Vanishing of Ext groups in functor categories

T. Pirashvili

Mathematical Institute
Academy of Sciences
Rukhadze st. 1.. Tbilisi,
Georgia, 380093

USSR

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI / 91-19

§ 1. Introduction

The aim of this paper is the following result.

1.1 Theorem. *Let* $\underline{\underline{A}}$ *be a small additive category and* $\underline{\underline{F}}$ *be the category of all functors from* $\underline{\underline{A}}$ *to the category of modules over the ring* R . *Let*

$$N : \underline{\underline{A}}^n \longrightarrow R - mod\,,\ n \geq 2\,,$$

be a functor with the property that

(1.2) $$N(X_1, \cdots, X_n) = 0 \ \ if\ there\ is\ \ i \in \{1, \cdots, n\}\ \ with\ \ X_i = 0\,.$$

Let

$$N^d : \underline{\underline{A}} \longrightarrow R - mod$$

be the composition of N *with the diagonal embedding* $\underline{\underline{A}} \longrightarrow \underline{\underline{A}}_n$. *Then*

$$Ext^q_{\underline{\underline{F}}}\Big(M, N^d\Big) = 0 = Ext^q_{\underline{\underline{F}}}\Big(N^d, M\Big)\,,\ q \geq 0\,,$$

whenever F *has degree (in sense of Eilenberg and MacLane [EM] at most* $n-1$.

This theorem was proved in [P] in case $n = 2$. Using 3.9 [JP] we deduce from theorem 1.1 the

1.3 Corollary. Suppose that the values of M are projective R-modules. Then

$$H^*\Big(\underline{\underline{A}}, Hom_R\Big(M, N^d\Big)\Big) = 0$$

where $Hom_R(M, N^d)$ is the bifunctor on $\underline{\underline{A}}$ given by

$$(X, Y) \mapsto Hom_R(M(X),\ N(Y, \cdots, Y))\,.$$

§ 2. Proof of the theorem

We recall that for any functor $T : \underline{\underline{A}} \longrightarrow R - mod$ and any $\kappa \geq 0$ the $\kappa - th$ cross-effect of T is a functor

$$T_\kappa : \underline{\underline{A}}^\kappa \longrightarrow R - mod\,,$$

such that for $X_1, \cdots, X_m \in \mathrm{Ob}\,\underline{\underline{A}}$ there exists a natural decomposition (see [EM]

(2.1) $$T(X_1 \oplus \cdots \oplus X_m) = \oplus T_\kappa(X_{i_1}, \cdots, X_{i_n})\,,$$

where the sum is taken over all $1 \leq i_1 < \cdots < i_k \leq m$. The functor T has degree at most m if $T_{m+1} = 0$. We denote by $\eta_{T,\kappa}(X)$ the projection on the summand $T_\kappa(X, \cdots, X)$ of the map $T(X) \longrightarrow T\left(\bigoplus_{i=1}^{\kappa} X\right)$ which is induced by diagonal map $X \longrightarrow \bigoplus_{i=1}^{\kappa} X$. Then we obtain the natural transformation

$$\eta_{T,\kappa} : T \longrightarrow T^d_\kappa\,.$$

Here T^d_κ denotes the composition of $T_\kappa : \underline{\underline{A}}^\kappa \longrightarrow R-mod$ with the diagonal embedding $\underline{\underline{A}} \longrightarrow \underline{\underline{A}}^\kappa$. Let $\xi_{T,\kappa}(X)$ be the restriction on the summand $T^d_\kappa(X)$ of the map $T\left(\bigoplus_{i=1}^{\kappa} X\right) \longrightarrow T(X)$ which is induced by the codiagonal map $\bigoplus_{i=1}^{\kappa} X \longrightarrow X$. Then we have the natural transformation

$$\xi_{T,\kappa} : T^d_\kappa \longrightarrow T .$$

2.2 Lemma. Let $N : \underline{\underline{A}}^n \longrightarrow R-mod$ be the functor satisfying condition 1.2 and $T = N^d$. Then $\eta_{T,n}$ is a split monomorphism and $\xi_{T,n}$ is a split epimorphism.

