Projecting surfaces into $\mathbb{P}^4$.

by

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Hochgeehrter Herr Professor!

Schon seit recht langer Zeit strebten wir danach, Ihnen die Ergebnisse, die wir in der von Ihnen geschaffenen Theorie der topologischen Räume gefunden haben, mitzuteilen. Wir erlauben uns die Hoffnung auszusprechen, dass Sie die Gefälligkeit haben werden, uns zu gestatten, hier einige derselben zu nennen. Ein Teil der gewonnenen Resultate haben wir neuerdings in drei Noten („Bull. Internat. de l'Académie Polonaise", 1923) ohne Beweise formuliert; sie bilden die Anfangszüge einer Theorie, deren Darstellung die Redaktion der Zeitschrift „Fundamenta Mathematicae" von uns zu erhalten erwünscht hat.

Das Wesen der kompakten topologischen Räume ist das erste, was wir einer systematischen Untersuchung unterwerfen wollten. In dieser Hinsicht hatten wir zuerst die sogenannten bikompakten Räume herauszuheben, die durch eine jede der drei folgenden äquivalenten Eigenschaften charakterisiert werden können:

1°. Eine jede abnehmende wohlgeordnete Menge nichtleerer abgeschlossener Mengen besitzt einen nichtleeren Durchschnitt.

2°. Eine jede unendliche Menge $\mathfrak{M}$ besitzt wenigstens einen vollständigen Häufungspunkt ξ (d. i. dass $\mathfrak{D}(\mathfrak{M}, U_\xi)$ dieselbe Mächtigkeit wie $\mathfrak{M}$ hat, welche auch die Umgebung U_ξ von ξ sein möge).

3°. Der verschärfte Borelsche Satz (vgl. Satz VI, S. 272 Ihrer „Grundzüge").

Die bikompakten Räume besitzen mehrere bemerkenswerte Eigenschaften, sowohl mengentheoretischer, als topologischer Natur. Insbesondere sei auf Folgendes hingewiesen: Jede perfekte Menge besitzt daselbst die Mächtigkeit $\geq 2^{\aleph_0}$. Sie besitzt insbesondere genau die Mächtigkeit $2^{\aleph_0}$, wenn im Raume jedes F ein G_δ ist. Die letztere Bedingung (immer in bikompakten Räumen), die keineswegs dem Axiome „F" (II. Abzählbarkeitsaxiom) äquivalent ist, wohl aber aus dem letzten folgt, hat zur Folge das Axiom „E"; sie genügt um mehrere Mächtigkeitsfragen zu erledigen; z. B. lässt sich, unter der erwähnten Bedingung, jede abgeschlossene Menge in zwei Mengen zerspalten, deren eine perfekt,

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0. Introduction.

In this note we want to deal with projections of smooth surfaces in $\mathbb{P}^5 (= \mathbb{P}^5(\mathbb{C}))$ to $\mathbb{P}^4$. First of all we recall the following classical result, which was proved by F. Severi in 1901 (cf. [Se]).

<u>Theorem.</u> Let $Y \subset \mathbb{P}^5$ be a smooth, connected, non-degenerate surface. Then the secant variety of Y, Sec(Y), equals $\mathbb{P}^5$, unless Y is the Veronese surface.

Because the projection of $\mathbb{P}^5$ to $\mathbb{P}^4$ with center $y \in \mathbb{P}^5 \setminus Y$ gives rise to a closed embedding $\pi_y : Y \to \mathbb{P}^4$ iff y doesn't lie on a secant line of Y, this means that no smooth surface in $\mathbb{P}^5$ except the Veronese surface can be projected to a smooth surface in $\mathbb{P}^4$.

In this paper we want to study projections with <u>center on the surface</u> , i.e. we consider the following situation:
Let $\pi_y : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ be the projection from a point $y \in Y$. Then π_y induces a morphism $Y \hat{} (y) \to \mathbb{P}^4$ (which we also denote by π_y) from the blow-up $Y \hat{} (y)$ of Y in y to $\mathbb{P}^4$.
A natural question is now : when is $\pi_y : Y \hat{} (y) \to \mathbb{P}^4$ a closed embedding ?
For this we have the following criterion which is quite easy to verify.

