ETA INVARIANTS AND REGULARIZED DETERMINANTS FOR ODD DIMENSIONAL HYPERBOLIC MANIFOLDS WITH CUSPS

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Abstract. We study eta invariants of Dirac operators and regularized determinants of Dirac Laplacians over hyperbolic manifolds with cusps. We follow Werner Müller (see [18], [19]) and use relative traces to define the eta function and the zeta function. We show regularities of eta and zeta functions at the origin so that we can define the eta invariant and the regularized determinant. By the Selberg trace formula, we show an equality between the eta invariant and the Selberg zeta function of odd type. The complete analysis of the unipotent orbital integral allows us to show the vanishing of the unipotent contribution in this relation. We also show that the eta invariant and the Selberg zeta function of odd type satisfy certain functional equation. These results generalize the earlier work of John Millson (see [12]) to hyperbolic manifolds with cusps. We also get the corresponding relation and the functional equation for the regularized determinant and the Selberg zeta function of even type where the unipotent factor plays a non-trivial role.

0. Introduction

In the seminal paper [12], Millson proved a formula for the relation between the eta invariant of the odd signature operator and the Selberg zeta function of odd type for compact hyperbolic manifolds of $(4n-1)$ dimension . The proof of Millson's formula is an application of the Selberg trace formula, which expresses a relation between the spectral data and the geometric data. In [7], the corresponding work was done for the analytic torsion and the Ruelle zeta function for odd dimensional hyperbolic manifolds. These results are generalized to the case of compact locally symmetric spaces of higher ranks in [13], [14] .

It would be interesting problems to extend aforementioned results to the noncompact manifolds, in particular, to the noncompact locally symmetric spaces with finite volumes. However, we encounter several serious difficulties, when we discuss the extension of those results to the noncompact locally symmetric spaces. First of all, the heat operator of the Laplacian is not of trace class. Hence we can not use the trace of the heat operator as in the compact case. In [18] and [19], Müller introduced the relative trace to overcome this kind of difficulty and defined the relative eta invariant and the relative regularized determinant.

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In this paper, we follow Müller's approach to study the (relative) spectral invariants for noncompact hyperbolic manifolds with finite volumes. If the continuous spectrum of the Laplacian has a nonzero lower bound, the relative trace behaves as the usual trace in the compact case. This is the situation in the case of the Laplacian acting on functions. The regularized determinants of the Laplacian acting on functions and their relations with Selberg zeta functions are extensively studied for $2,3$-dimensional manifolds with cusps in [4], [5], [10], [11], [17]. However, if the continuous spectrum of the Laplacian reaches zero, we need more advanced analysis of the relative trace and the scattering theory appears on the scene. In this paper we study the spectral invariants of Dirac operators and Dirac Laplacians for odd dimensional hyperbolic manifolds with cusps. The continuous spectrum of these operators reaches the zero in this case, and we need the analysis of the scattering operator in order to treat the spectral invariants.

Now, we explain our result more precisely. Let $X = \Gamma\backslash\mathrm{Spin}(2n+1,1)/\mathrm{Spin}(2n+1)$ be a $(2n+1)$-dimensional hyperbolic manifold with cusps. We consider the Dirac operator $\mathcal{D}$ on the spin bundle E which is a locally homogeneous vector bundle defined by the spin representation τ_s of the maximal compact subgroup $\mathrm{Spin}(2n{+}1)$ of $\mathrm{Spin}(2n{+}1,1)$. Following [12], we decompose the space of sections for E into two subspaces which are given by the half spin representations $\sigma_\pm$ of $\mathrm{Spin}(2n) \subset \mathrm{Spin}(2n+1)$. Note that the representations σ_+, σ_- are unramified. Let $\mathcal{K}_t$ be a family of functions over $G = \mathrm{Spin}(2n+1,1)$ given by taking the local trace of the integral kernel of $e^{-t\tilde{D}^2}$ or $\tilde{D}e^{-t\tilde{D}^2}$ where $\tilde{D}$ is the lifting of $\mathcal{D}$ over the universal covering space of X. The Selberg trace formula applied to the scalar kernel function $\mathcal{K}_t$ gives

$$\begin{aligned}
&\sum_{\sigma=\sigma_\pm}\sum_{\lambda_k\in\sigma_p^\pm}\hat{\mathcal{K}}_t(\sigma, i\lambda_k) - \frac{1}{4\pi}\int_{-\infty}^{\infty}\mathrm{Tr}(C_\Gamma(\sigma_+,-i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+,i\lambda)\pi_\Gamma(\sigma_+,i\lambda)(\mathcal{K}_t))\ d\lambda \\
=\ & I_\Gamma(\mathcal{K}_t) + H_\Gamma(\mathcal{K}_t) + U_\Gamma(\mathcal{K}_t)
\end{aligned}$$

where $\sigma_p := \sigma_p^+ \cup \sigma_p^-$ gives the point spectrum of $\mathcal{D}$, $C_\Gamma(\sigma_+, i\lambda)$ is the intertwining operator and $I_\Gamma(\cdot)$, $H_\Gamma(\cdot)$, $U_\Gamma(\cdot)$ are the identity, hyperbolic and unipotent orbital integrals. Following [15], [18], [19], we define an operator $\mathcal{D}_0$, which is determined by $\mathcal{D}$, with the property that $e^{-t\mathcal{D}^2} - e^{-t\mathcal{D}_0^2}$ and $\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0e^{-t\mathcal{D}_0^2}$ are the trace class operators and show that the spectral side of the Selberg trace formula is equal to the relative traces $\mathrm{Tr}(e^{-t\mathcal{D}^2} - e^{-t\mathcal{D}_0^2})$, $\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0e^{-t\mathcal{D}_0^2})$. Then we have

$$\begin{aligned}
&\mathrm{Tr}(e^{-t\mathcal{D}^2} - e^{-t\mathcal{D}_0^2}) \\
&= \sum_{\sigma=\sigma_\pm} \sum_{\lambda_k \in \sigma_p^\pm} \hat{\mathcal{K}}_t(\sigma, i\lambda_k) - \frac{1}{4\pi} \int_{-\infty}^{\infty} \mathrm{Tr}(C_\Gamma(\sigma_+, -i\lambda) \frac{d}{d(i\lambda)} C_\Gamma(\sigma_+, i\lambda) \pi_\Gamma(\sigma_+, i\lambda)(\mathcal{K}_t)) \, d\lambda \\
&= \quad I_\Gamma(\mathcal{K}_t) + H_\Gamma(\mathcal{K}_t) + U_\Gamma(\mathcal{K}_t)
\end{aligned}$$

if $\mathcal{K}_t$ is give by $e^{-t\tilde{D}^2}$. A similar formula holds for $\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2})$ with the corresponding kernel function $\mathcal{K}_t$. If $\mathcal{K}_t$ is given by $\tilde{D}e^{-t\tilde{D}^2}$, we can easily see that $I_\Gamma(\mathcal{K}_t) = 0$ by the Fourier transformation of $\mathcal{K}_t$. For the unipotent orbital integral, we use result from [9] to compute it explicitly. By this explicit computation, we can show that $U_\Gamma(\mathcal{K}_t) = 0$ for $\mathcal{K}_t$ determined by $\tilde{D}e^{-t\tilde{D}^2}$. This fact simplifies many steps in our analysis and gives the vanishing of the contribution from the unipotent terms in the relation of the eta invariant and the Selberg zeta function of odd type for hyperbolic manifolds with cusps .

Following Müller's approach in [17], [18], [19], we define the eta function and the zeta function using the relative traces $\mathrm{Tr}(e^{-t\mathcal{D}^2} - e^{-t\mathcal{D}_0^2})$, $\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2})$. The continuous spectrum of $\mathcal{D}$ is the whole real line in our case, hence the large time contribution of the relative trace can also give the poles of the eta function and the zeta function. Therefore, we need to consider separately the small time contribution and the large time contribution for the meromorphic extensions of the eta function and the zeta function. We use the analytic expansion of the intertwining operator along the imaginary axis to get the large time contribution and we analyze all terms in the geometric side of the Selberg trace formula for the small time contribution. We prove the following pole structures of the eta and zeta functions,

Theorem 0.1. *The eta function $\eta_{\mathcal{D}}(z)$ and the zeta function $\zeta_{\mathcal{D}^2}(z)$ have the following pole structures*

$$\Gamma(z+1/2)\eta_{\mathcal{D}}(z) = \sum_{k=0}^{\infty} \frac{-2\gamma_k}{z-2k-2} + K(z),$$

$$\Gamma(z)\zeta_{\mathcal{D}^2}(z) = \sum_{k=-n}^{\infty} \frac{\beta_k}{z+k-\frac{1}{2}} + \frac{\beta_0'}{(z-\frac{1}{2})^2} - \frac{h}{z} + \sum_{k=0}^{\infty} \frac{-\gamma_k'}{z-k-\frac{1}{2}} + H(z)$$

where β_k, β_0' are locally computable constants, γ_k, γ_k' are constants which are determined by the intertwining operator $C_\Gamma(\sigma_+, i\lambda)$, h is the number of the zero eigenvalues of $\mathcal{D}$ and $K(z)$, $H(z)$ are holomorphic functions . In particular, $\eta_{\mathcal{D}}(z)$ and $\zeta_{\mathcal{D}^2}(z)$ are regular at $z = 0$.

It follows from Theorem 0.1 that we can define the eta invariant by

$$\eta(\mathcal{D}) := \eta_{\mathcal{D}}(0).$$

Following [12], we use the Selberg trace formula to show how the eta invariant is related to the Selberg zeta function of odd type. As we mentioned above, we show that all unipotent terms are vanishing in the Selberg trace formula applied to the odd heat kernel function. As the conclusion, the equality between the relative trace and the orbital integrals for the odd kernel function is exactly the same as in the case of compact hyperbolic manifolds. Therefore we can get the same formula for the eta invariant and the Selberg zeta function of odd type as in compact hyperbolic manifolds. On the other hand, in the functional equation for the eta invariant and the Selberg zeta function of odd type, the term determined by the intertwining operator appears.

Theorem 0.2. *For the Dirac operator $\mathcal{D}$ on a $(2n+1)$-dimensional hyperbolic manifold with cusps, the following equalities hold*

$$\eta(\mathcal{D}) = \frac{1}{\pi i} \log Z^o_H(n),$$
$$Z^o_H(n+s) Z^o_H(n-s) = \exp(2\pi i \eta(\mathcal{D})) \left(\frac{det\ C_+(s)C_-(0)}{det\ C_-(s)C_+(0)} \right)^{-2^{n-1}} \qquad for \quad s \in \mathbb{C}$$

where $Z^o_H(s)$ is the Selberg zeta function of odd type and $C_\pm(s)$ is a linear operator given by $C_\Gamma(\sigma_+, s) = \begin{pmatrix} 0 & C_-(s) \\ C_+(s) & 0 \end{pmatrix}$.

We can define the zeta function $\zeta_{\mathcal{D}^2}(z,s)$ where $\mathcal{D}^2 + s^2$ is the shifted Dirac Laplacian with a positive real number s . Now the continuous spectrum of $\mathcal{D}^2 + s^2$ does not reach 0 and the large time contribution does not create any poles of $\zeta_{\mathcal{D}^2}(z,s)$. Therefore we can see that $\zeta_{\mathcal{D}^2}(z,s)$ is regular at $z=0$ by Theorem 0.1. It follows that the regularized determinant

$$Det(\mathcal{D}^2, s) := exp(-\zeta'_{\mathcal{D}^2}(0,s))$$

is well-defined. We show that $Det(\mathcal{D}^2, s)$ can be extended to a meromorphic function of s on the whole complex plane. As such a meromorphic function, we have $Det(\mathcal{D}^2, s) \neq Det(\mathcal{D}^2, -s)$. This is because the meromorphic extension of $Det(\mathcal{D}^2, s)$ involves the intertwining operator $C_\Gamma(\sigma_+, s)$ which is not an even function of s. We use the Selberg trace formula to prove the relation between $Det(\mathcal{D}^2, s)$ and the geometric data which consists of the Selberg zeta function of even type, the identity factor and the unipotent factor. We also get the functional equation for $Det(\mathcal{D}^2, s)$ and the Selberg zeta function of even type where the unipotent factor plays a non-trivial role.

Theorem 0.3. *For the Dirac Laplacian $\mathcal{D}^2$ on a $(2n+1)$-dimensional hyperbolic manifold with cusps, the following equalities hold for any $s \in \mathbb{C}$,*

$$\begin{aligned} Det(\mathcal{D}^2, s) &= Z^e_H(s+n)\Gamma(s+\frac{1}{2})^{-2^n\kappa}\exp(2\pi\int_0^s 2p(\sigma_+, i\lambda) + 2P_U(i\lambda)\ d\lambda), \\ Det(\mathcal{D}^2, s)^2 &= (det\ C_\Gamma(\sigma_+, s))^{-2^{n-1}}\ Z^e_H(n+s)Z^e_H(n-s)\Big(\Gamma(s+\frac{1}{2})\Gamma(-s+\frac{1}{2})\Big)^{-2^n\kappa} \end{aligned}$$

where $Z^e_H(s)$ is the Selberg zeta function of even type, $p(\sigma_+, s)$ is Plancherel measure for σ_+, $P_U(s)$ is an even polynomial of degree $2n-4$ and κ denotes the number of the cusps of X .

If we compare Theorem 0.2 with Theorem 0.3, we can see that the eta invariant does not contain defect terms determined by cusps. But the defect terms, which play a non-trivial role in the functional equation, appear in the case of the regularized determinant.

The paper has the following structure. In Section 1, we define the Dirac operator $\mathcal{D}$ for the locally homogeneous vector bundle which is defined by the spin representation of maximal compact subgroup Spin$(2n+1)$. We introduce an operator $\mathcal{D}_0$ which is used to define the relative trace. In Section 2, we introduce the Selberg trace formula for nontrivial locally homogeneous vector bundles over noncompact locally symmetric spaces of $\mathbb{R}$-rank 1. In Section 3, we use results from [9] to analyze unipotent terms. In Section 4, we derive the relation between the relative trace and the spectral side of the Selberg trace formula following [15], [18]. In Section 5, we define the eta function and the zeta function and prove Theorem 0.1. In Sections 6 and 7 we prove Theorem 0.2 and Theorem 0.3 using results of Sections 2, 3 and 4.

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1. Dirac operators on odd dimensional hyperbolic manifolds with cusps

Let G be a noncompact connected semisimple Lie group with finite center Z_G and let K be a maximal compact subgroup of G. We assume that the $\mathbb{R}$-rank of the noncompact symmetric space G/K is 1. Let Γ be a discrete subgroup of G such that Γ is torsion free and the volume of $\Gamma\backslash G$ is finite. Let $G = KAN$ be the Iwasawa decomposition, then there is a Γ-cuspidal minimal parabolic subgroup P of G with Langlands decomposition $P = MAN$, where M is the centralizer of A in K. Let M^* be the normalizer of A in K and $W(A) = M^*/M$ be the Weyl group of the restricted roots for the pair (G, A). The order of $W(A)$ is 2 since the dimension of A is 1. The set of the restricted roots consists of at most two roots $\{\alpha_1, \alpha_2\}$ with $\alpha_2 = 2\alpha_1$ if there are two different elements. The root space corresponding to α_1, α_2 is denoted by $\mathfrak{n}_1$, $\mathfrak{n}_2$ respectively.

From now on, we assume that $G = \mathrm{Spin}(2n+1, 1)$ and $K = \mathrm{Spin}(2n+1)$. Let $\tilde{X}$ denote the noncompact symmetric space determined by $G/K = \mathrm{Spin}(2n+1,1)/\mathrm{Spin}(2n+1)$. We denote the Lie algebra of G by $\mathfrak{g}$. We have a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

where $\mathfrak{k}, \mathfrak{a}, \mathfrak{n}$ are the Lie algebras of K, A, N and $\mathfrak{p}$ can be identified with the tangent space of G/K at the identity coset. We observe that $\mathfrak{n} = \mathfrak{n}_1$ and $\mathfrak{n}_2 = 0$ in this case. The Killing form $C(\ ,\ )$ provides us with an invariant metric on G/K given by $\langle X, Y\rangle = \frac{1}{4n} C(X, \theta Y)$, where θ is the Cartan involution. This metric gives us the constant curvature -1. We construct a basis $\{X_i : 1 \le i \le 2n+1\}$ for $\mathfrak{p}$. First, we choose $H \in \mathfrak{a}$ with $\alpha_1(H) = 1$. Let $N_1, \cdots, N_{2n}$ be the root vectors for the root α_1 such that

$$[N_i, \theta(N_j)] = 0 \quad \text{if} \quad i \ne j, \quad [N_j, \theta(N_j)] = -2H.$$

Then, we define X_i in the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$ using the formula

$$X_i = \frac{1}{2}(N_i - \theta(N_i))$$

for $i = 1, \cdots, 2n$. Let $X_{2n+1} = H$. The basis $\{X_i : 1 \le i \le 2n+1\}$ is an orthonormal basis for $(\mathfrak{p}, \langle\ ,\ \rangle)$. We define $Y_i \in \mathfrak{k}$ for $i = 1, \cdots, 2n$ as follows

$$Y_i = \frac{1}{2}(N_i + \theta(N_i)).$$

We have the equalities $N_i = X_i + Y_i$, $[X_i, Y_i] = -H$ and X_i, Y_i, H have the unit length with respect to the inner product $\langle\ ,\ \rangle$.

