CLASSIFICATION OF STABLE HOMOTOPY TYPES WITH TORSION FREE HOMOLOGY

Hans-Joachim Baues and Yuri Drozd

The classification of homotopy types of finite polyhedra is a classical and fundamental task of topology which in particular is an inevitable step for the classification of manifolds. There are only a few explicit results on this problem in the literature; see [W1], [CH], [BH], [H], [B1], [B2], [BD1], [BD2]. A best possible solution is the classification by a complete list of indecomposable homotopy types as in the following surprising result.

Theorem A. *Let $n \geq 6$. There is a minimal list $X(\mathcal{L}_5)$ of 451 polyhedra such that each $(n-1)$-connected $(n+5)$-dimensional polyhedron X with finitely generated torsion free homology admits a homotopy equivalence*

$$X \simeq X_1 \vee X_2 \vee \ldots \vee X_t$$

with $X_i \in X(\mathcal{L}_5)$ for $1 \leq i \leq t$. Here $X_1 \vee \ldots \vee X_t$ denotes the one point union of the spaces X_i.

We describe the elementary polyhedra in the list $X(\mathcal{L}_5)$ explicitly in § 1. They turn out to be CW-complexes with at most 6 non-trivial cells.

Let $\mathbf{F}^k_n$ be the homotopy category of $(n-1)$-connected $(n+k)$-dimensional polyhedra or CW-complexes with finitely generated torsion free homology. In the stable range $n \geq k+2$ the category $\mathbf{F}^k = \mathbf{F}^k_n$ is an additive category which does not depend on n. Recall that the *isomorphism class group* $K_0(\mathbf{F}^k)$ is generated by the homotopy types $\{X\}$ in $\mathbf{F}^k$ with the relations

$$\{X\} + \{Y\} = \{X \vee Y\}$$

It is a remarkable result of Freyd [F] that $K_0(\mathbf{F}^k)$ is actually a countably generated free abelian group for all $k \geq 0$. It is well known that $\mathbf{F}^0$ is equivalent to the category of finitely generated free abelian groups so that $K_0(\mathbf{F}^0) = \mathbb{Z}$. We compute $K_0(\mathbf{F}^k)$ for all $k \geq 0$ and we compute the number $I(\mathbf{F}^k) \leq \infty$ of indecomposable homotopy types in $\mathbf{F}^k$ for all $k \geq 0$ as follows:

Theorem B.

k	0	1	2	3	4	5	$k \geq 6$
$K_0(\mathbf{F}^k)$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^7$	$\mathbb{Z}^{29}$	$\mathbb{Z}^{87}$	$\mathbb{Z}^\infty$
$I(\mathbf{F}^k)$	1	2	4	7	67	451	∞

Typeset by $\mathcal{AMS}$-TEX

Hence $k = 5$ is the maximal dimension k for which $K_0(\mathbf{F}^k)$ is finitely generated. We describe a list of generators of $K_0(\mathbf{F}^5)$ in (2.10) below. Theorem A, B and [BD2] imply the following result concerning the representation type of the additive category $\mathbf{F}^k$.

Theorem C. *The category $\mathbf{F}^k$ has finite representation type if and only if $k \leq 5$. Moreover for $k \geq 10$ the category $\mathbf{F}^k$ has wild representation type.*

Hence only the representation type of $\mathbf{F}^6, \mathbf{F}^7, \mathbf{F}^8, \mathbf{F}^9$ are unknown. The computation of the representation type of $\mathbf{F}^6$ involves a matrix problem given by a matrix with 328 rows and columns.

§ 1 The list $X(\mathcal{L})$ of elementary polyhedra

We need the following elements in stable homotopy groups of spheres, compare Toda [T]. Let $\eta_n = \eta \in \pi_{n+1}(S^n) = \mathbb{Z}/2$ be the Hopf map, and let $\eta_n^2 = \eta\eta \in \pi_{n+2}(S^n) = \mathbb{Z}/2$ be the double Hopf map, $n \geq 3$. Moreover let $\nu = \nu_n, \alpha = \alpha_n \in \pi_{n+3}(S^n) = \mathbb{Z}/24$ be the generator of order 8 and 3 respectively.

(1.1) Definition. We define a list $\mathcal{L}_5$ of 451 elements as follows. The *spherical elements* $S^0, S^1, S^2, S^3, S^4, S^5$ belong to $\mathcal{L}_5$ and the *Hopf elements* $\eta_0, \eta_1, \eta_2, \eta_3$ and $(\eta\eta)_0, (\eta\eta)_1, (\eta\eta)_2$ belong to $\mathcal{L}_5$. Moreover the following *words* consisting of letters η, v_0, w_1 belong to $\mathcal{L}_5$. Here v_0 and w_1 are numbers with an index in $\{0,1\}$; for example for $v = 3$ the letter v_0 is 3_0 and for $w = 6$ the letter w_1 is 6_1. The words are

$$\text{(1)} \qquad \eta\, v_0\, \eta, \quad \eta\, w_1\, \eta \quad \text{with} \quad v, w \in \{1, 2, 3\}, \quad \text{and}$$

$$\text{(2)} \qquad v_0, \quad w_1 \quad \text{with} \quad v, w \in \{1, \ldots, 12\},$$

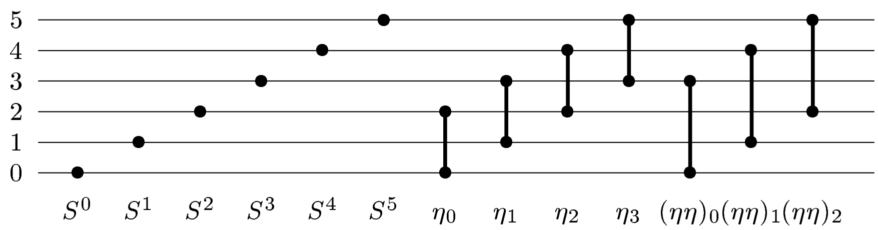
and for $v, w \in \{1, \ldots, 6\}$ all subwords of

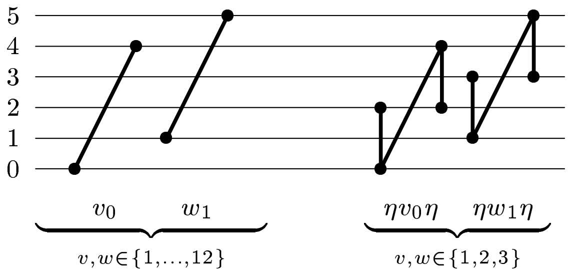
$$\text{(3)} \qquad \eta\eta\, v_0\, \eta\eta\, w_1\, \eta\eta$$

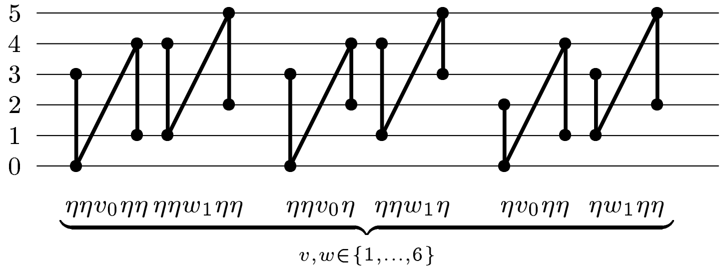
which are not of the form (1) or (2) or η or $\eta\eta$. Here a *subword* of $a_0 a_1 \ldots a_t$ is a connected subsequence $a_n a_{n+1} \ldots a_{n+k}$ with $0 \leq n \leq n + k \leq t$. We call a subword of (3) containing both letters v_0 and w_1 a *5-dimensional word.*

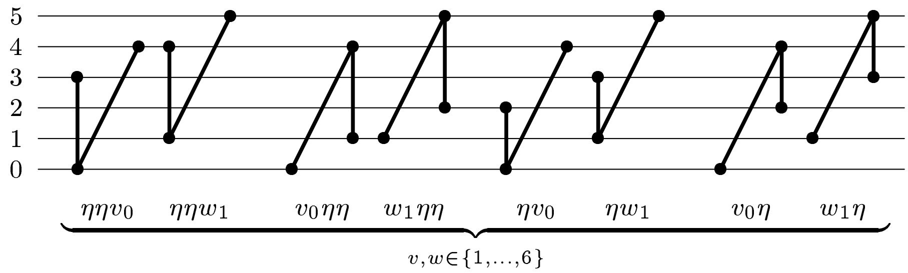
We can visualize the elements of $\mathcal{L}_5$ by *graphs* as follows. First we describe spherical elements and Hopf elements by points and vertical edges respectively. Then we describe the words in $\mathcal{L}_5$ by graphs consisting of such vertical edges and diagonal edges which represent the letters v_0 and w_1 respectively. This way we identify the elements of $\mathcal{L}_5$ with the following graphs.

Elementary polyhedra from dimension 4:

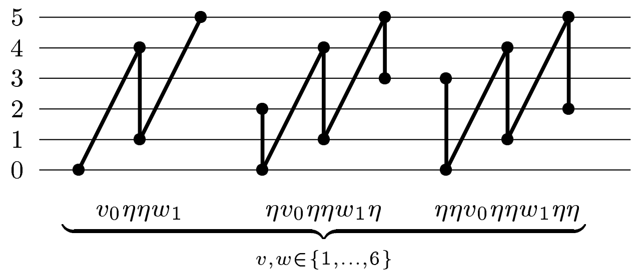


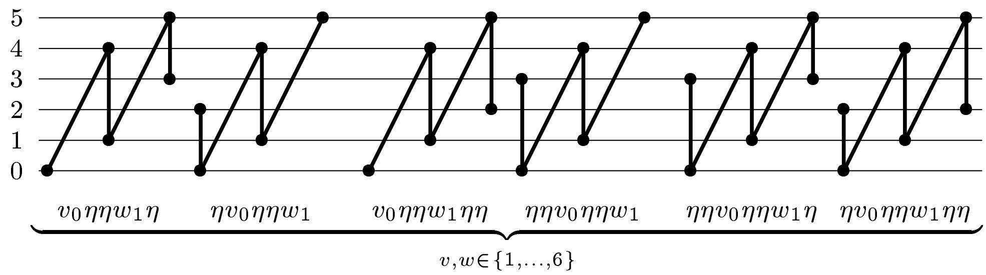






New elementary polyhedra of dimension 5:

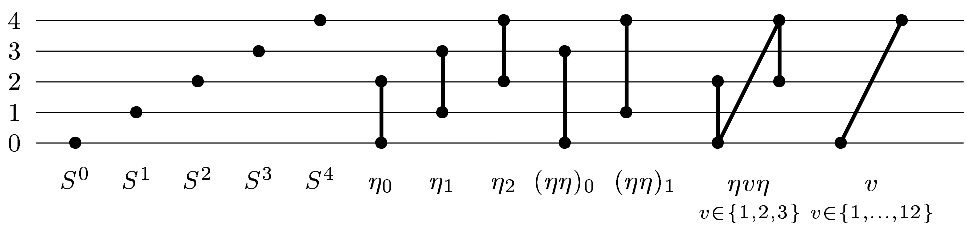


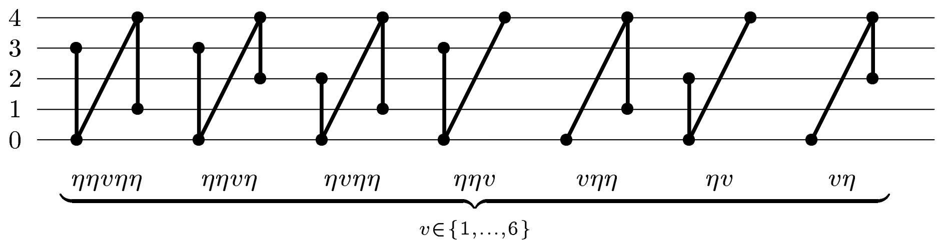


Only the 5-dimensional words correspond to graphs which have vertices in level 0 and level 5. Let $\mathcal{L}_4$ be the subset of all graphs in $\mathcal{L}_5$ which have no vertex of level

5. Then $\mathcal{L}_4$ coincides with the following list of 67 elements. (The list $\mathcal{L}_4 = \mathcal{L}$ was already achieved in our previous paper [BD2] and is needed in the proof below).

Elementary polyhedra of dimension 4:





Given an element g in $\mathcal{L}_4$ we obtain the *suspension* $\Sigma g \in \mathcal{L}_5$ by shifting all levels of g by $+1$; for example $\Sigma(\eta v_0 \eta\eta) = \eta v_1 \eta\eta$. Hence the set $\mathcal{L}_5$ is the union of the subsets $\mathcal{L}_4, \Sigma\mathcal{L}_4$ and the set of 5-dimensional words. The intersection $\mathcal{L}_4 \cap \Sigma\mathcal{L}_4$ contains only spherical elements and Hopf elements.