<u>Lemma.</u> $\pi_y : Y \hat{} (y) \to \mathbb{P}^4$ is a closed embedding if and only if y does not lie on a trisecant of Y.

This means that a smooth surface $Y \subset \mathbb{P}^5$ can be projected (with center on Y) to a smooth surface in $\mathbb{P}^4$ iff $\mathrm{Trisec}(Y) \cap Y \neq Y$.

The purpose of this note is to give a proof of the following conjecture of A. Van de Ven, which is in some sense an analogue to Severi's theorem.

Conjecture. There exist only finitely many families of smooth surfaces Y in $\mathbb{P}^5$ with $\mathrm{Trisec}(Y) \cap Y \neq Y$.

Besides the classical (in-)equalities (Severi's double point formula, Miyaoka-Yau inequality, Hodge index theorem,...) the main tool in our proof is a formula of P. Le Barz. He calculates there the degree of a certain zero cycle, which is in appropriate geometric situations just the number of trisecants of Y intersecting a general plane in $\mathbb{P}^5$, as an universal polynomial in the degree and the Chern numbers of Y.
We prove that in our case the degree of this cycle has the right geometric meaning and that in fact it is zero.
This additional equality allows us to bound the degree of Y under the assumption $\mathrm{Trisec}(Y) \cap Y \neq Y$. Actually it turns out that the degree of Y has to be smaller or equal to 11, and with the help of the list of smooth surfaces in $\mathbb{P}^4$ up to degree 10 (cf. Okonek, Ionescu, Decker /Schreyer, Aure, Ranestad) we give here a complete list of all the smooth surfaces in $\mathbb{P}^4$ which are projections in the above sense.

In the first part of this paper we essentially state the result of P. Le Barz (without proof), the second section is devoted to the proof of our main theorem.

Acknowledgements. This result is part of my Ph.D. thesis at the University of Bonn. I would like to thank my advisor Christian Okonek for suggesting this research, following its development, and for many stimulating discussions. I am grateful to F.-O. Schreyer, who pointed out to me a mistake in a previous version. It is a pleasure for me to acknowledge my indebtness to MPI in Bonn for providing an excellent environment for my work.

1. Background material.

In this section we want to recall some basic definitions and state a result of P. Le Barz, which is essential for the proof of our main theorem.

Throughout the paper Y shall be a smooth, connected, non-degenerate algebraic surface in $\mathbb{P}^5(=\mathbb{P}^5(\mathbb{C}))$, and we shall denote by K a canonical divisor and by H a hyperplane section.

By $\mathrm{Hilb}^3\mathbb{P}^5$, resp. Hilb^3Y, we denote the Hilbert scheme of zero dimensional subschemes of $\mathbb{P}^5$, respectively Y, of length three ("3-tuples ").

$\mathrm{Hilb}_c^3\mathbb{P}^5$ is the open subset of $\mathrm{Hilb}^3\mathbb{P}^5$ given by the 3-tuples lying (locally around every point of the support) on a smooth curve, $\mathrm{Hilb}_c^3Y := \mathrm{Hilb}^3Y \times_{\mathrm{Hilb}^3\,\mathbb{P}^5} \mathrm{Hilb}_c^3\mathbb{P}^5$.

(1.1) Remark. $\mathrm{Hilb}_c^3\mathbb{P}^5$ is smooth and has dimension 15 (compare $[LB]_1$).

$Al^3\mathbb{P}^5$ is the subvariety given by these elements of $\mathrm{Hilb}_c^3\mathbb{P}^5$,which are subscheme of some line in $\mathbb{P}^5$.

