Milne and Torus Universes Meet

A. Waldron

Department of Mathematics, University of California,
Davis, CA 95616, USA
wally@math.ucdavis.edu

Invited contribution to Deserfest: A celebration of the life and works of Stanley Deser

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Abstract

Three dimensional quantum gravity with torus universe, $T^2 \times \mathbb{R}$, topology is reformulated as the motion of a relativistic point particle moving in an $Sl(2,\mathbb{Z})$ orbifold of flat Minkowski spacetime. The latter is precisely the three dimensional Milne Universe studied recently by Russo as a background for Strings. We comment briefly on the dynamics and quantization of the model.

Stanley has made many fundamental contributions to our understanding of life in various dimensions. Notable amongst these is dimension three. Therefore, on the occasion of his seventy third birthday, I present him a reformulation of three dimensional gravity in terms of a point particle moving in a flat three dimensional spacetime.

1 Introduction

The limited tractability of quantum gravity means that minisuperspace reductions to quantum mechanical degrees of freedom are an important calculational tool. Moreover in three dimensions, where gravitons carry no field theoretic degrees of freedom, a minisuperspace reduction might be expected to faithfully represent the full theory. Indeed, at least for toroidal spatial topologies[1], this is the case[1]: on any given toroidal spatial slice, a Weyl transformation brings the 2-metric to a flat one. The space of conformally flat metrics on the torus is parameterized by the coset $Sl(2,\mathbb{R})/SO(2)$. Furthermore, a peculiarity of three dimensions is that the space dependence of this Weyl transformation can be gauged away. Therefore, as observed long ago by Martinec[1], all that remains is the quantum mechanical evolution of metric moduli in the space $\mathbb{R}_+ \times Sl(2,\mathbb{R})/SO(2)$. In our previous work[3] we made the additional observation that this system was in fact a relativistic particle with mass proportional to the cosmological constant. In addition we showed that this model enjoys a conformal (spectrum generating) symmetry and falls into a large class of novel conformal quantum mechanical models. In this short note we show that for the case of three dimensional spatially toroidal gravity, the metric moduli space is an $Sl(2,\mathbb{Z})$ orbifold of flat three dimensional Minkowski spacetime. The derivation of this model is given in Section 2. Dynamics and solutions are discussed in Section 4 while quantization and a discussion of possible physical computations may be found in Section 5.

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[1] Modulo questions of inequivalent quantizations discussed by Carlip[2].

celebration of Stanley's wide ranging scientific achievements which have guided not only the direction but style of modern theoretical physics.

2 Minisuperspace

Our model is obtained by rewriting three dimensional cosmological gravity in the limit where all spatial derivatives are discarded, so we parameterize the metric as

$$ds^2 = -N(t)^2 dt^2 + h_{ij}(t) dx^i dx^j \,, \tag{1}$$

where $i, j = 1, 2$. We study a toroidal topology $T^2 \times \mathbb{R}$ with $0 \le x^i < 1$. The extrinsic curvature is simply

$$K_{ij} = -\frac{1}{2N}\, \dot{h}_{ij} \,, \tag{2}$$

which implies $K \equiv h^{ij} K_{ij} = -\frac{1}{2N} \frac{d}{dt} \log h$ with $h \equiv \det h_{ij}$. The Einstein Hilbert action now reads

$$S = \int dt\ d^2x \sqrt{h}\ N\ \Big[- K^2 + K_{ij} K^{ij} - 2\Lambda \Big] \,. \tag{3}$$

