A family of Kähler-Einstein manifolds

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In this note we explain how the work of the second author described in [7],[11] can be used to canonically associate to each compact real-analytic Riemannian manifold a family of complete Kähler-Einstein metrics. The underlying manifolds of these metrics are open disc bundles in the tangent bundle of the original Riemannian manifold.

Let (M, g) be a complete real-analytic Riemannian manifold. If γ is a geodesic in M we can define a map $\psi_\gamma : \mathbb{C} \to TM$ by

$$\psi_\gamma : \sigma + i\tau \mapsto \tau\dot{\gamma}(\sigma).$$

If $S \in (0, \infty]$ we let $T^S M$ denote the open disc bundle in TM consisting of tangent vectors of norm less than S (note that we allow S to be infinite). A complex structure on $T^S M$ is said to be *adapted* with respect to g if ψ_γ is holomorphic on $\psi_\gamma^{-1}(T^S M)$ for each geodesic γ. We shall usually omit the phrase "with respect to g" if it is obvious which metric is being discussed.

Theorem 1 [11]

If (M, g) is a compact real-analytic Riemannian manifold then there exists $S \in (0, \infty]$ such that $T^S M$ carries an adapted complex structure. □

In fact, the adapted complex structure is uniquely determined by (M, g) [7]. The existence of adapted complex structures has also been shown (using a different approach) by Guillemin and Stenzel [5].

From now on we shall take (M, g) to be a compact Riemannian manifold, and use R to denote the *largest* element of $(0, \infty]$ such that $T^R M$ supports an adapted complex structure.

The second author and L. Lempert [7] have shown that the function F on TM which assigns to each tangent vector its norm-square with respect to the metric g is strictly plurisubharmonic on $T^R M$. It follows that $-\log(R^2 - F)$ is a strictly plurisubharmonic exhaustion function, so Grauert's theorem implies that $T^R M$ equipped with the adapted complex structure is a Stein manifold and that $T^S M$ is strictly pseudoconvex whenever $0 < S < R$.

Now it follows from work of Cheng and Yau **[3]** that a strictly pseudoconvex relatively compact domain Ω in a Stein manifold N admits a complete Kähler-Einstein metric. Moreover this metric on Ω is unique if we normalise its Einstein constant to be -1. (The existence result is not explicitly stated in **[3]** for Stein manifolds, although it is used in the later paper **[9]**. We can see that it is true by embedding N as a closed complex submanifold of $\mathbb{C}^n$ for some n, with the closure of the relatively compact domain Ω being contained in an open ball B of sufficiently large radius. The Poincaré metric on B induces a Kähler metric on some neighbourhood of $\bar{\Omega}$ in N whose Ricci curvature is bounded from above by a negative constant. The arguments of Cheng and Yau then give the required result).

We deduce the following theorem.

Theorem 2

Let (M, g) be a compact real-analytic Riemannian manifold. Then $T^S M$ carries a complete Kähler-Einstein metric h_S with Einstein constant -1 whenever $0 < S < R$. □

Remarks

(1) Since the adapted complex structure on $T^S M$ is unique, the natural map of $T^S M$ to itself induced by an isometry of (M, g) is a biholomorphism of $T^S M$ and hence an isometry of the Kähler-Einstein metric h_S (by the uniqueness result of Cheng and Yau **[3]**). It follows that the isometry group of (M, g) injects into the isometry groups of $(T^S M, h_S)$ and $(M, h_S|_M)$. We deduce that if (M, g) is an isotropy irreducible homogeneous space, then g and $h_S|_M$ are homothetic.

(2) It was shown in **[11]** that if (M, g) is a compact symmetric space U/K then an adapted complex structure exists on the entire tangent bundle. If the rank of the symmetric space is one the generic orbit of the action of U on TM has codimension one, so in this case it follows that for each $S \in (0, \infty)$ we have a complete Kähler-Einstein metric *of cohomogeneity one* on $T^S M$. When M is the two-sphere this metric was shown to exist in the course of a classification of Kähler-Einstein metrics of Bianchi IX type **[4]**.

We shall conclude by showing that, under certain topological assumptions on M, the metrics h_S are distinct for different S. We first need to establish some preliminary lemmas.

Lemma 3

Let X be a connected complex manifold of complex dimension n, admitting a smooth strictly plurisubharmonic bounded exhaustion function. Suppose that the nth. integral homology group $H_n(X;Z)$ is finitely generated and nonzero. Let $f : X \to X$ be a holomorphic injection inducing an isomorphism on $H_n(X;Z)$. Then f is a biholomorphism of X.

Proof

The result is essentially contained in Theorem 1 of a paper of N. Mok **[8]**, except that Mok also assumes that X is Caratheodory-hyperbolic. However the only place where he uses this extra condition is in the proof of Proposition 1.1 of his paper, and in fact to establish this result it is sufficient to know that X is *taut* (that is, for each complex manifold Y the set of holomorphic maps from Y to X is a normal family). The tautness of X under our assumptions follows from Corollary 5 of **[10]** and Theorem 2 of **[1]**. □

Given a Riemannian manifold (M, g) we may lift g to a metric $\tilde{g}$ on the universal cover $\tilde{M}$. If $T^S M$ supports an adapted complex structure, then $T^S \tilde{M}$ does also, and the covering map $\tilde{M} \to M$ induces a holomorphic covering $\pi : T^S \tilde{M} \to T^S M$.

Corollary 4

Let (M, g) be a compact connected real-analytic Riemannian manifold which is either orientable or has compact universal cover. Then for each S, Q with $0 < S < Q \leq R$ the spaces $T^S M$ and $T^Q M$ are neither biholomorphic nor antibiholomorphic.