(1.2) Remark. a) $Al^3\mathbb{P}^5$ is a smooth subvariety of $\mathrm{Hilb}_c^3\mathbb{P}^5$,

b) one has a canonical fibration a: $Al^3\mathbb{P}^5 \to \mathbb{G}(1,\mathbb{P}^5)$ (:= Grassmann manifold of lines in $\mathbb{P}^5$), where an element of $Al^3\mathbb{P}^5$ is mapped to the line on which it lies,

c) $Al^3\mathbb{P}^5$ is projective.

We denote by $[\mathrm{Hilb}_c^3Y]$ the cycle (of codimension 9) in the Chowring $Ch^{\cdot}(\mathrm{Hilb}_c^3\mathbb{P}^5)$ associated to the irreducible and smooth subscheme $\mathrm{Hilb}_c^3Y \subset \mathrm{Hilb}_c^3\mathbb{P}^5$.

Therefore, by considering the canonical inclusion i: $Al^3\mathbb{P}^5 \rightarrow Hilb_c{}^3\mathbb{P}^5$ we get a cycle $i^*[Hilb_c{}^3Y] \in Ch^9(Al^3\mathbb{P}^5)$. We call $i^*[Hilb_c{}^3Y]$ the trisecant cycle in $Al^3\mathbb{P}^5$.

Let $\sigma \in Ch^2(\mathbb{G}(1,\mathbb{P}^5))$ be the Schubert cycle of lines in $\mathbb{P}^5$, which intersect a fixed plane in $\mathbb{P}^5$.

With this set-up we are now able to formulate the theorem of P. Le Barz.

(1.3) Theorem ($[LB]_2$, Theoreme 3). Let $Y \subset \mathbb{P}^5$ be a smooth surface of degree n, δ the number of improper double points of a generic projection of Y to $\mathbb{P}^4$. Furthermore let d be the degree of the double curve and t the number of triple points of a generic projection of Y to $\mathbb{P}^3$. Then the degree of the zero cycle

$$a^*\sigma \cdot i^*[Hilb_c{}^3Y]$$

in $Ch^{\cdot}(Al^3\mathbb{P}^5)$ is

$$n(n-1)(n-2)/6 + 2t + (n-2)(\delta-d) .$$

(1.4) Remark. a) If one knows by geometric reasons that there exist only finitely many trisecants of Y intersecting a general plane in $\mathbb{P}^5$, then this number counted with appropriate multiplicities , is equal to $n(n-1)(n-2)/6 + 2t + (n-2)(\delta-d)$.
b) One can express the invariants t,δ,d of Y in terms of H, K, $c_2(Y)$ in the following way:

$$d = 1/2 (n(n-4) - H.K),$$

$$\delta = 1/2 (n(n-10) + c_2 - K^2 - 5H.K),$$

$$t = 1/6\ (n(n^2-12n+44) + 4K^2 - 2c_2 - 3H.K\ (n-8)),$$

(cf. $[LB]_2$, Annexe 6).

Finally we want to give a definition of the embedded trisecant variety of Y.

For this we denote by Z the closure of a(Al^3Y) in $\mathbb{G}(1,\mathbb{P}^5)$, where $Al^3Y := Al^3\mathbb{P}^5 \times_{Hilb_c{}^3\ \mathbb{P}^5} Hilb_c{}^3Y$. We consider the flag manifold $F := \{ (x,L) \in \mathbb{P}^5 \times \mathbb{G}(1,\mathbb{P}^5) : x \in L \}$ with the two projections

$$p : F \to \mathbb{G}(1,\mathbb{P}^5)\ ,$$

$$q : F \to \mathbb{P}^5.$$

<u>(1.5) Definition</u> (" embedded trisecant variety ").
$Trisec(Y) := q(p^{-1}(Z)) \subset \mathbb{P}^5$.

Obviously as a set Trisec(Y) is just the union of all trisecants of Y, where a trisecant is either a line contained in Y or a line in $\mathbb{P}^5$ which intersects Y in a zero dimensional subscheme of length at least three.

