A relative index theorem for families and applications

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Abstract

This paper is a supplement to [2] and [3]. We prove a relative index theorem for certain families of operators of Dirac type. We compute index bundles using the cutting and pasting procedure of [3].

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1 Relative index theory for families

1.1 Introduction

In this note we want to apply the results and techniques developed in [3] to families of operators of Dirac type. It will be demonstrated that the relative index theorem

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in its generalized form for families provides a rather powerful tool for computing index bundles.

Before considering the example we have to discuss families of operators of Dirac type on open manifolds. This requires certain uniform behaviour of the objects at infinity. Thus we will introduce the notion of a nice family which satisfies all conditions needed to construct index bundles and to apply relative index theorems.

Since the relative index theorem in [2] has been proven only for a single operator we have to modify it for families. It turns out that this is straight foreward if one interprets the objects in this reference in an appropriate way. Then one can in fact repeat the proofs word by word. We will obtain a relative index theorem for families over a compact parameter space.

Then we compute the index bundle of a family of real operators of Dirac type. Let M^n be a complete Riemannian spin manifold with associated real Clifford bundle $E \to M$ admitting a right $C^{0,n}$-action. Let $F : X \to C^\infty(M) \otimes \mathbf{R}^n$ be a family of maps parametrized by a compact space X such that $\|F\| \to 1$, $\|grad\ F\| \to 0$ at infinity, $F(x)$ is smooth and 0 is a regular value of $F(x)$ for any $x \in X$. In [3] we have seen how the index of a single operator of this type localizes at the zero-set of F. Here we obtain a close relation of the index bundle with the covering of X given by the set of zeros of F.

1.2 Trivial families

We will first describe the geometrical setting in the case of trivial families. Then will will globalize it in the obvious way.

A nice trivial family (of complete Riemannian manifolds with Clifford bundle) is given by a tuple

$$(M, X, S, \mu, K, I, h)$$

where

- M is a complete Riemannian manifold with metric g (called reference metric)
- X is a locally compact space
- S is a Clifford bundle with metric $(.,.)$ and connection ∇ compatible with the Riemannian structure g on M (called reference structure)
- $\mu \in C(X, C(M, Hom(TM \otimes S, S)))$
- $I \in C(X, C(M, Aut(S)))$ such that $I^{-1} \in C(X, C(M, Aut(S)))$, $I = I^*$ and $I \geq 0$.
- $h \in C(X, C(M, S^2T^*M))$ such that $h(x)$ is a Riemannian metric in the quasi-isometry class of the reference metric and $(S, \nabla + K(x), \mu(x), (.,.)_x)$ is a Clifford bundle compatible with the metric $h(x)$. Here $(.,.)_x := (I(x).,.)$.

Moreover we assume $\mu(x), I(x)$ and $h(x)$ to be smooth for any $x \in X$. We will need a further condition which allows us to compare the associated Sobolev spaces.

Let $\rho(x,m) := vol_g(m)/vol_{h(x)}(m)$. We define $J(x) : S \to S$ by

$$J(x)\phi := I^{-1/2}(x)\rho^{1/2}(x,m)\phi$$

where $\phi \in S_m$. This morphism will later be used to trivialize the associated bundle of Hilbert spaces. We define Sobolev spaces $H^k(M,S)$ using the reference structure with associated reference Dirac operator. For a nice trivial familiy we will require that the family $x \to J(x)^{-1}D(x)J(x)$ is norm continuous and uniformly bounded in $B(H^k(M,S), H^{k-1}(M,S))$ for $k = 1,2$ and that $x \to J(x)^{-1}(D^2(x)+1)^{-k}J(x)$ is norm continuous and uniformly bounded in $B(H^0(M,S), H^{2k}(M,S))$ for $k = 1/2, 1$. These conditions will make it easy to transfer the proofs from [2] and [3] to the case of families. They are probably not optimal.

