GEOMETRIC AND COMBINATORIAL REALIZATIONS OF CRYSTAL GRAPHS

ALISTAIR SAVAGE

Abstract. For irreducible integrable highest weight modules of the finite and affine Lie algebras of type A and D, we define an isomorphism between the geometric realization of the crystal graphs in terms of irreducible components of Nakajima quiver varieties and the combinatorial realizations in terms of Young tableaux and Young walls. For type $A_n^{(1)}$, we extend the Young wall construction to arbitrary level, describing a combinatorial realization of the crystals in terms of new objects which we call Young pyramids.

Introduction

In [3], the author and I. B. Frenkel drew a connection between two different approaches to representations of affine Lie algebras of type A. This yielded an explicit enumeration of the irreducible components of certain Nakajima quiver varieties in terms of Young and Maya diagrams while at the same time yielding an alternative and much simpler geometric proof of the main result of [2] on the construction of bases of representations of affine Lie algebras. The geometric proof involved arguments using commutative diagrams and the Young and Maya diagrams involved in the enumerations where realized in terms of these.

In [14], the author applied the techniques of [3] to the case of the spin representations of type D. Again, an explicit enumeration of the irreducible components in terms of Young diagrams was given. In [14] and in the simplest cases appearing in [3], the geometric action was explicitly computed. Furthermore, in [14] the geometric action of the entire Clifford algebra used in the classical construction of the spin representations was defined and explicitly computed.

In [7, 13], Kashiwara and Saito endowed the irreducible components of Lusztig and Nakajima quiver varieties with the structure of a crystal. In particular, the set of irreducible components of Nakajima's quiver varieties were given the structure of the crystals of highest weight irreducible representations. These crystals have also been realized in purely combinatorial ways. For instance, for the classical Lie algebras, the crystals have been realized on tableaux [6] and for basic representations of affine Lie algebras, the crystals have been realized on Young walls [5]. See [4] for a review of both of these realizations as well as a review of the theory of crystals. It is natural to ask what connection it is possible to make between the geometric and combinatorial realizations of crystals. This is the subject of the current paper. For finite type A, we are able to reinterpret the results of [3] in

Date: November 25, 2003.

2000 *Mathematics Subject Classification.* Primary 16G10, 17B37.

This research was supported in part by the Natural Sciences and Engineering Research Council (NSERC) of Canada and was partially conducted by the author for the Clay Mathematics Institute.

terms of Young tableaux. Specifically, we enumerate the irreducible components of Nakajima's quiver varieties by the Young tableaux appearing in the combinatorial crystal realization. Furthermore, we show this correspondence is actually a crystal isomorphism. This approach has the added advantage of being able to be extended to type D where we also enumerate the irreducible components by the tableaux appearing in the combinatorial crystal realization and show that this correspondence is a crystal isomorphism.

We also consider the affine case. For both type A and D, we obtain an enumeration of the irreducible components of Nakajima's quiver varieties for the basic representation by Young walls. For type A we are able to extend the combinatorial construction beyond level one to arbitrary level. We thus define a new combinatorial object which we call *Young pyramids*, inspired by the geometry of quiver varieties, realizing the crystals of arbitrary integrable irreducible highest weight representations of type $A_n^{(1)}$.

The connection between the geometry of quiver varieties and the combinatorial objects is very beautiful and has numerous benefits. First of all, we obtain a very explicit enumeration of irreducible components of Nakajima's quiver varieties in terms of classical combinatorial objects. Secondly, since Nakajima's theory gives us a realization of Lie algebra representations in terms of the homology of or constructible functions on the quiver varieties, we can translate the action of the Lie algebra to the vector space spanned by the combinatorial objects. Thus we obtain the full (rather than just the crystal) structure of the representation in terms of these objects. This action seems to be new and in some cases is the first realization of the entire structure on such objects. Finally, the geometry naturally explains many characteristics of the combinatorial crystal realizations. In particular, entries in a Young tableaux or columns in a Young wall correspond to specific representations of the quiver. Possible maps between these representations determine the ordering on the index set of the tableaux. Further notions of Young walls such as the ground state wall and the pattern for building the walls are also naturally explained by the geometry of quiver varieties. The geometric interpretation of classical objects such as Young tableaux also suggests further avenues of research. For instance, it would be interesting to interpret such results as the Littlewood-Richardson rule in terms of the geometry of Malkin-Nakajima tensor product varieties.

For other connections between quiver varieties and Young tableaux, we refer the reader to [9, 12] where Nakajima noted a relationship between the homology of quiver varieties and Young tableaux for types A and D using methods different than those employed in the current paper.

The organization of this paper is as follows. In Sections 1 and 2 we review the definition of Lusztig and Nakajima quiver varieties. In Sections 3 and 4 we recall the realization of various crystal graphs by the geometry of quiver varieties and the combinatorics of Young tableaux. We reinterpret some results of [3] on the enumeration of irreducible components of Nakajima quiver varieties in terms of Young tableaux in Section 5 and in Section 6 we show that this enumeration actually yields a crystal isomorphism. In Section 7 we define the crystal isomorphism between the geometric and Young tableaux realizations of type D crystals. We introduce the Young pyramid realization of the crystal graphs of arbitrary irreducible integrable highest weight representations of type $A_n^{(1)}$ in Section 8 and define a crystal isomorphism with the geometric realization. In Section 9 we define an isomorphism between the geometric and Young wall realizations of the crystal graphs of the basic representations of type $D_n^{(1)}$. Finally, in Section 10 we note the connection to the path model of [1, 2] and describe how our results yield a new action in the space of paths and Young walls/pyramids.

The author would like to thank I. B. Frenkel for useful discussions and suggestions and the Max-Planck-Institut für Mathematik where much of this research took place.

1. Lusztig's quiver variety

In this section, we will recount the description given in [8] of Lusztig's quiver variety and its irreducible components. See this reference for details, including proofs.

1.1. **General definitions.** Let I be the set of vertices of the Dynkin graph of a symmetric Kac-Moody Lie algebra $\mathfrak{g}$ and let H be the set of pairs consisting of an edge together with an orientation of it. For $h \in H$, let $\mathrm{in}(h)$ (resp. $\mathrm{out}(h)$) be the incoming (resp. outgoing) vertex of h. We define the involution $^- : H \to H$ to be the function which takes $h \in H$ to the element of H consisting of the same edge with opposite orientation. An *orientation* of our graph is a choice of a subset $\Omega \subset H$ such that $\Omega \cup \bar{\Omega} = H$ and $\Omega \cap \bar{\Omega} = \emptyset$.

Let $\mathcal{V}$ be the category of finite-dimensional I-graded vector spaces $\mathbf{V} = \oplus_{i \in I} \mathbf{V}_i$ over $\mathbb{C}$ with morphisms being linear maps respecting the grading. Then $\mathbf{V} \in \mathcal{V}$ shall denote that $\mathbf{V}$ is an object of $\mathcal{V}$. We identify the graded dimension $\mathbf{v}$ of $\mathbf{V}$ with the element $\sum_i \mathbf{v}_{i \in I} \alpha_i$ of the root lattice of $\mathfrak{g}$. Here the α_i are the simple roots corresponding to the vertices of our quiver (graph with orientation), whose underlying graph is the Dynkin graph of $\mathfrak{g}$.