2. The main theorem.

This section is devoted to formulate and prove our main result. For this let $Y \subset \mathbb{P}^5$ as usual be a smooth (connected, non-degenerate) surface. We consider the diagram:

$$\begin{array}{ccc} y \in Y & \subset & \mathbb{P}^5 \\ \sigma \uparrow & & \downarrow \pi_y \\ Y\,\hat{}\,(y) & \rightarrow & \mathbb{P}^4, \end{array}$$

where π_y is the projection of $\mathbb{P}^5$ to $\mathbb{P}^4$ with center y and σ is the blow-up of Y in y. Then we have the following:

<u>(2.1) Lemma.</u> $\pi_y: Y\,\hat{}\,(y) \rightarrow \mathbb{P}^4$ is a closed embedding if and only if y does not lie on a trisecant line of Y.

<u>Proof.</u> Let H be a hyperplane section of Y, then $\pi_y: Y\,\hat{}\,(y) \rightarrow \mathbb{P}^4$ is given by the linear system $| H - y |$. Using [Ha] II, 7.8.2 one checks easily that $| H - y |$ gives a closed embedding iff y is not an element of Trisec(Y). Q.E.D.

Now we state our main result.

<u>(2.2) Theorem.</u> Let $Y \subset \mathbb{P}^5$ be a smooth (connected, non-degenerate) surface with $\text{Trisec}(Y) \cap Y \neq Y$. Then the degree of Y is smaller or equal to 11.

As a consequence we get the conjecture of A. Van de Ven.

<u>(2.3) Corollary.</u> There exist only finitely many families of smooth surfaces $Y \subset \mathbb{P}^5$ with $\text{Trisec}(Y) \cap Y \neq Y$.

<u>Or equivalently:</u> There exist only finitely many families of smooth surfaces in $\mathbb{P}^4$ which are obtained by projection (in the above sense).

Before giving a proof of theorem (2.2) we need some auxiliary results.

<u>(2.4) Remark.</u> Let Y be as in (2.2), K a canonical divisor of Y, H a hyperplane section and e = e(Y) the topological Euler characteristic of Y. Then

$$K^2 - e = n^2 - 12n - 5H.K + 8 ,$$

where n := degY.

<u>Proof.</u> We choose a point $y \in Y$ which does not lie on a trisecant. Then $\pi_y : Y\,\hat{}\,(y) \cong Y' \subset \mathbb{P}^4$ is a closed embedding. Obviously we have then for the hyperplane section and canonical divisor of Y':

$$H' = H - E ,$$

$$K' = K + E \qquad (\text{where } E := \sigma^{-1}(y)),$$

therefore:

$$n' := \deg Y' = n - 1 ,$$

$$K'^2 = K^2 - 1 ,$$

$$H'.K' = H.K + 1 .$$

By Severi's double point formula for $Y' \subset \mathbb{P}^4$ (cf. [Ha], Appendix A, 4.1.3) we get:

$$0 = n'^2 - 10n' - 5\,H'.K' - 2K'^2 + 12(1 + p_a(Y')) =$$

$$= (n-1)^2 - 10(n-1) - 5(H.K + 1) - 2(K^2 - 1) + 12(1 + p_a(Y'))$$

$$= n^2 - 12n - 5H.K + 8 - 2K^2 + K^2 + e ,$$

which implies the assertion. Q.E.D.

<u>(2.5) Lemma.</u> Let $Y \subset \mathbb{P}^5$ be a smooth surface with $\mathrm{Trisec}(Y) \cap Y \neq Y$. Then the following holds (with the same notation as in (1.3):

$$n(n-1)(n-2)/6 + 2t + (n-2)(\delta-d) =$$

$$= 1/6(\ n^2 - 18n - 3nH.K + 22H.K + 4(K^2 + 20)\ ) .$$

Proof. This is just a straightforward calculation using (1.4)b) and (2.4). Q.E.D.

We are now going to prove that the degree of the zero cycle in (1.3) is in fact zero (under the assumption $\mathrm{Trisec}(Y) \cap Y \neq Y$). By (1.4)a) we have to verify that for a general plane $P \subset \mathbb{P}^5$ it holds: $\mathrm{Trisec}(Y) \cap P = \emptyset$, which is equivalent to the fact that the dimension of Trisec(Y) is smaller or equal to two.