Let

$$C_c^\infty(X \times M) := C_c(X, C_c^\infty(M)),$$
$$C_g^\infty(X \times M) := C_0(X, C_g^\infty(M))$$

and $C_0(X \times M)$ and $C_g(X \times M)$ be the norm closures in $C(X \times M)$. We define H^k to be the closure of the $C_0(X)$-Hilbert-C^*-pre-module $C_0(X, C_c^\infty(M,S))$ with respect to the $C_0(X)$-valued scalar product

$$(\psi,\phi)_k(x) := \sum_{i=1}^{k} \int_M (D(x)^i\psi(x)(m), D(x)^i\phi(x)(m))_x vol_{h(x)}(m). \qquad (1)$$

We set $H := H^0$

Note that J induces an isometry of $C_0(X)$-Hilbert-C^*-modules

$$J : L^2(M,S) \otimes C_0(X) \xrightarrow{\sim} H^0$$

and in general non-isometric topological isomorphisms $H^k(M,S) \otimes C_0(X) \xrightarrow{\sim} H^k$ for $k = 1,2$. From this follows

Lemma 1.1 *If $K \in B(H^k)$, $k = 0,1,2$ and $J^{-1}KJ$ is represented by a norm continuous family of compact operators in $B(H^k(M,S))$ with $\|J^{-1}KJ\| \in C_0(X)$ then K is compact.*

For a proof see Blackadar [1],13.5.

1.3 Globalization

We consider now the global situation. Let $P \to X$ be a locally trivial fibre bundle over a compact space X. Let h be a fibrewise Riemannian metric and S be a fibrewise Clifford bundle. We call such a family nice if for any $x \in X$ there is a neighbourhood $U \subset X$ of x, a tivialization of $P_{|U} = U \times M$, a reference metric g on

M and a reference Clifford bundle $S_U \to M$ with identification as complex vector bundles $pr_M^* S_U = S_{|P_{|U}}$ such that the induced trivial family is nice.

Let $\{U_\alpha\}$ be a finite cover of X by such open sets and $\{\chi_\alpha\}$ be a partition of unity subordinate to this cover. Let H_α^k be the $C_0(U_\alpha)$-Hilbert-C^*-module defined for the trivial family as above.

Let $C_c^\infty(P,S) \subset C_c(P,S)$ be the space of fibrewise smooth sections with compact support the restrictions of which are in $C(U_\alpha, C^\infty(M, S_{U_\alpha}))$ for any nice trivialization. Let H^k be the closure of $C_c^\infty(P,S)$ with respect to the $C(X)$-valued scalar product (1). H^k is the $C(X)$-Hilbert-C^*-module associated to the global family.

Lemma 1.2 *If $K \in B(H^k)$, $k = 0,1,2$ is represented by norm continuous families of compact operators in any nice trivialization then $K \in K(H^k)$.*

Proof: Form $\chi_\alpha K \in B(H^k)$. In fact we can consider $\chi_\alpha K \in B(H_\alpha^k)$ and by Lemma 1.1 $\chi_\alpha K \in K(H_\alpha^k)$. Hence it can be approximated by finite sums of operators $\theta_{\psi,\phi} \in K(H_\alpha^k)$ given by

$$\theta_{\psi,\phi} s := (\phi, s)_k \psi \quad \psi, \phi, s \in H_\alpha^k.$$

But there is a natural embedding $H_\alpha^k \to H^k$ such that we can consider $\theta_{\psi,\phi} \in K(H^k)$. Thus $\chi_\alpha K \in K(H^k)$ and hence $K \in K(H^k)$. □.

We regard P as a symbol for all objects defining a nice family over P. Let P be a nice family.

Definition 1.3 *A nice family of operators of Dirac type is given by $B = D + \Phi$ where D is the Dirac operator associated to a nice family P and $\Phi \in C(P, End(S))$.*

We define

$$C_g(P) := \sum_\alpha C_g(U_\alpha \times M) \subset C(X)$$

$$C_0(P) := \sum_\alpha C_0(U_\alpha \times M) \subset C(X).$$

1.4 Adaptation

In order to apply the results of [2] and [3] to families we have to interpret them appropriately. Especially we have to show compactness of various remainder terms. In the papers mentioned above we have expressed these terms as norm convergent integrals the integrand of which is compact. The same formulas hold in the case of families. The operators can be interpreted fibrewise using nice trivializations. If we start with nice families of operators of Dirac type we obtain norm continuous families of compact operators. We can conclude with Lemma 1.2 that such families give compact operators on H (or H^k). Having this in mind we can prove the following Lemmata appearing in [2] and [3] repeating the proofs there word by word.