Given $\mathbf{V} \in \mathcal{V}$, let

$$\mathbf{E_V} = \bigoplus_{h \in H} \mathrm{Hom}(\mathbf{V}_{\mathrm{out}(h)}, \mathbf{V}_{\mathrm{in}(h)}).$$

For any subset $H' \subset H$, let $\mathbf{E}_{\mathbf{V},H'}$ be the subspace of $\mathbf{E_V}$ consisting of all vectors $x = (x_h)$ such that $x_h = 0$ whenever $h \notin H'$. The algebraic group $G_{\mathbf{V}} = \prod_i \mathrm{Aut}(\mathbf{V}_i)$ acts on $\mathbf{E_V}$ and $\mathbf{E}_{\mathbf{V},H'}$ by

$$(g, x) = ((g_i), (x_h)) \mapsto (g_{\mathrm{in}(h)} x_h g_{\mathrm{out}(h)}^{-1}).$$

Define the function $\varepsilon : H \to \{-1, 1\}$ by $\varepsilon(h) = 1$ for all $h \in \Omega$ and $\varepsilon(h) = -1$ for all $h \in \bar{\Omega}$. The Lie algebra of $G_{\mathbf{V}}$ is $\mathbf{gl_V} = \prod_i \mathrm{End}(\mathbf{V}_i)$ and it acts on $\mathbf{E_V}$ by

$$(a, x) = ((a_i), (x_h)) \mapsto [a, x] = (x'_h) = (a_{\mathrm{in}(h)} x_h - x_h a_{\mathrm{out}(h)}).$$

Let $\langle \cdot, \cdot \rangle$ be the nondegenerate, $G_{\mathbf{V}}$-invariant, symplectic form on $\mathbf{E_V}$ with values in $\mathbb{C}$ defined by

$$\langle x, y \rangle = \sum_{h \in H} \varepsilon(h) \operatorname{tr}(x_h y_{\bar{h}}).$$

Note that $\mathbf{E_V}$ can be considered as the cotangent space of $\mathbf{E}_{\mathbf{V},\Omega}$ under this form.

The moment map associated to the $G_{\mathbf{V}}$-action on the symplectic vector space $\mathbf{E_V}$ is the map $\psi : \mathbf{E_V} \to \mathbf{gl_V}$ with i-component $\psi_i : \mathbf{E_V} \to \mathrm{End}\,\mathbf{V}_i$ given by

$$\psi_i(x) = \sum_{h \in H,\, \mathrm{in}(h) = i} \varepsilon(h) x_h x_{\bar{h}}.$$

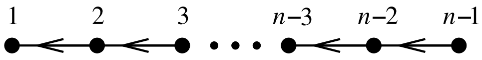


FIGURE 1. The quiver of type A_{n-1}.

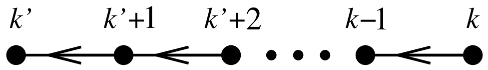


FIGURE 2. The string depicting the representation $(\mathbf{V}(k',k), x(k',k))$.

Definition 1.1 ([8]). *An element $x \in \mathbf{E}_\mathbf{V}$ is said to be* nilpotent *if there exists an $N \geq 1$ such that for any sequence $h_1', h_2', \ldots, h_N'$ in H satisfying $\mathrm{out}(h_1') = \mathrm{in}(h_2')$, $\mathrm{out}(h_2') = \mathrm{in}(h_3')$, ..., $\mathrm{out}(h_{N-1}') = \mathrm{in}(h_N')$, the composition $x_{h_1'} x_{h_2'} \ldots x_{h_N'} : \mathbf{V}_{\mathrm{out}(h_N')} \to \mathbf{V}_{\mathrm{in}(h_1')}$ is zero.*

Definition 1.2 ([8]). *Let $\Lambda_\mathbf{V}$ be the set of all nilpotent elements $x \in \mathbf{E}_\mathbf{V}$ such that $\psi_i(x) = 0$ for all $i \in I$.*

Proposition 1.3 ([8]). *For $\mathfrak{g}$ a symmetric Lie algebra of finite type, the irreducible components of $\Lambda_\mathbf{V}$ are the closures of the conormal bundles of the various $G_\mathbf{V}$-orbits in $\mathbf{E}_{\mathbf{V},\Omega}$.*

1.2. **Type** *A*. Let $\mathfrak{g} = \mathfrak{sl}_n$ be the simple Lie algebra of type A_{n-1}. Let $I = \{1, \ldots, n-1\}$ be the set of vertices of a graph with the set of oriented edges given by

$$H = \{h_{i,j} \mid i,j \in I,\ |i-j| = 1\}.$$

For two adjacent vertices i and j, $h_{i,j}$ is the oriented edge from vertex i to vertex j. Thus $\mathrm{out}(h_{i,j}) = i$ and $\mathrm{in}(h_{i,j}) = j$. We define the involution $^- : H \to H$ as the function that interchanges $h_{i,j}$ and $h_{j,i}$. Let $\Omega = \{h_{i,i-1} \mid 2 \leq i \leq n-1\}$. We picture this quiver as in Figure 1.

For two integers k', k such that $1 \leq k' \leq k \leq n-1$, define $\mathbf{V}(k',k) \in \mathcal{V}$ to be the vector space with basis $\{e_r \mid k' \leq r \leq k\}$. We require that e_r has degree $r \in I$. Let $x(k',k) \in \mathbf{E}_{\mathbf{V}(k',k),\Omega}$ be defined by $x(k',k) : e_r \mapsto e_{r-1}$ for $k' \leq r \leq k$, where $e_{k'-1} = 0$. It is clear that $(\mathbf{V}(k',k), x(k',k))$ is an indecomposable representation of our quiver (i.e. element of $\mathbf{E}_{\mathbf{V},\Omega}$). Conversely, any indecomposable finite-dimensional representation $(\mathbf{V}, x)$ of our quiver is isomorphic to some $(\mathbf{V}(k',k), x(k',k))$. We picture such an indecomposable representation as a string with endpoints k' and k (see Figure 2).

Let Z be the set of all pairs (k',k) of integers such that $1 \leq k' \leq k \leq n-1$ and let $\tilde{Z}$ be the set of all functions $Z \to \mathbf{N}$ with finite support.

It is easy to see that for $\mathbf{V} \in \mathcal{V}$, the set of $G_\mathbf{V}$-orbits in $\mathbf{E}_{\mathbf{V},\Omega}$ is naturally indexed by the subset $\tilde{Z}_\mathbf{V}$ of $\tilde{Z}$ consisting of those $f \in \tilde{Z}$ such that

$$\sum_{k' \leq i \leq k} f(k',k) = \dim \mathbf{V}_i$$

for all $i \in I$. Here the sum is over all k', k such that $1 \leq k' \leq i \leq k \leq n-1$. Corresponding to a given f is the orbit consisting of all representations isomorphic to a sum of the indecomposable representations $(V(k',k), x(k',k))$, each occuring with multiplicity $f(k',k)$. Denote by $\mathcal{O}_f$ the $G_\mathbf{V}$-orbit corresponding to $f \in \tilde{Z}_\mathbf{V}$.

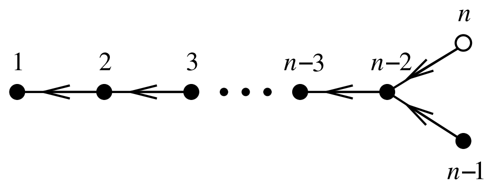


FIGURE 3. The quiver of type D_n. We represent the nth vertex by an open dot to distinguish it from the $(n-1)$st vertex.

Let $\mathcal{C}_f$ be the conormal bundle to $\mathcal{O}_f$ and let $\bar{\mathcal{C}}_f$ be its closure. We then have the following proposition.

Proposition 1.4. *The map $f \to \bar{\mathcal{C}}_f$ is a one-to-one correspondence between the set $\tilde{Z}_{\mathbf{V}}$ and the set of irreducible components of $\Lambda_{\mathbf{V}}$.*

Proof. This follows immediately from Proposition 1.3. □

1.3. **Type** D. Now let $\mathfrak{g} = \mathfrak{so}_{2n}\mathbb{C}$ be the simple Lie algebra of type D_n. Let $I = \{1, 2, \ldots, n\}$ be the set of vertices of the Dynkin graph of $\mathfrak{g}$, labelled and oriented as in Figure 3. As for type A, we label each oriented edge by its incoming and outgoing vertices. That is, if vertices i and j are connected by an edge, $h_{i,j}$ denotes the oriented edge with $\mathrm{out}(h) = i$, $\mathrm{in}(h) = j$.

