leq b \leq c,\ 0 \leq r \leq 1) .$$

Assume, at first, that $b = c$, i.e. B_r are ellipsoids of revolution. Denote by M and M′ the central hyperplanes, given by the equations $z = 0$ and $x = 0$ respectively. Clearly, $M' \cap \partial B_1$ is a circle with radius b. Suppose γ is a symmetric curve in ∂B_1. The intersection $(M' \cap \partial B_1) \cap \gamma$ contains antipodal points $\{P,Q\}$. Denote by π half a hyperplane, passing through P and Q and perpendicular to M′. Let Θ be the map, defined as in Proposition 4.1. It is easy to see that $\Theta(\partial B_1) \subset \pi \backslash B_1$. Therefore, $\pi \cap \partial B_1$ is length minimizing in ∂B_1 by force of Corollary 4.4. Hence,
length $(\gamma) \geq 2\text{length}(\pi \cap \partial B_1) = \text{length}(M \cap \partial B_1)$. Consequently, we have

(4.5) $$\text{length}(M \cap \partial B_1) = \inf_{\gamma} \text{length}(\gamma),$$

where γ is an arbitrary symmetric curve in ∂B_1. Without lost generality we can assume that π is contained in the coordinate plane M. Set $\alpha_1 = M \cap \partial B_1 \cap \{x>0,y<0\}$, $\alpha_2 = M \cap \partial B_1 \cap \{x>0,y>0\}$ and denote by γ_1 (respectively, γ_2) the connected component of $\gamma \cap \{x>0,y<0\}$ (respectively, $\gamma \cap \{x>0,y>0\}$), containing the point (0,–1,0) (respectively, (0,1,0)). Fist of all we note that $\text{length}(\gamma_1) \geq \text{length}(\alpha_1)$. Really, suppose not. One can consider the curve γ', consisting of γ_1 and its image under the symmetry with respect to the coordinate plane y=0. Obviously, length $(\gamma') \geq 2$ length(α_1) = length $(\pi \cap \partial B_1)$ and this is contradict to the fact that $\pi \cap \partial B_1$ is volume minimizing. We define the mapping $\tau : \alpha_1 \longrightarrow \gamma_1$ by the following requirement: for any $P \in \alpha_1$ the length of the part of α_1 between $(0,-1,0)$ and P and the length of the part of γ_1 between $(0,-1,0)$ and $\tau(P)$ are equal.

Lemma 4.5. For any $P \in \alpha_1$ we have

$$|\mathrm{grad}f(P)| \geq |\mathrm{grad}f(\tau(P))| \tag{4.6}$$

Proof. We have

$$\mathrm{grad}f = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{b^2}\right).$$

Therefore,

$$|\mathrm{grad}\ f|^2 = 4\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{b^4}\right) \tag{4.7}$$

$$= \frac{4}{b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2}\right) + \frac{4}{a^2}\left(\frac{X^2}{a^2} - \frac{x^2}{b^2}\right)$$

$$= \frac{4}{b^2} + \frac{4x^2}{a^4b^2}(b^2 - a^2).$$

On the other hand, from the statement c) of Proposition 4.1 it follows that the x–coordinate of $\tau(p)$ does not exceed the x–coordinate of P. Combining this fact with (4.7) proves (4.6). Now taking (4.6) into account, we obtain

$$H_1(\alpha_1) = \int_{\alpha_1} \frac{ds(P)}{|\mathrm{grad}\ f(P)|} = \int_{\alpha_1} \frac{ds(\tau(P))}{|\mathrm{grad}\ f(P)|}$$

$$\leq \int_{\alpha_1} \frac{ds(\tau(P))}{|\mathrm{grad}\ f(\tau(p))|} = \int_{\tau(\alpha_1)} \frac{ds(P)}{|\mathrm{grad}\ f(P)|}$$

$$\leq \int_{\gamma_1} \frac{ds(P)}{|\mathrm{grad} f(p)|} = H_1(\gamma_1) .$$

Similarly, one can get $H_1(\alpha_2) \leq H_1(\gamma_2)$. Consequently,

(4.8) $$H_1(\pi \cap \partial B_1) \leq H_1(\alpha_1)+H_1(\alpha_2) \leq H_1(\gamma_1) + H_1(\gamma_2) \leq \tfrac{1}{2}H_1(\gamma)$$

for any symmetric curve γ in ∂B_1. By using Theorem 3.2, we can conclude that

(4.9) $$\mathrm{vol}_2(X \cap B) \geq \mathrm{vol}_2(M \cap B) ,$$

where X is an arbitrary simply connected volume minimizing hypersurface in B with boundary on ∂B, which contains 0 as an isolated singularity, satisfying the condition (1.1).

Finally, since every ellipsoid with axes a,b,c ($a \leq b \leq c$) contains an ellipsoid of revolution with axes a,b,b, the inequality (4.9) is also true in general case of an arbitrary two–dimensional ellipsoid. Thus, in this case Fomenko's conjecture is true.

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