(2.6) Proposition. Let $Y \subset \mathbb{P}^5$ be a smooth surface with $\mathrm{Trisec}(Y) \cap Y \neq Y$. Then every irreducible component of Trisec(Y) has dimension smaller or equal to two.

Proof. Because $\mathrm{Trisec}(Y) \cap Y \neq Y$ we see that $\mathrm{Trisec}(Y) \cap Y =: C$ has dimension smaller or equal to one. Let $C = \cup_{1 \leq i \leq r} C_i \ \cup \ \cup_{1 \leq j \leq s} \{x_j\}$ be the decomposition of C in its irreducible components.
Assume that there exists an irreducible component T of Trisec(Y) (cf. (1.5)) with $\dim T = 3$. Then we have the following possibilities for T:

1) $T = \cup_{x \in C_i} T_x Y$, where C_i is an irreducible component of C and each of these trisecants is a tangent line at $x \in C_i$ meeting Y in a third point. Since $\dim T = 3$, this third point is not fixed. The union of these points must then be an irreducible component C_j ($j \neq i$) of C.

2) $T = \cup_{x \in C_i} T_x Y$, where C_i is an irreducible component of C and every tangent line in each $T_x Y$ meets Y in x of order at least three.

3) $T = \mathrm{Sec}(C_i)$, where C_i is a reduced, irreducible component of C, which is not contained in a plane.

4) $T = C_i * C_j$, where C_i, C_j are reduced, irreducible components of C, and $C_i \cup C_j$ is not contained in a plane.

We are going to exclude step by step all these possibilities.

1) Note that for all $x \in C_i$ it holds $C_j \subset T_xY$. Therefore C_j must be a line, because otherwise T_xY would be an unique plane for all $x \in C_i$ contradicting $\dim T = 3$. We state the following

Lemma. Let $C \subset \mathbb{P}^n$ be an irreducible,reduced curve. If there exists a line $L \subset \mathbb{P}^n$, s. th. each tangent line of C meets L, then C must be a plane curve.

Proof. Let t be a local parameter of C, and v(t) a local lift to $\mathbb{C}^{n+1}$. Furthermore let $V \subset \mathbb{C}^{n+1}$ be the rank 2 vector subspace ,such that $L = \mathbb{P}(V)$. Then setting $w(t) = p(v(t))$, where $p: \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}/V$ is the natural projection, we have that w(t) and w'(t) are always proportional, whence w(t) is constant. Q.E.D.

From the proof of the previous lemma we get that C_i is contained in a plane π containing the line C_j . But then $T_xY = \pi$ for all $x \in C_i$, since T_xY is the span of C_j and the tangent line to C_i at x, which again contradicts $\dim T = 3$.

2) We consider the Gauss map

$$\varphi: Y \to \mathbb{G}(2,\mathbb{P}^5) , \quad y \longmapsto T_yY .$$

It is easy to verify that $(d\varphi)_y = 0$ for a point $y \in Y$ where every tangent line has contact of order at least three and so $\varphi|C_i$ is constant. Therefore also in this case the dimension of T must be smaller than three.

The cases 3),4) can be treated simultaneously. The argument we use is the same as in [A-C-G-H] p.110.

$T = \mathrm{Sec}(C_i)$ (resp. $C_i * C_j$), so for $p,q \in C_i$ (resp. $p \in C_i$, $q \in C_j$) the line $p*q$ meets C in a third point $v = u(p,q)$. After choosing appropriate local parameters s,t,u around p,q,v and viewing $p(s)$, $q(t)$, $v(u)$ as functions with values in $\mathbb{C}^6$, we can assume:

$$p(s) \wedge q(t) \wedge v(u(s,t)) \equiv 0.$$

Differentiation with respect to s and t gives:

$$p'(s) \wedge q(t) \wedge v(u(s,t)) + p(s) \wedge q(t) \wedge v'(u(s,t))(\partial u/\partial s)(s,t) \equiv 0,$$

$$p(s) \wedge q'(t) \wedge v(u(s,t)) + p(s) \wedge q(t) \wedge v'(u(s,t))(\partial u/\partial t)(s,t) \equiv 0.$$