2. Nakajima's quiver variety

We introduce here a description of the quiver varieties first presented in [10]. See [10] and [11] for details.

Definition 2.1 ([10])**.** *For $\mathbf{v}, \mathbf{w} \in (\mathbb{Z}_{\geq 0})^I$, choose I-graded vector spaces $\mathbf{V}$ and $\mathbf{W}$ of graded dimensions $\mathbf{v}$ and $\mathbf{w}$ respectively. We associate $\mathbf{w}$ with the element $\sum_i \mathbf{w}_i \omega_i$ of the root lattice of $\mathfrak{g}$, where the ω_i are the fundamental weights of $\mathfrak{g}$. Recall that we identified $\mathbf{v}$ with the weight $\sum_i \mathbf{v}_i \alpha_i$. Then define*

$$\Lambda \equiv \Lambda(\mathbf{v}, \mathbf{w}) = \Lambda_{\mathbf{V}} \times \sum_{i \in I} \mathrm{Hom}(\mathbf{V}_i, \mathbf{W}_i).$$

Now, suppose that $\mathbf{S}$ is an I-graded subspace of $\mathbf{V}$. For $x \in \Lambda_{\mathbf{V}}$ we say that $\mathbf{S}$ is *x-stable* if $x(\mathbf{S}) \subset \mathbf{S}$.

Definition 2.2 ([10])**.** *Let $\Lambda^{st} = \Lambda(\mathbf{v}, \mathbf{w})^{st}$ be the set of all $(x, t) \in \Lambda(\mathbf{v}, \mathbf{w})$ satisfying the following condition: If $\mathbf{S} = (\mathbf{S}_i)$ with $\mathbf{S}_i \subset \mathbf{V}_i$ is x-stable and $t_i(\mathbf{S}_i) = 0$ for $i \in I$, then $\mathbf{S}_i = 0$ for $i \in I$.*

The group $G_{\mathbf{V}}$ acts on $\Lambda(\mathbf{v}, \mathbf{w})$ via

$$(g, (x, t)) = ((g_i), ((x_h), (t_i))) \mapsto ((g_{\mathrm{in}(h)} x_h g_{\mathrm{out}(h)}^{-1}), t_i g_i^{-1}).$$

and the stabilizer of any point of $\Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}}$ in $G_{\mathbf{V}}$ is trivial (see [11, Lemma 3.10]). We then make the following definition.

Definition 2.3 ([10])**.** *Let $\mathcal{L} \equiv \mathcal{L}(\mathbf{v}, \mathbf{w}) = \Lambda(\mathbf{v}, \mathbf{w})^{st}/G_{\mathbf{V}}$.*

Let $\mathrm{Irr}\,\mathcal{L}(\mathbf{v}, \mathbf{w})$ (resp. $\mathrm{Irr}\,\Lambda(\mathbf{v}, \mathbf{w})$) be the set of irreducible components of $\mathcal{L}(\mathbf{v}, \mathbf{w})$ (resp. $\Lambda(\mathbf{v}, \mathbf{w})$). Then $\mathrm{Irr}\,\mathcal{L}(\mathbf{v}, \mathbf{w})$ can be identified with

$$\{Y \in \mathrm{Irr}\,\Lambda(\mathbf{v}, \mathbf{w}) \mid Y \cap \Lambda(\mathbf{v}, \mathbf{w})^{\mathrm{st}} \neq \emptyset\}.$$

Specifically, the irreducible components of $\operatorname{Irr}\mathcal{L}(\mathbf{v},\mathbf{w})$ are precisely those

$$X_f \stackrel{\text{def}}{=} \left(\left(\bar{\mathcal{C}}_f \times \sum_{i\in I} \operatorname{Hom}(\mathbf{V}_i, \mathbf{W}_i)\right) \cap \Lambda(\mathbf{v},\mathbf{w})^{\text{st}}\right)/G_{\mathbf{V}}$$

which are nonempty.

The following will be used in the sequel.

Lemma 2.4. *One has*

$$\Lambda^{st} = \left\{ (x,t) \in \Lambda \;\middle|\; \left(\bigcap_{h\,:\,\operatorname{out}(h)=i} \ker x_h\right) \cap \ker t_i = 0 \ \forall i \right\}.$$

Proof. Since each $\left(\bigcap_{h\,:\,\operatorname{out}(h)=i} \ker x_h\right) \cap \ker t_i$ is x-stable, the lefthand side is obviously contained in the righthand side. Now suppose x is an element of the righthand side. Let $\mathbf{S} = (\mathbf{S}_i)$ with $\mathbf{S}_i \subset \mathbf{V}_i$ be x-stable and $t_i(\mathbf{S}_i) = 0$ for $i \in I$. Assume that $\mathbf{S} \neq 0$. Since all elements of Λ are nilpotent, we can find a minimal value of N such that the condition in Definition 1.1 is satisfied. Then we can find a $v \in \mathbf{S}_i$ for some i and a sequence $h'_1, h'_2, \dots, h'_{N-1}$ (empty if $N = 1$) in H such that $\operatorname{out}(h'_1) = \operatorname{in}(h'_2)$, $\operatorname{out}(h'_2) = \operatorname{in}(h'_3)$, $\dots$, $\operatorname{out}(h'_{N-2}) = \operatorname{in}(h'_{N-1})$ and $v' = x_{h'_1} x_{h'_2} \dots x_{h'_{N-1}}(v) \neq 0$. Now, $v' \in \mathbf{S}_{i'}$ for some $i' \in I$ by the stability of $\mathbf{S}$ (hence $t_{i'}(v') = 0$) and $v' \in \bigcap_{h\,:\,\operatorname{out}(h)=i'} \ker x_h$ by our choice of N. This contradicts the fact that x is an element of the righthand side. $\square$

3. Crystal action on quiver varieties

In this section, we review the realization of the crystal graph of integrable highest weight representations of a Kac-Moody algebra $\mathfrak{g}$ with symmetric Cartan matrix via quiver varieties. See [7, 13] for details, including proofs.

Let $\mathbf{w}, \mathbf{v}, \mathbf{v}', \mathbf{v}'' \in (\mathbb{Z}_{\geq 0})^I$ be such that $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$. Consider the maps

$$\Lambda(\mathbf{v}'', \mathbf{0}) \times \Lambda(\mathbf{v}', \mathbf{w}) \stackrel{p_1}{\leftarrow} \tilde{\mathbf{F}}(\mathbf{v}, \mathbf{w}; \mathbf{v}'') \stackrel{p_2}{\to} \mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v}'') \stackrel{p_3}{\to} \Lambda(\mathbf{v}, \mathbf{w}), \tag{3.1}$$

where the notation is as follows. A point of $\mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v}'')$ is a point $(x,t) \in \Lambda(\mathbf{v},\mathbf{w})$ together with an I-graded, x-stable subspace $\mathbf{S}$ of $\mathbf{V}$ such that $\dim \mathbf{S} = \mathbf{v}' = \mathbf{v} - \mathbf{v}''$. A point of $\tilde{\mathbf{F}}(\mathbf{v}, \mathbf{w}; \mathbf{v}'')$ is a point $(x,t,\mathbf{S})$ of $\mathbf{F}(\mathbf{v}, \mathbf{w}; \mathbf{v}'')$ together with a collection of isomorphisms $R'_i : \mathbf{V}'_i \cong \mathbf{S}_i$ and $R''_i : \mathbf{V}''_i \cong \mathbf{V}_i/\mathbf{S}_i$ for each $i \in I$. Then we define $p_2(x,t,\mathbf{S},R',R'') = (x,t,\mathbf{S})$, $p_3(x,t,\mathbf{S}) = (x,t)$ and $p_1(x,t,\mathbf{S},R',R'') = (x'',x',t')$ where x'', x', t' are determined by

$$\begin{aligned} R'_{\operatorname{in}(h)} x'_h &= x_h R'_{\operatorname{out}(h)} : \mathbf{V}'_{\operatorname{out}(h)} \to \mathbf{S}_{\operatorname{in}(h)}, \\ t'_i &= t_i R'_i : \mathbf{V}'_i \to \mathbf{W}_i \\ R''_{\operatorname{in}(h)} x''_h &= x_h R''_{\operatorname{out}(h)} : \mathbf{V}''_{\operatorname{out}(h)} \to \mathbf{V}_{\operatorname{in}(h)}/\mathbf{S}_{\operatorname{in}(h)}. \end{aligned}$$