Let B be a nice selfadjoint family of Dirac-type operators over a nice family P. We assume that there is a $S \in K(H^0, H^1)$ such that $B+S \in B(H^1, H^0)$ is boundedly invertible and such that S is given by a norm-continuous family of compact operators $S(x) \in K(H^0(M, S_{U_\alpha}), H^1(M, S_{U_\alpha}))$ in nice trivializations over the U_α.

Let $A := B + S$.

Lemma 1.4 $(A^*A)^{-1/2} - (AA^*)^{-1/2} \in K(H^0, H^2)$

Lemma 1.5 $[f, (AA^*)^{-1/2}] \in K(H^0, H^1) \quad \forall f \in C_g(P)$

Now assume that S is $\mathbf{Z}_2$-graded and that $deg\, B = 1$. Then all trivializations should be compatible with the grading.

Lemma 1.6 $[B(AA^*)^{-1/2}]^{ev} \in K(H)$

Let $F := [B(AA^*)^{-1/2}]^{odd}$.

Lemma 1.7 *(H, F) is a Kasparov module over $(C_g(P), C(X))$.*

We denote by $[P]$ the element represented by (H, F) in $\mathbf{KK}(C_g(P), C(X))$ and by $\{P\} \in \mathbf{KK}(\mathbf{C}, C(X)) = \mathbf{K}^0(X)$ the restriction of $[P]$.

Lemma 1.8 *If B_t, $t \in I$ is a continuous family in $B(H^1, H^0)$ such that for every t there exists such $S \in K(H^0, H^1)$ as above then $\{P_t\} = \{P_s\}$ for any $s, t \in I$.*

1.5 Invertibility at infinity

A key notion in [3] was invertibility at infinity. It generalizes to families in an obvious fashion. Let B be a nice family of operators of Dirac type on a nice family P. B is invertible at infinity if there is a positive $s \in C_c(P)$ such that $B^2 + s$ is invertible. Let the Clifford bundle be $\mathbf{Z}_2$-graded by z and B be selfadjoint and of degree one.

Lemma 1.9 *If B is invertible at infinity then there is a $S \in K(H^0, H^1)$ such that $B + S$ is invertible in $B(H^1, H^0)$.*

Proof: Since X is compact there is a $\epsilon > 0$ such that for all $x \in X$

$$\sigma_{ess}(B_x) \cap [-\epsilon, \epsilon] = \emptyset.$$

Let $\chi \in C_c^\infty(\mathbf{R})$, $\chi \geq 0$ with $\chi = 0$ on $\mathbf{R} \setminus [-\epsilon/2, \epsilon/2]$ and $\chi = 1$ on $[-\epsilon/4, \epsilon/4]$. Then $S := z\chi(B)$ is a norm continuous family of compact operators between the appropriate Sobolev spaces in any nice trivialization. Moreover $(B + S)^2 = B^2 + \chi(B)^2 > \epsilon^2/16$. □

We can now introduce admissible endomorphisms as in [3]. Let S be $\mathbf{Z}_2$-graded. An endomorphism $\Phi \in C(P, S)$ is admissible if $\Phi^* = \Phi$, $deg\, \Phi = 1$, $\Phi D + D\Phi + \Phi^2$ is bounded of zero order and there is a compact subset $K \subset P$ such that $\Phi D + D\Phi + \Phi^2 \geq c > 0$ on $P \setminus K$ for some constant c.

Lemma 1.10 *If Φ is admissible then $B := D + \Phi$ is invertible at infinity.*

An essential support of Φ is a compact subset $K \subset P$ such that $\Phi^2 \geq c > 0$ on $P \setminus K$.