We use the spin representation (τ_s, V_{τ_s}) of $\mathrm{Spin}(2n+1)$ to define a homogeneous vector bundle $\tilde{E}$ over $\mathrm{Spin}(2n+1,1)/\mathrm{Spin}(2n+1)$ by $\tilde{E} = G \times V_{\tau_s}/\sim$ where

$$(g, v) \sim (g', v') \qquad \text{if} \quad (g', v') = (gk, \tau_s(k^{-1})v)$$

for $g, g' \in G,\ k \in K$. The action of the group $G = \mathrm{Spin}(2n+1,1)$ on the section s of $\tilde{E}$ is given by the standard formula

$$g \cdot s(x) = gs(g^{-1}x) \qquad \text{for} \quad g \in G\ ,\ x \in G/K.$$

This procedure allows us to associate every unitary representation τ of K with a unitary representation of G. We denote this induced representation of G by $\mathrm{Ind}_K^G \tau$. If we restrict the spin representation τ_s of $K = \mathrm{Spin}(2n+1)$ to $M = \mathrm{Spin}(2n)$, then this representation decomposes into two half spin representations σ_+, σ_- of $\mathrm{Spin}(2n)$. Two representations $(\sigma_+, H_{\sigma_+}), (\sigma_-, H_{\sigma_-})$ are unramified and $w\sigma_+ = \sigma_-, w\sigma_- = \sigma_+$ for the nontrivial element $w \in W(A)$.

Let $H \in \mathfrak{a}$ be the frame vector with $\alpha_1(H) = 1$ as above. Then the spinor bundle $\tilde{E}$ decomposes into the eigenspaces for the eigenvalues $i, -i$ of $c(H)$. Since the Clifford multiplication $c(\cdot)$ is K-equivariant, in particular M-equivariant, we have

$$\tau_s(m)c(X)\tau_s(m^{-1}) = c(mXm^{-1})$$

for $m \in M$ and $X \in \mathfrak{p}$. Since the group M is the centralizer of A in K, $c(mHm^{-1}) = c(H)$. Hence the restriction of τ_s to M preserves each $\pm i$-eigenspace of $c(H)$ and this means that the eigenspaces of $c(H)$ are the representation spaces of σ_+, σ_- of $M = \mathrm{Spin}(2n)$.

We define the Dirac operator $\tilde{D} : C^\infty(\tilde{X}, \tilde{E}) \ \rightarrow\ C^\infty(\tilde{X}, \tilde{E})$ by

$$\tilde{D} = \sum_{i=1}^{2n+1} c(X_i)\nabla_{X_i}$$

where $\{X_i : 1 \le i \le 2n+1\}$ is the orthonormal frame constructed above and $c(\cdot)$ is the Clifford multiplication.

We consider a locally symmetric space given by $X = \Gamma\backslash\mathrm{Spin}(2n+1,1)/\mathrm{Spin}(2n+1)$ where Γ is a discrete subgroup of $\mathrm{Spin}(2n+1,1)$ with unipotent elements and no elliptic elements. We assume that $\Gamma \cap P = \Gamma \cap N$ so that $\Gamma \cap P \backslash N = \Gamma \cap N \backslash N$. Then X is a $(2n+1)$-dimensional hyperbolic manifold with cusps. We denote by E the quotient space of $\Gamma \backslash \tilde{E}$. This is a locally homogeneous vector bundle over X. Moreover the Dirac operator $\tilde{D}$ can be pushed down to X. We denote this operator by D.

For the Iwasawa decomposition $G = KAN$, we can identify A with the real line $\mathbb{R}$ by the map $a \longrightarrow |\alpha_1|^{-1}\alpha_1(\log\ a)$. For a Γ-cuspidal minimal parabolic subgroup P, we define $P^+ := MA^+N$ where A^+ is the inverse image of the above identification of the nonnegative real part $\mathbb{R}^+$. We denote by $P_1, \cdots, P_\kappa$ the representatives of the set of Γ-cuspidal parabolic subgroups, which are not Γ-conjugate to each other. We have the corresponding decomposition for P_i. Hence there exists the identification map I_i for each cuspidal part

$$I_i :\ \mathbb{R}^+ \times T^{2n} \quad \to \quad \Gamma \cap P_i \backslash P_i^+ / K \cap M_i = \Gamma \cap N_i \backslash P_i^+ / M_i$$

and under this map we have the induced metric $dg^2(r,x) = dr^2 + e^{-2r} dn^2(x)$ for $(r,x) \in \mathbb{R}^+ \times T^{2n}$.

By the identification map I_i, we may regard that the restriction of the Dirac operator D to $\Gamma \cap P_i \backslash P_i^+ / K \cap M_i$ is defined over $\mathbb{R}^+ \times T^{2n}$. Let B be the Dirac operator acting on $C^\infty(T^{2n}, E|_{T^{2n}})$ given by $\sum_{i=1}^{2n} c(X_i)c(H)\nabla_{X_i}$. Note that B has the property $c(H)B = -Bc(H)$. We have the following expression for D over cusps

$$D = c(H)(\nabla_H + B - nI) \tag{1}$$

(see (7) in [2]). Let $0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots \to \infty$ be positive eigenvalues of B, each eigenvalue repeated according to its multiplicity with corresponding eigensections ψ_j. We denote by E_{μ_j} the eigenspace corresponding to μ_j. We decompose

$$L^2(\mathbb{R}^+ \times T^{2n}, E|_{\mathbb{R}^+ \times T^{2n}}, dg^2(r,x))$$

into

$$L^2(\mathbb{R}^+, E_0, e^{-2nr}dr) \oplus \oplus_{j=1}^\infty L^2(\mathbb{R}^+, E_{\mu_j} \oplus c(H)E_{\mu_j}, e^{-2nr}dr).$$

Note that $E_0 \cong V_{\tau_s}$ is a symplectic vector space with a symplectic structure $\langle c(H)\ \ ,\ \rangle$. We fix a Lagrangian subspace L of $E_0 \cong V_{\tau_s}$ such that $E_0 = L \oplus c(H)L$. Then the map

$$\alpha_j \psi_j + \alpha_{-j} c(H)\psi_j \to\ e^{-nr}\begin{pmatrix} \alpha_j \\ \alpha_{-j} \end{pmatrix}$$

gives a unitary equivalence

$$L^2(\mathbb{R}^+, E_0, e^{-2nr}dr) \oplus \oplus_{j=1}^\infty L^2(\mathbb{R}^+, E_{\mu_j} \oplus c(H)E_{\mu_j}, e^{-2nr}dr)$$

$$\cong \quad \oplus_{j=1}^{d(\sigma_+)} L^2(\mathbb{R}^+, \mathbb{C}^2, dr) \oplus \oplus_{j=1}^{\infty} L^2(\mathbb{R}^+, \mathbb{C}^2, dr)$$

under which the Dirac operator D over the cusp decomposes as follows

$$D \to \quad \oplus_{i=1}^{d(\sigma_+)} D_0 \oplus \oplus_{j=1}^{\infty} D_{\mu_j} \tag{2}$$

where

$$D_\mu = \begin{pmatrix} 0 & -\frac{d}{dr} + \mu \\ \frac{d}{dr} + \mu & 0 \end{pmatrix}.$$

Hence the Dirac Laplacian D^2 is transformed into $\oplus_{\mu \in spec(B)} D_\mu^2$ where $D_\mu^2 = -\frac{d^2}{dr^2} + \mu^2$ on $L^2(\mathbb{R}^+, \mathbb{C}, dr)$.

Now we consider the operator

$$c(H)(\frac{d}{dr} - nI) : C_0^\infty(\mathbb{R}^+, E_0) \to C_0^\infty(\mathbb{R}^+, E_0)$$

whose L^2-extension (with respect to $e^{-2nr} dr$) is transformed to $d(\sigma_+)$-copies of D_0 in (2). The operator

$$c(H)(\frac{d}{dr} - nI) : C_0^\infty(\mathbb{R}^+, E_0, L) := \{ \ \phi \in C_0^\infty(\mathbb{R}^+, E_0) \mid \phi(0) \in L \ \} \to L^2(\mathbb{R}^+, E_0, e^{-2nr} dr)$$

is essentially self adjoint. By the natural embedding and the identification map I_i, we have the embedding map J_i defined by

$$J_i : \mathbb{R}^+ \quad \hookrightarrow \quad \mathbb{R}^+ \times T^{2n} \quad \cong \quad \Gamma \cap N_i \backslash P_i^+ / M_i \quad \hookrightarrow \quad X. \tag{3}$$

By the embedding map J_i we can consider κ-copies of $L^2(\mathbb{R}^+, E_0, L, e^{-2nr} dr)$ as a subspace of $L^2(X, E)$ and κ-copies of $c(H)(\frac{d}{dr} - nI)$ can be extended to the self adjoint operator $\mathcal{D}_0$ on $L^2(X, E)$ by zero map over the orthogonal complement of this subspace in $L^2(X, E)$. We can see easily that $\mathcal{D}_0$ has no point spectrum. Let $\mathcal{D}$ be the unique selfadjoint extension of D on $L^2(X, E)$.

Proposition 1.1. *For $t > 0$, operators $e^{-t\mathcal{D}^2} - e^{-t\mathcal{D}_0^2}$ and $\mathcal{D} e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2}$ are trace class operators.*

Proof. Let F_0 be the union of $\Gamma \cap N_i \backslash P_i^+ / M_i$ for $i = 1, \cdots, \kappa$. Pick $f \in C^\infty(X)$ such that $0 < f \leq 1$, $f(x) = 1$ for $x \in X - F_0$ and $f((r, \cdot)) = e^{-\frac{r}{4}}$ for $(r, \cdot) \in F_0$ with $r \geq 1$. Denote by U_f an operator in $L^2(X, E)$ defined by the multiplication by f. Then we may write

$$\begin{aligned}e^{-t\mathcal{D}^2}-e^{-t\mathcal{D}_0^2}=&(e^{-\frac{t}{2}\mathcal{D}^2}-e^{-\frac{t}{2}\mathcal{D}_0^2})\circ U_f^{-1}\circ U_f\circ e^{-\frac{t}{2}\mathcal{D}^2}\\&+e^{-\frac{t}{2}\mathcal{D}_0^2}\circ U_f\circ U_f^{-1}\circ(e^{-\frac{t}{2}\mathcal{D}^2}-e^{-\frac{t}{2}\mathcal{D}_0^2}).\end{aligned}$$

The heat kernel estimates for $e^{-t\mathcal{D}^2}$ and $e^{-t\mathcal{D}_0^2}$ show that $(e^{-\frac{t}{2}\mathcal{D}^2}-e^{-\frac{t}{2}\mathcal{D}_0^2})\circ U_f^{-1}$, $U_f\circ e^{-\frac{t}{2}\mathcal{D}^2}$, $e^{-\frac{t}{2}\mathcal{D}_0^2}\circ U_f$ and $U_f^{-1}\circ(e^{-\frac{t}{2}\mathcal{D}^2}-e^{-\frac{t}{2}\mathcal{D}_0^2})$ are Hilbert-Schmidt operators. The composition of Hilbert-Schmidt operators is of trace class, hence $e^{-t\mathcal{D}^2}-e^{-t\mathcal{D}_0^2}$ is a trace class operator. The remaining case is proved in the same way. □

2. The Selberg trace formula

Let R_Γ be the right regular representation of G on $L^2(\Gamma\backslash G)$ and f be a right K-finite function in the Harish-Chandra L^p-Schwartz space $\mathcal{C}^p(G)$ where $0<p<1$. The trace of the restriction $R_\Gamma(f)$ to the discrete part $L^2_d(\Gamma\backslash G)$ of $L^2(\Gamma\backslash G)$ can be written as

$$Tr(R_\Gamma(f)|_{L^2_d(\Gamma\backslash G)})=I_\Gamma(f)+H_\Gamma(f)+U_\Gamma(f)+S_\Gamma(f)+T_\Gamma(f).$$

Here

$$\begin{aligned}I_\Gamma(f)&=\mathrm{Vol}(\Gamma\backslash G)f(1),\\H_\Gamma(f)&=\sum_{\{\gamma:\text{hyperbolic}\}}\mathrm{Vol}(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G}f(x^{-1}\gamma x)dx,\\U_\Gamma(f)&=\frac{1}{|\alpha_1|}\{C_1(\Gamma)T_1(f)+C_2(\Gamma)T_2(f)+C_1'(\Gamma)T_1'(f)\},\\S_\Gamma(f)&=\frac{1}{4\pi i}\sum_{\sigma\in\hat{M}}\int_{\mathrm{Re}(\nu)=0}\mathrm{Tr}(C_\Gamma(\sigma,\nu)^{-1}\frac{d}{d\nu}C_\Gamma(\sigma,\nu)\pi_\Gamma(\sigma,\nu)(f))d\nu,\\T_\Gamma(f)&=-\frac{1}{4}\sum_{\sigma\in\hat{M}}\mathrm{Tr}(C_\Gamma(\sigma,0)\pi_\Gamma(\sigma,0)(f))\end{aligned}$$

where Γ_γ, G_γ are centralizers of γ in Γ, G respectively, constants $C_1(\Gamma)$, $C_2(\Gamma)$, $C_1'(\Gamma)$ and tempered distributions T_1, T_2, T_1' will be discussed in the next section and $C_\Gamma(\sigma,\nu)$ denote the intertwining operator. We refer to [1], [21], [22], [23] for detailed expositions of the Selberg trace formula. The various invariant measures that we use are normalized as in [21].