It follows that x' and x'' are nilpotent.

Lemma 3.1 ([10, Lemma 10.3])**.** *One has*

$$(p_3 \circ p_2)^{-1}(\Lambda(\mathbf{v},\mathbf{w})^{st}) \subset p_1^{-1}(\Lambda(\mathbf{v}'',\mathbf{0}) \times \Lambda(\mathbf{v}',\mathbf{w})^{st}).$$

Thus, we can restrict (3.1) to Λ^{st}, forget the $\Lambda(\mathbf{v}'',\mathbf{0})$-factor and consider the quotient by $G_{\mathbf{V}}$, $G_{\mathbf{V}'}$. This yields the diagram

$$\mathcal{L}(\mathbf{v}', \mathbf{w}) \stackrel{\pi_1}{\leftarrow} \mathcal{F}(\mathbf{v}, \mathbf{w}; \mathbf{v} - \mathbf{v}') \stackrel{\pi_2}{\to} \mathcal{L}(\mathbf{v}, \mathbf{w}), \tag{3.2}$$

where

$$\mathcal{F}(\mathbf{v},\mathbf{w};\mathbf{v}-\mathbf{v}') \stackrel{\text{def}}{=} \{(x,t,\mathbf{S}) \in \mathbf{F}(\mathbf{v},\mathbf{w};\mathbf{v}-\mathbf{v}') \mid (x,t) \in \Lambda(\mathbf{v},\mathbf{w})^{\text{st}}\}/G_{\mathbf{V}}.$$

For $i \in I$ define $\varepsilon_i : \Lambda(\mathbf{v},\mathbf{w}) \to \mathbb{Z}_{\geq 0}$ by

$$\varepsilon_i((x,t)) = \dim_{\mathbb{C}} \operatorname{Coker}\left(\bigoplus_{h\,:\,\text{in}(h)=i} V_{\text{out}(h)} \xrightarrow{(x_h)} V_i\right).$$

Then, for $c \in \mathbb{Z}_{\geq 0}$, let

$$\mathcal{L}(\mathbf{v},\mathbf{w})_{i,c} = \{[x,t] \in \mathcal{L}(\mathbf{v},\mathbf{w}) \mid \varepsilon_i((x,t)) = c\}$$

where $[x,t]$ denotes the $G_{\mathbf{V}}$-orbit through the point (x,t). $\mathcal{L}(\mathbf{v},\mathbf{w})_{i,c}$ is a locally closed subvariety of $\mathcal{L}(\mathbf{v},\mathbf{w})$.

Assume $\mathcal{L}(\mathbf{v},\mathbf{w})_{i,c} \neq \emptyset$ and let $\mathbf{v}' = \mathbf{v} - c\mathbf{e}^i$ where $\mathbf{e}^i_j = \delta_{ij}$. Then

$$\pi_1^{-1}(\mathcal{L}(\mathbf{v}',\mathbf{w})_{i,0}) = \pi_2^{-1}(\mathcal{L}(\mathbf{v},\mathbf{w})_{i,c}).$$

Let

$$\mathcal{F}(\mathbf{v},\mathbf{w};c\mathbf{e}^i)_{i,0} = \pi_1^{-1}(\mathcal{L}(\mathbf{v}',\mathbf{w})_{i,0}) = \pi_2^{-1}(\mathcal{L}(\mathbf{v},\mathbf{w})_{i,c}).$$

We then have the following diagram.

$$\mathcal{L}(\mathbf{v}',\mathbf{w})_{i,0} \xleftarrow{\pi_1} \mathcal{F}(\mathbf{v},\mathbf{w};c\mathbf{e}^i)_{i,0} \xrightarrow{\pi_2} \mathcal{L}(\mathbf{v},\mathbf{w})_{i,c} \tag{3.3}$$

The restriction of π_2 to $\mathcal{F}(\mathbf{v},\mathbf{w};c\mathbf{e}^i)_{i,0}$ is an isomorphism since the only possible choice for the subspace $\mathbf{S}$ of $\mathbf{V}$ is to have $\mathbf{S}_j = \mathbf{V}_j$ for $j \neq i$ and $\mathbf{S}_i$ equal to the sum of the images of the x_h with $\text{in}(h) = i$. $\mathcal{L}(\mathbf{v}',\mathbf{w})_{i,0}$ is an open subvariety of $\mathcal{L}(\mathbf{v}',\mathbf{w})$.

Lemma 3.2 ([13]). (1) *For any $i \in I$,*

$$\mathcal{L}(\mathbf{0},\mathbf{w})_{i,c} = \begin{cases} pt & \text{if } c = 0 \\ \emptyset & \text{if } c > 0 \end{cases}.$$

(2) *Suppose $\mathcal{L}(\mathbf{v},\mathbf{w})_{i,c} \neq \emptyset$ and $\mathbf{v}' = \mathbf{v} - c\mathbf{e}^i$. Then the fiber of the restriction of π_1 to $\mathcal{F}(\mathbf{v},\mathbf{w};c\mathbf{e}^i)_{i,0}$ is isomorphic to a Grassmanian variety.*

Corollary 3.3. *Suppose $\mathcal{L}(\mathbf{v},\mathbf{w})_{i,c} \neq \emptyset$. Then there is a 1-1 correspondence between the set of irreducible components of $\mathcal{L}(\mathbf{v}-c\mathbf{e}^i,\mathbf{w})_{i,0}$ and the set of irreducible components of $\mathcal{L}(\mathbf{v},\mathbf{w})_{i,c}$.*

Let $B(\mathbf{v},\mathbf{w})$ denote the set of irreducible components of $\mathcal{L}(\mathbf{v},\mathbf{w})$ and let $B(\mathbf{w}) = \bigsqcup_{\mathbf{v}} B(\mathbf{v},\mathbf{w})$. For $X \in B(\mathbf{v},\mathbf{w})$, let $\varepsilon_i(X) = \varepsilon_i((x,t))$ for a generic point $[x,t] \in X$. Then for $c \in \mathbb{Z}_{\geq 0}$ define

$$B(\mathbf{v},\mathbf{w})_{i,c} = \{X \in B(\mathbf{v},\mathbf{w}) \mid \varepsilon_i(X) = c\}.$$

Then by Corollary 3.3, $B(\mathbf{v}-c\mathbf{e}^i,\mathbf{w})_{i,0} \cong B(\mathbf{v},\mathbf{w})_{i,c}$.