(1.2) Definition. The *duality operator* D on $\mathcal{L}_5$ is the function

$$D : \mathcal{L}_5 \to \mathcal{L}_5$$

with $DD =$ identity defined as follows:

$$\begin{aligned} D(S^i) &= S^{5-i} && \text{for} \quad i \in \{0, \ldots, 5\} \\ D(\eta_i) &= \eta_{3-i} && \text{for} \quad i \in \{0,1,2,3\} \\ D(\eta\eta)_i &= (\eta\eta)_{2-i} && \text{for} \quad i \in \{0,1,2\} \end{aligned}$$

Moreover for a word g in $\mathcal{L}$ let $D(g)$ be obtained by reversing the order of g and by replacing v_0 by v_1 and w_1 by w_0; that is, for example $D(\eta\eta\, v_0\, \eta\eta\, w_1) = w_0\, \eta\eta\, v_1\, \eta\eta$ with $w, v \in \{1, \ldots, 6\}$. Hence if we look at the graph g then the graph $D(g)$ is obtained by turning g around.

(1.3) Definition. Let $n \geq 6$. We associate with each element $g \in \mathcal{L}_5$ a CW-complex $X(g)$. The vertices of the graph given by g correspond exactly to the non-trivial cells of $X(g)$. We call $X(\mathcal{L}_5) = \{(X(g), g \in \mathcal{L}_5\}$ the list of elementary polyhedra associated to $\mathcal{L}_5$. For the spherical elements we set

$$X(S^i) = S^{n+i} \quad \text{with} \quad i \in \{0,1,2,3,4,5\}$$

where S^{n+i} is the $(n+i)$-sphere. For the Hopf elements we obtain the following 2-cell complexes where Σ denotes the suspension and η the Hopf map

$$X(\eta_i) = \Sigma^i(S^n \cup_\eta e^{n+2}) \quad \text{for} \quad i \in \{0,1,2,3\}$$
$$X(\eta\eta)_i = \Sigma^i(S^n \cup_{\eta\eta} e^{n+3}) \quad \text{for} \quad i \in \{0,1,2\}$$

Moreover for the words g in $\mathcal{L}_4$ we get the following CW-complexes with attaching maps corresponding to the edges of the graph g. In the following definitions the attaching map v at the right hand side is obtained by identifying the number v with the $v \cdot (v + \alpha) \in \pi_{n+3}(S^n)$ where ν and α are the generators described above. Moreover i_k denotes the inclusion of S^{n+k} into a wedge of spheres with $k \in \{0,1,2,3\}$.

$$v \in \{1,2,3\}, \qquad X(\eta v \eta) = S^n \vee S^{n+2} \cup_{i_0\eta} e^{n+2} \cup_{i_0 v + i_2\eta} e^{n+4}$$

$$v \in \{1,\dots,6\}, \begin{cases} X(\eta\eta v \eta\eta) = S^n \vee S^{n+1} \cup_{i_0\eta\eta} e^{n+3} \cup_{i_0 v + i_1\eta\eta} e^{n+4} \\ X(\eta\eta v \eta) = S^n \vee S^{n+2} \cup_{i_0\eta\eta} e^{n+3} \cup_{i_0 v + i_2\eta} e^{n+4} \\ X(\eta v \eta\eta) = S^n \vee S^{n+1} \cup_{i_0\eta} e^{n+2} \cup_{i_0 v + i_1\eta\eta} e^{n+4} \\ X(\eta\eta v) = S^n \cup_{\eta\eta} e^{n+3} \cup_v e^{n+4} \\ X(v\eta\eta) = S^n \vee S^{n+1} \cup_{i_0 v + i_1\eta\eta} e^{n+4} \\ X(\eta v) = S^n \cup_\eta e^{n+2} \cup_v e^{n+4} \\ X(v\eta) = S^n \vee S^{n+2} \cup_{i_0 v + i_2\eta} e^{n+4} \end{cases}$$

$$v \in \{1,\dots,12\}, \qquad X(v) = S^n \cup_v e^{n+4}$$

Moreover for elements $\Sigma g \in \Sigma\mathcal{L}_4$, we define

$$X(\Sigma g) = \Sigma X(g)$$

Finally we define for the 5-dimensional words g in $\mathcal{L}_5$ the following CW-complexes with $v, w \in \{0,\dots,6\}$. Here we identify v with $v(\nu + \alpha) \in \pi_{n+3}S^n$ and w with $w(\nu + \alpha) \in \pi_{n+4}S^{n+1}$.

$$\begin{aligned}
X(v_0\eta\eta w_1) &= S^n \vee S^{n+1} \cup_{i_0 v + i_1\eta\eta} e^{n+4} \cup_{i_1 w} e^{n+5} \\
X(\eta v_0\eta\eta w_1\eta) &= S^n \vee S^{n+1} \vee S^{n+3} \cup_{i_0\eta} e^{n+2} \cup_{i_0 v + i_1\eta\eta} e^{n+4} \cup_{i_1 w + i_3\eta} e^{n+5} \\
X(\eta\eta v_0\eta\eta w_1\eta\eta) &= S^n \vee S^{n+1} \vee S^{n+2} \cup_{i_0\eta\eta} e^{n+3} \cup_{i_0 v + i_1\eta\eta} e^{n+4} \cup_{i_1 w + i_2\eta\eta} e^{n+5} \\
X(v_0\eta\eta w_1\eta) &= S^n \vee S^{n+1} \vee S^{n+3} \cup_{i_0 v + i_1\eta\eta} e^{n+4} \cup_{i_1 w + i_3\eta} e^{n+5} \\
X(\eta v_0\eta\eta w_1) &= S^n \vee S^{n+1} \cup_{i_0\eta} e^{n+2} \cup_{i_0 v + i_1\eta\eta} e^{n+4} \cup_{i_1 w} e^{n+5} \\
X(v_0\eta\eta w_1\eta\eta) &= S^n \vee S^{n+1} \vee S^{n+2} \cup_{i_0 v + i_1\eta\eta} e^{n+4} \cup_{i_1 w + i_2\eta\eta} e^{n+5} \\
X(\eta\eta v_0\eta\eta w_1) &= S^n \vee S^{n+1} \cup_{i_0\eta\eta} e^{n+3} \cup_{i_0 v + i_1\eta\eta} e^{n+4} \cup_{i_1 w} e^{n+5} \\
X(\eta\eta v_0\eta\eta w_1\eta) &= S^n \vee S^{n+1} \vee S^{n+3} \cup_{i_0\eta\eta} e^{n+3} \cup_{i_0 v + i_1\eta\eta} e^{n+4} \cup_{i_1 w + i_3\eta} e^{n+5} \\
X(\eta v_0\eta\eta w_1\eta\eta) &= S^n \vee S^{n+1} \vee S^{n+2} \cup_{i_0\eta} e^{n+2} \cup_{i_0 v + i_1\eta\eta} e^{n+4} \cup_{i_1 w + i_2\eta\eta} e^{n+5}
\end{aligned}$$

The cells of the CW-complexes $X(w)$ with $w \in \mathcal{L}_5$ correspond exactly to the vertices of the graph w above and the edges in the graph w show precisely how these cells are attached.

§ 2 Decomposition and congruence of spaces with torsion free homology

Let $\mathbf{C}$ be an additive category with zero object $*$ and biproducts $A \oplus B$. An object X in $\mathbf{C}$ is decomposable if there exists an isomorphism $X \cong A \oplus B$ where A and B are not isomorphic to $*$. A decomposition of X is an isomorphism

$$(2.1) \qquad X = A_1 \oplus \ldots \oplus A_n, n < \infty,$$

where A_i is indecomposable for all $i \in \{1, \ldots, n\}$. The decomposition of X is unique if $B_1 \oplus \ldots \oplus B_m \cong X \cong A_1 \oplus \ldots A_n$ implies that $m = n$ and that there is a permutation σ with $B_{\sigma(i)} \cong A_i$. The decomposition problem in $\mathbf{C}$ can be described by the following task: find a complete list of indecomposable isomorphism types in $\mathbf{C}$ and describe the possible decompositions of objects in $\mathbf{C}$. This problem is considered by representation theory. We say that the decomposition problem in $\mathbf{C}$ is wild or equivalently that $\mathbf{C}$ has wild representation type if the solution of the decomposition problem would imply a solution of the following problem.

(2.2) Problem. Let k be a field and consider the following additive category $\mathbf{V}^{\alpha,\beta}$. Objects are finite dimensional k-vector spaces V together with two endomorphisms $\alpha_V, \beta_V : V \to V$. Morphisms are k-linear maps $f : V \to W$ satisfying $f\alpha_V = \alpha_W f$ and $f\beta_V = \beta_W f$. The decomposition problem in $\mathbf{V}^{\alpha,\beta}$ for any field k is termed a "wild problem of representation theory".

If the list of all indecomposable objects of $\mathbf{C}$ is finite then $\mathbf{C}$ has *finite* representation type. If the representation type of $\mathbf{C}$ is neither finite nor wild then $\mathbf{C}$ is of *tame representation type.* In representation theory there are in general means to compute an explicit list of all indecomposable objects in $\mathbf{C}$ if $\mathbf{C}$ has finite or tame representation type.

Next we describe our decomposition problem of homotopy theory. Let $\mathbf{Top}^*/\simeq$ be the homotopy category of pointed topological spaces. The set of morphisms $X \to Y$ in $\mathbf{Top}^*/\simeq$ is the set of homotopy classes $[X, Y]$. Isomorphisms in $\mathbf{Top}^*/\simeq$ are called homotopy equivalences and isomorphism types in $\mathbf{Top}^*/\simeq$ are homotopy types. Let $\mathbf{F}_n^k$ be the full subcategory of $\mathbf{Top}^*/\simeq$ consisting of $(n-1)$ -connected $(n+k)$ -dimensional CW-complexes which have finitely generated torsion free homology. The objects of $\mathbf{F}_n^k$ are special A_n^k -polyhedra, see [W1]. The suspension Σ gives us sequences of functors

$$(2.3) \qquad \mathbf{F}_1^k \xrightarrow{\Sigma} \mathbf{F}_2^k \to \ldots \to \mathbf{F}_n^k \xrightarrow{\Sigma} \mathbf{F}_{n+1}^k \to \ldots$$

with $k \geq 0$. The Freudenthal suspension theorem shows that these sequences stabelize in the sense that for $k+1 < n$ the functor $\Sigma : \mathbf{F}_n^k \to \mathbf{F}_{n+1}^k$ is an equivalence of additive categories so that

$$(1) \qquad \mathbf{F}^k = \mathbf{F}_n^k \qquad \text{with} \qquad k+1 < n$$

does not depend on n. This is the stable homotopy category of (-1) -connected k-dimensional spectra with finitely generated torsion free homology. The biproduct in the additive category $\mathbf{F}^k$ is the one point union of spaces. We point out that for $k+1=n$ the functor Σ is full and a 1-1 correspondence of homotopy types. The Spanier-Whitehead duality is a contravariant functor

$$D : \mathbf{F}^k \to \mathbf{F}^k \tag{2}$$

satisfying $DD = 1$ and $D(S^{n+i}) = S^{n+k-i}$ for $i \in \{0, \ldots, k\}$; compare for example [Co].

Each nontrivial element α in the $(k-1)$ -stem, $\alpha \in \pi_{n+k-1}(S^n)$, yields the canonical 2-cell complex $S^n \cup_\alpha e^{n+k} \in \mathbf{F}^k_n, n \geq 2$, which is indecomposable. Hence elements in homotopy groups of spheres can essentially be identified with special indecomposable objects in $\mathbf{F}^k_n, k \geq 2$. The decomposition in $\mathbf{F}^k_n$ is not unique. For example Freyd [F] points out that for $n \geq 5$ there is a homotopy equivalence

$$S^n \vee (S^n \cup_\nu e^{n+4}) \simeq S^n \vee (S^n \cup_{3\nu} e^{n+4}) \tag{2.4}$$

in $\mathbf{F}^4_n$ where, however, the CW-complexes $S^n \cup_\nu e^{n+4}$ and $S^n \cup_{3\nu} e^{n+4}$ are not homotopy equivalent. Here $\nu \in \pi_{n+3}(S^n)$ is a generator of order 8 as in § 1.