The kernel $\tilde{K}_t(x:y)$ of the integral operator $e^{-t\tilde{D}^2}$ (or $\tilde{D}e^{-t\tilde{D}^2}$) over G/K is a section of $\tilde{E}\boxtimes\tilde{E}^*$, the external tensor product of $\tilde{E}$ and $\tilde{E}^*$ over $G\times G$. The bundle E is a trivial bundle, hence the kernel $\tilde{K}_t(x:y)$ is an element of $(C^\infty(G\times G)\otimes \mathrm{End}(V_{\tau_s}))^K$, that is an endomorphism valued function on $G\times G$, invariant under K. It follows that there exists a function $\tilde{K}_t(x): G\to \mathrm{End}(V_{\tau_s})$ such that

$$\tilde{K}_t(x:y)=\tilde{K}_t(x^{-1}y) \quad \text{and} \quad \tilde{K}_t(k_1^{-1}xk_2)=\tau_s(k_1^{-1})\tilde{K}_t(x)\tau_s(k_2)$$

for $x,y\in G, k_1,k_2\in K$. We define the local trace of $\tilde{K}_t(x)$ to be the scalar function on G by $K_t(x):=\mathrm{tr}(\tilde{K}_t(x))$. We denote by $\hat{K}_t(\sigma,\nu)$ the Fourier transform of K_t for the unitary principal representation $\pi_{\sigma,\nu}$ of G, that is,

$$\hat{K}_t(\sigma,\nu):=\mathrm{Tr}(\pi_{\sigma,\nu}(K_t)).$$

By the Plancherel theorem, we have

$$\mathrm{Ind}_K^G\tau_s=\int_{\mathrm{Re}(\nu)=0}\pi_{\sigma_+,\nu}\ p(\sigma_+,\nu)\ d\nu\oplus\int_{\mathrm{Re}(\nu)=0}\pi_{\sigma_-,\nu}\ p(\sigma_-,\nu)\ d\nu$$

where $p(\sigma_\pm,\nu)$ is the Plancherel measure. Hence $\hat{K}_t(\sigma,\nu)$ is zero unless $\sigma=\sigma_\pm$. From now on, we will denote by K_t^e, K_t^o the scalar functions corresponding to $e^{-t\tilde{D}^2}$, $\tilde{D}e^{-t\tilde{D}^2}$. By (4.5) in [13], we have

Proposition 2.1. *For $\lambda\in\mathbb{R}$, we have*

$$\hat{K}_t^e(\sigma_\pm,i\lambda)=e^{-t\lambda^2},\qquad \hat{K}_t^o(\sigma_+,i\lambda)=\lambda e^{-t\lambda^2},\qquad \hat{K}_t^o(\sigma_-,i\lambda)=-\lambda e^{-t\lambda^2}.$$

Since the representation $\sigma_\pm$ is unramified, the intertwining operator $C_\Gamma(\sigma_+,\nu)=C_\Gamma(\sigma_-,\nu)$ acting on $\mathcal{H}_\Gamma(\sigma_+,\nu)=\mathcal{H}_\Gamma(\sigma_-,\nu)$ switches the subspaces induced by the representations σ_+,σ_- . Hence we have the following form of $C_\Gamma(\sigma_+,\nu)$,

$$C_\Gamma(\sigma_+,\nu)=\begin{pmatrix}0 & C_-(\nu)\\ C_+(\nu) & 0\end{pmatrix}$$

with respect to the decomposition of $\mathcal{H}_\Gamma(\sigma_+,\nu)$. We refer to p. 166–167 in [1] or p. 9–10 in [22] for more details on $C_\Gamma(\sigma,\nu),\mathcal{H}_\Gamma(\sigma,\nu)$. Therefore we have

$$C_\Gamma(\sigma_+,\nu)^{-1}\frac{d}{d\nu}C_\Gamma(\sigma_+,\nu)=\begin{pmatrix}C_-(-\nu)\frac{d}{d\nu}C_+(\nu) & 0\\ 0 & C_+(-\nu)\frac{d}{d\nu}C_-(\nu)\end{pmatrix}$$

and

$$T_\Gamma(\cdot) = -\frac{1}{4}\mathrm{Tr}(C_\Gamma(\sigma_+,0)\pi_\Gamma(\sigma_+,0)(\cdot)) = 0.$$

Now the Selberg trace formulas applied to K_t^e, K_t^o are given by

$$\begin{aligned}
(4)\quad & \sum_{\sigma=\sigma_\pm}\sum_{\lambda_k\in\sigma_p^\pm} \hat{K}_t^e(\sigma,i\lambda_k) - \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{Tr}(C_\Gamma(\sigma_+,-i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+,i\lambda)\pi_\Gamma(\sigma_+,i\lambda)(K_t^e))\ d\lambda \\
= \quad & \sum_{\lambda_k\in\sigma_p} e^{-t\lambda_k^2} - \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} e^{-t\lambda^2}\mathrm{tr}\ (C_\Gamma(\sigma_+,-i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+,i\lambda))\ d\lambda \\
= \quad & I_\Gamma(K_t^e) + H_\Gamma(K_t^e) + U_\Gamma(K_t^e)
\end{aligned}$$

$$\begin{aligned}
(5)\quad & \sum_{\sigma=\sigma_\pm}\sum_{\lambda_k\in\sigma_p^\pm} \hat{K}_t^o(\sigma,i\lambda_k) - \frac{1}{4\pi}\int_{-\infty}^{\infty} \mathrm{Tr}(C_\Gamma(\sigma_+,-i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+,i\lambda)\pi_\Gamma(\sigma_+,i\lambda)(K_t^o))\ d\lambda \\
= \quad & \sum_{\lambda_k\in\sigma_p} \lambda_k e^{-t\lambda_k^2} - \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} \lambda e^{-t\lambda^2}\mathrm{tr}(C_-(-i\lambda)\frac{d}{d(i\lambda)}C_+(i\lambda))\ d\lambda \\
& \qquad + \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} \lambda e^{-t\lambda^2}\mathrm{tr}(C_+(-i\lambda)\frac{d}{d(i\lambda)}C_-(i\lambda))\ d\lambda \\
= \quad & \sum_{\lambda_k\in\sigma_p} \lambda_k e^{-t\lambda_k^2} - \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} \lambda e^{-t\lambda^2}\mathrm{tr}_s(C_\Gamma(\sigma_+,-i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+,i\lambda))\ d\lambda \\
= \quad & I_\Gamma(K_t^o) + H_\Gamma(K_t^o) + U_\Gamma(K_t^o)
\end{aligned}$$

where $d(\sigma_+)$ is the degree of σ_+. Although we know $d(\sigma_+) = 2^{n-1}$, we will keep to use the notation $d(\sigma_+)$ instead of 2^{n-1}. This indicates the origin of the constant factor. And

$$\mathrm{tr}_s(C_\Gamma(\sigma_+,-\nu)\frac{d}{d\nu}C_\Gamma(\sigma_+,\nu)) := \mathrm{tr}(C_-(-\nu)\frac{d}{d\nu}C_+(\nu)) - \mathrm{tr}(C_+(-\nu)\frac{d}{d\nu}C_-(\nu)).$$

For $I_\Gamma(K_t)$, we have

$$I_\Gamma(K_t) = \mathrm{Vol}(\Gamma\backslash G)\Big(\int_{-\infty}^{\infty} \hat{K}_t(\sigma_+,i\lambda)p(\sigma_+,i\lambda)\ d\lambda + \int_{-\infty}^{\infty} \hat{K}_t(\sigma_-,i\lambda)p(\sigma_-,i\lambda)\ d\lambda\Big)$$

where $p(\sigma_\pm,i\lambda)$ is the Plancherel measure which is an even polynomial with respect to λ. The equalities $\hat{K}_t^e(\sigma_\pm,i\lambda) = e^{-t\lambda^2}$ and $p(\sigma_+,i\lambda) = p(\sigma_-,i\lambda)$ give

$$(6)\qquad I_\Gamma(K_t^e) = 2\mathrm{Vol}(\Gamma\backslash G)\int_{-\infty}^{\infty} e^{-t\lambda^2}p(\sigma_+,i\lambda)\ d\lambda$$

and $\hat{K}_t^o$ is odd with respect to λ so that

$$I_\Gamma(K_t^o) = 0. \tag{7}$$

It is well known that

$$H_\Gamma(K_t^e) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma:\text{hyperbolic}} l(C_\gamma) j(\gamma)^{-1} D(\gamma)^{-1} (\overline{\chi_{\sigma_+}(m_\gamma)} + \overline{\chi_{\sigma_-}(m_\gamma)}) e^{-\frac{l(C_\gamma)^2}{4t}}, \tag{8}$$

$$H_\Gamma(K_t^o) = \frac{2\pi i}{(4\pi t)^{\frac{3}{2}}} \sum_{\gamma:\text{hyperbolic}} l(C_\gamma)^2 j(\gamma)^{-1} D(\gamma)^{-1} (\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)}) e^{-\frac{l(C_\gamma)^2}{4t}} \tag{9}$$

where $l(C_\gamma)$ is the length of the closed geodesic C_γ, $j(\gamma)$ is the positive integer such that $\gamma = \gamma_0^{j(\gamma)}$ for a primitive element γ_0, $D(\gamma) = e^{nl(C_\gamma)} |\det(\mathrm{Ad}(m_\gamma a_\gamma)^{-1} - I|_{\mathfrak{n}})|$ for the element $m_\gamma a_\gamma \in MA^+$ which is conjugate to γ and χ_σ is the character of σ.

3. Unipotent terms

In this Section we compute unipotent terms $U_\Gamma(K_t^e)$, $U_\Gamma(K_t^o)$. We use formula obtained by Hoffmann (see [9]) to compute these terms explicitly. By this explicit computation and Proposition 2.1, it follows that $U_\Gamma(K_t^o) = 0$. This simplifies many steps related to the application of the Selberg trace formula for K_t^o.

The unipotent term for a right K-finite function f in $\mathcal{C}^p(G)$, $0 < p < 1$ is given by

$$U_\Gamma(f) = \frac{1}{|\alpha_1|} \{ C_1(\Gamma) T_1(f) + C_2(\Gamma) T_2(f) + C_1'(\Gamma) T_1'(f) \}.$$

Here α_1 is the unique positive restricted root for $(\mathfrak{g}, \mathfrak{a})$, the constants $C_1(\Gamma), C_2(\Gamma), C_1'(\Gamma)$ which depend on Γ are computed in [3] and

$$\begin{aligned} T_1(f) &= \frac{1}{A(\mathfrak{n}_1)} \int_N \int_K f(k^{-1} n k) dk dn \\ T_2(f) &= \frac{|\alpha_1|}{2} \{ \int_{G/G_{n_0}} f(x n_0 x^{-1}) dx + \int_{G/G_{n_0^{-1}}} f(x n_0^{-1} x^{-1}) dx \} \\ T_1'(f) &= \frac{m_1 + 2m_2}{A(\mathfrak{n}_1)} \int_{N_1} \int_{N_2} \int_K f(k^{-1} n_1 n_2 k) \log|\log(n_1)| dk dn_1 dn_2 \end{aligned}$$

where n_0 is a representative of the nontrivial unipotent orbit in $\mathfrak{n}_2$, $m_i = dim(\mathfrak{n}_i)$ and $A(\mathfrak{n}_1)$ is the volume of the unit sphere in $\mathfrak{n}_1$ and G_n is the centralizer of an element n in G.

In our case $G = \mathrm{Spin}(2n+1,1)$, the second term $T_2(f) = 0$ and $m_2 = 0$ in the third term $T_1'(f)$ since $N = N_1$. The following formula for $C_1'(\Gamma)$ results from the computations in [3] ,

$$\frac{C_1'(\Gamma)}{|\alpha_1|}\frac{m_1}{A(\mathfrak{n}_1)} = \frac{\kappa}{2}. \tag{10}$$

The first term T_1 for K_t is given by

$$T_1(K_t) = \frac{1}{2\pi A(\mathfrak{n}_1)}\Big(\int_{-\infty}^{\infty} \hat{K}_t(\sigma_+, i\lambda)\, d\lambda + \int_{-\infty}^{\infty} \hat{K}_t(\sigma_-, i\lambda)\, d\lambda\Big).$$

We have $\hat{K}^e_t(\sigma_\pm, i\lambda) = e^{-t\lambda^2}$ and $\hat{K}^o_t(\sigma_\pm, i\lambda) = \pm\lambda e^{-t\lambda^2}$, which implies

$$T_1(K^e_t) = \frac{1}{\pi A(\mathfrak{n}_1)}\int_{-\infty}^{\infty} e^{-t\lambda^2}\, d\lambda \quad \text{and} \quad T_1(K^o_t) = 0. \tag{11}$$

The third term $T_1'(f)$ is more complicated and we need to introduce some notations. Let T_M be a Cartan subgroup in M , so that $T = T_M \cdot A$ is a Cartan subgroup of G. Let Σ_M denote the set of positive roots for $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{t}_{\mathfrak{m}\mathbb{C}})$. Let ρ_{Σ_M} be the half sum of elements in Σ_M. We denote by Σ_A the set of positive roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ which do not vanish on $\mathfrak{a}_{\mathbb{C}}$. The union of Σ_M with Σ_A gives the set of positive roots for $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ denoted by Σ_G. Let $H_\alpha \in \mathfrak{t}_{\mathbb{C}}$ be the coroot corresponding to $\alpha \in \pm\Sigma_G$,that is, $\alpha(H_\alpha) = 2, \alpha'(H_\alpha) \in \mathbb{Z}$ for all $\alpha, \alpha' \in \pm\Sigma_G$ and $\Pi := \prod_{\alpha\in\Sigma_M} H_\alpha$ be the element of the symmetric algebra $S(\mathfrak{t}_{\mathfrak{m}\mathbb{C}})$. We denote the simple reflection corresponding to α by s_α for $\alpha \in \Sigma_G$. Now we define

$$I(f) := \frac{1}{2\pi}\sum_{\sigma\in\hat{M}}\int_{-\infty}^{\infty} \Omega(\sigma, -i\lambda)\hat{f}(\sigma, i\lambda)\, d\lambda$$

where

$$\Omega(\sigma, i\lambda) = 2d(\sigma)\psi(1) - \frac{1}{2}\sum_{\alpha\in\Sigma_A}\lambda(H_\alpha)\frac{\Pi(s_\alpha\lambda_\sigma)}{\Pi(\rho_{\Sigma_M})}\times\Big(\psi(1+\lambda_\sigma(H_\alpha)) + \psi(1-\lambda_\sigma(H_\alpha))\Big)$$

where $d(\sigma)$ is the degree of σ, ψ is the logarithmic derivative of the Gamma function and $\lambda_\sigma - \rho_{\Sigma_M}$ is the highest weight of $(\sigma, i\lambda) \in \hat{M} \times i\mathfrak{a}$.

The principal series representation $(\pi_{\sigma,\nu}, \mathcal{H}_{\sigma,\nu})$ depends on the Langlands decomposition and we denote the dependence on P by $(\pi_{\sigma,\nu}(P), \mathcal{H}_{\sigma,\nu}(P))$. The intertwining operator

$$J_{\bar{P}|P}(\sigma,\nu) : \mathcal{H}_{\sigma,\nu}(P) \to \mathcal{H}_{\sigma,\nu}(\bar{P})$$

is defined by

$$(J_{\bar{P}|P}(\sigma,\nu)\phi) = \int_{\bar{N}} \phi(x\bar{n})\, d\bar{n}$$

and satisfies

$$J_{\bar{P}|P}(\sigma,\nu)\pi_{\sigma,\nu}(P) = \pi_{\sigma,\nu}(\bar{P})J_{\bar{P}|P}(\sigma,\nu).$$

The restriction to K defines an isomorphism from $\mathcal{H}_{\sigma,\nu}(P)$ to $\mathcal{H}_{\sigma}(P)$. Then $J_{\bar{P}|P}(\sigma,\nu)$ can be considered as a family of operators from $\mathcal{H}_{\sigma}(P)$ to $\mathcal{H}_{\sigma}(\bar{P})$. Here $\mathcal{H}_{\sigma}(P)$ is a space of all measurable functions $v : K \to H_\sigma$ such that

$$v(km) = \sigma(m)^{-1}v(k)$$

for all $m \in M$, $k \in K$. Let

$$J_P(\sigma,\nu : f) := -\mathrm{Tr}(\pi_{\sigma,\nu}(f)J_{\bar{P}|P}(\sigma,\nu)^{-1}\partial_\nu J_{\bar{P}|P}(\sigma,\nu))$$

where the derivative ∂_ν is taken with respect ν for the family of operators $J_{\bar{P}|P}(\sigma,\nu)$ acting on $\mathcal{H}_\sigma(P)$. Then there exits the Harish Chandra C-function $C_\tau(\sigma,\nu)$ such that

$$T_\tau J_{\bar{P}|P}(\sigma,\nu)^{-1}\partial_\nu J_{\bar{P}|P}(\sigma,\nu) \;=\; C_\tau(\sigma,\nu)^{-1}\partial_\nu C_\tau(\sigma,\nu)T_\tau$$

where T_τ is the projection to the τ-isotypic component of $\mathcal{H}_{\sigma,\nu}(P)$. We refer to [6] for more detail. We have the following proposition for $C_{\tau_s}(\sigma_\pm,\nu)$,

Proposition 3.1. *For the half spin representation $\sigma_\pm$ of Spin$(2n)$,*

$$C_{\tau_s}(\sigma_+, i\lambda) = C_{\tau_s}(\sigma_-, i\lambda) = \frac{(2n-1)!}{(n-1)!}\frac{\Gamma(i\lambda+\frac{1}{2})}{\Gamma(i\lambda+n+\frac{1}{2})}.$$

Proof. This is a special case of the theorem 8.2 in [6]. □

The weighted orbital integral T_1' is given by

$$T_1'(f) = \frac{m_1}{A(\mathfrak{n}_1)}\Big(I(f) + \frac{1}{2\pi}\mathrm{p.v.}\sum_{\sigma\in\hat{M}} d(\sigma)\int_{-\infty}^{\infty} J_P(\sigma, i\lambda : f)\, d\lambda + \sum_{\sigma\in\hat{M}} d(\sigma)\frac{n(\sigma)}{2}\hat{f}(\sigma,0)\Big)$$

where p.v. means the Cauchy principal value and $2n(\sigma)$ is the order of the zero of $p(\sigma,\nu)$ at $\nu=0$. Using $n(\sigma_\pm)=0$ and Proposition 3.1 we obtain

$$T_1'(K_t^e)=\frac{m_1}{2\pi A(\mathfrak{n}_1)}\int_{-\infty}^{\infty}e^{-t\lambda^2}\Big(\Omega(\sigma_+,-i\lambda)+\Omega(\sigma_-,-i\lambda)-2d(\sigma_+)\frac{d}{d(i\lambda)}\log C_{\tau_s}(\sigma_+,i\lambda)\Big)\,d\lambda,$$

$$T_1'(K_t^o)=\frac{m_1}{2\pi A(\mathfrak{n}_1)}\int_{-\infty}^{\infty}\lambda e^{-t\lambda^2}\Big(\Omega(\sigma_+,-i\lambda)-\Omega(\sigma_-,-i\lambda)\Big)\,d\lambda.$$

Now we consider $\Omega(\sigma_\pm,i\lambda)$ in the following proposition,

Proposition 3.2. *For the half spin representation $\sigma_\pm$ of Spin$(2n)$, the following equalities hold*

$$\begin{aligned}\Omega(\sigma_+,i\lambda) &= \Omega(\sigma_-,i\lambda)\\ = -\frac{d(\sigma_+)}{2}\Big(\psi(i\lambda-n+\frac12)+\psi(-i\lambda-n+\frac12)&+\psi(i\lambda+\frac12)+\psi(-i\lambda+\frac12)\Big)+P^n(\lambda)\end{aligned}$$

where $P^n(\lambda)$ is an even polynomial of $2n-4$ degree for $n\ge 2$ and $P^1(\lambda)$ is a constant.