Extracting the spatial volume

$$h_{ij} = h^{1/2}\ \widehat{h}_{ij} \,, \tag{4}$$

the unit volume metric $\widehat{h}_{ij}$ is parameterized by the coset $Sl(2,\mathbb{R})/SO(2)$. Requiring, in addition, that spatial sections are toroidal, means that we must identify metrics related by the left action of $Sl(2,\mathbb{Z})$. The resulting $Sl(2,\mathbb{Z})\backslash Sl(2,\mathbb{R})/SO(2)$ orbifold is just the usual upper half plane modded out by modular transformations. Introducing coset coordinates U^M $(M = 1, 2)$ and invariant metric

$$\dot{U}^M G_{MN} \dot{U}^N \equiv -\frac{1}{2}\ \dot{\widehat{h}}_{ij}\ \dot{\widehat{h}}^{ij} \tag{5}$$

as well as the field redefinition

$$\eta \equiv RN \,, \qquad R \equiv h^{1/2} \,, \tag{6}$$

yields

$$S = \frac{1}{2} \int dt\ \Big\{ \frac{1}{\eta} \Big[- \dot{R}^2 + R^2 \dot{U}^M G_{MN} \dot{U}^N \Big] - 4\Lambda\eta \Big\} \,. \tag{7}$$

This is the action of a relativistic particle in three dimensions, $(\text{mass})^2 = 4\Lambda$ (tachyonic in AdS!) with metric

$$ds^2_{\mathfrak{M}} = -dR^2 + R^2\ dU^M G_{MN} dU^N \,. \tag{8}$$

Hence, the time evolution of the metric moduli U^M and R in this minisuperspace truncation is described by the dynamics of a fictitious particle moving in a three dimensional Lorentzian spacetime which we will denote $\mathfrak{M}$.

3 Geometry of the Metric Moduli Space

To understand the space $\mathfrak{M}$ better, write the unit volume spatial metric in terms of a zweibein

$$\widehat{h}_{ij} = (ee^t)_{ij} \tag{9}$$

where the $Sl(2)$ valued zweibein e has Iwasawa decomposition

$$e = \begin{pmatrix} 1 & \tau_1 \\ & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\tau_2} & \\ & 1/\sqrt{\tau_2} \end{pmatrix} \tag{10}$$

so that

$$ds^2_{\mathfrak{M}} = -dR^2 + R^2\, \frac{d\tau_1^2 + d\tau_2^2}{\tau_2^2}\,. \tag{11}$$

This space is the three dimensional Milne Universe. Here $-\infty < \tau_1 < \infty$, $0 < \tau_2$ and $0 < R$. Now change coordinates

$$R = \sqrt{Z^2 - X^2 - Y^2}\,, \qquad \tau_1 = \frac{Y}{Z - X}\,, \qquad \tau_2 = \frac{R}{Z - X}\,. \tag{12}$$

The metric becomes the flat three dimensional Minkowski one

$$ds^2_{\mathfrak{M}} = -dZ^2 + dX^2 + dY^2 \tag{13}$$

and our fictitious particle action is simply

$$S = \frac{1}{2} \int dt\, \Big\{ \frac{1}{\eta} \Big[-\dot{Z}^2 + \dot{X}^2 + \dot{Y}^2 \Big] - 4\Lambda\eta \Big\}\,. \tag{14}$$

To complete the discussion, we still need to identify the topology of the space $\mathfrak{M}$. Firstly note that the inverse of the above coordinate transformation is

$$X = \frac{R}{2}\Big(\frac{\tau\overline{\tau}}{\tau_2} - \frac{1}{\tau_2}\Big), \quad Y = R\,\frac{\tau_1}{\tau_2}, \quad Z = \frac{R}{2}\Big(\frac{\tau\overline{\tau}}{\tau_2} + \frac{1}{\tau_2}\Big)\,. \tag{15}$$

(The second solution with an overall $-$ is ruled out by positivity of R and τ_2 which requires $Z > X$.) Surfaces of constant R are hyperboloids in the forward lightcone (since $Z > X$) isomorphic to the upper half plane $\mathbb{H}$. This is simply Lobacevskii's *ur* non-Euclidean geometry depicted in Figure 1. The boundary $\partial\mathbb{H}$ at $\tau_2 = 0$ maps to the boundary of each hyperboloid while the cusp at $\tau = i\infty$