In [BD2] we solved the decomposition problem in the additive category $\mathbf{F}^4$ by showing that $X(\mathcal{L}_4)$ is a complete list of indecomposable objects in $\mathbf{F}^4$. Our main purpose in this paper is the solution of the decomposition problem in $\mathbf{F}^5$. We show that $\mathbf{F}^5$ is again of finite representation type. On the other hand we have seen in the Appendix of [BD2] that $\mathbf{F}^k$ is of wild representation type for $k \geq 10$. Hence only the representation types of $\mathbf{F}^6, \mathbf{F}^7, \mathbf{F}^8, \mathbf{F}^9$ remain unknown.

(2.5) Theorem. *The list $X(\mathcal{L}_5)$ of 451 elementary polyhedra in § 1 is a complete list of all indecomposable spaces in $\mathbf{F}^5$. Hence for $n \geq 6$ each $(n-1)$-connected $(n+5)$-dimensional CW-complex X with finitely generated torsion free homology admits a homotopy equivalence*

$$X \simeq X_1 \vee \ldots \vee X_t$$

with $X_i \in X(\mathcal{L}_5)$ for $1 \leq i \leq t$. Moreover the Spanier-Whitehead duality functor $D : \mathbf{F}^5 \to \mathbf{F}^5$ is completely understood on objects since we have

$$D(X(g)) = X(D(g))$$

where $D(g)$ is defined by the duality operator D in (1.2) with $g \in \mathcal{L}_5$.

Following Freyd [F] and Cohen [Co] 4.26 we use the following notation.

(2.6) Definition. We say that two spaces X, Y in $\mathbf{F}^k$ are congruent and we write $X \equiv Y$ if (a) or equivalently (b) is satisfied.

(a) There exists a space Z in $\mathbf{F}^k$ such that $X \vee Z \simeq Y \vee Z$ are homotopy equivalent

(b) There exists a homotopy equivalence $X \vee B_X \simeq Y \vee B_X$ where B_X is the unique one point union of spheres which has the same Betti numbers as X, that is $H_*(X)/\text{torsion} = H_*(B_X)$.

(2.7) Definition. Let p be a prime. A space X in $\mathbf{F}^k$ is a p-primary space if there exists a homotopy commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{p^N \cdot 1_X} & X \\ & \searrow \quad \nearrow & \\ & B & \end{array}$$

where B is a one point union of spheres. Here p^N is a power of the prime p and $p^N \cdot 1_X$ is a multiple of the identity of X in the abelian group of homotopy classes $[X, X]$ in $\mathbf{F}^k$ and N can not be chosen to be $N = 0$. This implies that X is not a one point union of spheres.

(2.8) Lemma. *An elementary polyhedron $X(g)$ with $g \in \mathcal{L}_5$ is 2-primary if and only if g is a Hopf element or g is a word with letters in the set $\{\eta, v_0, w_1; v$ and w divisible by 3$\}$. The only congruences between 2-primary polyhedra in $X(\mathcal{L}_5)$ are given by (2.4) that is $X(3_0) \equiv X(9_0)$ and $X(3_1) \equiv X(9_1)$. Moreover $X(g)$ is 3-primary if and only if $g = 8_0$ or $g = 8_1$. For a prime $p > 3$ there are no p-primary spaces in $X(\mathcal{L}_5)$.*

Recall that for any small additive category $\mathbf{C}$ (for example $\mathbf{C} = \mathbf{F}^k, k \geq 0$) we have the isomorphism class group $K_0(\mathbf{C})$. This is the abelian group with one generator [A] for each isomorphism class of objects $A \in \mathbf{C}$ with relations $[A]+[B] = [A \oplus B]$. This is just the Grothendieck group of $\mathbf{C}$ as defined by Bass [Ba]. A typical element of $K_0(\mathbf{C})$ is a formal difference $[A] - [B]$ with $[A] - [B] = [A'] - [B']$ if and only if there exists an isomorphism in $\mathbf{C}$ of the form $A \oplus B' \oplus C \cong A' \oplus B \oplus C$ for some object C in $\mathbf{C}$. The following result is due to Freyd [F]; see also Cohen [Co] 4.44.

(2.9) Theorem of Freyd. *Let $k \geq 0$. Then $K_0(\mathbf{F}^k)$ is a free abelian group generated by the spheres in $\mathbf{F}^k$ and by the congruence classes of indecomposable p-primary spaces in $\mathbf{F}^k$ where p runs through all primes.*

Such a wonderful result yields the crucial task to compute the generators of $K_0(\mathbf{F}^k)$ explicitly. For the category $\mathbf{F}^k$ of torsion free polyhedra we get accordingly:

- $K_0(\mathbf{F}^0) = \mathbb{Z}$ generated by S^n
- $K_0(\mathbf{F}^1) = \mathbb{Z}^2$ generated by S^n, S^{n+1}
- $K_0(\mathbf{F}^2) = \mathbb{Z}^4$ generated by $S^n, S^{n+1}, S^{n+2}, X(\eta_0)$
- $K_0(\mathbf{F}^3) = \mathbb{Z}^7$ generated by $S^n, S^{n+1}, S^{n+2}, S^{n+3}$, and $X(\eta_0), X(\eta_1), X(\eta\eta)_0$.

In [BD2] we show that

- $K_0(\mathbf{F}^4) = \mathbb{Z}^{29}$

is generated by the 5-spheres $S^n, \dots, S^{n+4}$ in $\mathbf{F}^4$, the 23 congruence classes of 2-primary polyhedra in $X(\mathcal{L}_4)$ and by the unique 3-primary polyhedron in $X(\mathcal{L}_4)$.

Using (2.8) and (2.5) we get accordingly:

(2.10) Theorem. *The group $K_0(\mathbf{F}^5) = \mathbb{Z}^{87}$ is generated by the 6-spheres S^n, $\dots$, S^{n+5}, the two 3-primary polyhedra in $X(\mathcal{L}_5)$ and the 79 congruence classes of 2-primary polyhedra in $X(\mathcal{L}_5)$.*

In § 6 we show that $K_0(\mathbf{F}^k) = \mathbb{Z}^\infty$ for $k \geq 6$.

§ 3 The algebraic classification of homotopy types in $\mathbf{F}^5$

Let X be a CW-complex in $\mathbf{F}^5 = \mathbf{F}^5_n$ with $n \geq 7$. We use *stable notation* so that we are allowed to omit n. Hence S^i corresponds to the sphere S^{n+i}, moreover e^i corresponds to the cell e^{n+i}, and the homotopy group $\pi_j\, S^i$ corresponds to $\pi_{n+j} S^{n+i}$, etc. We may assume that X has a cell structure given by the homology decomposition; see [B1]. This implies (since the homology of X is free abelian) that the cells of X are in 1-1 correspondence with free generators of the homology of X. Let c_i be the number of i-cells in X so that $H_i(X) = \mathbb{Z}^{c_i}$. For a space A and a natural number $d \geq 0$ let

$$dA = \underbrace{A \vee \ldots \vee A}_{d-times} \tag{3.1}$$

be the d-fold one point union of A. The attaching map of (stable) 5-cells of X is a map

$$f : c_5 S^4 \to X^4 \tag{3.2}$$

with $X^4 \in \mathbf{F}^4$. Here X^4 is the 4-skeleton of X. Since the cell structure is given by a homology decomposition we know that f admits a factorization

$$f : c_5 S^4 \to X^3 \subset X^4 \tag{1}$$

with $X^3 \in \mathbf{F}^3$. Let

$$\Gamma_4 X = \text{image}\{\pi_4 X^3 \to \pi_4 X^4\}$$

Then the homotopy class of f in (3.2) is determined by a homomorphism

$$f : H_5(X) = \mathbb{Z}^{c_5} \to \Gamma_4(X^4) \tag{2}$$

This is the secondary boundary in J.H.C. Whitehead's exact sequence [W2]. In [BD2] we classified the homotopy types in $\mathbf{F}^4$ showing that X^4 is a one point union of spaces $X(g)$ with $g \in \mathcal{L}_4$. We say that f in (2) is in *normal form* if X is a one point union of spaces $X(g)$ with $g \in \mathcal{L}_5$ so that f is canonically given by the attaching maps of 5-cells in $X(g)$ with $g \in \mathcal{L}_5$, see (1.3).

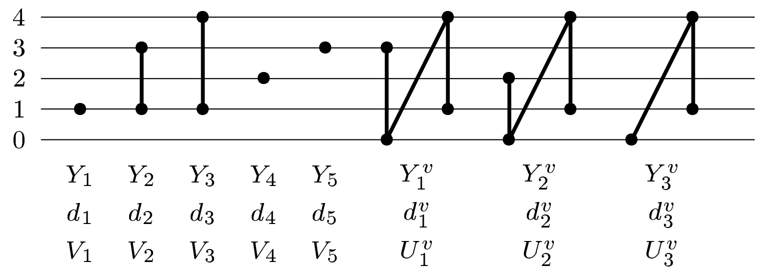
Since we are in the stable range the functor Γ_4 is additive. Therefore we can compute $\Gamma_4\, X^4$ as in proposition (3.4) below which determines $\Gamma_4 X(g)$ for $g \in \mathcal{L}_4$. We define a subset $\mathcal{L}'_4$ of $\mathcal{L}_4$ by

$$\mathcal{L}'_4 = \left\{ \begin{matrix} S^0, & S^4, & \eta_0, & \eta_2, & (\eta\eta)_0, & \eta\eta v\eta, \\ \eta v\eta, & \eta\eta v, & \eta v, & v\eta, & v & \end{matrix} \right\}$$

Moreover we define the spaces Y_i and Y_j^v with $i \in \{1, \dots, 5\}$, $j \in \{1,2,3\}, v \in \{1, \dots, 6\}$ by

$$(3.3) \qquad \begin{cases} Y_1 = S^1 = X(S^1) \\ Y_2 = S^1 \cup_\eta e^3 = X(\eta_1) \\ Y_3 = S^1 \cup_{\eta\eta} e^4 = X(\eta\eta)_1 \\ Y_4 = S^2 = X(S^2) \\ Y_5 = S^3 = X(S^3) \\ Y_1^v = X(\eta\eta v\eta\eta) \\ Y_2^v = X(\eta v\eta\eta) \\ Y_3^v = X(v\eta\eta) \end{cases}$$

These spaces form the set $X(\mathcal{L}_4 - \mathcal{L}_4')$; they are given by the following graphs



Here the numbers d_i, d_j^v and the corresponding free abelian groups V_i and U_j^v are chosen in (3.5) and (3.8) below.

(3.4) Proposition. $\Gamma_4 X(g) = 0$ *for* $g \in \mathcal{L}_4'$ *and*

$$\begin{cases} \Gamma_4(Y_1) = \mathbb{Z}/24 \\ \Gamma_4(Y_4) = \Gamma_4(Y_5) = \mathbb{Z}/2 \\ \Gamma_4(Y_2) = \Gamma_4(Y_3) = \Gamma_4(Y_j^v) = \mathbb{Z}/12 \end{cases}$$

Proof. We describe three examples. Consider $g = \eta_0 \in \mathcal{L}_4'$ so that $X(\eta_0) = S^0 \cup_\eta e^2$. Then we have the stable cofiber sequence

$$S^1 \xrightarrow{\eta} S^0 \to X(\eta_0) \to S^2 \xrightarrow{\eta} S^1$$

which induces the exact sequence

$$\pi_4 S^0 \to \pi_4 X(\eta_0) \to \pi_4 S^2 \xrightarrow{\eta_*} \pi_4 S^1$$

Here $\pi_4 S^0 = 0$ by [T] and η_* is injective by [T]. Hence $\pi_4 X(\eta_0) = 0$. In the same way we see $\pi_4 X(\eta\eta)_0 = 0$. Moreover consider $g = v\eta \in \mathcal{L}_4'$. Then we have

$$\Gamma_4 X(v\eta) = \mathrm{image}\{\pi_4(S^0 \vee S^2) = \pi_4 S^2 \to \pi_4 X(v\eta)\}$$

where $i_2 : S^2 \to X(v\eta)$ satisfies $i_0 v + i_2 \eta = 0$ by definition of $X(v\eta)$ in (1.3). Hence we get $i_2\eta\eta = i_0 v\eta = 0$ since $\nu\eta = 0$ by [T]. Hence $\Gamma_4 X(v\eta) = 0$. Moreover we use similar arguments for the computation of $\Gamma_4 X(g)$ with $g \notin \mathcal{L}_4'$. Here we need the relation $12(\nu+\alpha)=\eta\eta\eta$ in $\pi_4 S^1=\mathbb{Z}/24$; compare [T]. For example we have the cofiber sequence

$$S^2 \xrightarrow{\eta} S^1 \to X(\eta_1)$$

inducing the exact sequence

$$\pi_4 S^2 \to \pi_4 S^1 \to \pi_4 X(\eta_1)$$

so that $\Gamma_4 X(\eta_1)=\operatorname{cokernel}(\eta_* : \pi_4 S^2 \to \pi_4 S^1)=\mathbb{Z}/12$ since $\pi_4 S^2=\mathbb{Z}/2$ is generated by $\eta\eta$ and $\pi_4 S^1=\mathbb{Z}/24$ is generated by $\nu+\alpha$. We point out that the generator of $\mathbb{Z}/12$ in the proposition is the composite $i_1(\nu+\alpha)$ where i_1 is the inclusion of S^1 into Y_2, Y_3 or Y_j^v. q.e.d.