Suppose that $\bar{X} \in B(\mathbf{v}-c\mathbf{e}^i,\mathbf{w})_{i,0}$ corresponds to $X \in B(\mathbf{v},\mathbf{w})_{i,c}$ by the above isomorphism. Then we define maps

$$\tilde{f}_i^c : B(\mathbf{v}-c\mathbf{e}^i,\mathbf{w})_{i,0} \to B(\mathbf{v},\mathbf{w})_{i,c}, \quad \tilde{f}_i^c(\bar{X}) = X,$$
$$\tilde{e}_i^c : B(\mathbf{v},\mathbf{w})_{i,c} \to B(\mathbf{v}-c\mathbf{e}^i,\mathbf{w})_{i,0}, \quad \tilde{e}_i^c(X) = \bar{X}.$$

We then define the maps

$$\tilde{e}_i, \tilde{f}_i : B(\mathbf{w}) \to B(\mathbf{w}) \sqcup \{0\}$$

by

$$\tilde{e}_i : B(\mathbf{v},\mathbf{w})_{i,c} \xrightarrow{\tilde{e}_i^c} B(\mathbf{v}-c\mathbf{e}^i,\mathbf{w})_{i,0} \xrightarrow{\tilde{f}_i^{c-1}} B(\mathbf{v}-\mathbf{e}^i,\mathbf{w})_{i,c-1},$$
$$\tilde{f}_i : B(\mathbf{v},\mathbf{w})_{i,c} \xrightarrow{\tilde{e}_i^c} B(\mathbf{v}-c\mathbf{e}^i,\mathbf{w})_{i,0} \xrightarrow{\tilde{f}_i^{c+1}} B(\mathbf{v}+\mathbf{e}^i,\mathbf{w})_{i,c+1}.$$

We set $\tilde{e}_i(X) = 0$ for $X \in B(\mathbf{v},\mathbf{w})_{i,0}$ and $\tilde{f}_i(X) = 0$ for $X \in B(\mathbf{v},\mathbf{w})_{i,c}$ with $B(\mathbf{v},\mathbf{w})_{i,c+1} = \emptyset$. We also define

$$\mathrm{wt} : B(\mathbf{w}) \to P, \quad \mathrm{wt}(X) = \sum_{i\in I} (\mathbf{w}_i\omega_i - \mathbf{v}_i\alpha_i) \text{ for } X \in B(\mathbf{v},\mathbf{w}),$$
$$\varphi_i(X) = \varepsilon_i(X) + \langle h_i, \mathrm{wt}(X)\rangle .$$

Recall that we can consider $\mathbf{w}$ to be an dominant integral weight by $\mathbf{w} = \sum_i \mathbf{w}_i\omega_i$.

Proposition 3.4 ([13]). *$B(\mathbf{w})$ is a crystal and is isomorphic to the crystal of the highest weight $U_q(\mathfrak{g})$-module with highest weight $\mathbf{w}$.*

4. Crystal action on tableaux

We now review the realization of the crystal graph of finite dimensional representations of Lie algebras of type A and D via Young tableaux. See [6, 4] for details, including proofs.

Given a set of crystal graphs, the tensor product rule for crystals gives a very explicit description of the action of the Kashiwara operators on the multifold tensor product of these graphs. Let $\mathcal{B}_j$ be $\mathfrak{g}$-crystals for $i = 1,\dots,N$. Fix $i \in I$ and let $b = b_1 \otimes \cdots \otimes b_N \in \mathcal{B}_1 \otimes \cdots \otimes \mathcal{B}_N$. To each b_j, assign a series of $-$'s and $+$'s with as many $-$'s as $\varepsilon_i(b_j)$ followed by as many $+$'s as $\varphi_i(b_j)$. In the sequence obtained by concatenating the series for the individual b_j's, cancel all $(+,-)$ pairs to obtain a sequence i-sgn(b) of $-$'s followed by $+$'s. Then the tensor product rule tells us that $\tilde{e}_i$ acts on the b_j corresponding to the rightmost $-$ in i-sgn(b) and $\tilde{f}_i$ acts on the b_j corresponding to the leftmost $+$ in i-sgn(b).

4.1. **Type A.** Let $\mathfrak{g} = \mathfrak{sl}_n$ be the Lie algebra of type A_{n-1}. Recall that $\mathfrak{g}$ acting on the space $\mathbb{C}^n$ by left multiplication yields the *vector representation.* It has crystal $\mathbf{B} = \{\boxed{j} \mid j = 1,\dots,n\}$ with crystal graph

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-2} \boxed{n{-}1} \xrightarrow{n-1} \boxed{n}\ .$$

and $\mathrm{wt}(\boxed{j}) = \epsilon_j$ for $j = 1,\dots,n$. Here the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is the set of traceless diagonal matrices and ϵ_j is the element of $\mathfrak{h}^*$ such that $\epsilon_j(e_{i,i}) = \delta_{i,j}$.

A tableau with m boxes represents an element of the tensor product crystal $\mathbf{B}^{\otimes m}$ by the *Far-Eastern reading* which proceeds down columns from top to bottom and from right to left. For example

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3&4\\\cline{1-3} 4&4\\\cline{1-2} 5\\\cline{1-1}\end{array} = \boxed{3}\otimes\boxed{2}\otimes\boxed{4}\otimes\boxed{1}\otimes\boxed{3}\otimes\boxed{4}\otimes\boxed{1}\otimes\boxed{2}\otimes\boxed{4}\otimes\boxed{5}\ .$$

From now on, when we say that one box or entry in a tableau is earlier or later than another, we are using this ordering. Earlier means "to the left of" and later means "to the right of." We avoid the words left and right to prevent confusion with the spatial location of boxes in the tableau. To denote the spatial arrangement of boxes or entries in a tableau, we will use the compass directions north, south, east and west. North is up on the page, east is right, etc. Now since

$$\varepsilon_i(\boxed{i{+}1}) = 1,\ \varphi_i(\boxed{\ i\ }) = 1$$

and ε_i and φ_i take the value zero on all other elements of $\mathbf{B}$, the tensor product rule tells us that to compute the action of the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on a given tableau we neglect all entries not equal to i or $i+1$ and cancel $(i,i+1)$ pairs (in the ordering given by the Far-Eastern reading). We will say that such a pair of entries are *i-matched.* Then $\tilde{f}_i$ acts by changing the earliest remaining i to an $i+1$ and $\tilde{e}_i$ acts by changing the latest remaining $i+1$ to an i. If no i (resp. $i+1$) entries remain, then $\tilde{f}_i$ (resp. $\tilde{e}_i$) kills the tableau.

As an example, consider the action of $\tilde{e}_2$ and $\tilde{f}_2$ on the tableau

$$T = \begin{array}{|c|c|c|c|c|}\hline 1&2&2&3&3\\\hline 2&3&4\\\cline{1-3} 4&5\\\cline{1-2} 5\\\cline{1-1}\end{array}\ .$$

Neglecting all entries but 2 and 3, we obtain

$$\begin{array}{|c|c|c|c|c|}\hline \ &2&2&3&3\\\hline 2&3&\ \\\cline{1-3} \ &\ \\\cline{1-2} \ \\\cline{1-1}\end{array}\ .$$

Read off in the Far-Eastern reading, the entries are $(3,3,2,2,3,2)$. We then cancel the 2-matched $(2,3)$ pair to leave the tableau

$$\begin{array}{|c|c|c|c|c|}\hline \ &\ &2&3&3\\\hline 2&\ &\ \\\cline{1-3} \ &\ \\\cline{1-2} \ \\\cline{1-1}\end{array}\ .$$

Thus

$$\tilde{f}_2(T) = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&3&3\\\hline 2&3&4\\\cline{1-3} 4&5\\\cline{1-2} 5\\\cline{1-1}\end{array}\ ,\quad \tilde{e}_2(T) = \begin{array}{|c|c|c|c|c|}\hline 1&2&2&2&3\\\hline 2&3&4\\\cline{1-3} 4&5\\\cline{1-2} 5\\\cline{1-1}\end{array}\ .$$