Since $\dim T = 3$, it is clear that $\partial u/\partial s$ and $\partial u/\partial t$ are not identically zero. Hence:

$$p' \wedge q \wedge v = \lambda(p \wedge q' \wedge v),$$

with $\lambda \neq 0$, which implies that p,p',q,q' lie in a $\mathbb{C}^3$. So we have shown that any two tangent lines of C_i (resp. a tangent line of C_i and a tangent line of C_j always meet in a point. Because C_i (resp. $C_i \cup C_j$) is not contained in a plane this point must be the same for all tangents (in fact, if l,l' are tangent lines of C_i , L is a tangent line of C_i (resp. C_j), then if l' does not pass through $l \cap L$, it is contained in the plane $l*L$). Projection from this point (we call it c) in $\mathbb{P}^4$ gives a map

$$f: C_i \setminus \{c\} \text{ (resp. } C_i \cup C_j \setminus \{c\} \text{) } \rightarrow \mathbb{P}^4$$

with $df = 0$ and so C_i must be a line containing c (resp. $C_i \cup C_j$ must be the union of two lines through c) which is a contradiction.

Alltogether we have shown that the dimension of T must be strictly smaller than three. Q.E.D.

<u>(2.7) Corollary.</u> Let $Y \subset \mathbb{P}^5$ be as in (2.6). Then

$$n^2 - 18n - 3nH.K + 22H.K + 4(K^2 + 20) = 0 ,$$

where n is again the degree of Y.

Proof. This follows by combining (1.3), (1.4)a), (2.5) and (2.6). Q.E.D.

Proof of (2.2). The assumption $Trisec(Y) \cap Y \neq Y$ implies

(1) $K^2 - e = n^2 - 12n - 5H.K + 8$ (cf. (2.4)) ,

(2) $n^2 - 18n - 3nH.K + 22H.K + 4(K^2 + 20) = 0$ (cf (2.7)).

1. case. $kod(Y) \geq 0$ or Y is rational.
Then it follows by the Miyaoka-Yau inequality :

$$K^2 = -1/4(n^2 - 18n - 3nH.K + 22H.K + 80) \leq 3e =$$

$$= -3/4 (5n^2 - 66n - 3nH.K + 2H.K + 112)$$

and therefore one gets :

$$H.K \geq \frac{7n^2 - 90n + 128}{3n + 8} \tag{3}$$

2. case. Y is a birationally ruled surface.
Using the inequality $K^2 \leq 2e$ we get by the same calculation as above :

$$H.K \geq \frac{9n^2 - 114n + 144}{3n + 18} \tag{4}$$

From the Hodge index theorem it follows now in both cases :

$$K^2 = -1/4(n^2 - 18n - 3nH.K + 22H.K + 80) \leq (H.K)^2/n ,$$

which implies :

$$H.K\,(\,3n^2 - 22n\ - 4H.K\,) \le\ n^3 - 18n^2 + 80n\,. \qquad (5)$$

We assume from now on that the <u>degree of Y is bigger or equal to 12</u> and distinguish again between the two cases :

<u>1. case.</u> kod(Y) $\ge$ 0 or Y is rational.

a) $3n^2 - 22n\ - 4H.K \le 0$.
This implies

$$H.K \ge 3/4n^2 - 11/2n\,. \qquad (6)$$

b) $3n^2 - 22n\ - 4H.K > 0$.
Then we have

$$\frac{7n^2 - 90n + 128}{3n + 8}\,(\,3n^2 - 22n\ - 4H.K\,) \le$$

$$\le H.K\,(\,3n^2 - 22n\ - 4H.K\,) \le\ n^3 - 18n^2 + 80n\,.$$

Because $7n^2 - 90n + 128 > 0$ for $n \ge 12$ this implies

$$3n^2 - 22n\ - 4H.K\ \le\ \frac{(3n + 8)(n^3 - 18n^2 + 80n)}{7n^2 - 90n + 128}$$

and therefore

$$H.K\ \ge\ 3/4n^2 - 11/2n - \frac{(3n + 8)(n^3 - 18n^2 + 80n)}{4(7n^2 - 90n + 128)} \qquad (7)$$