Lemma 1.11 *If an essential support of Φ is empty then $\{P\} = 0$ where $\{P\} \in \mathbf{K}^0(X)$ is the class defined with $B = D + \Phi$.*

1.6 The relative index theorem

Let B be a nice family of Dirac type operators beeing invertible at infinity over a nice family $P \to X$ with compact base. Let

$$\begin{array}{ccc} Q & \subset & P \\ \downarrow & & \downarrow \\ X & \to & X \end{array}$$

be a subbundle of codimension one compact submanifolds cutting a neighbourhood $U(Q) = U(Q)^+ \cup_Q U(Q)^-$ into two pieces. Let

$$\begin{array}{cccccc} \Gamma & : & S_{|U(Q)^-} & \to & S_{|U(Q)^-} \\ & & \downarrow & & \downarrow \\ \gamma & : & U(Q)^- & \to & U(Q)^- \\ & & \downarrow & & \downarrow \\ id & : & X & \stackrel{\sim}{\to} & X \end{array}$$

be an isomorphism of fibre bundles where γ is a fibrewise isometry and Γ is an isomorphism of Clifford bundles intertwining with B. Then we can cut at Q and glue together again using γ and Γ obtaining a new nice family $\tilde{P}$ and a nice family of operators of Dirac type $\tilde{B}$.

Proposition 1.12 (Relative index theorem for families) $\{P\} = \{\tilde{P}\}$

The proof goes as in [2] using the interpretation explained above.

All notions and results described above have a real counterpart where the Clifford bundles and the Hilbert-C^*-modules admit a $C^{0,n}$-right action ($C^{0,n}$ is the real Clifford algebra generated by $\mathbf{R}^n$) for some $n \geq 0$. All morphisms then are assumed to be compatible with this action. æ

2 Computation of an index bundle

2.1 The family of real operators of Dirac type

In this section we stay in the real framework and demonstrate by example how the method developed in the last section can be applied to compute index bundles for families of operators of Dirac type.

Let M^n be a complete Riemannian spin manifold and $E \to M$ be the real $\mathbf{Z}_2$-graded Clifford bundle associated to the spin structure of M admitting a right $C^{0,n}$-action. Let $V(M)$ be the space of all smooth maps $F : M \to \mathbf{R}^n$ such that

- $\|F\| \to 1$ at infinity,
- $\|dF\| \to 0$ at infinity and

- 0 is a regular value of F.

We topologize $V(M)$ with the uniform C^0-topology. Let X be some compact space and $F : X \to V(M)$. Then we can consider the family of operators of Dirac type over $P := X \times M$ given by $B(x) = D + zF(x)$. We consider $\mathbf{R}^n \subset C^{0,n}$ and thus $F(x)$ acts by right multiplication. Of course the trivial family of Clifford bundles $pr_M^* E \to P$ is nice. Moreover since F is C^0-continuous the family of Dirac type operators B is nice. $\Phi := zF$ is admissible and thus B is invertible at infinity. Thus we can form $\{P\} \in \mathbf{KK}(\mathbf{R}, C(X)) = \mathbf{KO}^0(X)$. Note that Φ destroys the equivariance with respect to the Clifford action.

2.2 Application of the relative index theorem

In [3] we have shown that the index of $B(x)$ can be localized near the zero-set of F. The same is true for the family. Let $x \in X$. Then the zero-set of $F(x)$ consists of points $p_1(x), \dots, p_m(x) \in M$. We can put labels $o(p_i(x)) \in \{1, -1\}$ to the points $p_i(x)$ (or better to the index i) depending on wether $dF(p_i(x))$ does or does not change the orientation. Then by the real index theorem of [3]

$$ind\, B(x) = \sum_{i=1}^{m} o(p_i(x)).$$

This is independent of $x \in X$.