Since X^4 is a one point union of spaces $X(g)$ with $g\in\mathcal{L}_4$ we can write

$$X^4 = L \vee \tilde{X}^4. \tag{3.5}$$

Here L is a one point union of spaces $X(g)$ with $g\in\mathcal{L}_4'$ and $\tilde{X}^4$ is a one point union of spaces in (3.3), that is,

$$\tilde{X}^4 = d_1Y_1 \vee d_2Y_2 \vee Z_3 \vee d_4Y_4 \vee d_5Y_5 \tag{1}$$

where Z_3 is a one point union of spaces Y_3, Y_1^v, Y_2^v, Y_3^v namely

$$Z_3 = d_3Y_3 \vee \bigvee_{v\in\{1,\dots,6\}} (d_1^vY_1^v \vee d_2^vY_2^v \vee d_3^vY_3^v) \tag{2}$$

Here d_i and d_j^v are numbers ≥ 0. Using (3.4) we see that $\Gamma_4(L)=0$ so that the attaching map (3.2) factors through the inclusion $\tilde{X}^4\subset X^4$. Hence we get the following result which simplifies the proof of the decomposition theorem (2.5) a lot.

(3.6) Proposition. *For X in $\mathbf{F}^4$ there is a homotopy equivalence $X\simeq L\vee\tilde{X}$. Here L is a one point union of spaces $X(g)$ with $g\in\mathcal{L}_4'$ and $\tilde{X}$ is the cofiber of a map $\tilde{f}: c_5S^4\to\tilde{X}^4$ with $\tilde{X}^4$ as in (3.5).*

Hence in order to find all indecomposable spaces in $\mathbf{F}^5$ we only have to consider decompositions of spaces $\tilde{X}$ as in (3.6). We therefore assume in the following that $L=*$ so that $X=\tilde{X}$ satisfies $X^4=\tilde{X}^4$ with $\tilde{X}^4$ as in (3.5) and $f=\tilde{f}$. In this case we say that X is *special*.

(3.7) Proposition. *The space $Z=d_1Y_1\vee d_2Y_2\vee Z_3$ in (3.5) admits a surjection*

$$H_1(Z)\otimes\mathbb{Z}/24 \to \Gamma_4(Z)$$

which is natural in Z.

This is an easy consequence of (3.4) and the definition of Z. We associate with a special space X the following free abelian groups; see (3.5).

$$
(3.8) \qquad \begin{cases} V_i = \mathbb{Z}^{d_i} \\ V = \bigoplus_{i=1}^{5} V_i \\ U_j^v = \mathbb{Z}^{d_j^v}, \quad U_j = \bigoplus_{v=1}^{6} U_j^v, \quad U^v = \bigoplus_{j=1}^{3} U_j^v \\ U = \bigoplus_{j=1}^{3} \bigoplus_{v=1}^{6} U_j^v = \bigoplus_{j=1}^{3} U_j = \bigoplus_{v=1}^{6} U^v \end{cases}
$$

Then we obtain the homology groups $H_i = H_i(X)$ of the special space X by the formulas:

$$
(1) \qquad \begin{cases} H_0 = U \\ H_1 = V_1 \oplus V_2 \oplus V_3 \oplus U \\ H_2 = V_4 \oplus U_2 \\ H_3 = V_2 \oplus V_5 \oplus U_1 \\ H_4 = V_3 \oplus U \\ H_5 = \mathbb{Z}^{c_5} \end{cases}
$$

We also obtain for $\Gamma_4 = \Gamma_4(X^4)$ the formula

$$
(2) \qquad \Gamma_4 = V_1 \otimes \mathbb{Z}/24 \oplus (V_2 \oplus V_3 \oplus U) \otimes \mathbb{Z}/12 \oplus (V_4 \oplus V_5) \otimes \mathbb{Z}/2
$$

as a consequence of (3.4).

(3.9) Sketch of proof. Let X be special. In order to prove the decomposition theorem (2.5) we show that there exist a homotopy equivalence $\alpha : X^4 \simeq X^4$ and an automorphism φ of H_5X such that the composite

$$
(1) \qquad \alpha_* f \varphi^{-1} : H_5(X) \overset{\varphi}{\cong} H_5(X) \xrightarrow{f} \Gamma_4(X^4) \overset{\alpha_*}{\cong} \Gamma_4(X^4)
$$

is in normal form; see (3.2). This implies that a special space admits a decomposition as in (2.5). Here a map $\alpha : X^4 \to X^4$ is a homotopy equivalence if and only if α induces an isomorphism in homology. Since X^4 is a one point union of spaces $X(g)$ with $g \in \mathcal{L}_4 - \mathcal{L}_4'$ as in (3.5) (1) we have to consider algebraic maps

$$
(2) \qquad n_{g'}^{g} : H_* X(g) \to H_* X(g')
$$

with $g, g' \in \mathcal{L}_4 - \mathcal{L}_4'$ such that there exists a map

$$
(3) \qquad \alpha_{g'}^{g} : X(g) \to X(g')
$$

inducing $n^g_{g'}$ in homology. Moreover we determine $\Gamma_4(\alpha^g_{g'})$ in terms of $n^g_{g'}$. Using (2) we can describe all automorphisms n_* of the homology $H_*(X^4)$ in (3.9) for which there exists a map $\alpha : X^4 \to X^4$ with $n_* = H_*(\alpha)$ and we can compute $\alpha_* = \Gamma_4(\alpha)$ since we know $\Gamma_4(\alpha^g_{g'})$ in terms of $n^g_{g'}$. This leads to the following definitions and theorem.

(3.10) Definition. We say that an automorphism $\varphi : \Gamma_4(X^4) \cong \Gamma_4(X^4)$ is *realizable* if there exists a homotopy equivalences $\alpha : X^4 \simeq X^4$ which induces φ; that is $\Gamma_4(\alpha) = \varphi$.

The next result yields by the remarks in (3.9) a purely algebraic classification of all special homotopy types in $\mathbf{F}^5$. Proper automorphisms are defined in (3.12) below.

(3.11) Theorem. *An automorphism φ of $\Gamma_4(X^4)$ is realizable if and only if there exists a proper automorphism M of the free abelian group*

$$W = V_1 \oplus V_2 \oplus (V_3 \oplus U) \oplus V_4 \oplus V_5$$

for which the following diagram commutes

$$\begin{array}{ccc} W & \xrightarrow{M} & W \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \Gamma_4(X^4) & \xrightarrow{\varphi} & \Gamma_4(X^4) \end{array}$$

Here p is the quotient map given by (3.8) (2).

(3.12) Definition. Let $W_3 = V_3 \oplus U$ so that

$$W = V_1 \oplus V_2 \oplus W_3 \oplus V_4 \oplus V_5$$

is a direct sum of 5 summands. An automorphism M of the free abelian group W is *proper* if and onyl if M is given by a 5×5 matrix of the form

$$(1) \qquad M = \begin{pmatrix} a_{11} & 2a_{12} & 2a_{13} & 12a_{14} & 12a_{15} \\ a_{21} & a_{22} & a_{23} & 12a_{24} & 6a_{25} \\ a_{31} & 2a_{32} & a_{33} & 12a_{34} & 12a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix}$$

with the following properties. The coordinates of the matrix M correspond to the direct sum decomposition of W above, for example $a_{15} \in \mathrm{Hom}\,(V_5, V_1)$, $a_{23} \in \mathrm{Hom}\,(W_3, V_2)$, $a_{33} \in \mathrm{Hom}\,(W_3, W_3)$. The submatrix

$$(2) \qquad \mathcal{H}_1 = \begin{pmatrix} a_{11} & 2a_{12} & 2a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 2a_{32} & a_{33} \end{pmatrix}$$

of M is an automorphism of $V_1 \oplus V_2 \oplus W_3$. Recall that $W_3 = V_3 \oplus U$ where $U = U^1 \oplus \ldots \oplus U^6$ as in (3.8). We say that for

$$F, G \in \mathrm{Hom}(U, U)$$

the homomorphism F is $\mathbb{Z}/12$-*related* to G if the coordinates $F_{vw}, G_{vw} \in \mathrm{Hom}\,(U^w, U^v)$ of F and G satisfy for $v, w \in \{1, \dots, 6\}$ the equation

$$(w \cdot F_{vw}) \otimes \mathbb{Z}/12 = (v \cdot G_{vw}) \otimes \mathbb{Z}/12 \tag{3}$$

We require that there exist automorphisms

$$\begin{cases} \mathcal{H}_0 \in \mathrm{Aut}\,(U) = \mathrm{Aut}\,(U_1 \oplus U_2 \oplus U_3) \\ \mathcal{H}_4 \in \mathrm{Aut}\,(W_3) = \mathrm{Aut}\,(V_3 \oplus U) \end{cases} \tag{4}$$

as follows. With respect to the direct sum decomposition $U = U_1 \oplus U_2 \oplus U_3$ in (3.8) the automorphism $\mathcal{H}_0$ is given by a matrix of the form

$$\mathcal{H}_0 = \begin{pmatrix} b_{11} & 2b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 2b_{31} & 2b_{32} & b_{33} \end{pmatrix} \tag{5}$$

Moreover with respect to the direct sum decomposition $W_3 = V_3 \oplus U$ the automorphism $\mathcal{H}_4$ is given by a matrix

$$\mathcal{H}_4 = \begin{pmatrix} c_{33} & c_{3U} \\ c_{U3} & c_{UU} \end{pmatrix} \tag{6}$$

We require that

$$\mathcal{H}_0 \quad \text{is } \mathbb{Z}/12\text{-related to} \quad c_{UU} \in \mathrm{Hom}\,(U, U) \tag{7}$$

and $c_{U3} \in \mathrm{Hom}\,(V_3, U)$ has coordinates $c_{U^v 3} \in \mathrm{Hom}\,(V_3, U^v)$ with $v \in \{1, \dots, 6\}$ which satisfy the equation

$$(v \cdot c_{U^v 3}) \otimes \mathbb{Z}/12 = 0. \tag{8}$$

Finally we require that $\mathcal{H}_4$ in (6) satisfies

$$\mathcal{H}_4 \otimes \mathbb{Z}/2 = a_{33} \otimes \mathbb{Z}/2 \tag{9}$$

where $a_{33} \in \mathrm{Hom}\,(W_3, W_3)$ is the coordiante of M in (1) above.

We point out that by (3.8) (1) we have the homology groups

$$\begin{cases} H_0 = U \\ H_1 = V_1 \oplus V_2 \oplus W_3 \\ H_4 = W_3 = V_3 \oplus U \end{cases} \tag{3.13}$$

so that the automorphisms $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_4$ in (3.12) are automorphisms of H_0, H_1 and H_4 respectively. Clearly a homotopy equivalence $X^4 \simeq X^4$ induces such automorphisms $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_4$ of homology and in addition automorphisms of H_2 and H_3 in (3.8) (1) which we obtain as follows.

(3.14) Definition. Given $M, \mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_4$ as in (3.12) we choose automorphism $\mathcal{H}_2$ on $\mathcal{H}_3$ as follows. Let $\mathcal{H}_2$ be an automorphism of $H_2 = V_4 \oplus U_2$ which is given by a matrix

$$\mathcal{H}_2 = \begin{pmatrix} d_{44} & d_{42} \\ 2d_{24} & d_{22} \end{pmatrix}$$

satisfying the equations (1) and (2).

$$d_{44} \otimes \mathbb{Z}/2 = a_{44} \otimes \mathbb{Z}/2 \tag{1}$$

$$d_{22} \otimes \mathbb{Z}/2 = b_{22} \otimes \mathbb{Z}/2 \tag{2}$$

Here a_{44} and b_{22} are coordinates of Γ and $\mathcal{H}_0$ respectively. The properties of M and $\mathcal{H}_0$ readily show that $a_{44} \otimes \mathbb{Z}/2$ and $b_{22} \otimes \mathbb{Z}/2$ are automorphisms. Hence we can choose $\mathcal{H}_2$ since $GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/2)$ is surjective for all n; choose for example $d_{42} = 0$.