On the other hand

$$\frac{n^3 - 18n^2 + 80n}{28n^2 - 360n + 512} \leq 1/28n$$

for $n \geq 12$ as one easily checks and so we get finally

$$H.K \geq 3/4n^2 - 11/2n - 1/28n(3n + 8) = 9/14n^2 - 81/14n .$$

Alltogether we obtain in the first case (under the assumption $n \geq 12$) :

$$H.K \geq \min (3/4n^2 - 11/2n , 9/14n^2 - 81/14n)$$

$$= 9/14n^2 - 81/14n . \qquad (8)$$

<u>2. case.</u> Y is a birationally ruled surface.
The same calculation as in case 1 (just replacing (3) by (4)) gives rise to the following inequality (again for $n \geq 12$) :

$$H.K \geq 2/3n^2 - 6n . \qquad (9)$$

Using the Castelnuovo inequality (cf. [A-C-G-H] p. 116) for the genus π of H (considered as a smooth curve in $\mathbb{P}^4$) we obtain :

$$H.K = 2\pi - 2 - n \leq 2((3m(m-1))/2 + m(n-1-3m)) - 2 - n ,$$

where $m := [(n-1)/3]$.

As an easy calculation shows, this implies :

$$H.K \leq 1/3n^2 - 8/3n + 7/3 . \qquad (10)$$

Combining now (10) and (8) in the first case (resp. (9) in the second case) we get the following inequalities :

<u>1. case:</u> $9/14n^2 - 81/14n \leq H.K \leq 1/3n^2 - 8/3n + 7/3$,

and

<u>2. case:</u> $2/3n^2 - 6n \le H.K \le 1/3n^2 - 8/3n + 7/3$.

Checking that these two inequalities are never fulfilled for $n \ge 12$ we get a contradiction, and so it follows that the degree of Y has to be smaller or equal to 11. Q.E.D.

<u>(2.8) Remark.</u> 1) If $Y' \subset \mathbb{P}^4$ is a smooth surface which comes from $\mathbb{P}^5$ by projection, theorem (2.2) says that $\deg Y' \le 10$.

2) If $Y' \subset \mathbb{P}^4$ contains an exceptional line and $H^1(Y',\mathcal{O}(H')) = 0$, then Y' arises from a smooth surface $Y \subset \mathbb{P}^5$ by projection with center on Y.

<u>Proof of 2).</u> Because $H^1(Y',\mathcal{O}(H')) = 0$ we have the exact sequence

$$0 \to H^0(Y',\mathcal{O}(H')) \to H^0(Y',\mathcal{O}(H'+E)) \to H^0(E,\mathcal{O}(H'+E)|E) \to 0,$$

which shows that $|H+E|$ gives an embedding φ of $Y'\setminus E$ to $\mathbb{P}^5$ and maps E to a point p not contained in $\varphi(Y'\setminus E)$. That p is a smooth point follows since

$$H^0(Y',\mathcal{O}(H')) \twoheadrightarrow H^0(E,\mathcal{O}(H')|E) \text{ . Q.E.D.}$$

Although it is not needed in the following we would like to mention that in the case of rational surfaces also the converse is valid.

<u>(2.9) Lemma.</u> Let $Y' \subset \mathbb{P}^4$ be a smooth rational surface. Then Y' is projection of a smooth surface $Y \subset \mathbb{P}^5$ with center on Y iff Y' contains an exceptional line and $H^1(Y',\mathcal{O}(H')) = 0$.