In order to compute the index bundle $\{P\}$ we want to 'simplify' P. Let $Z(F) \subset P$ be the zero-set of F. Then $Z(F) \to X$ is a m-fold covering. Moreover $Z(F) = Z(F)^+ \cup Z(F)^-$ where $Z(F)^\pm$ consists of points with label ± 1. Let

$$P_1 := Z(F)^+ \times \mathbf{R}^n \cup Z(F)^- \times -\mathbf{R}^n$$

where $-$ stands for opposite orientation. We consider a map $F_1 \in C^\infty(\mathbf{Z}(F)^\pm \times \mathbf{R}^n, \mathbf{R}^n)$ such that $F_1(p, x) = x/\|x\|$ for $\|x\| \geq 1$. The spin structure on P_1 is obtained by pulling back the spin structure of a tubular neighbourhood $U \subset X \times M$ of $Z(F)$ using $F^{-1} : Z(F) \times D^n \xrightarrow{\sim} U$. Let $E_1 \to P_1$ be the family of real Clifford bundles associated to that spin structure and $B_1 := D_1 + zF_1$ be the family associated to P_1 and F_1. This family is also nice. Let $\{P_1\} \in \mathbf{KO}^0(X)$ be the associated index bundle.

Lemma 2.1 $\{P\} = \{P_1\}$

Proof: We can deform the fibrewise Riemannian metric on P such that F induces an isometry I of a neighbourhood U of $Z(F)$ onto $Z(F) \times D^n$. Furthermore we can deform F to $\tilde{F}$ such that $\tilde{F} = F_1 \circ I$ on U. All this does not change $\{P\}$. Now we can apply the relative index theorem for families. We take $P_2 = P \cup -P_1$ (- stands for opposite fibrewise orientation), cut at the two copies of $Z(F) \times S^{n-1}$ and glue together again interchanging boundary components obtaining P_3. We glue the

Clifford bundles using the multiplication with the unit normal vector field at S^{n-1} in every fibre. We claim $\{P_2\} = 0$. In fact P_2 is the union of the bundle of compact manifolds

$$P_4 := Z(F) \times D^n \cup_{Z(F)\times S^{n-1}} -(Z(F) \times D^n)$$

and

$$P_5 := P \setminus Z(F) \times D^n \cup_{Z(F)\times S^{n-1}} -(Z(F) \times (\mathbf{R}^n \setminus D^n)).$$

But $\{P_4\} = 0$ since the fibres are compact and we can deform the admissible endomorphism to zero there such that the operator becomes equivariant to a larger Clifford algebra. $\{P_5\} = 0$ since there an essential support of the admissible endomorphism is empty. This proves the claim and hence $\{P\} = \{P_1\}$. □

2.3 Computation on the simplified family

We compute $\{P_1\}$. Let $pr_{Z(F)} : P_1 \to Z(F)$ be the projection and $\theta \to Z(F)$ be the flat $\mathbf{R}$-bundle such that $pr^*_{Z(F)}\theta \otimes E_1$ is the Clifford bundle associated to the spin structure induced by $pr_{\mathbf{R}^n} : P_1 \to \mathbf{R}^n$. Let $L^\pm$ be the push down $(pr_X)_*\theta_{|Z(F)^\pm}$. $L^\pm$ is a flat bundle of dimension $dim\, L^\pm = \sharp\{p \in Z(F)(x) \mid o(p) = \pm 1\}$.

Theorem 2.2 $\{P\} = \{P_1\} = [L^+] - [L^-] \in \mathbf{KO}^0(X)$

Proof: This theorem follows since B_1 is the pull-back of an operator of Dirac type with index 1 on $\mathbf{R}^n$ to P_1 twisted with θ. □