Moreover let $\mathcal{H}_3$ be an automorphism of $H_3 = V_5 \oplus V_2 \oplus U_1$ which is given by a matrix

$$\mathcal{H}_3 = \begin{pmatrix} e_{55} & e_{52} & e_{51} \\ 2e_{25} & e_{22} & 2e_{21} \\ 2e_{15} & e_{12} & e_{11} \end{pmatrix}$$

satisfying the equations (3) ... (4).

$$e_{55} \otimes \mathbb{Z}/2 = a_{55} \otimes \mathbb{Z}/2 \tag{3}$$

$$e_{22} \otimes \mathbb{Z}/2 = a_{22} \otimes \mathbb{Z}/2 \tag{4}$$

$$e_{11} \otimes \mathbb{Z}/2 = b_{11} \otimes \mathbb{Z}/2 \tag{5}$$

$$e_{25} \otimes \mathbb{Z}/2 = a_{25} \otimes \mathbb{Z}/2 \tag{6}$$

Here again a_{55}, a_{22}, a_{25} are coordinates of M in (3.12) (1) and b_{11} is a coordinate of $\mathcal{H}_0$ in (3.12) (5). Since a_{55} and $a_{22} \otimes \mathbb{Z}/2$ and $b_{11} \otimes \mathbb{Z}/2$ are automorphisms it is possible to choose $\mathcal{H}_3$. For example take $e_{52} = e_{51} = e_{21} = 0$ and use the surjection $GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/2)$.

Equation (6) above corresponds to the equation in 4.6 (13) of [BD2].

§ 4 Proof of theorem (3.11)

Let $X^4 = \tilde{X}^4$ be a one point union as in (3.5) (1), that is,

$$
(4.1)\qquad \begin{aligned} X^4 &= Z \vee d_4Y_4 \vee d_5Y_5, \\ Z &= d_1Y_1 \vee d_2Y_2 \vee Z_3, \\ Z_3 &= d_3Y_3 \vee \bigvee_{v\in\{1,\dots,6\}} (d_1^vY_1^v \vee d_2^vY_2^v \vee d_3^vY_3^v). \end{aligned}
$$

Here Y_i and Y_j^v are given by (3.3). With respect to the decomposition (4.1) a map $\alpha : X^4 \to X^4$ is given by a matrix

$$
(4.2)\qquad \alpha = \begin{pmatrix} \alpha_{ZZ} & \alpha_{Z4} & \alpha_{Z5} \\ \alpha_{4Z} & \alpha_{44} & \alpha_{45} \\ \alpha_{5Z} & 0 & \alpha_{55} \end{pmatrix}
$$

with $\alpha_{Z5} : d_5Y_5 \to Z$, etc. We clearly have $\alpha_{54} = 0$ since there are no essential maps from $Y_4 = S^2$ to $Y_5 = S^3$. Moreover $\alpha_{44} = a_{44}$ and $\alpha_{55} = a_{55}$ are determined by $H_\alpha(\alpha)$ and $H_3(\alpha)$ respectively and α_{45} is given by a homomorphism $qa_{45} : V_5 \to V_4 \to V_4 \otimes \mathbb{Z}/2$ where q is the quotient map. Clearly a_{45} is not well defined by α_{45}.

(4.3) Proposition. *The maps α_{4Z} and α_{5Z} induce the trivial homomorphism on Γ_4, that is $\Gamma_4(\alpha_{4Z}) = \Gamma_4(\alpha_{5Z}) = 0$. Moreover α_{Z4} has a coordinate $\alpha_{Z_34} : d_4Y_4 \to Z_3$ and α_{Z5} has a coordinate $\alpha_{Z_35} : d_5Y_5 \to Z_3$ with $\Gamma_4(\alpha_{Z_34}) = \Gamma_4(\alpha_{Z_35}) = 0$.*

Proof. We obtain $\Gamma_4(\alpha_{4Z}) = \Gamma_4(\alpha_{5Z}) = 0$ easily from (3.7) and (3.4) since composites $S^1 \to Z \to S^2, S^3$ are trivial. Moreover $\Gamma_4(\alpha_{Z_34}) = \Gamma_4(\alpha_{Z_35}) = 0$ is a consequence of the fact that Γ_4S^2 and Γ_4S^3 are generated by $\eta\eta$ and η respectively and that $\eta\eta\eta$ is trivial in Γ_4Y_3 and $\Gamma_4Y_j^v$ by (3.4). q.e.d.

Moreover we deduce from (3.7):

(4.4) Proposition. *$\Gamma_4(\alpha_{ZZ}) = H_1(\alpha_{ZZ})_*$ is induced by $H_1(\alpha_{ZZ})$ and with respect to the decomposition $Z = d_1Y_1 \vee d_2Y_2 \vee Z_3$ the automorphism $H_1(\alpha_{ZZ})$ is given by a matrix of the form*

$$
H_1(\alpha) = \begin{pmatrix} a_{11} & 2a_{12} & 2a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & 2a_{32} & a_{33} \end{pmatrix}
$$

Proof. One readily checks that each realizable map $H_1X(\eta_1) \to H_1(S^1)$ is divisible by 2 so that we obtain $2a_{12}$. Similarly one readily checks for $\tilde{Y} = Y_3, Y_j^v$ that realizable maps $H_1X(\eta_1) \to H_1(\tilde{Y})$ and $H_1(\tilde{Y}) \to H_1(S^1)$ are divisible by 2. Hence we obtain $2a_{32}$ and $2a_{13}$. These facts are easy consequences of the attaching maps q.e.d.

Using (4.4) nd (4.3) we see that with respect to the decomposition

$$
\Gamma_4(X^4) = \Gamma_4(d_1Y_1) \oplus \Gamma_4(d_2Y_2) \oplus \Gamma_4(Z_3) \oplus \Gamma_4(d_4Y_4) \oplus \Gamma_4(d_5Y_5)
$$

the homomorphism $\Gamma_4\alpha$ is given by a matrix of the following form:

$$
(4.5) \qquad \Gamma_4\alpha = \begin{pmatrix} \cdot & \cdot & \cdot & 12a_{14} & 12a_{15} \\ \cdot & \cdot & \cdot & 0 & 6a_{25} \\ \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix}
$$

Here the dotted 3×3-matrix is induced by $H_1(\alpha)$ in (4.4). The coordinates a_{14}, a_{15} and a_{25} are obtained as in the proof of [BD2].

If the map α is a homotopy equivalence then clearly $H_1(\alpha)$ is an automorphism and by the form (4.5) of the automorphism $\Gamma_4\alpha$ we see that also $a_{44} \otimes \mathbb{Z}/2$ and $a_{55} \otimes \mathbb{Z}/2$ are automorphisms. Using the surjection $GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/2)$ we may assume that a_{44} and a_{55} are automorphisms over $\mathbb{Z}$ and hence we obtain by (4.5) a matrix M as in (3.12) for which the diagram (3.11) commutes with $\varphi = \Gamma_4(\alpha)$.

In order to prove (3.11) we have to show the following two lemmas.

(4.6) Lemma. *Let α be a homotopy equivalence as above. Then $\mathcal{H}_1 = H_1(\alpha), \mathcal{H}_0 = H_0(\alpha)$, and $\mathcal{H}_4 = H_4(\alpha)$ together with M given by $\Gamma_4(\alpha)$ as above have properties as described in (3.12)*

Hence for each homotopy equivalence α there is a proper automorphism M such that diagram (3.11) with $\varphi = \Gamma_4(\alpha)$ commutes.

(4.7) Lemma. *Let M be a proper automorphism as in (3.12) so that we can choose $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4$ as in (3.12) and (3.14). Then there exists a map $\alpha : X^4 \to X^4$ with $H_i(\alpha) = \mathcal{H}_i$ such that diagram (3.11) with $\varphi = \Gamma_4(\alpha)$ commutes.*

Since $\mathcal{H}_i$ are automorphisms we see that α in (4.7) is a homotopy equivalence. Therefore each proper automorphism M induces a realizable automorphism φ on $\Gamma_4(\alpha)$.

In order to prove (4.6) and (4.7) we repeat the classification theorem of Unsöld [U1], [U2]. He defines the following algebraic category $\mathbf{SF}^4$.

(4.8) Definition. Objects in $\mathbf{SF^4}$ are tuple of abelian groups

$$
\mathcal{H} = (H_0, H_1, H_2, H_3, H_4, \pi_1, \pi_2) \in \mathbf{Ab}^7
$$

where H_i with $i \in \{0, \ldots, 4\}$ is finitely generated and free abelian together with the following diagrams (1), (2), (3) in $\mathbf{Ab}$:

(1) An exact sequence

$$
H_3 \to \pi_1 \otimes \mathbb{Z}/2 \to \pi_2 \to H_2 \xrightarrow{b} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta^1} \pi_1 \to H_1 \to 0
$$

(2) Let $P = \ker(H_0 \xrightarrow{q} H_0 \otimes \mathbb{Z}/2 \xrightarrow{\eta^1} \pi_1)$ where q is the quotient map. Then

$$
\begin{tikzcd}
0 \arrow[r] & H_0 \arrow[r, "2"] \arrow[d, "q\eta^1 q"'] & P \arrow[r] \arrow[d, "T"] & \ker(\eta^1) \arrow[r] \arrow[d, "\Omega"] & 0 \\
& \pi_1 \otimes \mathbb{Z}/2 \arrow[r] & \pi_2 \otimes \mathbb{Z}/2 \arrow[r] & \ker(b) \otimes \mathbb{Z}/2 &
\end{tikzcd}
$$

commutes where $q\eta^1 q$ is given by (1) and where Ω is determined by the extension

$$0 \to \ker(b) \to H_2 \to \ker(\eta^1) \to 0$$

given by (1). The top row of the diagram is short exact.

(3) Moreover for the abelian group

$$\Gamma_3 = (H_0 \otimes \mathbb{Z}/24 \oplus \pi_2 \otimes \mathbb{Z}/2)/\{(\xi \otimes 6, T(\xi)); \xi \in P \subset H_0\}$$

defined by T in (2) a homomorphism

$$b_4 : H_4 \to \Gamma_3$$

is given.

A morphism between such objects in $\mathbf{SF}^4$ is a tuple of homomorphisms $\mathcal{H} \to \mathcal{H}'$ in $\mathbf{Ab}^7$ which is compatible with all arrows in the diagrams (1), (2) and (3). Clearly $\mathbf{SF}^4$ is an additive category with the direct sum of objects given by the direct sum of abelian groups and morphisms.

In [U1], [U2] one finds the proof of the following result.

(4.9) Theorem. *There is an additive functor $\lambda : \mathbf{F}^4 \to \mathbf{SF}^4$ which is full and representative and which reflects isomorphisms.*

The functor carries a space X to the certain exact sequence of J.H.C. Whitehead [W2] of X together with the secondary homotopy operation T which was introduced by Unsöld.

Theorem (4.9) allows the computation of all realizable homology homomorphisms

$$n_* : H_*(Y) \to H_*(Y') \tag{4.10}$$

with $Y, Y' \in X(\mathcal{L}_4 - \mathcal{L}_4')$; see (3.3). For this we use the fact that the functor λ in (4.9) is full. One readily checks that $H_i(Y)$ is either 0 or $\mathbb{Z}$ so that n_* is given by $n_i \in \mathbb{Z}$ with $i \in \{0, \dots, 4\}$. Below we describe the non-trivial n_i for which $n_* = (n_0, n_1, n_2, n_3, n_4)$ is realizable by a map $Y \to Y'$. For example if Y or Y' are spheres with $Y \neq Y'$ one can check that there are only the following possibilities (1)...(5).