Proof. Let $X_1, \cdots, X_n \in Ob\, \underline{\underline{A}}$ and let

$$\alpha(X_1,\cdots,X_n) : \bigoplus_{i=1}^{n} T\left(X_1 \oplus \cdots \oplus \hat{X}_i \oplus \cdots \oplus X_n\right) \longrightarrow T\left(\bigoplus_{i=1}^{n} X_i\right)$$

be induced by the inclusion

$$X_1 \oplus \cdots \oplus \hat{X}_i \oplus \cdots \oplus X_n \longrightarrow \bigoplus_{i=1}^{n} X_i .$$

Then 2.1 implies that the column in the following diagramm is exact.

$$\begin{array}{ccccccc}
 & & & & \bigoplus_{i=1}^{n} T(X\oplus\cdots\oplus\hat{X}\oplus\cdots\oplus X) & & \\
 & & & & \downarrow \alpha & & \\
T(X) & = & N(X,\cdots,X) & \underset{N(p_1,\cdots,p_n)}{\overset{T(diag)}{\rightleftarrows}} & T\left(\bigoplus_{i=1}^{n} X\right) & = & N\left(\bigoplus_{i=1}^{n} X,\cdots,\bigoplus_{i=1}^{n} X\right) \\
 & & \eta_{T,n}\searrow & & \downarrow pr & & \\
 & & & & T^d_n(X) & & \\
 & & & & \downarrow & & \\
 & & & & 0 . & &
\end{array}$$

Here pr is the projection in the decomposition 2.1 and

$$p_i : \bigoplus_{i=1}^{n} X \longrightarrow X$$

is the projection on the i-th summand. Hence we get

$$\eta_{T,n} = pr \circ T(diag) ,$$
$$N p_1, \cdots, p_n) \circ T(diag) = 1_{T(X)} ,$$
$$N(p_1, \cdots, p_n) \circ \alpha = 0 .$$

This shows that there is a retraction of $\eta_{T,n}$. A similar argument shows that $\xi_{T,n}$ has a section.

Lemma 2.3 Let $\underline{\underline{F}}(\underline{\underline{n}})$ be the category of all functors from $\underline{\underline{A}}^n$ to $R-mod$. Then there is a functor

$$\psi : \underline{\underline{F}}(\underline{\underline{n}}) \longrightarrow \underline{\underline{F}}(\underline{\underline{n}})$$

with the following properties

i) ψ is an exact functor

ii) $\psi(W)$ satisfies property 1.2

iii) if W satisfies property 1.2 then $\psi(W) = W$

iv) if N satisfies property 1.2 and W is an arbitrary object in $\underline{\underline{F}}(\underline{n})$ then

$$Hom_{\underline{\underline{F}}(\underline{n})}(W, N) = Hom_{\underline{\underline{F}}(\underline{n})}(\psi(W), N)$$

and

$$Hom_{\underline{\underline{F}}(\underline{n})}(N, W) = Hom_{\underline{\underline{F}}(\underline{n})}(N, \psi(W))$$

v) ψ sends injective and projective objects to injective and projective objects resepectively.

Proof. We define ψ by

$$(\psi W)(X_1, \cdots, X_n) = Ker\left(W(X_1, \cdots, X_n) \xrightarrow{\gamma} \bigoplus_{i=1}^{n} W(X_1, \cdots, 0, \cdots, X_n) \right) .$$

Here γ is induced by $X_i \to 0$. Then ii) and iii) follows from the definition. Since γ has a section ψW is a direct summand of W and we have

$$(\psi W)(X_1, ..., X_n) = Coker\left(\bigoplus_{i=1}^{n} W(X_1, \cdots, 0, \cdots, X_n) \to W(X_1, \cdots, X_n) \right) .$$

From this follows i). We remark that v) formally follows from i) and iii). Let $f : W \to N$ be a natural transformation. Then the commutative diagramm

$$\begin{array}{ccc} W(X_1, \cdots, X_n) & \xrightarrow{f(X_1, \cdots, X_n)} & N(X_1, \cdots, X_n) \\ \uparrow & & \uparrow \\ \bigoplus_{i=1}^{n} W(X_1, \cdots, 0, \cdots, X_n) & \xrightarrow{\oplus f(X_1, \cdots, 0, \cdots, X_n)} & \bigoplus_{i=1}^{n} N(X_1, \cdots, 0, \cdots, X_n) = 0 \end{array}$$

shows that $f(X_1, \cdots, X_n)$ uniquely factors through $\psi(W)$ and we obtain the first isomorphism in iii). The second is proved by the dual argument.