Corollary 2.3 *All rational Pontrjagin classes of $\{P\}$ vanish.*

2.4 A universal index bundle

Let $C_k(M)$ be the configuration space of unordered k-tuples of pairwise different points of M (let $C_0(M)$ be the one-point space) and $\tilde{C}_k(M) \to C_k(M)$ be the covering according to the permutation group $S_k \subset \pi_1(C_k(M))$. Let $P_{Spin} \to P_{SO} \to M$ be the spin structure of M which is a two-fold covering of the bundle of oriented frames and let P_k be the restriction of $\times_k P_{Spin}$ to $\tilde{C}_k(M)$. There is an $(\mathbf{Z}_2)^k$-action on P_k such that $Q_k := P_k/(\mathbf{Z}_2)^k$ is the restriction of $\times_k P_{SO}$ to $\tilde{C}_k(M)$. Let ρ be the representation of $(\mathbf{Z}_2)^k$ on $\mathbf{R}$ given by

$$\rho(\epsilon) := \Pi_{i=1}^k \epsilon_i, \quad \epsilon = (\epsilon_1, \dots, \epsilon_k), \quad \epsilon_i \in \{\pm 1\}$$

and $\theta_k := P_k \times_{(\mathbf{Z}_2)^k} \mathbf{R}$ be the associated line bundle over Q_k. There is an action of S_k on Q_k by permutations of the factors. Let $R_k := Q_k/S_k$ (here R_0 is defined as the one-point space) and $V_k \to R_k$ be the push down of θ_k to R_k (and V_0 be zero-dimensional). Then V_k is a flat k-dimensional real vector bundle. There are maps

$$h^\pm : V(M) \to R := \cup_{k=0}^\infty R_k$$

defined as follows. Fix oriented orthonormal bases $b, -b$ in both orientations of R^n. Let $F \in V(M)$ and $\{p_i\}$ be the zero-set of F. Then

$$h^{\pm}(F) := \times_{o(p_i)=\pm 1} \left(dF(p_i)^{-1}(\pm b)\right)_{ES} \in R$$

where $()_{ES}$ means applying a canonical orthogonalization procedure. Let $V := \cup_{k=0}^{\infty} V_k$ and form $\tau := [(h^+)^* V] - [(h^-)^* V] \in \mathbf{KO}^0(V(M))$. We can interpret our result as

Corollary 2.4 *τ is the universal index bundle in the sense that for any compact space X and any map $F : X \to V(M)$ the index bundle of the induced family $\{B(x) := D + zF(x)\}_{x \in X}$ is given by $F^* \tau \in \mathbf{KO}^0(X)$.*

For defining the index bundle of such families one can of course forget the condition that the zeros of F are non-degenerate. But this assumption is no restriction in the sense that one can have this property after a small index bundle preserving deformation of the family.

A in some sense similar result has been obtained by Cohen/Jones [4] for families of operators of Dirac type parametrized by monopoles.

2.5 An explicit example

We consider $M := \mathbf{R}^2$ and a cut-off function $\chi \in C^\infty(\mathbf{R})$ with $\chi(r) = 0$ for $r \geq 2$ and $\chi(r) = 1$ for $r \leq 1$. Let $F(0)$ be the vector field on M

$$F(0)(x,y) = (\chi(x^2+y^2) + \frac{1-\chi(x^2+y^2)}{(x^2+y^2)^{1/2}})(x\partial_x - y\partial_y).$$

Then $F(0)$ is invariant with respect to the rotation $R(\pi)$ over π around $(0,0)$. We consider $F(0)$ as a map $\mathbf{R}^2 \to \mathbf{R}^2$ using the fixed basis ∂_x, ∂_y. Let $B(0) := D + zF(0)$ where D is the Dirac operator associated to the spin structure of $\mathbf{R}^2$. We have $ind\, B = -1$. We can now consider the family parametrized by S^1 given by rotation

$$F(\alpha) = dR(\alpha/2)F(0)$$

where $\alpha \in [0, 2\pi]$. Note that $F(0) = F(2\pi)$. Then $Z(F) = Z(F)^- = S^1$ is the trivial one-fold covering of S^1 and the line bundle θ is the generator of $\mathbf{KO}^0(S^1)$ since the path $\alpha \to dF^{-1}(\alpha)(0,0)(-b)$ rotates a basis by 2π which lifts to a non-closed path in the spin-structure. Hence the index bundle of the family is the generator of $\mathbf{KO}^0(S^1) = \mathbf{Z}_2$.

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