(1) $$Y_1 \xrightarrow{n_1} Y_3 \xrightarrow{n_1 \equiv 0(2)} Y_1$$

(2) $$Y_1 \xrightarrow{n_1} Y_2 \xrightarrow{n_1 \equiv 0(2)} Y_1, \quad Y_5 \xrightarrow{n_3 \equiv 0(2)} Y_2 \xrightarrow{n_3} Y_5$$

(3) $$Y_1 \xrightarrow{n_1} Y_1^v \xrightarrow{n_1 \equiv 0(2)} Y_1, \quad Y_5 \xrightarrow{n_3 \equiv 0(2)} Y_1^v \xrightarrow{n_3} Y_5$$

(4) $$Y_1 \xrightarrow{n_1} Y_2^v \xrightarrow{n_1 \equiv 0(2)} Y_1, \quad Y_4 \xrightarrow{n_2 \equiv 0(2)} Y_2^v \xrightarrow{n_2} Y_4$$

(5) $$Y_1 \xrightarrow{n_1} Y_3^v \xrightarrow{n_1 \equiv 0(2)} Y_1$$

Moreover if Y or Y' are Hopf elements we get the following possibilities (6)...(14).

$$(6) \qquad Y_2 \overset{n_1\equiv n_2(2)}{\longrightarrow} Y_2$$

$$(7) \qquad Y_3 \overset{n_1\equiv n_4(2)}{\longrightarrow} Y_3$$

$$(8) \qquad Y_2 \overset{n_1\equiv 0(2)}{\longrightarrow} Y_3 \overset{n_1}{\longrightarrow} Y_2$$

$$(9) \qquad Y_2 \underset{n_3}{\overset{n_1\equiv 0(2)}{\longrightarrow}} Y_1^v \underset{n_3\equiv 0(2)}{\overset{n_1}{\longrightarrow}} Y_2$$

$$(10) \qquad Y_3 \underset{n_4 v\equiv 0(12)}{\overset{n_4\equiv n_1(2)}{\longrightarrow}} Y_1^v \overset{n_1\equiv n_4(2)}{\longrightarrow} Y_3$$

$$(11) \qquad Y_2 \overset{n_1\equiv 0(2)}{\longrightarrow} Y_2^v \overset{n_1}{\longrightarrow} Y_2$$

$$(12) \qquad Y_3 \underset{n_4 v\equiv 0(12)}{\overset{n_1\equiv n_4(2)}{\longrightarrow}} Y_2^v \overset{n_1\equiv n_4(2)}{\longrightarrow} Y_3$$

$$(13) \qquad Y_2 \overset{n_1\equiv 0(2)}{\longrightarrow} Y_3^v \overset{n_1}{\longrightarrow} Y_2$$

$$(14) \qquad Y_3 \underset{n_4 v\equiv 0(12)}{\overset{n_1\equiv n_4(2)}{\longrightarrow}} Y_3^v \overset{n_1\equiv n_4(2)}{\longrightarrow} Y_3$$

Finally we get for $Y, Y' \in \{Y_1^v, Y_2^v, Y_3^v\}$ the following possibilities (15)...(23) with $v, w \in \{1, \dots, 6\}$.

$$(15) \qquad Y_1^w \underset{n_0 w\equiv n_4 v(12)}{\overset{n_0\equiv n_3(2), n_1\equiv n_4(2)}{\longrightarrow}} Y_1^v$$

$$(16) \qquad Y_1^w \underset{n_0 w\equiv n_4 v(12)}{\overset{n_4\equiv n_1(2)}{\longrightarrow}} Y_2^v$$

$$(17) \qquad Y_1^w \underset{n_4 v\equiv n_0 w(12)}{\overset{n_0\equiv 0(2), n_4\equiv n_1(2)}{\longrightarrow}} Y_3^v$$

$$(18) \qquad Y_2^w \underset{n_0 w\equiv n_4 v(12)}{\overset{n_0\equiv 0(2), n_1\equiv n_4(2)}{\longrightarrow}} Y_1^v$$

$$(19) \qquad Y_2^w \underset{n_0 w\equiv n_4 v(12)}{\overset{n_2\equiv n_0(2), n_1\equiv n_4(2)}{\longrightarrow}} Y_2^v$$

$$(20) \qquad Y_2^w \underset{n_0 w\equiv n_4 v(12)}{\overset{n_0\equiv 0(2), n_4\equiv n_1(2)}{\longrightarrow}} Y_3^v$$

$$(21) \qquad Y_3^w \underset{n_0 w\equiv n_4 v(12)}{\overset{n_4\equiv n_1(2)}{\longrightarrow}} Y_1^v$$

$$(22) \qquad Y_3^w \underset{n_0 w\equiv n_4 v(12)}{\overset{n_1\equiv n_4(2)}{\longrightarrow}} Y_2^v$$

$$(23) \qquad Y_3^w \underset{n_0 w\equiv n_4 v(12)}{\overset{n_1\equiv n_4(2)}{\longrightarrow}} Y_3^v$$

One can prove the conditions of realizability of $n_* = (n_1, \dots, n_4)$ in (1)...(23) either directly by use of the cell structure of $Y, Y' \in X(\mathcal{L}_4 - \mathcal{L}'_4)$ or by use of theorem (4.9). In order to apply (4.9) we point out that the objects in $\mathbf{SF}^4$ corresponding to Y_3, Y_1^v, Y_2^v, Y_3^v are given by the following list.

$$
\begin{array}{ccccccccccccccc}
 & H_3 & \to & \pi_1\otimes\mathbb{Z}/2 & \to & \pi_2 & \to & H_2 & \to & H_0\otimes\mathbb{Z}/2 & \to & \pi_1 & \twoheadrightarrow & H_1, & H_0 \\
Y_3 & 0 & & \mathbb{Z}/2 & & \mathbb{Z}/2 & & 0 & & 0 & & \mathbb{Z} & & \mathbb{Z} & 0 \\
Y_1^v & \mathbb{Z} & \xrightarrow{(1,0)} & \mathbb{Z}/2\oplus\mathbb{Z}/2 & \xrightarrow{(0,1)} & \mathbb{Z}/2 & & 0 & & \mathbb{Z}/2 & & \mathbb{Z}/2\oplus\mathbb{Z} & & \mathbb{Z} & \mathbb{Z} \\
Y_2^v & 0 & & \mathbb{Z}/2 & & \mathbb{Z}/2\oplus\mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}/2 & & \mathbb{Z} & & \mathbb{Z} & \mathbb{Z} \\
Y_3^v & 0 & & \mathbb{Z}/2\oplus\mathbb{Z}/2 & & \mathbb{Z}/2\oplus\mathbb{Z}/2 & & 0 & & 0 & & \mathbb{Z} & & \mathbb{Z} & 0
\end{array}
$$

$$
\begin{array}{cccccccccc}
 & H_0 & \supset & P & \xrightarrow{T} & \pi_2\otimes\mathbb{Z}/2, & H_4 & \to & \Gamma_3 & \twoheadleftarrow & H_0\otimes\mathbb{Z}/24\oplus\pi_2\otimes\mathbb{Z}/2 \\
Y_3 & 0 & & 0 & & \mathbb{Z}/2 & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z}/2 & \leftarrow & 0\oplus\mathbb{Z}/2 \\
 & & & & & & & & y & & y \\
Y_1^v & \mathbb{Z} & & 2\mathbb{Z} & \xrightarrow{0} & \mathbb{Z}/2 & \mathbb{Z} & \xrightarrow{(v,1)} & \mathbb{Z}/12\oplus\mathbb{Z}/2 & \leftarrow & \mathbb{Z}/24\oplus\mathbb{Z}/2 \\
 & & & & & & & & (x,y) & & (x,y) \\
Y_2^v & \mathbb{Z} & = & \mathbb{Z} & \xrightarrow{(0,1)} & \mathbb{Z}/2\oplus\mathbb{Z}/2 & \mathbb{Z} & \xrightarrow{(v,1)} & \mathbb{Z}/12\oplus\mathbb{Z}/2 & \leftarrow & \mathbb{Z}/24\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2 \\
 & & & & & & & & (x+6z,y) & & (x,y,z) \\
Y_3^v & \mathbb{Z} & & 2\mathbb{Z} & \xrightarrow{(1/2,0)} & \mathbb{Z}/2\oplus\mathbb{Z}/2 & \mathbb{Z} & \xrightarrow{(v,1)} & \mathbb{Z}/24\oplus\mathbb{Z}/2 & \leftarrow & \mathbb{Z}/24\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2 \\
 & & & & & & & & (x+12z,z) & & (x,y,z)
\end{array}
$$

Using these objects of $\mathbf{SF}^4$ corresponding to Y_3, Y_1^v, Y_2^v, Y_3^v it is an easy but inevitable calculation to obtain the conditions on n_* in (1)...(23) above. For example if we consider maps $Y_2^w \to Y_1^v$ we obtain $n_* = (n_0, n_1, n_4)$ such that the following diagrams commute (these diagrams describe a morphism $\lambda(Y_2^w) \to \lambda(Y_1^v)$ in $\mathbf{SF}^4$).

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r, "{(1,0)}"] \arrow[d, "{(\alpha,n_1)}"] & \mathbb{Z}/2\oplus\mathbb{Z} \arrow[r] \arrow[d, "{(n_1,\beta)}"] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r, "0"] \arrow[d, "n_0"] & \mathbb{Z} \arrow[r] \arrow[d, "{(\alpha,n_1)}"] & \mathbb{Z}, \arrow[d, "n_1"] & \mathbb{Z} \arrow[d, "n_0"] \\
\mathbb{Z} \arrow[r, "{(1,0)}"] & \mathbb{Z}/2\oplus\mathbb{Z}/2 \arrow[r, "{(0,1)}"] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r, "{(1,0)}"] & \mathbb{Z}/2\oplus\mathbb{Z} \arrow[r] & \mathbb{Z}, & \mathbb{Z}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{Z} \arrow[r, phantom, "="] \arrow[d, "n_0"'] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/2\oplus\mathbb{Z}/2; \arrow[d, "{(n_1,\beta)}"] \\
\mathbb{Z} \arrow[r, phantom, "\supset"] & 2\mathbb{Z} \arrow[r, "0"] & \mathbb{Z}/2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z} \arrow[r, "{(w,1)}"] \arrow[d, "n_4"'] & \mathbb{Z}/12\oplus\mathbb{Z}/2 \arrow[d, "n_0\oplus n_1"] & \mathbb{Z}/24\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2 \arrow[l] \arrow[d, "{n_0\oplus(n_1,0)}"'] \\
\mathbb{Z} \arrow[r, "{(v,1)}"] & \mathbb{Z}/12\oplus\mathbb{Z}/2 & \mathbb{Z}/24\oplus\mathbb{Z}/2 \arrow[l]
\end{tikzcd}
$$

The first diagram shows $n_0 \equiv 0(2)$ and the last diagram shows $n_1 \equiv n_4(2)$ and $n_0 w \equiv n_4 v(12)$. If $n_* = (n_0, n_1, n_4)$ satisfies these conditions then α and $\beta = 0$ can be chosen such that the diagrams commute.

Homology homomorphisms (4.10) are realizable if and only if they satisfy these conditions. This proves (4.6) and (4.7) by checking that the conditions (1)...(23) correspond exactly to the assumptions used in the definition of a proper automorphism in (3.12) and (3.14).

The condition describing a proper automorphism in (3.12) can be considerably simplified as follows.

(5.1) Proposition. *An automorphism M of W is proper if and only if with respect to the decomposition*

$$W = V_1 \oplus V_2 \oplus V_3 \oplus U^1 \oplus U^2 \oplus U^3 \oplus U^4 \oplus U^5 \oplus U^6 \oplus V_4 \oplus V_5$$

the matrix M is a matrix of the form

$$\left(\begin{array}{cc|ccccccc|cc}
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 12 & 12 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 12 & 6 \\ \cline{3-9}
1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 12 & 12 \\
1 & 2 & 2 & \bigstar & 2 & \bigstar & 2 & \bigstar & 2 & 12 & 12 \\
1 & 2 & 2 & 1 & \bigstar & 1 & 2 & 1 & \bigstar & 12 & 12 \\
1 & 2 & 2 & \bigstar & 2 & \bigstar & 2 & \bigstar & 2 & 12 & 12 \\
1 & 2 & 1 & 1 & 1 & 1 & \bigstar & 1 & 1 & 12 & 12 \\
1 & 2 & 2 & \bigstar & 2 & \bigstar & 2 & \bigstar & 2 & 12 & 12 \\
1 & 2 & 2 & 1 & \bigstar & 1 & 2 & 1 & \bigstar & 12 & 12 \\ \cline{3-9}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

Here an integer m at some place of a matrix means that the corresponding block equals ma where a can be any integral matrix. Moreover $\bigstar$ means that this block with respect to the decomposition

$$U^v = U_1^v \oplus U_2^v \oplus U_3^v$$

is a matrix of the form

$$\bigstar = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}$$

Remark. If all U^i with $i \in \{1, \dots, 6\}$ are trivial then the square in the matrix of (5.1) shrinks to a 1×1-matrix and in this case we get the matrix problem for $\mathbf{F}^4$ which was solved in [BD2].