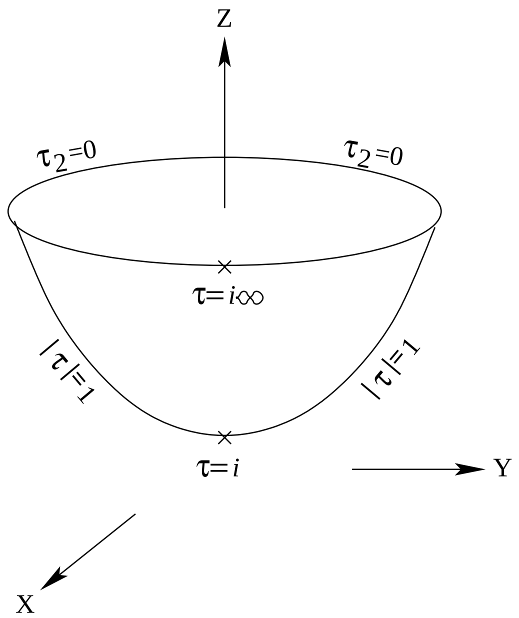

Figure 1: Lobacevskii's non-Euclidean plane realized as a hyperboloid in the forward lightcone.

maps to $(X,Y,Z)=(\infty,0,\infty)$. The unit circle $|\tau|=1$ corresponds to the line $\{X=0\}\cap\{R^2=Z^2-X^2-Y^2 \mid Z>0\}$. Therefore, before modding out by modular transformations, the metric moduli space is simply the interior of the forward lightcone in three dimensions with usual flat Minkowski metric.

Now we must mod out the upper half plane $PSl(2,\mathbb{R})/SO(2)=\mathbb{H}$ valued torus moduli U^M by the left action $PSl(2,\mathbb{Z})$ which acts independently on each hyperboloid. This group is a discrete subgroup of the three dimensional Lorentz group generated by isometries $T:\tau\mapsto\tau+1$ and $S:\tau\mapsto -1/\tau$. In terms of the new variables (X,Y,Z) these isometries leave R invariant and act as

$$\begin{aligned} T:(X,Y,Z) &\mapsto \frac{1}{2}\Big(X+2Y+Z,-2[X-Y-Z],-X+2Y+3Z\Big)\,,\\ S:(X,Y,Z) &\mapsto (-X,-Y,Z)\,. \end{aligned} \tag{16}$$

It is easy to verify that an infinitesimal transformation $\tau\to\tau+t$ is generated by

$$\frac{\partial}{\partial\tau_1}=Y\partial_X-X\partial_Y+Y\partial_Z+Z\partial_Y=M_{YX}+M_{YZ}\,, \tag{17}$$

the sum of a rotation in the (X,Y)-plane and an (Y,Z)-boost (or better, a lightcone boost). The resulting orbifold is precisely the three dimensional Milne universe studied by Russo[4] in a String theoretic context.

Finally, note that the generators of the $Sl(2,\mathbb{R})$ isometry subgroup are

$$\begin{aligned} e_+=\partial_{\tau_1}=M_{YZ}+M_{YX}\,,\quad h=2\tau_1\partial_{\tau_1}+2\tau_2\partial_{\tau_2}=-2M_{XZ}\,,\\ e_-=-\tau_1(\tau_1\partial_{\tau_1}+2\tau_2\partial_{\tau_2})+\tau_2^2\partial_{\tau_1}=M_{YZ}-M_{YX}\,. \end{aligned} \tag{18}$$

Hence, we may also view this subgroup as the $SO(2,1)$ Lorentz group in the natural way.

4 Dynamics

The time coordinate t in the relativistic particle model (14) plays no preferred *rôle*, since the the "einbein" η ensures reparameterization invariance. Instead classically, we may only predict trajectories in the space $\mathfrak{M}$. The beauty of this model is that for a free relativistic particle these are simply straight lines with slopes subject to Einstein's relativistic dispersion relation.