Proof. The case for $n=1$ can be computed by the same way as in the other cases for $n\ge 2$. Hence we assume that $n\ge 2$ in the following proof. The highest weight of the half spin representation $\sigma_\pm$ of $\mathrm{Spin}(2n)\subset\mathrm{Spin}(2n+1)$ is given by

$$\frac12(e_2+e_3+\cdots+e_n\pm e_{n+1})$$

with respect to the standard basis $\{e_i\}$. This implies

$$i\lambda e_1+\sigma_\pm+\rho_{\Sigma_M}=i\lambda e_1+(n-\frac12)e_2+(n-\frac32)e_3+\cdots+\frac32 e_n\pm\frac12 e_{n+1}.$$

The positive roots $\alpha\in\Sigma_A$ are given by e_1-e_j, e_1+e_j for $2\le j\le n+1$. Then we can see that $s_{(e_1-e_j)}(i\lambda e_1+\sigma_\pm+\rho_{\Sigma_M})$ is

$$\begin{aligned}&i\lambda e_j+(n-\frac12)e_2+\cdots+(n-j+\frac32)e_1+\cdots+\frac32 e_n\pm\frac12 e_{n+1}\quad\text{if}\quad 2\le j\le n,\\ &i\lambda e_{n+1}+(n-\frac12)e_2+\cdots+\frac32 e_n\pm\frac12 e_1\quad\text{if}\quad j=n+1\end{aligned}$$

and $s_{(e_1+e_j)}(i\lambda e_1+\sigma_\pm+\rho_{\Sigma_M})$ is

$$
\begin{aligned}
&- i\lambda e_j + (n-\frac{1}{2})e_2 + \cdots - (n-j+\frac{3}{2})e_1 + \cdots + \frac{3}{2}e_n \pm \frac{1}{2}e_{n+1} \quad \text{if} \quad 2 \le j \le n,\\
&- i\lambda e_{n+1} + (n-\frac{1}{2})e_2 + \cdots + \frac{3}{2}e_n \mp \frac{1}{2}e_1 \quad \text{if} \quad j = n+1.
\end{aligned}
$$

These give us the following formula,

$$
\begin{aligned}
&\Pi(s_{(e_1 \pm e_j)}(i\lambda e_1 + \sigma_\pm + \rho_{\Sigma_M}))\\
=&C_j(\lambda^2 + (n-\frac{1}{2})^2)\cdot(\lambda^2 + (n-\frac{3}{2})^2)\cdots(\lambda^2 + (n-j+\frac{5}{2})^2)\\
&\cdot(-\lambda^2 - (n-j+\frac{1}{2})^2)\cdots(-\lambda^2 - (\frac{3}{2})^2)\cdot(-\lambda^2 - (\frac{1}{2})^2)
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
C_j &= \prod_{1\le k<l\le n, k\ne j, l\ne j} ((n-k+\frac{1}{2})^2 - (n-l+\frac{1}{2})^2)\\
&= \prod_{1\le k<l\le n, k\ne j, l\ne j} (l-k)(2n-k-l+1)
\end{aligned}
$$

for $2 \le j \le n+1$. In particular, we obtain

$$
\begin{aligned}
\Pi(s_{e_1\pm e_j}(i\lambda e_1 + \sigma_+ + \rho_{\Sigma_M})) &= \Pi(s_{e_1\pm e_j}(i\lambda e_1 + \sigma_- + \rho_{\Sigma_M}))\\
\Pi(s_{e_1 - e_j}(i\lambda e_1 + \sigma_\pm + \rho_{\Sigma_M})) &= \Pi(s_{e_1 + e_j}(i\lambda e_1 + \sigma_\pm + \rho_{\Sigma_M}))
\end{aligned}
$$

for $2 \le j \le n+1$. From now on, we denote by $P_j(\lambda)$ this common $2(n-1)$ degree even polynomial of λ in (12) for $2 \le j \le n+1$.

On the other hand, $(i\lambda e_1 + \sigma_\pm + \rho_{\Sigma_M})(H_\alpha)$ is given by

$$
\begin{aligned}
&i\lambda - (n-j+\frac{3}{2}) \quad \text{if} \quad \alpha = e_1 - e_j, \quad 2 \le j \le n\\
&i\lambda \mp \frac{1}{2} \quad \text{if} \quad \alpha = e_1 - e_{n+1}\\
&i\lambda + (n-j+\frac{3}{2}) \quad \text{if} \quad \alpha = e_1 + e_j, \quad 2 \le j \le n\\
&i\lambda \pm \frac{1}{2} \quad \text{if} \quad \alpha = e_1 + e_{n+1}.
\end{aligned}
$$

Then the pair $\Big(\psi(1+(i\lambda e_1+\sigma_\pm+\rho_{\Sigma_M})(H_\alpha)), \psi(1-(i\lambda e_1+\sigma_\pm+\rho_{\Sigma_M})(H_\alpha))\Big)$ is given by

$$\begin{aligned}
&\psi(i\lambda-n+j-\frac{1}{2}), \psi(-i\lambda+n-j+\frac{5}{2}) \quad \text{for} \quad e_1-e_j, \quad 2\le j\le n\\
&\psi(i\lambda+\frac{1}{2}), \psi(-i\lambda+\frac{3}{2}) \quad \text{for} \quad e_1-e_{n+1}, \quad \sigma=\sigma_+\\
&\psi(i\lambda+\frac{3}{2}), \psi(-i\lambda+\frac{1}{2}) \quad \text{for} \quad e_1-e_{n+1}, \quad \sigma=\sigma_-\\
&\psi(i\lambda+n-j+\frac{5}{2}), \psi(-i\lambda-n+j-\frac{1}{2}) \quad \text{for} \quad e_1+e_j, \quad 2\le j\le n\\
&\psi(i\lambda+\frac{3}{2}), \psi(-i\lambda+\frac{1}{2}) \quad \text{for} \quad e_1+e_{n+1}, \quad \sigma=\sigma_+\\
&\psi(i\lambda+\frac{1}{2}), \psi(-i\lambda+\frac{3}{2}) \quad \text{for} \quad e_1+e_{n+1}, \quad \sigma=\sigma_-.
\end{aligned}$$

Now we compare the following two sets

$$\Big\{\Pi(s_\alpha(i\lambda e_1+\sigma_++\rho_{\Sigma_M})), 1+(i\lambda e_1+\sigma_++\rho_{\Sigma_M})(H_\alpha), 1-(i\lambda e_1+\sigma_++\rho_{\Sigma_M})(H_\alpha)\Big\},$$
$$\Big\{\Pi(s_\alpha(i\lambda e_1+\sigma_-+\rho_{\Sigma_M})), 1+(i\lambda e_1+\sigma_-+\rho_{\Sigma_M})(H_\alpha), 1-(i\lambda e_1+\sigma_-+\rho_{\Sigma_M})(H_\alpha)\Big\}$$

and see that they are equal to each other so that $\Omega(\sigma_+, i\lambda)=\Omega(\sigma_-, i\lambda)$. We compute the exact form of $\Omega(\sigma_+, i\lambda)=\Omega(\sigma_-, i\lambda)$. Using the relations $\psi(z+1)=\frac{1}{z}+\psi(z)$, we have

$$\begin{aligned}
&\psi(i\lambda-n+j-\frac{1}{2})+\psi(-i\lambda-n+j-\frac{1}{2})+\psi(i\lambda+n-j+\frac{5}{2})+\psi(-i\lambda+n-j+\frac{5}{2})\\
=&\frac{1}{\lambda^2+(\frac{1}{2})^2}+\cdots+\frac{2(n-j+\frac{1}{2})}{\lambda^2+(n-j+\frac{1}{2})^2}+\frac{-2(n-j+\frac{5}{2})}{\lambda^2+(n-j+\frac{5}{2})^2}+\cdots+\frac{-2(n-\frac{1}{2})}{\lambda^2+(n-\frac{1}{2})^2}\\
&+\psi(i\lambda-n+\frac{1}{2})+\psi(-i\lambda-n+\frac{1}{2})+\psi(i\lambda+\frac{1}{2})+\psi(-i\lambda+\frac{1}{2}).
\end{aligned}$$

Now using the formula (see the last line of p.95 in [9]),

$$\sum_{\alpha\in\Sigma_A}\Pi(s_\alpha\lambda_\sigma)=2\Pi(\lambda_\sigma),$$

we decompose

$$\frac{1}{2}\sum_{\alpha\in\Sigma_A}\frac{\Pi(s_\alpha\lambda_\sigma)}{\Pi(\rho_{\Sigma_M})}\times\Big(\psi(1+\lambda_\sigma(H_\alpha))+\psi(1-\lambda_\sigma(H_\alpha))\Big)$$

into

$$\frac{d(\sigma_\pm)}{2}\Big(\psi(i\lambda-n+\frac{1}{2})+\psi(-i\lambda-n+\frac{1}{2})+\psi(i\lambda+\frac{1}{2})+\psi(-i\lambda+\frac{1}{2})\Big)$$

and

$$\frac{1}{4\Pi(\rho_{\Sigma_M})}\sum_{j=2}^{n+1} P_j(\lambda)R_j(\lambda)$$

where

$$R_j(\lambda) = \frac{1}{\lambda^2+(\frac{1}{2})^2} + \cdots + \frac{2(n-j+\frac{1}{2})}{\lambda^2+(n-j+\frac{1}{2})^2} + \frac{-2(n-j+\frac{5}{2})}{\lambda^2+(n-j+\frac{5}{2})^2} + \cdots + \frac{-2(n-\frac{1}{2})}{\lambda^2+(n-\frac{1}{2})^2}.$$

We can see, from the definitions of $P_j(\lambda)$ and $R_j(\lambda)$, that $P_j(\lambda)R_j(\lambda)$ is an even polynomial of degree $2n-4$. This ends the proof. □

Now we have the following corollary

Corollary 3.3.

$$U_\Gamma(K^o_t) = 0, \qquad U_\Gamma(K^e_t) = \frac{2}{2\pi}\int_{-\infty}^{\infty} e^{-t\lambda^2}(P_U(\lambda)+Q(\lambda))\ d\lambda$$

where $P_U(\lambda)$ is an even polynomial of degree $2n-4$ and $Q(\lambda)$ is given by

$$-\kappa\frac{d(\sigma_+)}{2}\Big(\psi(i\lambda+\frac{1}{2})+\psi(-i\lambda+\frac{1}{2})\Big).$$

Proof. The first claim follows easily from (10), (11), Proposition 3.1 and 3.2. For the second claim, $Q(\lambda)$ is a priori given by

$$\begin{aligned}-\frac{\kappa d(\sigma_+)}{2}\Big(&\frac{1}{2}(\psi(i\lambda-n+\frac{1}{2})+\psi(-i\lambda-n+\frac{1}{2}))\\ &-\frac{1}{2}(\psi(i\lambda+n+\frac{1}{2})+\psi(-i\lambda+n+\frac{1}{2}))+(\psi(i\lambda+\frac{1}{2})+\psi(-i\lambda+\frac{1}{2}))\Big).\end{aligned}$$

We use the relation $\psi(z+1) = \frac{1}{z}+\psi(z)$, then we can reduce the above formula to the claimed one for $Q(\lambda)$. □

4. Relative Traces and Spectral Sides

In this Section we study relations of $\mathrm{Tr}(e^{-t\mathcal{D}^2} - e^{-t\mathcal{D}_0^2})$, $\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2})$ with the spectral sides of the Selberg trace formulas applied to the test functions K_t^e, K_t^o. The formula of this type was proved by Müller in the case of the Laplacian acting on functions over the manifold with cusps (see [15]) and for the Dirac operator over the manifold with cylindrical end (see [18]). Following [15], [18], we prove the corresponding formula for the Dirac operator acting on spinor bundle over the hyperbolic manifold with cusps.

We can identify $L^2(X, E)$ with the space

$$(L^2(\Gamma\backslash G) \otimes V_{\tau_s})^K = \{\ f \in L^2(\Gamma\backslash G) \otimes V_{\tau_s} \mid f(xk) = \tau_s(k)^{-1} f(x) \quad \text{for} \quad k \in K, x \in G\ \}.$$

Then the decomposition of $L^2(\Gamma\backslash G) = L^2_d(\Gamma\backslash G) \oplus L^2_c(\Gamma\backslash G)$ allows us to decompose

$$(L^2(\Gamma\backslash G) \otimes V_{\tau_s})^K = (L^2_d(\Gamma\backslash G) \otimes V_{\tau_s})^K \oplus (L^2_c(\Gamma\backslash G) \otimes V_{\tau_s})^K. \tag{13}$$

Recall that the continuous part $(L^2_c(\Gamma\backslash G) \otimes V_{\tau_s})^K$ is spanned by the wave packets with the Eisenstein series. Therefore we need to know how D acts on the Eisenstein series. For $\Phi \in (\mathcal{H}_\Gamma(\sigma_+, \nu) \otimes V_{\tau_s})^K$, the Eisenstein series attached to Φ is defined by

$$E(\Phi : \nu : x) := \sum_{\gamma \in \Gamma \cap P \backslash \Gamma} \Phi(\gamma x).$$

Assume that $\Phi_{j,\pm}$ is in the $\pm i$-eienspace of $c(H)$ of $(\mathcal{H}_\Gamma(\sigma_+, \nu) \otimes V_{\tau_s})^K$ for $j = 1, \cdots, d(\sigma_+)$. Then the Eisenstein series $E(\Phi_{j,\pm} : i\lambda : x)$ for $\nu = i\lambda$ satisfies

$$DE(\Phi_{j,\pm} : i\lambda : x) = \pm\lambda E(\Phi_{j,\pm} : i\lambda : x). \tag{14}$$

For $\phi \in C_0^\infty(X, E)$, the decomposition (13) provides us with the formula

$$\begin{aligned} \phi(x) = &\sum_k (\phi, \phi_k)\phi_k(x) \\ &+ \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^{\infty} E(\Phi_{j,+} : i\lambda : x) \int_X E(\Phi_{j,+} : -i\lambda : y)\phi(y)\ dy\ d\lambda \\ &+ \frac{1}{4\pi} \sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^{\infty} E(\Phi_{j,-} : i\lambda : x) \int_X E(\Phi_{j,-} : -i\lambda : y)\phi(y)\ dy\ d\lambda \end{aligned} \tag{15}$$

where $\{\phi_k\}$ is the orthonormal basis of $(L^2_d(\Gamma\backslash G)\otimes V_{\tau_s})^K$. Since $\mathcal{D}$ preserves the decomposition of (13), we may assume that each ϕ_k is the eigensections of $\mathcal{D}$. Therefore (14) and (15) implies

$$\begin{aligned}\mathcal{D}\phi(x) = &\sum_{\lambda_k}\lambda_k(\phi,\phi_k)\phi_k(x)\\ &+\frac{1}{4\pi}\sum_{j=1}^{d(\sigma_+)}\int_{-\infty}^{\infty}\lambda E(\Phi_{j,+}:i\lambda:x)\int_X E(\Phi_{j,+}:-i\lambda:y)\phi(y)\;dy\;d\lambda\\ &-\frac{1}{4\pi}\sum_{j=1}^{d(\sigma_+)}\int_{-\infty}^{\infty}\lambda E(\Phi_{j,-}:i\lambda:x)\int_X E(\Phi_{j,-}:-i\lambda:y)\phi(y)\;dy\;d\lambda.\end{aligned}$$