Proof. The necessity of this condition follows readily from the definition of a proper automorphism. To prove sufficiency consider the submatrix N of M corresponding to the direct summand $W_3 = V_3 \oplus U$. The submatrix N is indicated by the rectangle in the matrix M above. Using the notation $U^0 = V_3$ the matrix N has blocks $N_{ij} : U^j \to U^i$ given with respect to the decomposition

$$W_3 = U^0 \oplus U^1 \oplus U^2 \oplus U^3 \oplus U^4 \oplus U^5 \oplus U^6.$$

Since M is invertible also N is invertible modulo 2. Hence the following submatrices of N are also invertible modulo 2.

$$N_0 = \begin{pmatrix} N_{00} & N_{04} \\ N_{40} & N_{44} \end{pmatrix}$$

$$N_2 = \begin{pmatrix} N_{22} & N_{26} \\ N_{62} & N_{66} \end{pmatrix}$$

$$N_1 = \begin{pmatrix} N_{11} & N_{13} & N_{15} \\ N_{31} & N_{33} & N_{35} \\ N_{51} & N_{53} & N_{55} \end{pmatrix}$$

Consider the ring R_0 of integral matrices of the form

$$B_0 = \begin{pmatrix} B_{00} & B_{04} \\ 3B_{40} & B_{44} \end{pmatrix}$$

where B_{vw} has the same size as N_{vw} and B_{44} has the form $\bigstar$. Then we get

$$R_0/2R_0 = \mathrm{Mat}\,(n, \mathbb{F}_2)$$

where n is the number of rows in N_0. As $GL(n, \mathbb{F}_2) = SL(n, \mathbb{F}_2)$ is generated by elementary matrices the natural homomorphism

$$R_0^\times \to GL(n, \mathbb{F}_2)$$

is surjective. In particular, there exists an invertible matrix $B_0 \in R_0$ such that $B_0 \equiv N_0 (\mathrm{mod}\, 2)$. Moreover the matrix B_{44} is invertible modulo 3.

Next let R be the ring of all matrices of the form $\bigstar$. Then

$$R/3R = \mathrm{Mat}\,(r, \mathbb{F}_3)$$

The group $GL(r, \mathbb{F}_3)$ is generated by elementary matrices and the matrix diag $(-1, 1, \ldots, 1)$. Hence the natural homomorphism

$$R^\times \to GL(r, \mathbb{F}_3)$$

is also surjective; in particular, there exists an invertible matrix $A_{44} \in R$ such that $A_{44} \equiv B_{44} (\mathrm{mod}\, 3)$.

Similar observations show that tere exist invertible matrices

$$B_2 = \begin{pmatrix} B_{22} & 3B_{26} \\ B_{62} & B_{66} \end{pmatrix}$$

$$B_1 = \begin{pmatrix} B_{11} & 3B_{13} & B_{15} \\ B_{31} & B_{33} & B_{35} \\ B_{51} & 3B_{53} & B_{55} \end{pmatrix}$$

where all blocks B_{wv} are of the form $\bigstar$ and for which in addition $B_2 \equiv N_2\,(\mathrm{mod}\, 2)$ and $B_1 \equiv N_1\,(\mathrm{mod}\, 2)$. Now put

$$B_1' = \begin{pmatrix} B_{11} & B_{13} & 5B_{15} \\ 3B_{31} & B_{33} & 3B_{35} \\ 5B_{51} & 5B_{53} & B_{55} \end{pmatrix}$$

so that

$$B_1' \equiv \begin{pmatrix} I & 0 & 0 \\ 0 & 3I & 0 \\ 0 & 0 & 5I \end{pmatrix} B_1 \begin{pmatrix} I & 0 & 0 \\ 0 & \frac{1}{3}I & 0 \\ 0 & 0 & 5I \end{pmatrix} \pmod{12}$$

and $\det B_1' \equiv \det B_1 \equiv \pm 1 \pmod{12}$. Therefore just as above there is an invertible matrix

$$A_1 = \begin{pmatrix} A_{11} & A_{13} & 5A_{15} \\ 3A_{31} & A_{33} & 3A_{35} \\ 5A_{51} & 5A_{53} & A_{55} \end{pmatrix}$$

such that $A_1 \equiv B_1' \pmod{12}$. Moreover this implies that all blocks A_{wv} have the form $\bigstar$.

On the other hand as B_2 is invertible its conjugate

$$A_2 = \begin{pmatrix} I & 0 \\ 0 & 3I \end{pmatrix} B_2 \begin{pmatrix} I & 0 \\ 0 & \frac{1}{3}I \end{pmatrix} = \begin{pmatrix} B_{22} & B_{26} \\ 3B_{62} & B_{66} \end{pmatrix}$$

is also invertible. Put $A_{wv} = B_{wv}$ for $v, w \in \{2, 6\}$.

We are now ready to define $\mathcal{H}_0$ and $\mathcal{H}_4$ associated to M such that the properties in (3.12) hold. This proves that M is proper and hence completes the proof of (5.1). Let $\mathcal{H}_4$ be the matrix

$$\mathcal{H}_4 = \left(\begin{array}{c|cccccc} B_{00} & B_{01} & B_{02} & B_{03} & B_{04} & B_{05} & B_{06} \\ \cline{2-7} 0 & B_{11} & 0 & 3B_{13} & 0 & B_{15} & 0 \\ 0 & B_{21} & B_{22} & 3B_{23} & 0 & B_{25} & B_{26} \\ 0 & B_{31} & 0 & B_{33} & 0 & B_{35} & 0 \\ 3B_{40} & B_{41} & B_{42} & 3B_{43} & B_4^4 & B_{45} & 3B_4^6 \\ 0 & B_{51} & 0 & 3B_{53} & 0 & B_{55} & 0 \\ 0 & B_{61} & B_{62} & B_{63} & 0 & B_{65} & B_{66} \end{array}\right)$$

Here all blocks B_{wv} which have not yet been defined coincide with N_{wv}. Then $\mathcal{H}_4$ is invertible and $\mathcal{H}_4 \equiv N \pmod 2$. Next let

$$\mathcal{H}_0 = \begin{pmatrix} A_{11} & 0 & A_{13} & 0 & 5A_{15} & 0 \\ 2A_{21} & A_{22} & 2A_{23} & 0 & 2A_{25} & A_{26} \\ 3A_{31} & 0 & A_{33} & 0 & 3A_{35} & 0 \\ 4A_{41} & 2A_{42} & 4A_{43} & A_{44} & 4A_{45} & 2A_{46} \\ 5A_{51} & 0 & 5A_{53} & 0 & A_{55} & 0 \\ 6A_{61} & 3A_{62} & 2A_{63} & 0 & 6A_{65} & A_{66} \end{pmatrix}$$

where all blocks A_{wv} which have not yet been defined coincide with B_{wv}. Then A is also invertible and, in fact, A is $\mathbb{Z}/12$-related to the lower rigth part of $\mathcal{H}_4$ corresponding to the direct summand U and indicated by the rectangle in $\mathcal{H}_4$ above. Hence this proves that M is proper; compare (3.12). q.e.d.

§ 6 Proof of the decomposition theorem (2.5)

Let Λ be the ring of all integral (23×23)-matrices of the form (5.1) whose rows and columns are numbered by the indexes $i = 1, \dots, 5$ and pairs $\binom{v}{j}$ with $v = 1, \dots, 6$; $j = 1, \dots, 3$. Let

$$\begin{cases} \mathcal{U}_1 = \mathbb{Z}/24 \\ \mathcal{U}_2 = \mathcal{U}_3 = \mathcal{U}_{\binom{v}{j}} = \mathbb{Z}/12 \\ \mathcal{U}_4 = \mathcal{U}_5 = \mathbb{Z}/2 \end{cases}$$

Then the direct sum

$$\mathcal{U} = \bigoplus_{i=1}^{5} \mathcal{U}_i \oplus \bigoplus_{v=1}^{6} \bigoplus_{j=1}^{3} \mathcal{U}_{\binom{v}{j}}$$

is in the obvious way a $\Lambda - \mathbb{Z}$-bimodule denoted by $\mathcal{U} = {}_\Lambda \mathcal{U}_\mathbb{Z}$. Recall that a *matrix over* $\mathcal{U}$ is by definition (cf. [D]) an element of $P \otimes_\Lambda \mathcal{U} \otimes H^*$ where P and H are finitely generated right projective modules over Λ and $\mathbb{Z}$ respectively. It is more convenient to identify this tensor product with

$$\mathcal{U}(H, P) = \mathrm{Hom}\,(H, P \otimes_\Lambda \mathcal{U}).$$

Two matrices $u \in \mathcal{U}(H, P), u' \in \mathcal{U}(H', P')$ are *isomorphic* if there are isomorphisms $\alpha : H \to H'$ and $\beta : P \to P'$ such that $\beta u = u' \alpha$.

Let $e_i = e_{ii}$ and $e_{\binom{v}{j}} = e_{\binom{u}{j}\binom{v}{j}}$ be matrix units and

$$P_i = e_i \Lambda, P_{\binom{v}{j}} = e_{\binom{v}{j}} \Lambda$$

Then P can be uniquely decomposed as

$$P = (\bigoplus_{i=1}^{5} V_i \otimes P_i) \oplus \bigoplus_{v=1}^{6} \bigoplus_{j=1}^{3} U_j^v \otimes P_{\binom{v}{j}}$$

for some free abelian groups V_i, U_j^v. Therefore

$$P \otimes_\Lambda \mathcal{U} \cong \bigoplus_{i=1}^{5} V_i \otimes \mathcal{U}_i \oplus \bigoplus_{v=1}^{6} \bigoplus_{j=1}^{3} U_j^v \otimes \mathcal{U}_{\binom{v}{j}}$$

This shows that isomorphism classes of matrices u above are in 1-1 correspondence with homotopy types in $\mathbf{F}^5$ (for this we use (3.11) and (5.1)).

We shall write the elements of $\mathcal{U}(H, P)$ as families of matrices (u_i, u_j^v) with u_i being of size $d_i \times c_5$ and u_j^v of size $d_j^u \times c_5$ if

$$\begin{cases} H = \mathbb{Z}^{c_5} \\ V_i = \mathbb{Z}^{d_i} \\ U_j^v = \mathbb{Z}^{d_j^v} \end{cases}$$

as in (3.5) and (3.8). The entries of u_i, u_j^v are in the corresponding groups $\mathcal{U}_i$ and $\mathcal{U}_{\binom{v}{j}}$ respectively. The matrices $M \in \Lambda$ define the "admissible transformations" of rows in these matrices. For instance, as we have a_{21} in M we can add any multiple of a row of the matrix u_1 to any row of the matrix u_2. On the other hand, as we have $2a_{12}$ in M, we can add only even multiples of rows of u_2 to the rows of u_1, etc.

We point out that the ring Λ can be "Morita-reduced" since Λ is Morita-equivalent to the following ring Λ'. The ring Λ' consists of all matrices of the same form as in Λ for which rows and columns, however, correspond only to the indexes $i = 1, \dots, 5$ and pairs $\binom{v}{j}$ with $j = 1, 2, 3$ and $v \in \{3, 4, 6\}$ except $\binom{4}{3}$. Hence the restriction of the bimodule $\mathcal{U}$ to Λ' yields the $\Lambda' - \mathbb{Z}$-bimodule $\mathcal{U}' = _{\Lambda'} \mathcal{U}_{\mathbb{Z}}$. The categories of $\mathcal{U}$-matrices and $\mathcal{U}'$-matrices are equivalent.

Moreover $\mathcal{U}'$ is also a $\bar{\Lambda} - \bar{\mathbb{Z}}$-bimodule with $\bar{\mathbb{Z}} = \mathbb{Z}/24$ and $\bar{\Lambda} = \Lambda'/24$ which we denote by $\bar{\mathcal{U}} = _{\bar{\Lambda}} \mathcal{U}_{\bar{\mathbb{Z}}}$. The elements of $\mathcal{U}$ and $\bar{\mathcal{U}}$ are the same and also the matrices from $\mathcal{U}(P, H)$ coincide with those from $\bar{\mathcal{U}}(\bar{P}, \bar{H})$ with $\bar{P} = P/24$ and $\bar{H} = H/24$, but non-isopmorphic $\mathcal{U}$-matrices might be isomorphic as $\bar{\mathcal{U}}$-matrices.

Consider the 2-primary part $\tilde{\mathcal{U}}$ of $\bar{\mathcal{U}}$ and let $\tilde{\Lambda}$ be the ring of matrices with entries in $\mathbb{Z}/8$ satisfying the same conditions as matrices in Λ' above (i.e., in the corresponding congruences we replace $(\bmod 24)$ by $(\bmod 8)$, $(\bmod 12)$ by $(\bmod 4)$ and $(\bmod 6)$ by $(\bmod 2)$. Then $\tilde{\mathcal{U}}$ is a $\tilde{\Lambda} - \mathbb{Z}/8$-bimodule $\tilde{\mathcal{U}} = _{\tilde{\Lambda}} \tilde{\mathcal{U}}_{\mathbb{Z}/8}$.