4.2. **Type D.** Let $\mathfrak{g} = \mathfrak{so}_{2n}\mathbb{C}$ be the simple Lie algebra of type D_n. Let $\epsilon_i : M_{2n\times 2n}(\mathbb{C}) \to \mathbb{C}$ be the linear functional defined by $\epsilon_i(T) = T_{ii}$. Then the simple roots and fundamental weights are

$$\begin{aligned}
\alpha_i &= \epsilon_i - \epsilon_{i+1}, \quad 1 \le i \le n-1,\\
\alpha_n &= \epsilon_{n-1} + \epsilon_n,\\
\omega_i &= \epsilon_1 + \cdots + \epsilon_i, \quad 1 \le i \le n-2,\\
\omega_{n-1} &= \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n),\\
\omega_n &= \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n).
\end{aligned}$$

Let $\mathbf{N} = \{1, 2, \ldots, n, \bar{n}, \ldots, \bar{2}, \bar{1}\}$. We put a linear ordering on $\mathbf{N}$ by

$$1 \prec 2 \prec \cdots \prec \begin{matrix} n \\ \bar{n} \end{matrix} \prec \cdots \prec \bar{2} \prec \bar{1}$$

where the order between n and $\bar{n}$ is not defined. Will will use roman characters (e.g. i and j) to refer to the elements $1, \dots, n \in \mathbf{N}$ and greek characters (e.g α and β) to refer to arbitrary elements of $\mathbf{N}$. Recall that $\mathfrak{g}$ acting on the space $\mathbb{C}^{2n}$ by left multiplication yields the *vector representation.* It has crystal $\mathbf{B} = \{\boxed{\alpha} \mid \alpha \in \mathbf{N}\}$ with the following crystal graph.

$$\begin{tikzcd}
 & & & & \boxed{n} \arrow[dr, "n"] & & & & \\
\boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \cdots \arrow[r, "n-2"] & \boxed{n-1} \arrow[ur, "n-1"] \arrow[dr, "n"'] & & \boxed{\overline{n-1}} \arrow[r, "n-2"] & \cdots \arrow[r, "2"] & \boxed{\bar{2}} \arrow[r, "1"] & \boxed{\bar{1}} \\
 & & & & \boxed{\bar{n}} \arrow[ur, "n-1"'] & & & &
\end{tikzcd}$$

We have that

$$\mathrm{wt}(\boxed{j}) = \epsilon_j, \quad \mathrm{wt}(\boxed{\bar{j}}) = -\epsilon_j \quad 1 \le j \le n.$$

Now let $\mathbf{V}_{sp}^{+}$ and $\mathbf{V}_{sp}^{-}$ be the spin representations $V(\omega_n)$ and $V(\omega_{n-1})$ respectively. The crystal graphs $\mathbf{B}_{sp}^{\pm}$ of these representations can be realized using the (generalized) Young tableaux consisting of half boxes:

$$\mathbf{B}_{sp}^{+} = \left\{ \begin{array}{|c|}\hline t_1 \\ \hline \vdots \\ \hline t_n \\ \hline \end{array} \;\middle|\; \begin{array}{c} t_i \in \mathbf{N},\ t_1 \prec \cdots \prec t_n, \\ i \text{ and } \bar{i} \text{ do not appear simultaneously,} \\ \text{if } t_k = n, \text{ then } n-k \text{ is even,} \\ \text{if } t_k = \bar{n}, \text{ then } n-k \text{ is odd.} \end{array} \right\},$$

$$\mathbf{B}_{sp}^{-} = \left\{ \begin{array}{|c|}\hline t_1 \\ \hline \vdots \\ \hline t_n \\ \hline \end{array} \;\middle|\; \begin{array}{c} t_i \in \mathbf{N},\ t_1 \prec \cdots \prec t_n, \\ i \text{ and } \bar{i} \text{ do not appear simultaneously,} \\ \text{if } t_k = n, \text{ then } n-k \text{ is odd,} \\ \text{if } t_k = \bar{n}, \text{ then } n-k \text{ is even.} \end{array} \right\}.$$

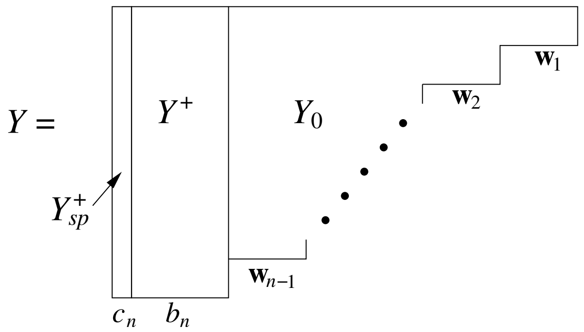
The action of the Kashiwara operators is as follows.

$$\begin{array}{|c|}\hline \\ \hline i \\ \hline \vdots \\ \hline \overline{i+1} \\ \hline \\ \hline \end{array} \xrightarrow{i} \begin{array}{|c|}\hline \\ \hline i+1 \\ \hline \vdots \\ \hline \bar{i} \\ \hline \\ \hline \end{array}, \; i \neq n, \qquad \begin{array}{|c|}\hline \\ \hline n-1 \\ \hline n \\ \hline \\ \hline \end{array} \xrightarrow{n} \begin{array}{|c|}\hline \\ \hline \bar{n} \\ \hline \overline{n-1} \\ \hline \\ \hline \end{array}. \tag{4.1}$$

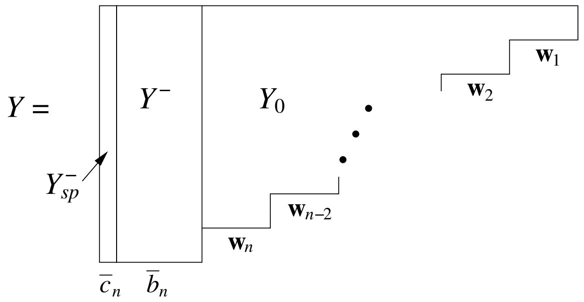
Let $\lambda = \mathbf{w}_1\omega_1 + \cdots + \mathbf{w}_n\omega_n$ ($\mathbf{w}_i \in \mathbb{Z}_{\geq 0}$) be a dominant integral weight. Then $\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n$ where

$$\begin{aligned}
\lambda_1 &= \mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_{n-2} + \frac{1}{2}(\mathbf{w}_{n-1} + \mathbf{w}_n),\\
\lambda_2 &= \mathbf{w}_2 + \cdots + \frac{1}{2}(\mathbf{w}_{n-1} + \mathbf{w}_n),\\
&\vdots\\
\lambda_{n-1} &= \frac{1}{2}(\mathbf{w}_{n-1} + \mathbf{w}_n),\\
\lambda_n &= \frac{1}{2}(\mathbf{w}_n - \mathbf{w}_{n-1}).
\end{aligned}$$

We then associate a (generalized) Young diagram Y to λ in the following manner. If $\mathbf{w}_n \geq \mathbf{w}_{n-1}$, let $\mathbf{w}_n - \mathbf{w}_{n-1} = 2b_n + c_n$ with $b_n \in \mathbb{Z}_{\geq 0}$ and $c_n = 0$ or 1 and set