The integral kernels of $e^{-t\mathcal{D}^2}$, $\mathcal{D}e^{-t\mathcal{D}^2}$ are given by the formulas

$$\mathcal{K}^e_t(x:y) := \sum_{\gamma\in\Gamma}\tilde{K}^e_t(\gamma\tilde{x}:\tilde{y}),\quad \mathcal{K}^o_t(x:y) := \sum_{\gamma\in\Gamma}\tilde{K}^o_t(\gamma\tilde{x}:\tilde{y})$$

where $x=\Gamma\tilde{x}$, $y=\Gamma\tilde{y}$, $\tilde{x},\tilde{y}\in G$ and $\tilde{K}^e_t$, $\tilde{K}^o_t$ are integral kernels of $e^{-t\tilde{D}^2}$, $\tilde{D}e^{-t\tilde{D}^2}$. The following equalities hold, when we consider those kernels as distributions

$$\begin{aligned}\mathcal{K}^e_t(x:y) = &\sum_{\lambda_k\in\sigma_p}e^{-t\lambda_k^2}\phi_k(x)\otimes\phi_k^*(y)\\ &+\frac{1}{4\pi}\sum_{j=1}^{d(\sigma_+)}\int_{-\infty}^{\infty}e^{-t\lambda^2}E(\Phi_{j,+}:i\lambda:x)\otimes E(\Phi_{j,+}:-i\lambda:y)^*\;d\lambda\\ &+\frac{1}{4\pi}\sum_{j=1}^{d(\sigma_+)}\int_{-\infty}^{\infty}e^{-t\lambda^2}E(\Phi_{j,-}:i\lambda:x)\otimes E(\Phi_{j,-}:-i\lambda:y)^*\;d\lambda,\end{aligned}\tag{16}$$

$$\begin{aligned}\mathcal{K}^o_t(x:y) = &\sum_{\lambda_k\in\sigma_p}\lambda_k e^{-t\lambda_k^2}\phi_k(x)\otimes\phi_k^*(y)\\ &+\frac{1}{4\pi}\sum_{j=1}^{d(\sigma_+)}\int_{-\infty}^{\infty}\lambda e^{-t\lambda^2}E(\Phi_{j,+}:i\lambda:x)\otimes E(\Phi_{j,+}:-i\lambda:y)^*\;d\lambda\\ &-\frac{1}{4\pi}\sum_{j=1}^{d(\sigma_+)}\int_{-\infty}^{\infty}\lambda e^{-t\lambda^2}E(\Phi_{j,-}:i\lambda:x)\otimes E(\Phi_{j,-}:-i\lambda:y)^*\;d\lambda.\end{aligned}\tag{17}$$

We can regard $[R,\infty)\times T^{2n}\subset\mathbb{R}^+\times T^{2n}$ as a subset of X by the map I_i for $i=1,\cdots,\kappa$. We denote this subset by $F_{i,R}$ and let $F_R:=\cup_{i=1}^{\kappa}F_{i,R}$ and $X_R:=X-F_R$. The constant term of the Eisenstein series $E(\Phi:i\lambda:x)$ over $F_{i,0}$ has the following form,

$$e^{(-i\lambda+n)r}\Phi_i+e^{(i\lambda+n)r}((C_\Gamma(\sigma_+,i\lambda)\otimes I)\Phi)_i. \tag{18}$$

Here Φ_i is the component of Φ over $F_{i,0}$ and note that the operator $C_\Gamma(\sigma_+,i\lambda)\otimes I$ acts on $(\mathcal{H}_\Gamma(\sigma_+,i\lambda)\otimes V_{\tau_s})^K$. From now on, we assume that $||\Phi_i||=1$ for $i=1,\cdots,\kappa$. We now discuss the *Maass-Selberg relation* in our context.

Proposition 4.1. *(Maass-Selberg) We have*

$$\begin{aligned}\int_{X_R}|E(\Phi_{j,\pm}:i\lambda:x)|^2dx &= 2\kappa R-tr(C_\pm(i\lambda)\frac{d}{d(i\lambda)}C_\mp(-i\lambda))+O(e^{-cR})\\ &= 2\kappa R-tr(C_\mp(-i\lambda)\frac{d}{d(i\lambda)}C_\pm(i\lambda))+O(e^{-cR})\end{aligned} \tag{19}$$

where $C_\pm(i\lambda)_{ik}$ is a component of $C_\pm(i\lambda)$ and c is a positive constant.

Proof. We will consider only the case of Φ_+ so that we use the notation Φ instead of Φ_+ in the following proof. The case of Φ_- can be done by the same way. It follows from the Green's formula that we have an equality

$$\begin{aligned}&(\lambda-\lambda')\langle E(\Phi:i\lambda:x),E(\Phi:i\lambda':x)\rangle_{X_R}\\ &=\langle DE(\Phi:i\lambda:x),E(\Phi:i\lambda':x)\rangle_{X_R}-\langle E(\Phi:i\lambda:x),DE(\Phi:i\lambda':x)\rangle_{X_R}\\ &=\langle c(H)E(\Phi:i\lambda:x),E(\Phi:i\lambda':x)\rangle_{\partial X_R}.\end{aligned} \tag{20}$$

By (18), we have

$$\begin{aligned}&\langle c(H)E(\Phi:i\lambda:x),E(\Phi:i\lambda':x)\rangle_{\partial F_{i,R}}\\ &=i\langle\ e^{(-i\lambda+n)R}\Phi_i\ ,\ e^{(-i\lambda'+n)R}\Phi_i\ \rangle_{\partial F_{i,R}}\\ &\quad -i\ \langle\ e^{(i\lambda+n)R}((C_\Gamma(\sigma_+,i\lambda)\otimes I)\Phi)_i\ ,\ e^{(i\lambda'+n)R}((C_\Gamma(\sigma_+,i\lambda')\otimes I)\Phi)_i\ \rangle_{\partial F_{i,R}}+O(e^{-cR})\\ &=ie^{-i(\lambda-\lambda')R}-ie^{i(\lambda-\lambda')R}\sum_{k=1}^{\kappa}C_+(i\lambda)_{ik}C_-(-i\lambda')_{ki}+O(e^{-cR}).\end{aligned}$$

Now we use the functional equation

$$C_\Gamma(\sigma_+, i\lambda)C_\Gamma(\sigma_+, -i\lambda) = \begin{pmatrix} 0 & C_-(i\lambda) \\ C_+(i\lambda) & 0 \end{pmatrix} \begin{pmatrix} 0 & C_-(-i\lambda) \\ C_+(-i\lambda) & 0 \end{pmatrix} = I,$$

which implies the following equality,

$$\begin{aligned} & \langle c(H)E(\Phi : i\lambda : x), E(\Phi : i\lambda' : x)\rangle_{\partial F_{i,R}} \\ (21) \qquad & = ie^{-i(\lambda-\lambda')R} - ie^{i(\lambda-\lambda')R} \\ & - ie^{i(\lambda-\lambda')R} \sum_{k=1}^{\kappa} C_+(i\lambda)_{ik} C_-(-i\lambda')_{ki} \\ & + ie^{i(\lambda-\lambda')R} \sum_{k=1}^{\kappa} C_+(i\lambda)_{ik} C_-(-i\lambda)_{ki} + O(e^{-cR}). \end{aligned}$$

Combining (20) and (21), we have

$$\begin{aligned} (22) \qquad (\lambda - \lambda')\langle E(\Phi : i\lambda : x), E(\Phi : i\lambda' : x)\rangle_{X_R} \; & = \; i\kappa e^{-i(\lambda-\lambda')R} - i\kappa e^{i(\lambda-\lambda')R} \\ + ie^{i(\lambda-\lambda')R} \sum_{i,k=1}^{\kappa} C_+(i\lambda)_{ik}(C_-(-i\lambda)_{ki} - C_-(-i\lambda')_{ki}) & + O(e^{-cR}). \end{aligned}$$

If we pass to the limit $\lambda \to \lambda'$ after dividing each side of (22) by $\lambda - \lambda'$, we get the equality (19).

□

The next task is to consider the corresponding formula for $\mathcal{D}_0$. We fix a Lagrangian subspace L in V_{τ_s} as before. For the orthonormal basis $\{\psi_j\} \in L$, we define

$$\phi_{j,+} := \frac{1}{\sqrt{2}}\Big(\psi_j - ic(H)\psi_j\Big), \quad \phi_{j,-} := \frac{1}{\sqrt{2}}\Big(\psi_j + ic(H)\psi_j\Big)$$

and

$$\begin{aligned} e(\phi_{j,+} : i\lambda : x) &:= e^{(n-i\lambda)r}\phi_{j,+} + e^{(n+i\lambda)r}\phi_{j,-}, \\ e(\phi_{j,-} : i\lambda : x) &:= e^{(n-i\lambda)r}\phi_{j,-} + e^{(n+i\lambda)r}\phi_{j,+}. \end{aligned}$$

Note that $e(\phi_{j,\pm} : i\lambda : x)$ lies in $C^\infty(\mathbb{R}^+, V_{\tau_s}, L)$ and we have

$$\begin{aligned} c(H)(\frac{d}{dr} - nI)e(\phi_{j,+} : i\lambda : x) &= \lambda e(\phi_{j,+} : i\lambda : x), \\ c(H)(\frac{d}{dr} - nI)e(\phi_{j,-} : i\lambda : x) &= -\lambda e(\phi_{j,-} : i\lambda : x). \end{aligned}$$

As before, by the map J_i we regard $e(\phi_{j,\pm} : i\lambda : x)$ lying in $C^\infty(X, E)$ and define a section $E(\phi_{j,\pm} : i\lambda : x)$ in $C^\infty(X, E)$ by the sum of these for $j = 1, \cdots, d(\sigma_+)$. For $\phi \in C_0^\infty(X, E)$, we have

$$\begin{aligned} \mathcal{D}_0\phi &= \frac{1}{4\pi}\sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^{\infty} \lambda E(\phi_{j,+} : i\lambda : x) \int_X E(\phi_{j,+} : -i\lambda : y)\phi(y)dy\ d\lambda \\ &- \frac{1}{4\pi}\sum_{j=1}^{d(\sigma_+)} \int_{-\infty}^{\infty} \lambda E(\phi_{j,-} : i\lambda : x) \int_X E(\phi_{j,-} : -i\lambda : y)\phi(y)dy\ d\lambda. \end{aligned} \tag{23}$$

The following equality is proved in the same way as in the proof of Proposition 4.1,

$$\int_{X_R} |E(\phi_{j,\pm} : i\lambda : x)|^2 dx = 2\kappa R + O(e^{-cR}) \tag{24}$$

for some positive constant c. It follows from (19) and (24) that we have

$$\begin{aligned} &\int_X |E(\Phi_{j,\pm} : i\lambda : x)|^2 - |E(\phi_{j,\pm} : i\lambda : x)|^2 dx \\ &= \lim_{R\to\infty} \int_{X_R} |E(\Phi_{j,\pm} : i\lambda : x)|^2 - |E(\phi_{j,\pm} : i\lambda : x)|^2 dx \\ &= -\operatorname{tr}(C_\mp(-i\lambda)\frac{d}{d(i\lambda)}C_\pm(i\lambda)). \end{aligned} \tag{25}$$

Finally, the following proposition is the result of (4), (5), (7), (16), (17), (25) and Corollary 3.3,

Proposition 4.2. *We have*

$$\begin{aligned} &Tr(e^{-t\mathcal{D}^2} - e^{-t\mathcal{D}_0^2}) = \int_X |\mathcal{K}_t^e(x : x)|^2 - |k_t^e(x, x)|^2 dx \\ &= \sum_{\lambda_k} e^{-t\lambda_k^2} - \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} e^{-t\lambda^2}\, tr(C_\Gamma(\sigma_+, -i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+, i\lambda))\ d\lambda \\ &= I_\Gamma(K_t^e) + H_\Gamma(K_t^e) + U_\Gamma(K_t^e), \end{aligned} \tag{26}$$

$$
\begin{aligned}
(27) \qquad & Tr(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2}) = \int_X |\mathcal{K}_t^o(x:x)|^2 - |k_t^o(x,x)|^2 dx \\
&= \sum_{\lambda_k} \lambda_k e^{-t\lambda_k^2} - \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} \lambda e^{-t\lambda^2} tr_s(C_\Gamma(\sigma_+, -i\lambda) \frac{d}{d(i\lambda)} C_\Gamma(\sigma_+, i\lambda)) \, d\lambda \\
&= \sum_{\lambda_k} \lambda_k e^{-t\lambda_k^2} - \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} \lambda e^{-t\lambda^2} tr(C_-(-i\lambda) \frac{d}{d(i\lambda)} C_+(i\lambda)) \, d\lambda \\
&\qquad + \frac{d(\sigma_+)}{4\pi} \int_{-\infty}^{\infty} \lambda e^{-t\lambda^2} tr(C_+(-i\lambda) \frac{d}{d(i\lambda)} C_-(i\lambda)) \, d\lambda \\
&= H_\Gamma(K_t^o)
\end{aligned}
$$

where $k_t^e(x,y), k_t^o(x,y)$ are the integral kernels of $e^{-t\mathcal{D}_0^2}, \mathcal{D}_0 e^{-t\mathcal{D}_0^2}$.

Remark 4.3. *Although the definition of $\mathcal{D}_0$ depends on the choices of the Lagrangian subspace L and the embedding in (3), the relative traces in the left hand sides of (26), (27) do not depend on these choices. First, we observe that the right hand side in (24) does not depend on the choice of the Lagrangian subspace L. Hence the relative traces do not depend on the choice of L. Second, for the embedding in (3), we may use the different embedding $\mathbb{R}^+ \to \mathbb{R}^+ \times T^{2n}$ given by $r \to (r + r_0, \cdot)$ for a positive r_0. If we recall the proof of (24), we can see that the boundary contribution of $\{r_0\} \times T^{2n}$ is trivial in the step corresponding to (20) since the boundary condition is given by the Lagrangian subspace L. Hence the relative traces are independent on the choice of the embedding in (3) too.*

5. Meromorphic continuations of the eta and the zeta functions

In this Section, we prove Theorem 0.1, which provides us with the pole structures of the eta function and the zeta function over $\mathbb{C}$.

We follow [18] and [19] and define

$$\eta_{\mathcal{D},1}(z) := \frac{1}{\Gamma(z+1/2)} \int_0^1 t^{\frac{z-1}{2}} \ \mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2}) \ dt,$$

$$\zeta_{\mathcal{D}^2,1}(z) := \frac{1}{\Gamma(z)} \int_0^1 t^{z-1} \ [\ \mathrm{Tr}(e^{-t\mathcal{D}^2} - e^{-t\mathcal{D}_0^2}) - h \ ] \ dt$$

for $\mathrm{Re}(z) \gg 0$ and

$$\eta_{\mathcal{D},2}(z) := \frac{1}{\Gamma(z+1/2)} \int_1^\infty t^{\frac{z-1}{2}} \operatorname{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2})\, dt,$$

$$\zeta_{\mathcal{D}^2,2}(z) := \frac{1}{\Gamma(z)} \int_1^\infty t^{z-1} \left[\operatorname{Tr}(e^{-t\mathcal{D}^2} - e^{-t\mathcal{D}_0^2}) - h \right] dt$$

for $\mathrm{Re}(z) \ll 0$ where h is the number of zero eigenvalues of $\mathcal{D}^2$. To introduce the eta invariant and the regularized determinant we need to study the meromorphic extension of $\eta_{\mathcal{D},i}(z)$ and $\zeta_{\mathcal{D}^2,i}(z)$ near $z = 0$ for $i = 1, 2$. The difficulty not present for the closed manifold is the presence of the continuous spectrum. Moreover, the continuous spectrum of $\mathcal{D}$ is equal to the whole real line in our case. Hence the meromorphic extensions of $\eta_{\mathcal{D},2}(z)$ and $\zeta_{\mathcal{D}^2,2}(z)$ have the nontrivial poles. We use analytic expansion of $C_\Gamma(\sigma_+, i\lambda)$ along the imaginary axis and the analysis of the geometric terms in the left hand side of the Selberg trace formula for the meromorphic extensions of $\eta_{\mathcal{D},i}(z)$ and $\zeta_{\mathcal{D}^2,i}(z)$ over $\mathbb{C}$.

We start with $\eta_{\mathcal{D},1}(z)$. It follows from (9) and (27) that we have

$$\begin{aligned} (28) \qquad & \operatorname{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2}) = H_\Gamma(K_t^o) \\ &= \frac{2\pi i}{(4\pi t)^{\frac{3}{2}}} \sum_{\gamma:\text{hyperbolic}} l(C_\gamma)^2 j(\gamma)^{-1} D(\gamma)^{-1} (\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)}) e^{-\frac{l(C_\gamma)^2}{4t}}. \end{aligned}$$

The number $c_0 := \min_{\{\gamma:\text{hyperbolic}\}} l(C_\gamma)$ is a positive real number, hence

$$(29) \qquad \frac{2\pi i}{(4\pi t)^{\frac{3}{2}}} \sum_{\gamma:\text{hyperbolic}} l(C_\gamma)^2 j(\gamma)^{-1} D(\gamma)^{-1} (\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)}) e^{-\frac{l(C_\gamma)^2}{4t}} \sim a e^{-c_0^2/t}$$

as $t \to 0$ for a constant a. Now (28) and (29) give

$$\operatorname{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2}) \sim a e^{-c_0^2/t} \quad \text{as} \quad t \to 0.$$

This means that $\eta_{\mathcal{D},1}(z)$ can be extended to the whole complex plane without poles.