Now denote by z_i (resp. $z_{\binom{v}{j}}$) the image of $z \in \mathbb{Z}$ in $\tilde{\mathcal{U}}_i$ (resp, in $\tilde{\mathcal{U}}_{\binom{v}{j}}$). For elements $u, v \in \tilde{\mathcal{U}}$ we write $u < v$ if there exists $a \in \tilde{\Lambda}$ with $au = v$. Then, in fact, all 2-primary elements from $\tilde{\mathcal{U}}_i$ (resp. $\tilde{\mathcal{U}}_{\binom{v}{j}}$) are linearly ordered as follows:

$$\begin{aligned}
&1_1 < 1_{\binom{3}{3}} < 1_{\binom{3}{1}} < 1_{\binom{3}{2}} < 1_{\binom{6}{3}} < 1_{\binom{6}{1}} < 1_{\binom{6}{2}} \\
&< 1_3 < 1_{\binom{4}{1}} < 1_{\binom{4}{3}} < 1_2 \\
&< 2_1 < 2_{\binom{3}{3}} < 2_{\binom{3}{1}} < 2_{\binom{3}{2}} < 2_{\binom{6}{3}} < 2_{\binom{6}{1}} < 2_{\binom{6}{2}} \\
&< 2_3 < 2_{\binom{4}{1}} < 2_{\binom{4}{3}} < 2_2 < 4_1
\end{aligned}$$

On the other hand $1_5 < 1_4 < 4_1$ and $1_5 < 2_2$ and there are no other relations $<$ between 2-primary elements. Therefore we can procede as in [BD2] and [D] to obtain the following list of indecomposable $\tilde{\mathcal{U}}$-matrices:

$$\begin{aligned}
&(a_1) \text{ with } a = 1, 2, 4. \\
&(a_i) \text{ with } i = 2, 3 \text{ and } a = 1, 2. \\
&(1_i) \text{ with } i = 4, 5. \\
&(a_{\binom{v}{j}}) \text{ with } a = 1, 2 \text{ and } \tbinom{v}{j} \text{ satisfying } v \in \{3, 4, 6\}, j = 1, 2, 3 \text{ except } \tbinom{4}{3}. \\
&\begin{pmatrix} a_i \\ 1_k \end{pmatrix} \text{ with } a = 1, 2 \text{ and } i = 1, 2, 3 \text{ and } k = 4, 5 \text{ except } \begin{pmatrix} 2_2 \\ 1_5 \end{pmatrix}. \\
&\begin{pmatrix} a_{\binom{v}{j}} \\ 1_k \end{pmatrix} \text{ with } a = 1, 2 \text{ and } k = 4, 5 \text{ and } \tbinom{v}{j} \text{ satisfying } v \in \{3, 4, 6\} \\
&\qquad \text{and } j \in \{1, 2, 3\} \text{ except } \tbinom{4}{3}.
\end{aligned}$$

If we consider the 3-part of $\bar{\mathcal{U}}$-matrices the answer is easy. There is only one non-trivial indecomposable matrix (1_1) (isomorphic to (1_i) and $(1_{\binom{v}{j}})$ for all possible values of i, v, j). The Chinese remainder theorem implies that we can glue any 2-primary element from $\tilde{\mathcal{U}}(P, H)$ with any 3-primary element with the same values of P and H. This gives us the following indecomposable $\mathcal{U}$-matrices (taking into account those "crossed out" under Morita-reduction).

$$
\begin{aligned}
&(w_1) \text{ with } w = 1, \ldots, 12. \\
&(w_i) \text{ with } i = 2,3 \text{ and } w = 1, \ldots, 6. \\
&(w_{\binom{v}{j}}) \text{ with } v = 1, \ldots, 6 \text{ and } j = 1,2,3 \text{ and } w = 1, \ldots, 6 \\
&(1_4) \text{ and } (1_5) \\
&\begin{pmatrix} w_i \\ 1_4 \end{pmatrix} \text{ with } i = 1,3 \text{ and } w = 1, \ldots, 6. \\
&\begin{pmatrix} w_{\binom{v}{j}} \\ 1_4 \end{pmatrix} \text{ with } v = 1, \ldots, 6 \text{ and } j = 1,2,3 \text{ and } w = 1, \ldots, 6. \\
&\begin{pmatrix} w_2 \\ 1_5 \end{pmatrix} \text{ with } w = 1,2,3. \\
&\begin{pmatrix} w_i \\ 1_5 \end{pmatrix} \text{ with } i = 1,3 \text{ and } w = 1, \ldots, 6. \\
&\begin{pmatrix} w_{\binom{v}{j}} \\ 1_5 \end{pmatrix} \text{ with } v = 1, \ldots, 6 \text{ and } j = 1,2,3 \text{ and } w = 1, \ldots, 6. \\
&\begin{pmatrix} w_2 \\ 1_4 \end{pmatrix} \text{ with } w = 1,2,3.
\end{aligned}
$$

These indecomposables are in 1-1 correspondence with all elements in $\mathcal{L}_5 - \mathcal{L}_4 - \{S^5\}$. The correspondence is given as follows; compare the list of graphs following (1.1). The correspondence is easily deduced from the notation in (3.3) since the matrices describe the attaching map of the top cell.

$$
\begin{gathered}
(1_5) = \eta_3, \qquad (1_4) = (\eta\eta)_2, \qquad (w_1) = w_1, \\
\begin{pmatrix} w_2 \\ 1_5 \end{pmatrix} = \eta w_1 \eta, \qquad \begin{pmatrix} w_3 \\ 1_4 \end{pmatrix} = \eta\eta w_1 \eta\eta, \qquad \begin{pmatrix} w_3 \\ 1_5 \end{pmatrix} = \eta\eta w_1 \eta, \\
\begin{pmatrix} w_2 \\ 1_4 \end{pmatrix} = \eta w_1 \eta\eta, \qquad (w_2) = \eta\eta w_1, \qquad \begin{pmatrix} w_1 \\ 1_4 \end{pmatrix} = w_1 \eta\eta, \\
(w_3) = \eta w_1, \qquad \begin{pmatrix} w_1 \\ 1_5 \end{pmatrix} = w_1 \eta, \qquad (w_{\binom{v}{3}}) = v_0 \eta\eta w_1, \\
\begin{pmatrix} w_{\binom{v}{2}} \\ 1_5 \end{pmatrix} = \eta v_0 \eta\eta w_1 \eta, \qquad \begin{pmatrix} w_{\binom{v}{1}} \\ 1_4 \end{pmatrix} = \eta\eta v_0 \eta\eta w_1 \eta\eta, \\
\begin{pmatrix} w_{\binom{v}{3}} \\ 1_5 \end{pmatrix} = v_0 \eta\eta w_1 \eta, \qquad (w_{\binom{v}{2}}) = \eta v_0 \eta\eta w_1, \\
\begin{pmatrix} w_{\binom{v}{3}} \\ 1_4 \end{pmatrix} = v_0 \eta\eta w_1 \eta\eta, \qquad (w_{\binom{v}{1}}) = \eta\eta v_0 \eta\eta w_1, \\
\begin{pmatrix} w_{\binom{v}{1}} \\ 1_5 \end{pmatrix} = \eta\eta v_0 \eta\eta w_1 \eta, \qquad \begin{pmatrix} w_{\binom{v}{2}} \\ 1_4 \end{pmatrix} = \eta v_0 \eta\eta w_1 \eta\eta,
\end{gathered}
$$

This completes the proof of the decomposition theorem (2.5). q.e.d.

§ 6 On the representation type of $\mathbf{F}^6$

Since we classified above the indecomposable homotopy types of $\mathbf{F}^5$ we can proceed to classify the homotopy types in $\mathbf{F}^6$. The method is similar to the computation in § 3 for $\mathbf{F}^5$. Similarly as in (3.1) (2) we now obtain for $X \in \mathbf{F}^6$ a homomorphism

$$f : H_6X \to \Gamma_5(X^5) \tag{6.1}$$

where we may assume that the skeleton X^5 again is given by a homology decomposition of X. Moreover X^5 is a one point union of indecomposable objects in $X(\mathcal{L}_5)$ by (2.5). Hence we can compute $\Gamma_5(Y)$ for each object Y in $X(\mathcal{L}_5)$ in order to obtain an explicit form of $\Gamma_5(X^5)$. Then we have to understand the action of the group of homotopy equivalences of X^5 on the Γ_5X^5 and using this action we have to construct a "normal form" of (6.1).

(6.2) Theorem. *For $Y \in X(\mathcal{L}_5)$ the group Γ_5Y is either given by (3.4) or by the following list:*

$\Gamma_5X(g) = 0$ *for* $g = S^0, v_0\eta\eta, v_0\eta\eta w_1, v_0\eta\eta w_1\eta$.
$\Gamma_5X(g) = \mathbb{Z}/24$ *for* $g = \eta_0, \eta v_0\eta\eta, \eta v_0\eta\eta w_1\eta, \eta v_0\eta\eta w_1$.
$\Gamma_5X(g) = \mathbb{Z}/12$ *for* $g = v_0\eta, v_0\eta\eta w_1\eta\eta$.
$\Gamma_5X(g) = \mathbb{Z}/2$ *for* $g = (\eta\eta)_0, v_0, \eta\eta v_0\eta\eta, \eta\eta v_0\eta\eta w_1, \eta\eta v_0\eta\eta w_1\eta$.
$\Gamma_5X(g) = \mathbb{Z}/24 \oplus \mathbb{Z}/12$ *for* $g = \eta v_0\eta, \eta v_0\eta\eta w_1\eta\eta$.
$\Gamma_5X(g) = \mathbb{Z}/2 \oplus \mathbb{Z}/12$ *for* $g = \eta\eta v_0\eta, \eta\eta v_0\eta\eta w_1\eta\eta$.
$\Gamma_5X(g) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ *for* $g = \eta\eta v_0$.
$\Gamma_5X(g) = \mathbb{Z}/24 \oplus \mathbb{Z}/2$ *for* $g = \eta v_0$

The theorem shows that there are exactly 328 elements $g \in \mathcal{L}_5$ for which $\Gamma_5(g)$ is non zero. This shows that the 23×23 matrix in (5.1) has to be replaced by a 328×328 matrix with the additional complication that various $\Gamma_5(Y)$ in (6.2) are given by the direct sum of two cyclic groups. We now consider the special case that X^5 in (6.1) is a one point union of d copies of $X(\eta\eta v_0)$ with $v_0 \in \{1, \dots, 6\}$. In this case we get the following problem.

Let $H = \mathbb{Z}^h$ and $V = \mathbb{Z}^d$ be finitely generated free abelian groups and consider homomorphisms

$$f, f' : H \to V \otimes \mathbb{Z}/2 \oplus V \otimes \mathbb{Z}/2 \tag{6.3}$$

Then f is equivalent to f' if there exist automorphisms $n_0, n_3, n_4 \in \mathrm{Aut}(V)$ and $h \in \mathrm{Aut}(H)$ with

$$\begin{cases} v_0\, n_0 \equiv v_0\, n_4 \pmod{12} \\ \quad n_0 \equiv n_3 \pmod{2} \end{cases} \tag{1}$$

such that $f' = (n_3 \oplus n_4) \otimes \mathbb{Z}/2 \circ f \circ h$. Equivalence classes of such homomoprhisms are in 1-1 correspondence with homotopy types of CW-complexes of the form

$$\underbrace{X(\eta\eta v_0) \vee \ldots \vee X(\eta\eta v_0)}_{d-\text{times}} \cup_f \underbrace{e^6 \cup \ldots \cup e^6}_{h-\text{times}}. \tag{2}$$

(6.4) Proposition. *There are infinitely many indecomposable homotopy types in* $\mathbf{F}^6$ *and* $K_0(\mathbf{F}^6) = \mathbb{Z}^\infty$.

Proof. We consider the decomposition of homotopy types as in (6.3) (2) for $v_0 = 2$. Then (6.3) (1) shows that the 2-primary part of this problem has the representation type of the well known Kroneker quiver

$$\bullet \rightrightarrows \bullet$$

which has tame representation type and infinitely many indecomposable representations over $\mathbb{Z}/2$. q.e.d.

Proposition (6.4) together with theorem (2.5) yields a proof of Theorem B in the introduction.

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