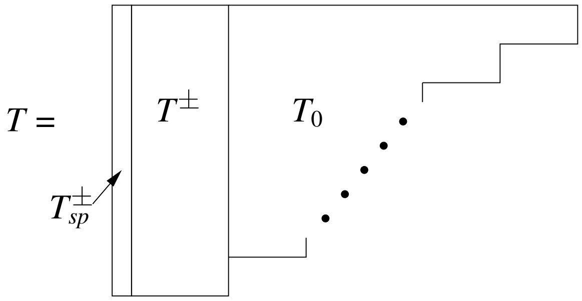
If $\mathbf{w}_n \leq \mathbf{w}_{n-1}$, let $\mathbf{w}_{n-1} - \mathbf{w}_n = 2\bar{b}_n + \bar{c}_n$ with $\bar{b}_n \in \mathbb{Z}_{\geq 0}$ and $\bar{c}_n = 0$ or 1 and set



The column $Y_{sp}^{\pm}$ consists of half-boxes. We identify the (generalized) Young diagram with the sequence of half-integers $Y = (\lambda_1, \ldots, \lambda_n)$. A *$D_n$-tableau of shape* Y is a tableau obtained from Y by filling in the boxes with entries from $\mathbf{N}$. A D_n-tableau is said to be *semistandard* if

(1) the entries in each row are weakly increasing and n and $\bar{n}$ do not appear simultaneously,
(2) the entries in each column of Y_0 and $Y^{\pm}$ are strictly increasing, but n and $\bar{n}$ can appear successively,
(3) the entries in the column $Y_{sp}^{\pm}$ are strictly increasing, and i and $\bar{i}$ do not appear simultaneously.

For a D_n-tableau T, we write



and we define its weight to be

$$\mathrm{wt}(T) = \sum_{i=1}^{n} (k_i - \overline{k_i})\epsilon_i + \frac{1}{2}\sum_{i=1}^{n}(l_i - \overline{l_i})\epsilon_i,$$

where k_i (respectively $\overline{k_i}$) is the number of i's (respectively $\bar{i}$'s) occurring in T_0 and $T^{\pm}$, and l_i (respectively $\overline{l_i}$) is the number of i's (respectively $\bar{i}$'s) occurring in $T_{sp}^{\pm}$. We define $\mathcal{B}(Y)$ to be the set of all semistandard D_n-tableaux T of shape Y satisfying conditions (D1)-(D7) of [4].

Proposition 4.1 ([4, Thm 8.5.2])**.** *For a (generalized) Young diagram Y associated with a dominant integral weight λ, $\mathcal{B}(Y)$ is isomorphic to the crystal graph of the $U_q(\mathfrak{g})$-module of irreducible highest weight λ under the Far-Eastern reading.*

Note that when we compute the action of the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ for $1 \le i \le n-1$, the entries i and $\overline{i+1}$ contribute a $+$ sign and the entries $i+1$ and $\bar{i}$ contribute a $-$ sign. For $\tilde{e}_n$ and $\tilde{f}_n$, the entries $n-1$ and n contribute a $+$ sign and the entries $\overline{n-1}$ and $\bar{n}$ contribute a - sign. As for the type A case, we will say that two entries are i-matched if they correspond to a $(+,-)$ pair. Of course, we can also have a $+$ or $-$ sign associated to the half-column $Y_{sp}^{\pm}$ (which comes last in the Far-Eastern reading) and this column can thus be i-matched to an entry in the rest of the tableau.

5. Enumeration of components in type A

In [3], the irreducible components of Nakajima's quiver variety for type A were enumerated by certain sets of Maya diagrams. For type $A_n^{(1)}$, this enumeration matched that of a basis for irreducible representations of the Lie algebra given in [2]. In this section we will define for type A_n a natural 1-1 correspondence between the irreducible components of Nakajima's quiver variety and the semistandard Young tableaux of shape given by the highest weight.

Let $\mathfrak{g} = \mathfrak{sl}_n$ be the Lie algebra of type A_{n-1}. Then $\mathfrak{g}$ is the space of all traceless $n \times n$ matrices. The Cartan subalgebra $\mathfrak{h}$ is spanned by the matrices

$$H_i = e_{i,i} - e_{i+1,i+1},\ 1 \le i \le n-1$$

where $e_{i,j}$ is the matrix with a one in entry (i,j) and zeroes everywhere else. Thus the dual space $\mathfrak{h}^*$ is spanned by the simple roots

$$\alpha_i = \epsilon_i - \epsilon_{i+1},\ 1 \le i \le n-1$$

where $\epsilon_i(e_{j,j}) = \delta_{ij}$ and the fundamental weights are given by

$$\omega_i = \epsilon_1 + \cdots + \epsilon_i,\ 1 \le i \le n-1.$$

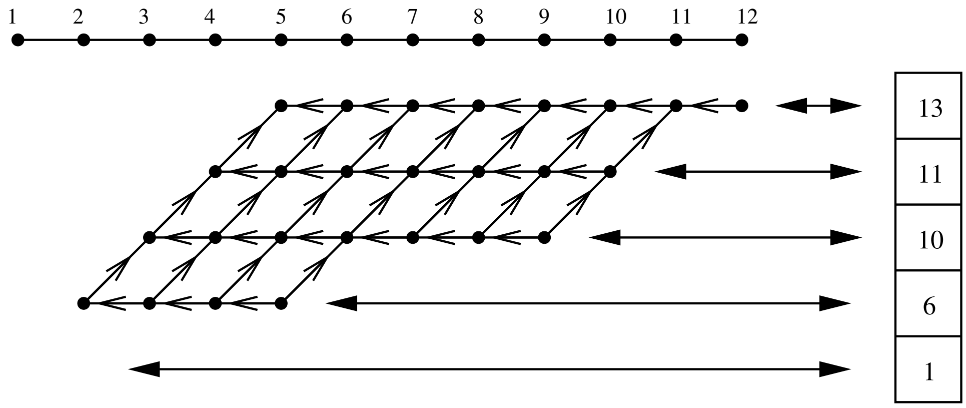


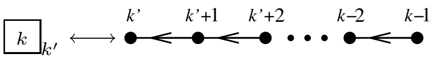
FIGURE 4. The correspondence between strings/rows in a Maya diagram of charge 5 and the entries in a 5 box column in a Young tableau for the case of A_{12}. The entry is one greater than the degree of the right endpoint of the string/row of the Maya diagram. An empty string/row corresponds to an entry equal to the where the left endpoint of the string would have been. The column of the Young tableau has been inverted to better show the correspondence.

Consider a dominant weight $\mathbf{w} = \mathbf{w}_1\omega_1 + \cdots + \mathbf{w}_{n-1}\omega_{n-1}$. Then

$$\mathbf{w} = \lambda_1\epsilon_1 + \cdots + \lambda_{n-1}\epsilon_{n-1}$$

where $\lambda_i = \mathbf{w}_1 + \cdots + \mathbf{w}_i$ and so $\mathbf{w}$ corresponds to a partition (or Young diagram) $\lambda(\mathbf{w}) = (\lambda_1 \geq \cdots \geq \lambda_{n-1})$.

Let $\mathcal{T}(\lambda)$ be the set of semistandard Young tableaux of shape λ with entries chosen from the set $\{1, \ldots, n\}$. Thus the entries are weakly increasing from left to right along each row and strictly increasing down columns. For $T \in \mathcal{T}(\lambda)$, let $f_T(k', k)$ equal the number of entries of T in row k' equal to $k+1$. Thus, intuitively speaking, an entry k in row k' corresponds to a string with endpoints k' and $k-1$ or the representation $(\mathbf{V}(k', k-1), x(k', k-1))$ (where $k = k'$ yields an empty string or the zero representation).