For $\zeta_{\mathcal{D}^2,1}(z)$, we use (26) to get

$$\operatorname{Tr}(e^{-t\mathcal{D}^2} - e^{-t\mathcal{D}_0^2}) = I_\Gamma(K_t^e) + H_\Gamma(K_t^e) + U_\Gamma(K_t^e).$$

Recall that

$$H_\Gamma(K_t^e) = \frac{1}{\sqrt{4\pi t}} \sum_{\gamma:\text{hyperbolic}} l(C_\gamma) j(\gamma)^{-1} D(\gamma)^{-1} (\overline{\chi_{\sigma_+}(m_\gamma)} + \overline{\chi_{\sigma_-}(m_\gamma)}) e^{-\frac{l(C_\gamma)^2}{4t}}. \tag{30}$$

As $H_\Gamma(K_t^o)$, we have

$$H_\Gamma(K_t^e) \sim a' e^{-c_0^2/t} \quad \text{as} \quad t \to 0 \tag{31}$$

for a constant a'. We can see that $H_\Gamma(K_t^e)$ does not give any poles in the meromorphic continuation of $\zeta_{\mathcal{D}^2,1}(z)$. Next we consider $I_\Gamma(K_t^e)$ and $U_\Gamma(K_t^e)$. An elementary computation shows

$$I_\Gamma(K_t^e) = \sum_{k=0}^{n} a_k t^{-k-\frac{1}{2}} \tag{32}$$

for some constants a_k. By Corollary 3.3, we have

$$U_\Gamma(K_t^e) = \frac{2}{2\pi} \int_{-\infty}^{\infty} e^{-t\lambda^2} (P_U(\lambda) + Q(\lambda))\, d\lambda$$

where $P_U(\lambda)$ is the even polynomial of degree $2n-4$ and $Q(\lambda)$ is given by

$$-\kappa \frac{d(\sigma_+)}{2} \Big(\psi(i\lambda + \frac{1}{2}) + \psi(-i\lambda + \frac{1}{2}) \Big).$$

An elementary computation leads to

$$\int_{-\infty}^{\infty} e^{-t\lambda^2} P_U(\lambda)\, d\lambda = \sum_{k=0}^{n-2} b_k t^{-k-\frac{1}{2}} \tag{33}$$

for some constants b_k. Using the relations

$$\psi(x+1) = \frac{1}{x} + \psi(x), \qquad \psi(x) + \psi(x + \frac{1}{2}) = 2(\psi(2x) - \log 2),$$

we can reduce $Q(\lambda)$ to the sum of following terms:

$$\psi(i\lambda + 1) + \psi(-i\lambda + 1), \quad \psi(2i\lambda + 1) + \psi(-2i\lambda + 1).$$

We use the Stirling-Binet formula to deal with the terms $\psi(i\lambda + 1) + \psi(-i\lambda + 1)$

$$\mathrm{Re}(\psi(i\lambda + 1)) = \log \lambda + \sum_{k=1}^{N} \alpha_k \lambda^{-2k} + R_N(\lambda)$$

where α_k are constants and $|R_N(\lambda)| \leq M_N/\lambda^{2N+1}$ for some constant M_N and $\lambda \geq \frac{1}{2}$. This implies the following expansion

$$
\begin{aligned}
&\int_1^\infty e^{-t\lambda^2}(\psi(i\lambda+1)+\psi(-i\lambda+1))\,d\lambda \\
&\sim \sum_{k=0}^\infty c_k t^{k-\frac{1}{2}} + d_0 t^{-\frac{1}{2}}\log t \qquad \text{as} \quad t \to 0
\end{aligned} \tag{34}
$$

for some constants c_k and d_0. We deal with terms $\psi(2i\lambda+1)+\psi(-2i\lambda+1)$ in a similar way. By (31), (32), (33), (34), we have

$$
I_\Gamma(K_t^e) + H_\Gamma(K_t^e) + U_\Gamma(K_t^e) \sim \sum_{k=-n}^\infty \beta_k t^{k-\frac{1}{2}} + \beta_0' t^{-\frac{1}{2}}\log t \qquad \text{as} \quad t \to 0
$$

for constants β_k, β_0'. Therefore $\zeta_{\mathcal{D}^2,1}(z)$ is well defined for $\mathrm{Re}(z) > n+\frac{1}{2}$ and we can extend $\zeta_{\mathcal{D}^2,1}(z)$ to a meromorphic function on $\mathbb{C}$, with poles determined by

$$
\Gamma(z)\zeta_{\mathcal{D}^2,1}(z) = \sum_{k=-n}^\infty \frac{\beta_k}{z+k-\frac{1}{2}} + \frac{\beta_0'}{(z-\frac{1}{2})^2} - \frac{h}{z} + H_1(z) \tag{35}
$$

where β_k, β_0' are constants and $H_1(z)$ is a holomorphic function.

To deal with the meromorphic extensions of $\eta_{\mathcal{D},2}(z)$ and $\zeta_{\mathcal{D}^2,2}(z)$, we consider the right hand sides of the following equations

$$
\begin{aligned}
\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2}) &= \sum_{\lambda_k} \lambda_k e^{-t\lambda_k} \\
&- \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^\infty \lambda e^{-t\lambda^2}\mathrm{tr}_s(C_\Gamma(\sigma_+,-i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+,i\lambda))\,d\lambda,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}(e^{-t\mathcal{D}^2} - e^{-t\mathcal{D}_0^2}) - h &= \sum_{\lambda_k \neq 0} e^{-t\lambda_k^2} \\
&- \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^\infty e^{-t\lambda^2}\mathrm{tr}(C_\Gamma(\sigma_+,-i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+,i\lambda))\,d\lambda.
\end{aligned}
$$

The discrete eigenvalues give

$$
\sum_{\lambda_k} \lambda_k e^{-t\lambda_k^2} \sim e^{-ct}, \quad \sum_{\lambda_k \neq 0} e^{-t\lambda_k^2} \sim e^{-ct} \qquad \text{as} \ \ t \to \infty
$$

for a positive constant c. The operator $C_\Gamma(\sigma_+, i\lambda)$ is analytic along the imaginary axis and

$$\mathrm{tr}_s(C_\Gamma(\sigma_+, -i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+, i\lambda)) = -\mathrm{tr}_s(C_\Gamma(\sigma_+, i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+, -i\lambda)),$$
$$\mathrm{tr}(C_\Gamma(\sigma_+, -i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+, i\lambda)) = \mathrm{tr}(C_\Gamma(\sigma_+, i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+, -i\lambda)),$$

hence we have the following analytic expansion at $\lambda = 0$

$$\mathrm{tr}_s(C_\Gamma(\sigma_+, -i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+, i\lambda)) = \sum_{k=0}^{\infty} f_{2k+1}\lambda^{2k+1},$$
$$\mathrm{tr}(C_\Gamma(\sigma_+, -i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+, i\lambda)) = \sum_{k=0}^{\infty} g_{2k}\lambda^{2k}$$

for some constants f_{2k+1}, g_{2k}. Therefore we have

$$\int_{-1}^{1} \lambda e^{-t\lambda^2}\mathrm{tr}_s(C_\Gamma(\sigma_+, -i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+, i\lambda))\, d\lambda \quad \sim \quad \sum_{k=0}^{\infty} \gamma_k t^{-(k+\frac{3}{2})}, \tag{36}$$

$$\int_{-1}^{1} e^{-t\lambda^2}\mathrm{tr}(C_\Gamma(\sigma_+, -i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+, i\lambda))\, d\lambda \quad \sim \quad \sum_{k=0}^{\infty} \gamma'_k t^{-(k+\frac{1}{2})} \tag{37}$$

as $t \to \infty$. The equality (36) shows that $\eta_{\mathcal{D},2}(z)$ is well defined for $\mathrm{Re}(z) < 1$ and extends to the whole complex plane with the following pole structure

$$\Gamma(z + 1/2)\eta_{\mathcal{D},2}(z) = \sum_{k=0}^{\infty} \frac{-2\gamma_k}{z - 2k - 2} + K_2(z) \tag{38}$$

for the constants γ_k and a holomorphic function $K_2(z)$. The equation (37) implies that the $\zeta_{\mathcal{D}^2,2}(z)$ is well defined for $\mathrm{Re}(z) < \frac{1}{2}$ and extends to the whole complex plane with the poles determined by the equality

$$\Gamma(z)\zeta_{\mathcal{D}^2,2}(z) = \sum_{k=0}^{\infty} \frac{-\gamma'_k}{z - k - \frac{1}{2}} + H_2(z) \tag{39}$$

for the constants γ'_k and a holomorphic function $H_2(z)$.

We define the eta and zeta functions by

$$\eta_{\mathcal{D}}(z) := \eta_{\mathcal{D},1}(z) + \eta_{\mathcal{D},2}(z), \quad \zeta_{\mathcal{D}^2}(z) := \zeta_{\mathcal{D}^2,1}(z) + \zeta_{\mathcal{D}^2,2}(z).$$

The equalities (35), (38), (39) give the following theorem

Theorem 5.1. *The poles of the eta function $\eta_{\mathcal{D}}(z)$ and the zeta function $\zeta_{\mathcal{D}^2}(z)$ are determined by the following equations*

$$\Gamma(z+1/2)\eta_{\mathcal{D}}(z) = \sum_{k=0}^{\infty} \frac{-2\gamma_k}{z-2k-2} + K(z),$$
$$\Gamma(z)\zeta_{\mathcal{D}^2}(z) = \sum_{k=-n}^{\infty} \frac{\beta_k}{z+k-\frac{1}{2}} + \frac{\beta'_0}{(z-\frac{1}{2})^2} - \frac{h}{z} + \sum_{k=0}^{\infty} \frac{-\gamma'_k}{z-k-\frac{1}{2}} + H(z)$$

where $K(z)$ and $H(z)$ are holomorphic. In particular, $\eta_{\mathcal{D}}(z)$ and $\zeta_{\mathcal{D}^2}(z)$ are regular at $z=0$.

6. Eta invariants, Zeta functions of odd type and functional equations

In this Section, we study the eta invariant and its relation with the Selberg zeta function of odd type. We use the Selberg trace formula to prove a generalization of Millson's theorem in [12] for hyperbolic manifolds with cusps. We showed the vanishing of the unipotent term in our situation (see Corollary 3.3), hence we obtain the same formula as in the case of Millson in [12]. We also prove the functional equation for the eta invariant and the Selberg zeta function of odd type.

By the result in Section 5, the eta function $\eta_{\mathcal{D}}(z)$ is regular at $z=0$. In particular, this implies that the we can put $z=0$ in the following equality,

$$\eta_{\mathcal{D}}(z) \;=\; \frac{1}{\Gamma(\frac{z+1}{2})} \int_0^\infty t^{\frac{z-1}{2}} \operatorname{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2})\, dt.$$

We define the eta invariant of $\mathcal{D}$ by

$$\eta(\mathcal{D}) := \;\eta_{\mathcal{D}}(0) \;=\; \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \operatorname{Tr}(\mathcal{D}e^{-t\mathcal{D}^2} - \mathcal{D}_0 e^{-t\mathcal{D}_0^2})\, dt.$$

Let us recall that

$$\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2}-\mathcal{D}_0e^{-t\mathcal{D}_0^2}) = \frac{2\pi i}{(4\pi t)^{\frac{3}{2}}}\sum_{\gamma:\text{hyperbolic}} l(C_\gamma)^2 j(\gamma)^{-1}D(\gamma)^{-1}(\overline{\chi_{\sigma_+}(m_\gamma)}-\overline{\chi_{\sigma_-}(m_\gamma)})e^{-\frac{l(C_\gamma)^2}{4t}}.$$

Using the elementary equality $\int_0^\infty e^{-s^2t}\frac{e^{-r^2/4t}}{(4\pi t)^{\frac{3}{2}}}dt = \frac{e^{-sr}}{4\pi r}$, we have

$$\begin{aligned}
&\int_0^\infty e^{-s^2t}\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2}-\mathcal{D}_0e^{-t\mathcal{D}_0^2})\ dt\\
=&\frac{i}{2}\sum_{\gamma:\text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}D(\gamma)^{-1}(\overline{\chi_{\sigma_+}(m_\gamma)}-\overline{\chi_{\sigma_-}(m_\gamma)})e^{-sl(C_\gamma)}\\
=&\frac{i}{2}\sum_{\gamma:\text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}|\det(\mathrm{Ad}(m_\gamma a_\gamma)^{-1}-I|_{\mathfrak{n}})|^{-1}(\overline{\chi_{\sigma_+}(m_\gamma)}-\overline{\chi_{\sigma_-}(m_\gamma)})e^{-(s+n)l(C_\gamma)}.
\end{aligned}$$

We define the Selberg zeta function of odd type by

$$Z_H^o(s) := \exp(-\sum_{\gamma:\text{hyperbolic}} j(\gamma)^{-1}|\det(\mathrm{Ad}(m_\gamma a_\gamma)^{-1}-I|_{\mathfrak{n}})|^{-1}(\overline{\chi_{\sigma_+}(m_\gamma)}-\overline{\chi_{\sigma_-}(m_\gamma)})e^{-sl(C_\gamma)})$$

for $\mathrm{Re}(s)\gg 0$. We will show that $Z_H^o(s)$ has the meromorphic extension over $\mathbb{C}$ and $Z_H^0(s)$ is regular at $s=n$ in Proposition 6.2. Now we have

$$\int_0^\infty e^{-s^2t}\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2}-\mathcal{D}_0e^{-t\mathcal{D}_0^2})\ dt = \frac{i}{2}\frac{d}{ds}\log Z_H^o(s+n).$$

Following the argument in p. 27 of [12], we use the equality,

$$t^{\frac{z-1}{2}} = \frac{2}{\Gamma(\frac{1-z}{2})}\int_0^\infty s^{-z}e^{-s^2t}\ ds$$

for $Re(z)<1$. Then we have

$$\begin{aligned}
\eta_{\mathcal{D}}(z) &= \frac{2}{\Gamma(\frac{z+1}{2})\Gamma(\frac{1-z}{2})}\int_0^\infty\int_0^\infty s^{-z}e^{-s^2t}\ ds\ \mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2}-\mathcal{D}_0e^{-t\mathcal{D}_0^2})\ dt\\
&= \frac{2}{\Gamma(\frac{z+1}{2})\Gamma(\frac{1-z}{2})}\int_0^\infty s^{-z}\int_0^\infty e^{-s^2t}\mathrm{Tr}(\mathcal{D}e^{-t\mathcal{D}^2}-\mathcal{D}_0e^{-t\mathcal{D}_0^2})\ dt\ ds\\
&= \frac{i}{\Gamma(\frac{z+1}{2})\Gamma(\frac{1-z}{2})}\int_0^\infty s^{-z}\frac{d}{ds}\log Z_H^o(s+n)\ ds.
\end{aligned}$$

If we evaluate the above equality at $z=0$, we get the following theorem,

Theorem 6.1. *For the Dirac operator $\mathcal{D}$ over a $(2n+1)$-dimensional hyperbolic manifold with cusps, the eta invariant $\eta(\mathcal{D})$ and the Selberg zeta function of odd type $Z^o_H(s)$ satisfy the following equality*

$$\eta(\mathcal{D}) = \frac{1}{\pi i}\log Z^o_H(n). \tag{40}$$

Let us consider the meromorphic extension of $Z^o_H(s)$ over $\mathbb{C}$. We select a smooth odd function $g(u)$ such that $|g(u)| = 1$ if $|u| > c_0$, $g(u) = 0$ near 0 and $\int_0^\infty g'(u)du = 1$ where $c_0 = \min_{\{\gamma:\text{hyperbolic}\}} l(C_\gamma)$. We define

$$\begin{cases} H_s(\sigma_+,\lambda) := \int_{-\infty}^{\infty} g(u)e^{-s|u|}e^{i\lambda u}du, \\ H_s(\sigma_-,\lambda) := -H_s(\sigma_+,\lambda) \end{cases}$$

for a complex parameter s. By the Paley-Wiener theorem, (see the theorem 2.2 in [12]), there exists f_s over G with $\hat{f}_s(\sigma_\pm, i\lambda) = H_s(\sigma_\pm,\lambda)$, $\hat{f}_s(\sigma, i\lambda) = 0$ if $\sigma \neq \sigma_\pm$. We apply the Selberg trace formula to the one parameter family of functions f_s for $Re(s) \gg 0$ on G, then we get

$$\begin{aligned} &\sum_{\lambda_j\in\sigma_p^+} H_s(\sigma_+,\lambda_j) + \sum_{\lambda_j\in\sigma_p^-} H_s(\sigma_-,\lambda_j) \\ &\quad - \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} H_s(\sigma_+,\lambda)\mathrm{tr}_s(C_\Gamma(\sigma_+,-i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+,i\lambda))\ d\lambda \\ &= \sum_{\gamma:\text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}D(\gamma)^{-1}(\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)})e^{-sl(C_\gamma)} \\ &= \frac{d}{ds}\log Z^o_H(s+n). \end{aligned} \tag{41}$$

We use the above equality to get the meromorphic extension of

$$\mathcal{Z}(s) := \frac{d}{ds}\log Z^o_H(s+n)$$

over $\mathbb{C}$ and to investigate its poles.