Proof. It suffices to show that if $Y' \subset \mathbb{P}^4$ arises by projection, then $H^1(Y',\mathcal{O}(H')) = 0$. By Riemann-Roch we have

$$\chi(\mathcal{O}(H')) = 1/2\, H'.(H' - K') + 1.$$

Since Y' is linearly normal, it holds moreover:

$$\chi(\mathcal{O}(H')) = 5 - h^1(Y',\mathcal{O}(H')) + h^2(Y',\mathcal{O}(H')) =$$

$$= 5 - h^1(Y',\mathcal{O}(H')) .$$

This implies:

$$H'.K' = H'^2 + 2h^1(Y',\mathcal{O}(H')) - 8 =$$

$$= n' + 2h^1(Y',\mathcal{O}(H')) - 8 .$$

By [Al] Proposition (4.2) it holds :

$$K'^2 = 8 - m',$$

where $m' = -1/2(n'-3)(n'-12) + 5h^1(Y',\mathcal{O}(H'))$.
Putting these equalities together with (2.7) we obtain

$$0 = n^2 - 18n - 3nH.K + 22H.K + 4(K^2 + 20) =$$

$$= h^1(Y',\mathcal{O}(H'))\,(24 - 6n),$$

which implies $h^1(Y',\mathcal{O}(H')) = 0$ or $n' = 3$ (in which case also $h^1(Y',\mathcal{O}(H')) = 0$). Q.E.D.

(2.10) Theorem. The smooth, non-degenerate, connected surfaces $Y' \subset \mathbb{P}^4$, which are projections of a smooth surface $Y \subset \mathbb{P}^5$ (with center on Y) are exactly the following:

deg	π	p_g	q	kod	structure of the surface
3	0	0	0	-1	$\mathbb{P}_2$^(x), $\lvert H' \rvert = \lvert 2L - x \rvert$
4	1	0	0	-1	$\mathbb{P}_2$^$(x_1,\dots,x_5)$, $\lvert H' \rvert = \lvert 3L - \Sigma x_i \rvert$, Y' = compl. inters. of two quadrics
5	2	0	0	-1	$\mathbb{P}_2$^$(x_0,\dots,x_7)$, $\lvert H' \rvert = \lvert 4L - 2x_0 - \Sigma_{1\le i\le 7} x_i \rvert$
6	3	0	0	-1	$\mathbb{P}_2$^$(x_1,\dots,x_{10})$, $\lvert H' \rvert = \lvert 4L - \Sigma_{1\le i\le 10} x_i \rvert$
7	4	0	0	-1	$\mathbb{P}_2$^$(x_1,\dots,x_6,y_1,\dots,y_5)$, $\lvert H' \rvert = \lvert 6L - \Sigma_{1\le i\le 6} 2x_i - \Sigma_{1\le i\le 5} y_i \rvert$
7	5	1	0	0	K3-surface
8	5	0	0	-1	$\mathbb{P}_2$^$(x_0,\dots,x_{10})$, $\lvert H' \rvert = \lvert 7L - x_0 - \Sigma_{1\le i\le 10} 2x_i \rvert$
9	6	0	0	0	Enriques surface .

Here π is the sectional genus of Y', p_g is the geometric genus, q the irregularity and kod the Kodaira dimension of Y'. L is the strict transform of a line in $\mathbb{P}^2$.

Proof. Checking the list of smooth surfaces in $\mathbb{P}^4$ up to degree 10 (cf. [Al] for the rational surfaces, [Ok] for degree smaller or equal to 8, [A-Ra] for degree 9, [Ra] for degree 10) we see that all the surfaces except the ones in the above table are either minimal or don't fulfill (2.7), which means that they are not projections. The rational surfaces Y' in our list contain an exceptional line and it holds $H^1(Y',\mathcal{O}(H')) = 0$ (cf. [Al] Theoreme (1)) and so they arise by projection by (2.8). The Enriques surface is obtained by projection (cf. [Co-Ve], [Do-Rei]) and also the K3-surface (which is obtained by projecting the complete intersection of three quadrics in $\mathbb{P}_5$ into $\mathbb{P}^4$ (cf. [Ok]). Q.E.D.

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