Proof

If M is compact and orientable then $T^Q M$ satisfies the conditions of Lemma 3. Any biholomorphism from $T^Q M$ to $T^S M$ would define a holomorphic injective self-map of $T^Q M$ which was not onto. Moreover such a map would induce an isomorphism on the nth. homology of $T^Q M$, so we would have a contradiction to Lemma 3. Using the fact that the involution of $T^S M$ defined by changing the sign of tangent vectors to M is antiholomorphic [7], we see that there can be no antibiholomorphism from $T^Q M$ onto $T^S M$ either.

Any (anti)biholomorphism between $T^Q M$ and $T^S M$ will induce one between $T^S \tilde{M}$ and $T^Q \tilde{M}$, where $\tilde{M}$ is the universal cover of M. If $\tilde{M}$ is compact then we can argue as above to derive a contradiction (we need the compactness of $\tilde{M}$ to ensure the existence of an exhaustion function on $T^Q \tilde{M}$). □

Lemma 5

Let (M, g) be a compact connected real-analytic Riemannian manifold with universal cover $(\tilde{M}, \tilde{g})$. Then whenever $0 < S < R$, the complex manifold $T^S \tilde{M}$ is not biholomorphic to a product of complex manifolds.

Proof

A result of Huckleberry [6] shows that a connected strictly pseudoconvex relatively compact domain E in a complex manifold cannot be the total space of a locally trivial holomorphic fibre bundle, so in particular cannot be biholomorphic to a product of complex manifolds.

The only use made of relative compactness in his proof is as follows. There is given a sequence of holomorphic maps ϕ_j from a polydisc Δ to E, such that $\phi_j(0)$ converges to a point p_0 on the boundary ∂E. If E is relatively compact, then Montel's theorem implies the convergence of a subsequence of ϕ_j to a holomorphic function $\phi : \Delta \to \bar{E}$, where $\phi(0) = p_0$. An easy argument, involving peak functions (using the strict pseudoconvexity of ∂E at p_0) and the maximum principle, shows that ϕ is identically equal to p_0. It follows that for each $z \in \Delta$, a subsequence of $\phi_j(z)$ converges to p_0. It is this result that we must establish.

In our situation $E = T^S \tilde{M}$ will be connected and strictly pseudoconvex, but not relatively compact in $T^R \tilde{M}$ unless M has finite fundamental group. However, as explained earlier, we have a holomorphic covering $\pi : T^R \tilde{M} \to T^R M$; moreover this restricts to a covering $E \to H = T^S M$ mapping ∂E onto ∂H. Now H is relatively compact in $T^R M$ as well as connected and strictly pseudoconvex, so the argument of the preceding paragraph shows that a subsequence $\pi\phi_{j_k}$ of $\pi\phi_j$ converges to the constant map from Δ to ∂H with value πp_0. Let V be a neighbourhood of πp_0 in $T^R M$ such that $\pi^{-1}(V)$ is a disjoint union of open sets V_α, each containing precisely one element of $\pi^{-1}(\pi p_0)$. Let V_0 be the open set in this family containing p_0. As $\phi_j(\Delta)$ is connected, and as $\phi_j(0)$ converges to p_0 by hypothesis, we see that for sufficiently large k the subsequence ϕ_{j_k} has image contained in V_0. We can make V_0 arbitrarily small by shrinking V suitably, so we deduce that for each $z \in \Delta$ the subsequence $\phi_{j_k}(z)$ converges to p_0.

The remainder of the proof proceeds as in [6]. □

Theorem 6

Let (M, g) be a compact connected real-analytic Riemannian manifold. Suppose further that M is orientable or has compact universal cover, and, as before, let R be the greatest element of $(0, \infty]$ such that $T^R M$ supports an adapted complex structure. If S and Q are distinct positive numbers less than R, then h_S and h_Q are not isometric as Riemannian manifolds.

Proof

The metrics h_S and h_Q lift to Kähler metrics on $T^S\tilde{M}$ and $T^Q\tilde{M}$. If h_S or h_Q is locally reducible, then the associated Kähler metric on $T^Q\tilde{M}$ or $T^S\tilde{M}$ is reducible, contradicting Lemma 5. We see that h_S and h_Q are Kähler metrics with nonvanishing Ricci tensor (by definition) and are not even locally reducible.

Suppose now that h_S and h_Q are isometric. If they are locally symmetric then $T^Q\tilde{M}$ and $T^S\tilde{M}$ are irreducible symmetric spaces of negative Ricci curvature. The isometry between h_S and h_Q lifts to an isometry between the symmetric spaces, and this must be a biholomorphism or antibiholomorphism. It follows that $T^S M$ and $T^Q M$ are biholomorphic or antibiholomorphic, contradicting Corollary 4. If h_S and h_Q are not locally symmetric, then the preceding paragraph shows that for each metric the only covariant constant two-forms are multiples of the Kähler form. For Berger's classification of holonomy groups shows that the existence of other covariant constant two-forms would reduce the holonomy to the symplectic group, and this would force the Ricci tensor to vanish [**2**].

Note also that the volume forms of h_S and h_Q are powers of the Kähler forms (up to multiplication by a constant depending only on the dimension of M). We deduce that any isometry between h_S and h_Q will preserve the Kähler forms up to a sign, so is either a holomorphic or antiholomorphic bijection between $T^S M$ and $T^Q M$. This again contradicts Corollary 4, so no isometry can exist. □

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