Discrete eigenvalue term: The integration by part gives

$$H_s(\sigma_+,\lambda) = \pm\frac{1}{s-i\lambda}\int_0^\infty g'(u)e^{-(s-i\lambda)u}du \mp \frac{1}{s+i\lambda}\int_0^\infty g'(u)e^{-(s+i\lambda)u}du.$$

This equality provides the meromorphic extension of $H_s(\sigma_\pm, \lambda_j)$ over $\mathbb{C}$ and we see that $H_s(\sigma_\pm, \lambda_j)$ has the simple pole at $i\lambda_j$, $-i\lambda_j$ with residue $\pm m_j$, $\mp m_j$ for $\lambda_j \in \sigma_p^\pm$ where m_j is the multiplicity of λ_j.

Scattering term: We consider the scattering term

$$-\frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} H_s(\sigma_+, \lambda)\mathrm{tr}_s(C_\Gamma(\sigma_+, -i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+, i\lambda))\ d\lambda. \tag{42}$$

First we observe

$$\Phi(z) := \mathrm{tr}_s(C_\Gamma(\sigma_+, -z)\frac{d}{dz}C_\Gamma(\sigma_+, z)) \ = \ \frac{d}{dz}\bigg(\ \log\ \ det\ C_+(z) - \log\ \ det\ C_-(z)\ \bigg).$$

Using the functional equation $C_\Gamma(\sigma_+, z)C_\Gamma(\sigma_+, -z) = Id$ and the general formula for $\frac{d}{dz}\log\ \ det\ C_\Gamma(\sigma_+, z)$ given in [16], we have

$$\begin{aligned} det\ C_+(z) \ &= \ det\ C_+(0)\ p_+^z \prod_{Re(q_+)<0} \frac{z+q_-}{z-q_+}, \\ det\ C_-(z) \ &= \ det\ C_-(0)\ p_-^z \prod_{Re(q_-)<0} \frac{z+q_+}{z-q_-} \end{aligned}$$

for some constants p_+, p_-. Hence $\Phi(z)$ has the following form over $\mathbb{C}$,

$$\Phi(z) \ = \ -\sum_{Re(q_+)<0} \frac{1}{z-q_+} + \frac{1}{z+q_+} + \sum_{Re(q_-)<0} \frac{1}{z-q_-} + \frac{1}{z+q_-} + \log\ p_+ - \log\ p_-. \tag{43}$$

We observe that the infinite sum in the right hand side of (43) converges by the lemma 6.6 in [16]. Now we consider the contour integral,

$$\mathcal{L}_R := \frac{1}{4\pi i}\int_{L_R} H_s(\sigma_+, z)\ \mathrm{tr}_s(C_\Gamma(\sigma_+, -iz)\frac{d}{d(iz)}C_\Gamma(\sigma_+, iz))dz$$

where $L_R = [\ -R,\ R\ ]\ \cup \{\ Re^{i\theta}\ |\ 0 \le \theta \le \pi\ \}$. As in the proposition 3.10 of [8], we apply the Cauchy integral formula and obtain

$$\lim_{R\to\infty} \mathcal{L}_R \ = \ i\bigg(\sum_{\mathrm{Re}(q_+)<0} \frac{1}{s-q_+}\int_0^\infty g'(u)e^{-(s-q_+)u}du - \sum_{\mathrm{Re}(q_-)<0} \frac{1}{s-q_-}\int_0^\infty g'(u)e^{-(s-q_-)u}du\bigg)$$

for $Re(s) \gg 0$ and the right hand side gives us the meromorphic extension over $\mathbb{C}$. On the other hand, as in the proposition 3.10 in [8], we also show that

$$\lim_{R\to\infty}\mathcal{L}_R = \frac{1}{4\pi i}\int_{-\infty}^{\infty} H_s(\sigma_+,\lambda)\mathrm{tr}_s(C_\Gamma(\sigma_+,-i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+,i\lambda))\ d\lambda.$$

Therefore the scattering term has simple poles at $q_\pm$ for $\mathrm{Re}(q_\pm) < 0$ with residues $\pm d(\sigma_+)b_\pm$ where $b_\pm$ denotes the order of the pole of $det\ C_\Gamma(\sigma_+, z)$, $q_\pm$.

Combining the contributions of $H_s(\sigma_+,\lambda_j)$'s and the scattering term, we see that $\mathcal{Z}(s)$ has a meromorphic extension over $\mathbb{C}$ and its simple poles are located at $\pm i\lambda_j$, $\mp i\lambda_j$ for $\lambda_j \in \sigma_p^\pm$ and $q_\pm$ for $\mathrm{Re}(q_\pm) < 0$ with residues m_j, $-m_j$ and $\pm d(\sigma_+)b_\pm$. In particular, we can see that $\mathcal{Z}(s)$ is regular at $s = 0$. This implies the following proposition,

Proposition 6.2. *The Selberg zeta function of odd type $Z^o_H(s+n)$ has the meromorphic extension over $\mathbb{C}$ and is regular at $s = 0$.*

Remark 6.3. *The zeros of $Z^o_H(s+n)$ are located at $\pm i\lambda_j$ for $\lambda_j \in \sigma_p^\pm$, q_+ for Re$(q_+) < 0$ and their orders are m_j, $d(\sigma_+)b_+$. The poles of $Z^o_H(s+n)$ are located at $\mp i\lambda_j$ for $\lambda_j \in \sigma_p^\pm$, q_- for Re$(q_-) < 0$ and their orders are m_j, $d(\sigma_+)b_-$.*

Let us study the functional equation of $\eta(\mathcal{D})$ and $Z^o_H(s)$. We set

$$R(s) := \mathcal{Z}(s) - \mathcal{Z}(-s) + d(\sigma_+)\Phi(s).$$

Then $R(s)$ is an odd entire function of s. Now let $h(s)$ be an odd function which decreases sufficiently rapidly as $\mathrm{Im}(s) \to \infty$ in the strip $\{\ s \in \mathbb{C} \mid |\mathrm{Re}(s)| < n+\epsilon,\ \epsilon > 0\ \}$. Consider the contour integral,

$$\mathcal{L}_T := \frac{1}{2\pi i}\int_{L_T} h(s)\mathcal{Z}(s)ds$$

where L_T is the rectangle with the corners $a+iT, a-iT, -a+iT, -a-iT$ with $n < a < n+\epsilon$. Then we have

$$\lim_{T\to\infty}\mathcal{L}_T = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} h(s)(\mathcal{Z}(s)-\mathcal{Z}(-s))ds + 2\frac{1}{2\pi i}\int_{-a+i\infty}^{-a-i\infty} h(s)\mathcal{Z}(s)ds.$$

We apply the Cauchy integral theorem to get the equality

$$\lim_{T\to\infty}\mathcal{L}_T = 2\sum_{\lambda_j} m_j h(i\lambda_j) + \sum_{-a<\mathrm{Re}(q_k)<0} d(\sigma_+)b_k h(q_k). \tag{44}$$

Here we used the notations q_k, b_k instead of $q_\pm$, $\pm b_\pm$. By the simple pole of $\mathcal{Z}(s)$ located at q_k with residue $d(\sigma_+)b_k$ between $\mathrm{Re}(s) = -a$ and $\mathrm{Re}(s) = 0$, we have

$$\frac{1}{2\pi i}\int_{-a+i\infty}^{-a-i\infty} h(s)\mathcal{Z}(s)ds = \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} h(\lambda)\mathcal{Z}(\lambda)\ d\lambda + \sum_{-a<\mathrm{Re}(q_k)<0} d(\sigma_+)b_k h(q_k). \tag{45}$$

The simple poles at $-q_k$ of $\mathcal{Z}(-s)$ between $\mathrm{Re}(s) = a$ and $\mathrm{Re}(s) = 0$ produce the following equality

$$\begin{aligned} &\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} h(s)(\mathcal{Z}(s) - \mathcal{Z}(-s))ds \\ =&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} h(\lambda)(\mathcal{Z}(\lambda) - \mathcal{Z}(-\lambda))\ d\lambda + \sum_{0<\mathrm{Re}(-q_k)<a} d(\sigma_+)b_k h(-q_k). \end{aligned} \tag{46}$$

By (44), (45) and (46), we have

$$\begin{aligned} 2\sum_{\lambda_j} m_j h(i\lambda_j) = {}& 2\frac{1}{2\pi i}\int_{i\infty}^{-i\infty} h(\lambda)(\mathcal{Z}(\lambda) + \frac{d(\sigma_+)}{2}\Phi(\lambda))\ d\lambda \\ &+ \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} h(\lambda)(\mathcal{Z}(\lambda) - \mathcal{Z}(-\lambda) + d(\sigma_+)\Phi(\lambda))\ d\lambda. \end{aligned} \tag{47}$$

If we exchange $\lambda \to i\lambda$, we see that the first term of the right hand side in (47) is equal to

$$\begin{aligned} &2\frac{1}{2\pi}\int_{-\infty}^{\infty} h(i\lambda) \sum_{\gamma:\text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}D(\gamma)^{-1}(\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)})e^{-i\lambda l(C_\gamma)}\ d\lambda \\ &+ 2\frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} h(i\lambda)\Phi(i\lambda)\ d\lambda \\ = {}& 2\Big(\sum_{\gamma:\text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}D(\gamma)^{-1}(\overline{\chi_{\sigma_+}(m_\gamma)} - \overline{\chi_{\sigma_-}(m_\gamma)})\frac{1}{2\pi}\int_{-\infty}^{\infty} h(i\lambda)e^{-i\lambda l(C_\gamma)}d\lambda \\ &+ \frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} h(i\lambda)\Phi(i\lambda)\ d\lambda \Big). \end{aligned} \tag{48}$$

By (47), (48) and the Selberg trace formula applied to $h(i\lambda)$, we have

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} h(\lambda)R(\lambda)\ d\lambda = 0$$

for any odd holomorphic $h(s)$ satisfying suitable growth conditions. Since $R(s)$ is an entire function, this implies that $R(s) = 0$ over $\mathbb{C}$. Hence we have the equality

$$\frac{d}{ds}\log(Z^o_H(n+s)Z^o_H(n-s)) \;=\; -d(\sigma_+)\Phi(s)$$

for any $s \in \mathbb{C}$. Recalling the equalities,

$$\Phi(s) \;=\; \frac{d}{ds}\bigg(\log\; det\; C_+(s) - \log\; det\; C_-(s) \bigg),$$
$$\eta(\mathcal{D}) = \frac{1}{\pi i}\log Z^o_H(n),$$

we get the following theorem,

Theorem 6.4. *We have*

$$Z^o_H(n+s)Z^o_H(n-s) = \exp(2\pi i \eta(\mathcal{D}))\left(\frac{det\; C_+(s)C_-(0)}{det\; C_-(s)C_+(0)}\right)^{-d(\sigma_+)} \quad \textit{for} \quad s \in \mathbb{C}.$$

Remark 6.5. *In the above equality, $Z^o_H(n+s)Z^o_H(n-s)$ has the zeros at $q_+, -q_+$ of orders $d(\sigma_+)b_+$ for $Re(q_+) < 0$ and the poles at $q_-, -q_-$ of orders $d(\sigma_+)b_-$ for $Re(q_-) < 0$. These zeros and poles coincide with the zeros and poles of $(\frac{det\;\; C_+(s)C_-(0)}{det\;\; C_-(s)C_+(0)})^{-d(\sigma_+)}$.*

7. Cusp contributions for regularized determinants and functional equations

In this Section, we compute the unipotent factor in the relation between the regularized determinant and the Selberg zeta function of even type. We also prove the functional equation for the regularized determinant and the Selberg zeta function of even type where the unipotent factor plays a nontrivial role. This equation is the even type counterpart of the functional equation for the eta invariant and the Selberg zeta function of odd type proved in Section 6.

We define the zeta function with the factor e^{-ts^2} for a positive real number s by

$$\zeta_{\mathcal{D}^2}(z,s) := \frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}\;\mathrm{Tr}(e^{-t\mathcal{D}^2} - e^{-t\mathcal{D}_0^2})\; e^{-ts^2}dt.$$

Note that $\zeta_{\mathcal{D}^2}(z,0)=\zeta_{\mathcal{D}^2}(z)$ if the kernel of $\mathcal{D}^2$ is trivial. We proved that $\zeta_{\mathcal{D}^2}(z)$ is regular at $z=0$ in Section 5. Due to the factor s^2, the continuous spectrum of $\mathcal{D}^2+s^2$ has minimum s^2 and the large time part of the zeta function $\zeta_{\mathcal{D}^2}(z,s)$ does not produce any poles. Therefore $\zeta_{\mathcal{D}^2}(z,s)$ is regular at $z=0$. We define the regularized determinant of $\mathcal{D}^2+s^2$ by

$$Det(\mathcal{D}^2,s):=\exp(-\zeta'_{\mathcal{D}^2}(0,s))$$

for a positive real number s. Now we observe that $\zeta_{\mathcal{D}^2}(0,s)=0$ by the results of Section 5. Then we have

$$\frac{d}{dz}\bigg|_{z=0}\zeta_{\mathcal{D}^2}(z,s) \;=\; \int_0^\infty t^{-1}\,\mathrm{Tr}(e^{-t\mathcal{D}^2}-e^{-t\mathcal{D}_0^2})\;e^{-ts^2}dt.$$

Recall that

$$\begin{aligned}
&\mathrm{Tr}(e^{-t\mathcal{D}^2}-e^{-t\mathcal{D}_0^2}) \\
=&\sum_{\lambda_k}e^{-t\lambda_k^2}-\frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty}e^{-t\lambda^2}\mathrm{tr}(C_\Gamma(\sigma_+,-i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+,i\lambda))\;d\lambda \\
=&I_\Gamma(K_t^e)+H_\Gamma(K_t^e)+U_\Gamma(K_t^e).
\end{aligned}\tag{49}$$

We denote the Mellin transform of each term of the right side of (49) by

$$\begin{aligned}
I(z,s) &= \int_0^\infty t^{z-1}I_\Gamma(K_t^e)e^{-ts^2}dt,\\
H(z,s) &= \int_0^\infty t^{z-1}H_\Gamma(K_t^e)e^{-ts^2}dt,\\
U(z,s) &= \int_0^\infty t^{z-1}U_\Gamma(K_t^e)e^{-ts^2}dt.
\end{aligned}$$

Hence we have

$$-\log Det(\mathcal{D}^2,s) \;=\; \Big(I(z,s)+H(z,s)+U(z,s)\Big)\bigg|_{z=0}. \tag{50}$$

Note that each term $I(z,s)$, $H(z,s)$, $U(z,s)$ may not be finite at $z=0$. But the sum of these has the finite value at $z=0$ since the left hand side of (50) is finite.