2.4 Lemma. Let W be an injective (resp. projective) object in $\underline{\underline{F}}(\underline{n})$. Then W^d is an injective (resp. a projective) object in $\underline{\underline{F}}$.

Proof. We recall that for an arbitrary small category $\underline{\underline{C}}$ the following family of functors is a family of injective (resp. projective) generators in $(R - mod)^{\underline{\underline{C}}}$:

$$X \mapsto Maps(\underline{\underline{C}}(\underline{\underline{X}}, \underline{c}), I) \quad (resp. X \mapsto R[\underline{\underline{C}}(c, X)]) .$$

Here $c \in Ob\, \underline{\underline{C}}$, I is an injective generator in the category of R-modules and R[S] is the free R-module with base S . Therefore it is sufficient to consider the case

$$W(X_1, \cdots, X_n) = Maps(\underline{\underline{A}}(X_1, c_1) \times \cdots \times \underline{\underline{A}}(X_n, c_n), I) ,$$

resp. the case

$$W(X_1, \cdots, X_n) = R[\underline{\underline{A}}(c_1, X_1) \times \cdots \times \underline{\underline{A}}(c_n, X_n)]$$

where $c_1, \cdots, c_n \in Ob\ \underline{\underline{A}}$. Hence

$$W^d(X) = Maps\big(\underline{\underline{A}}(X, c_1 \times \cdots \times c_n), I\big)$$

resp.

$$W^d(X) = R\big[\underline{\underline{A}}(c_1 \oplus \cdots \oplus c_n, X)\big] .$$

Therefore W^d is injective (resp. projective) in $\underline{\underline{F}}$.

2.5 Proof of the theorem. First consider the case when $q = 0$. Let

$$\tau : M \to T = N^d$$

(resp.

$$\sigma : N^d = T \to F \ )$$

be a natural transformation with $deg\ M < n$ and assume N satisfies condition 1.2. Since $\eta_{T,n}$ and $\xi_{T,n}$ are natural with respect T , we obtain the following commutative diagrams.

$$\begin{array}{ccccccc} & & M & \overset{\sigma}{\to} & T \\ & & \eta_{M,n} \downarrow & & \downarrow \eta_{T,n} \\ 0 & = & M_n^d & \overset{\sigma_n}{\to} & T_n^d \end{array} \quad , \quad \begin{array}{ccccc} T_n^d & \overset{\sigma_n}{\to} & F_n^d & = & 0 \\ \xi_{T,n} \downarrow & & \downarrow \xi_{F,n} & & \\ T & \overset{\sigma}{\to} & F . & & \end{array}$$

By 2.2 $\eta_{T,n}$ is a monomorphism and $\xi_{T,n}$ is an epimorphism, therefore $\tau = \sigma = 0$. Let $0 \to N \to I^*$ be an injective resolution in $\underline{\underline{F}}(\underline{n})$. Then $0 \to N \to \psi(I^*)$ is also an injective resolution by 2.3. Therefore $0 \to N^d \to (\psi(I^*))^d$ is an injective resolution too (by 2.4). But $\psi(I^*)$ satisfies the property 2.1 by 2.3. Therefore

$$Ext^*_{\underline{\underline{F}}}\Big(M, N^d\Big) = H^*\Big(Hom_{\underline{\underline{F}}}\Big(M, (\psi(I^*))^d\Big)\Big) = H^*(0) = 0 .$$

By the dual argument we obtain the second equality

$$Ext^*_{\underline{\underline{F}}}\Big(N^d, M\Big) = 0 .$$

References

[EM] S. Eilenberg & S. MacLane. On the groups $H(\pi, n)$, II. Ann. Math. 60 (1954), 49–139.

[JP] M. Jibladze & T. Pirashvili. Cohomology of algebraic theories. To appear in J. Algebra.

[P] T. Pirashvili. Higher additivizations. Proc. Math. Inst. Tbilisi. v. 91 (1988), 44–54.