Explicitly, in the gauge $\eta=1$, straight line geodesics are

$$Z=Z_0+P_Zt\,,\quad X=X_0+P_Xt\,,\quad Y=Y_0+P_Yt\,, \tag{19}$$

subject to a mass-shell condition

$$-P_Z^2 + P_X^2 + P_Y^2 = 4\Lambda\,. \tag{20}$$

To convert these to metric solutions, note that the two-metric takes the compact form

$$\begin{pmatrix} h_{ij} \end{pmatrix} = \begin{pmatrix} Z+X & Y \\ Y & Z-X \end{pmatrix}\,. \tag{21}$$

Some fundamental solutions include:

Kasner

Setting $Y(t) = 0$ and $\Lambda = 0$ the mass shell constraint becomes

$$(\dot{Z} + \dot{X})(\dot{Z} - \dot{X}) = 0\,. \tag{22}$$

The Kasner solution to this constraint is $Z = t + \frac{1}{2}$, $X = t - \frac{1}{2}$ which yields the metric $ds^2 = -\frac{1}{2t}dt^2 + 2t(dx^1)^2 + (dx^2)^2$. The standard Kasner metric is obtained by changing time coordinates to "cosmological time" $\tau \equiv R = \sqrt{Z^2 - X^2}$ so that

$$ds^2 = -d\tau^2 + (\tau dx^1)^2 + (dx^2)^2\,, \tag{23}$$

which amounts to the gauge $\eta = \tau$ in the relativistic particle model.

de Sitter

Reintroducing the cosmological constant alias the relativistic mass $2\sqrt{\Lambda}$ we solve the mass shell constraint via $Z = 2\sqrt{\Lambda}t$, $X = Y = 0$ yielding metric $ds^2 = -\frac{1}{4\Lambda t^2}dt^2 + 2\sqrt{\Lambda}t\ [(dx^1)^2 + (dx^2)^2]$. Changing time coordinates $\tau = \tau(t)$ to the gauge $\eta = \exp\left(2\sqrt{\Lambda}\tau\right)$ yields the steady state de Sitter metric

$$ds^2 = -d\tau^2 + e^{2\sqrt{\Lambda}\tau}[(dx^1)^2 + (dx^2)^2]\,. \tag{24}$$

Anti de Sitter

We can also consider tachyonic trajectories corresponding to negative cosmological constant, $X = 2\sqrt{|\Lambda|}t$, $Y = 0$ and $Z = Z_0$ (say). This yields a novel Anti de Sitter metric

$$ds^2 = -\frac{dt^2}{Z_0^2 + 4\Lambda t^2} + (Z_0 + 2\sqrt{|\Lambda|}t)(dx^1)^2 + (Z_0 - 2\sqrt{|\Lambda|}t)(dx^2)^2\,. \tag{25}$$

This metric becomes singular at $t = \pm Z_0/(2\sqrt{|\Lambda|})$ at which points the volume of spatial slices $R = 0$. Therefore, it represents only a coordinate patch of Anti de Sitter space. It would be interesting to study the compatibility of the $\mathbb{Z}^2$ torus orbifold on the spatial coordinates (x^1, x^2) with the geodesic completion of the above metric.

5 Quantization and Conclusions

We have presented a simple three dimensional relativistic particle model that describes toroidal gravity in three dimensions. The model would be trivially solvable if it were not for the $Sl(2,\mathbb{Z})$ orbifold of the flat particle background necessary to identify equivalent gravity metrics. The Hilbert space of physical states in this model is dictated by the Hamiltonian constraint

$$\Big\{ -\partial_Z^2 + \partial_X^2 + \partial_Y^2 + 4\Lambda \Big\} \Psi = 0 \,. \tag{26}$$

Here we have a chosen a particular quantization corresponding to the natural operator ordering stated. An initial investigation of this Klein-Gordon equation has been conducted in[4], the key difficulty being to automorphize with respect to $Sl(2,\mathbb{Z})$. A more detailed study will appear[5].

Finally, an old observation[6] is that Hamiltonian constraints taking this Klein-Gordon form naturally imply a second quantization of the theory, jocularly dubbed "third quantization". A natural candidate for interactions would be ϕ^3 theory in three dimensions. Automorphic particle scattering amplitudes would then amount to probabilities of cobordisms with torus boundary in three dimensional quantum gravity.

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