We recall that the Selberg zeta function of even type,

$$Z_H^e(s):=\exp(-\sum_{\sigma\pm}\sum_{\gamma:\text{hyperbolic}}j(\gamma)^{-1}|\det(\mathrm{Ad}(m_\gamma a_\gamma)^{-1}-I|_{\mathfrak{n}})|^{-1}\overline{\chi_\sigma(m_\gamma)}e^{-sl(C_\gamma)})$$

is defined for $\mathrm{Re}(s) \gg 0$ and we will show that this has the meromorphic extension over $\mathbb{C}$. We note the following equality,

$$t^{z-1} \;=\; \frac{1}{\Gamma(1-z)} \int_0^\infty (x(x+2s))^{-z} e^{-tx(x+2s)} 2(x+s)\; dt$$

for $Re(z) < 0$. From (8), we have

$$H(z,s) \;=\; \int_0^\infty t^{z-1} \frac{1}{\sqrt{4\pi t}} \sum_{\gamma:\text{hyperbolic}} l(C_\gamma) j(\gamma)^{-1} D(\gamma)^{-1} (\overline{\chi_{\sigma_+}(m_\gamma)} + \overline{\chi_{\sigma_-}(m_\gamma)}) e^{-\frac{l(C_\gamma)^2}{4t}} e^{-ts^2} dt.$$

Using the elementary equality $\int_0^\infty \frac{1}{\sqrt{4\pi t}} \exp(-(\frac{r^2}{4t} + ts^2))dt = \frac{1}{2s} e^{-rs}$, we see that $H(z,s)$ is given by

$$H(z,s) \;=\; \frac{1}{\Gamma(1-z)} \int_0^\infty (x(x+2s))^{-z} \frac{d}{dx} \log\; Z_H^e(x+s+n)\; dx. \tag{51}$$

It follows from Corollary 3.3 that $U(z,s)$ is the Mellin transform with factor e^{-ts^2} of the following terms

$$\frac{2}{2\pi} \int_{-\infty}^\infty e^{-t\lambda^2} \Big(P_U(\lambda) - \kappa \frac{d(\sigma_\pm)}{2} (\psi(i\lambda + \frac{1}{2}) + \psi(-i\lambda + \frac{1}{2})) \Big)\; d\lambda$$

where $P_U(\lambda)$ is the even polynomial of degree $2n-4$. We repeat the method used for $H(z,s)$. We deal with the term $\psi(\pm i\lambda + \frac{1}{2})$ using the Cauchy integral formula

$$\begin{aligned}
&\frac{1}{2\pi} \int_0^\infty \int_{-\infty}^\infty e^{-t(\lambda^2 + (s+x)^2)} \psi(\pm i\lambda + \frac{1}{2})\; d\lambda\; dt \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \frac{1}{\lambda^2 + (s+x)^2} \psi(\pm i\lambda + \frac{1}{2})\; d\lambda = \frac{1}{2(s+x)} \psi(s+x+\frac{1}{2}) \\
&= \frac{1}{2(s+x)} \frac{d}{dx} \log \Gamma(s+x+\frac{1}{2}).
\end{aligned}$$

This gives us the following equality for a part of $U(z,s)$,

$$U_1(z,s) \;=\; \frac{1}{\Gamma(1-z)} \int_0^\infty (x(x+2s))^{-z} \frac{d}{dx} \log \Gamma(s+x+\frac{1}{2})^{-2\kappa d(\sigma_+)}\; dx. \tag{52}$$

We define

$$U_2(z,s) := \frac{2}{2\pi} \int_0^\infty t^{z-1} e^{-ts^2} \int_{-\infty}^\infty e^{-t\lambda^2} P_U(\lambda)\; d\lambda\; dt.$$

Then $U_2(z,s)$ is regular at $z=0$ and

$$U_2(0,s) \;=\; -\log\ \exp(2\int_0^s P_U(i\lambda)\ d\lambda) \tag{53}$$

by the lemma 3 in [7].

For $I(z,s)$, we can treat this as $U_2(z,s)$ since $p(\sigma_+,\lambda)=p(\sigma_-,\lambda)$ is an even polynomial of λ. By (50), (51), (52) and (53), we get the following theorem

Theorem 7.1. *The following equality holds for any $s\in\mathbb{C}$,*

$$Det(\mathcal{D}^2,s) \;=\; Z_H^e(s+n)\Gamma(s+\frac{1}{2})^{-2\kappa d(\sigma_+)}\exp(2\int_0^s P_I(i\lambda)+P_U(i\lambda)\ d\lambda) \tag{54}$$

where $P_I(\lambda)=2\pi Vol(\Gamma\backslash G)p(\sigma_+,\lambda)$.

Proof. The equality (54) holds for positive real numbers s, and the right hand side gives the meromorphic extension over $\mathbb{C}$ by Proposition 7.2. □

We consider an even smooth function $g(|u|)$ where $g(u)$ is the given function in Section 6. We set

$$H_s(\sigma_\pm,\lambda)=\int_{-\infty}^{\infty} g(|u|)e^{-s|u|}e^{i\lambda u}du$$

for a complex parameter s. The integration by parts gives

$$H_s(\sigma_\pm,\lambda)=\frac{1}{s-i\lambda}\int_0^\infty g'(u)e^{-(s-i\lambda)u}du+\frac{1}{s+i\lambda}\int_0^\infty g'(u)e^{-(s+i\lambda)u}du. \tag{55}$$

We apply the Selberg trace formula to the one parameter family of functions f_s for $Re(s)\gg 0$ on G with $\hat{f}_s(\sigma_\pm,i\lambda)=H_s(\sigma_\pm,\lambda)$, $\hat{f}_s(\sigma,i\lambda)=0$ if $\sigma\neq\sigma_\pm$. Then we get

$$\begin{aligned}
&\sum_{\lambda_j\in\sigma_p^+} H_s(\sigma_+,\lambda_j)+\sum_{\lambda_j\in\sigma_p^-} H_s(\sigma_-,\lambda_j)\\
&\quad-\frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} H_s(\sigma_+,\lambda)\ \mathrm{tr}(C_\Gamma(\sigma_+,-i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+,i\lambda))\ d\lambda\\
=&\sum_{\gamma:\text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}D(\gamma)^{-1}(\overline{\chi_{\sigma_+}(m_\gamma)}+\overline{\chi_{\sigma_-}(m_\gamma)})e^{-sl(C_\gamma)}\\
&\quad-\frac{\kappa d(\sigma_+)}{2\pi}\int_{-\infty}^{\infty} H_s(\sigma_+,\lambda)(\psi(i\lambda+\frac{1}{2})+\psi(-i\lambda+\frac{1}{2}))\ d\lambda\\
&\quad+\frac{2}{2\pi}\int_{-\infty}^{\infty} H_s(\sigma_+,\lambda)(P_I(\lambda)+P_U(\lambda))\ d\lambda.
\end{aligned} \tag{56}$$

We have the equality

$$\frac{d}{ds}\log Z^e_H(s+n) = \sum_{\gamma:\text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}D(\gamma)^{-1}(\overline{\chi_{\sigma_+}(m_\gamma)}+\overline{\chi_{\sigma_-}(m_\gamma)})e^{-sl(C_\gamma)},$$

and now we investigate the other terms in (56) to get the meromorphic extension of $\frac{d}{ds}\log Z^e_H(s+n)$ over $\mathbb{C}$ and to determine its poles.

Discrete eigenvalue term: Using (55) as before, we can see that $H_s(\sigma_\pm,\lambda_j)$ has the meromorphic extension over $\mathbb{C}$ and has the simple poles at $i\lambda_j$ and $-i\lambda_j$ for $\lambda_j \in \sigma_p^\pm$ with the residue m_j where m_j is the multiplicity of λ_j.

Scattering term: Now we consider the scattering term

$$-\frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} H_s(\sigma_+,\lambda)\ \mathrm{tr}(C_\Gamma(\sigma_+,-i\lambda)\frac{d}{d(i\lambda)}C_\Gamma(\sigma_+,i\lambda))\ d\lambda.$$

By the general formula given in [16], the function

$$\Psi(z) := \mathrm{tr}(C_\Gamma(\sigma_+,-z)\frac{d}{dz}C_\Gamma(\sigma_+,z))$$

has the following form over $\mathbb{C}$,

$$\Psi(z) \ = \sum_{Re(q_k)<0} -\frac{1}{z-q_k}+\frac{1}{z+q_k}+\log\ p$$

for some constant p. We repeat the method in Section 6 to prove that the scattering term has the meromorphic extension over $\mathbb{C}$ and has poles at q_k for $\mathrm{Re}(q_k) < 0$ with residues $d(\sigma_+)b_k$. As before b_k denotes the order of the pole of $det\ C_\Gamma(\sigma_+,z)$, q_k.

Identity and Unipotent term: It follows from the proposition 3.9 in [8], that we have an identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} H_s(\sigma_+,\lambda)(P_I(\lambda)+P_U(\lambda))\ d\lambda = 0.$$

We now turn our attention to other unipotent terms. As in the proposition 3.7 of [8], we use the Cauchy integral formula to get

$$\frac{\kappa d(\sigma_+)}{2\pi}\int_{-\infty}^{\infty} H_s(\sigma_+,\lambda)(\psi(i\lambda+\frac{1}{2})+\psi(-i\lambda+\frac{1}{2}))\ d\lambda \ = \ 2\kappa d(\sigma_+)\psi(s+\frac{1}{2})$$

for $Re(s) \gg 0$ and the right hand side gives us the meromorphic extension over $\mathbb{C}$ of the left hand side.

Considering the equality (56) and the above analysis of other terms, we can conclude

Proposition 7.2. *The Selberg zeta function of even type $Z^e_H(s)$ has the meromorphic extension over $\mathbb{C}$.*

Remark 7.3. *The zeros of $Z^e_H(s+n)\Gamma(s+\frac{1}{2})^{-2\kappa d(\sigma_+)}$ are located at $i\lambda_j$, $-i\lambda_j$ for $\lambda_j \in \sigma^\pm_p$ and q_k for $Re(q_k) < 0$ and their orders are m_j, $d(\sigma_+)b_k$.*

Now we prove the functional equation for $Det(\mathcal{D}^2, s)$ and $Z^e_H(s)$. First let us define

$$\mathcal{Z}(s) := \frac{d}{ds}\log Z^e_H(s+n) - 2\kappa d(\sigma_+)\psi(s+\frac{1}{2})$$

and we can see that

$$R(s) := \mathcal{Z}(s) + \mathcal{Z}(-s) + d(\sigma_+)\Psi(s)$$

is an even entire function of s. Now let $h(s)$ be an even function which decreases sufficiently rapidly as $\mathrm{Im}(s) \to \infty$ in the strip $\{\ s \in \mathbb{C} \mid |\mathrm{Re}(s)| < n+\epsilon,\ \epsilon > 0\ \}$. We follow Section 6 and consider the contour integral

$$\mathcal{L}_T := \frac{1}{2\pi i}\int_{L_T} h(s)\mathcal{Z}(s)ds$$

where L_T is the rectangle with the corners $a+iT, a-iT, -a+iT, -a-iT$ with $n < a < n+\epsilon$. Then we have

$$\lim_{T\to\infty}\mathcal{L}_T = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} h(s)(\mathcal{Z}(s)+\mathcal{Z}(-s))ds + \frac{2}{2\pi i}\int_{-a+i\infty}^{-a-i\infty} h(s)\mathcal{Z}(s)ds.$$

The Cauchy integral theorem gives

$$\lim_{T\to\infty}\mathcal{L}_T = 2\sum_{\lambda_j} m_j h(i\lambda_j) + \sum_{-a<\mathrm{Re}(q_k)<0} d(\sigma_+)b_k h(q_k). \tag{57}$$

The simple poles at q_k of $\mathcal{Z}(s)$ in the strip between $\mathrm{Re}(s) = -a$ and $\mathrm{Re}(s) = 0$ have residues $d(\sigma_+)b_k$ and give the following equality,

$$\frac{1}{2\pi i}\int_{-a+i\infty}^{-a-i\infty} h(s)\mathcal{Z}(s)ds = \frac{1}{2\pi i}\int_{i\infty}^{-i\infty} h(\lambda)\mathcal{Z}(\lambda)\ d\lambda + \sum_{-a<\mathrm{Re}(q_k)<0} d(\sigma_+)b_k h(q_k). \tag{58}$$

Similarly the simple poles at $-q_k$ of $\mathcal{Z}(-s)$ in a strip between $\mathrm{Re}(s) = a$ and $\mathrm{Re}(s) = 0$ have residues $-d(\sigma_+)b_k$ and provide us with

$$\begin{aligned}(59)\qquad &\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} h(s)(\mathcal{Z}(s)+\mathcal{Z}(-s))ds \\ =&\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} h(\lambda)(\mathcal{Z}(\lambda)+\mathcal{Z}(-\lambda))\, d\lambda + \sum_{0<\mathrm{Re}(-q_k)<a} -d(\sigma_+)b_k h(-q_k).\end{aligned}$$

We use (57), (58) and (59) to obtain

$$\begin{aligned}(60)\qquad 2\sum_{\lambda_j} m_j h(i\lambda_j) = 2\frac{1}{2\pi i}&\int_{i\infty}^{-i\infty} h(\lambda)(\mathcal{Z}(\lambda)+\frac{d(\sigma_+)}{2}\Psi(\lambda))\, d\lambda \\ &+\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} h(\lambda)(\mathcal{Z}(\lambda)+\mathcal{Z}(-\lambda)+d(\sigma_+)\Psi(\lambda))\, d\lambda.\end{aligned}$$

We replace λ by $i\lambda$ in the first term of the right hand side of the equality (60). Then this term is equal to

$$\begin{aligned}(61)\qquad &2\sum_{\gamma:\text{hyperbolic}} l(C_\gamma)j(\gamma)^{-1}D(\gamma)^{-1}(\overline{\chi_{\sigma_+}(m_\gamma)}+\overline{\chi_{\sigma_-}(m_\gamma)})\,\frac{1}{2\pi}\int_{-\infty}^{\infty} h(i\lambda)e^{-i\lambda l(C_\gamma)}\, d\lambda \\ &-2\frac{1}{2\pi}\int_{-\infty}^{\infty} h(i\lambda)(2\kappa d(\sigma_+)\psi(i\lambda+\frac{1}{2}))\, d\lambda + 2\frac{d(\sigma_+)}{4\pi}\int_{-\infty}^{\infty} h(i\lambda)\Psi(i\lambda)\, d\lambda.\end{aligned}$$

By (60), (61) and the Selberg trace formula applied to $h(i\lambda)$, we have

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} h(\lambda)(R(\lambda)+4P_I(-i\lambda)+4P_U(-i\lambda))\, d\lambda = 0$$

for any even holomorphic $h(s)$ satisfying suitable growth conditions. Since

$$R(s)+4P_I(-is)+4P_U(-is) = R(s)+4P_I(is)+4P_U(is)$$

is an entire function, we have

$$\mathcal{Z}(s)+\mathcal{Z}(-s)+4P_I(is)+4P_U(is) = -d(\sigma_+)\Psi(s)$$

over $\mathbb{C}$. Now we get the following equality

$$\frac{d}{ds}\log\Big(Z_H^e(n+s)\Gamma(s+\frac{1}{2})^{-2\kappa d(\sigma_+)}\Big) + 4P_I(is)+4P_U(is)$$

$$= \frac{d}{ds}\log\Big(Z^e_H(n-s)\Gamma(-s+\frac{1}{2})^{-2\kappa d(\sigma_+)}\Big) - d(\sigma_+)\frac{d}{ds}\log(det\ C_\Gamma(\sigma_+,s)).$$

This leads to the following formula,

$$\begin{aligned} &Z^e_H(n+s)\Gamma(s+\frac{1}{2})^{-2\kappa d(\sigma_+)}\exp(4\int_0^s P_I(i\lambda)+P_U(i\lambda)\ d\lambda) \\ &= (det\ C_\Gamma(\sigma_+,s))^{-d(\sigma_+)}\ Z^e_H(n-s)\Gamma(-s+\frac{1}{2})^{-2\kappa d(\sigma_+)}. \end{aligned} \tag{62}$$

This equality holds a priori up to constant C. As in the proof of the lemma 4.3 in [8], we multiply s^{-2m_0} to both sides of (62) where m_0 is the multiplicity of the zero eigenvalue of $\mathcal{D}$. Then this remove the spectral zero of $Z^e_H(n\pm s)$ at $s=0$. If we compare the remaining parts $s=0$, we can see that such a constant C should be equal to 1. From (54), we have

$$Det(\mathcal{D}^2,s)^2\ = Z^e_H(s+n)^2\Gamma(s+\frac{1}{2})^{-4\kappa d(\sigma_+)}\exp(4\int_0^s P_I(i\lambda)+P_U(i\lambda)\ d\lambda).$$

Finally we combine (62) and this equality, then we have

Theorem 7.4. *For any $s\in\mathbb{C}$, we have*

$$Det(\mathcal{D}^2,s)^2\ =\ (det\ C_\Gamma(\sigma_+,s))^{-d(\sigma_+)}\ Z^e_H(n+s)Z^e_H(n-s)\Big(\Gamma(s+\frac{1}{2})\Gamma(-s+\frac{1}{2})\Big)^{-2\kappa d(\sigma_+)}. \tag{63}$$

Remark 7.5. *The right hand side of (63) has the zeros at $i\lambda_j$, $-i\lambda_j$ of order $2m_j$ for $\lambda_j\in\sigma_p^{\pm}$ and at q_k of order $2d(\sigma_+)b_k$ for $Re(q_k)<0$. These are the zeros of $Det(\mathcal{D}^2,s)^2$ as we expected.*

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