Proposition 5.1. *The irreducible components of $\mathcal{L}(\mathbf{v}, \mathbf{w})$ are precisely the X_{f_T} for those $f_T \in \tilde{Z}_{\mathbf{V}}$ with $T \in \mathcal{T}(\lambda(\mathbf{w}))$. Denote the component corresponding to such a T by X_T. Thus, $T \leftrightarrow X_T$ is a natural 1-1 correspondence between the set $\mathcal{T}(\lambda(\mathbf{w}))$ and the irreducible components of $\cup_{\mathbf{v}}\mathcal{L}(\mathbf{v}, \mathbf{w})$.*

Proof. This follows from a natural generalization of Theorem 6.2 of [3] to the type A_{n-1} case. Each of the Maya diagrams in this theorem corresponds to a column of the Young tableau. The representations corresponding to rows in the Maya diagram now correspond to entries in the tableau (see Figure 4). The fact that we are in the A_{n-1} and not the A_∞ case results in the tableau entries being less than or equal to n (since the rows of the Maya diagrams cannot extend past the $(n-1)$st vertex). The fact that the lengths of the rows of the Maya diagrams weakly decrease

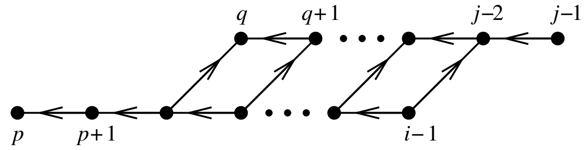


FIGURE 5. The string corresponding to an entry i in the pth row of a tableau can map into the string corresponding to an entry j in the qth row if and only if $p < q \le i < j$.

(and hence the degrees of their right endpoints strictly decrease) corresponds to the fact that the entries in the tableau must strictly decrease. The fact that the left endpoints of the strings in a Maya diagram move left by one vertex as we move down the diagram implies that we can have at most k rows where k is the charge of the Maya diagram. This explains the fact that the corresponding column in the Young tableau has k boxes. Then the ordering on the Maya diagrams imposed in Theorem 6.2 of [3] corresponds to the restriction that the entries in a tableau must weakly increase from left to right. $\square$

We now introduce some notation. We let $\boxed{i}_p$ denote an element i in the pth row of a Young tableau. Thus, as mentioned above, $\boxed{i}_p$ is identified with the representation $(\mathbf{V}(p, i-1), x(p, i-1))$ or the string with endpoints p and $i-1$. Recall that the case $i = p$ yields the zero representation or the empty string. We will often blur this distinction and refer to the string or representation $\boxed{i}_p$. We call the unique vector $e_{i-1} \in \boxed{i}_p$ of degree $i-1$ an *$(i-1)$-removable* vector. Sometimes, when we do not wish to specify the degree, we will simply say such a vector is removable.

Consider a representation $x_\Omega \in \mathbf{E}_{\mathbf{V},\Omega}$. It must be a sum of indecomposable representations of the form given in Section 1.2 (corresponding to strings). Let $(\mathbf{V}_1 = \mathbf{V}(k_1', k_1), x_1)$ and $(\mathbf{V}_2 = \mathbf{V}(k_2', k_2), x_2)$ be two of these representations or strings. Consider the conormal bundle to the $G_\mathbf{V}$-orbit through the point x_Ω. By the proof of Theorem 5.1 of [3], it contains points $(x_\Omega, x_{\bar\Omega})$ such that for some $v_1 \in \mathbf{V}_1$, $x_{\bar\Omega}(v_1)$ has non-zero v_2-component for some $v_2 \in \mathbf{V}_2$ if and only if $k_1' < k_2'$, $k_1 < k_2$, and $k_2' \le k_1 + 1$. In this case we say that the string corresponding to $(\mathbf{V}_1, x_1)$ maps into the string corresponding to $(\mathbf{V}_2, x_2)$. In other words, the string $\boxed{i}_p$ can map into the string $\boxed{j}_q$ if and only if $p < q \le i < j$. See Figure 5.

6. COINCIDENCE OF THE CRYSTAL ACTIONS: TYPE A

As we have seen, the crystal graph of an irreducible finite dimensional representation of $\mathfrak{g} = \mathfrak{sl}_n$ is realized on both the set of Young tableaux of a given shape and the set of all irreducible components of Nakajima's quiver varieties. In this section we show that, under the identification of tableaux with irreducible components given in Section 5, the two crystal structures are the same.

We say that two removable vectors are i-matched iff their associated entries in the tableau are i-matched.

Lemma 6.1. *For a Young tableau $T \in \mathcal{T}(\lambda(\mathbf{w}))$, a generic point of the irreducible component $X_T \in B(\mathbf{w})$ has a representative (x,t) such that $\boxed{i}_p$ maps into $\boxed{i{+}1}_q$ if and only if it is i-matched to it.*

Proof. Recall that the string $\boxed{i}_p$ can map into the string $\boxed{j}_q$ if and only if $p < q \leq i < j$. Suppose $p_1 \leq p_2 < q \leq i$ and T contains entries $\boxed{i}_{p_1}$, $\boxed{i}_{p_2}$ and $\boxed{i{+}1}_q$. For a generic point, both $\boxed{i}_{p_1}$ and $\boxed{i}_{p_2}$ map into $\boxed{i{+}1}_q$ with non-zero coefficient. Now, $\boxed{i}_{p_1}$ can map into any string $\boxed{i}_{p_2}$ can. Thus, by a change of basis (i.e. choosing a new representative of the $G_{\mathbf{V}}$-orbit), we can subtract a multiple of the basis vectors of $\boxed{i}_{p_2}$ from those of $\boxed{i}_{p_1}$ and assume that in our representative (x,t), $\boxed{i}_{p_1}$ does not map into $\boxed{i{+}1}_q$. Repeating this argument, we may assume that only the string $\boxed{i}_p$ with maximal p satisfying $p < q$ maps into $\boxed{i{+}1}_q$ (for multiple entries $\boxed{i}_p$ we choose the latest that is still earlier than $\boxed{i{+}1}_q$). An analogous argument then shows that we can also assume that $\boxed{i}_p$ only maps into the $\boxed{i{+}1}_q$ with minimal q satisfying $p < q$ (for multiple entries in the same row, we choose the earliest one that is still later than $\boxed{i}_p$).

Note that for two entries $\boxed{i}_{p_1}$ and $\boxed{i}_{p_2}$ with $p_1 < p_2$, the fact that our tableau is semistandard means that $\boxed{i}_{p_2}$ must occur southwest of $\boxed{i}_{p_1}$ in T. Also the entry $\boxed{i{+}1}_q$ for $q > p$ must either occur (weakly) southwest of $\boxed{i}_p$ or another entry $\boxed{i{+}1}_q$ does. Since one entry appearing southwest of another implies that it occurs later in the Far-Eastern reading of the tableau, we see from the above that our chosen $\boxed{i}_p$ and $\boxed{i{+}1}_q$ are i-matched. Removing them from further consideration and repeating the above process, the result follows. □

Proposition 6.2. *The value $\varepsilon_i(X_T)$ is equal to the number of $i+1$ entries in T which are not i-matched and $\tilde{e}_i^c(X_T) = X_{T'}$ where T' is obtained from T by changing all non-i-matched $i+1$ entries to i.*

Proof. Let $\varepsilon_i(X_T) = c$. Take a generic point (x,t) of X_T as in Lemma 6.1. Then $\bigoplus_{h:\, \mathrm{in}\, h = i} x_h$ is spanned by the degree i vectors of all $\boxed{j}_p$ with $j > i+1$ and the i-removable vectors of those $\boxed{i{+}1}_q$ which are i-matched to some $\boxed{i}_p$. The ith component $\mathbf{S}_i$ of the subspace $\mathbf{S}$ in the definition of the maps (3.3) must be equal to the span of these vectors and the result follows. □

Corollary 6.3. *If c is less than the number of non-i-matched i entries in T, then $\tilde{f}_i^c(X_T) = X_{T'}$ where T' is obtained from T by changing the earliest c non-i-matched i entries to $i+1$. Otherwise $\tilde{f}_i^c(X_T) = 0$.*

Proof. Since $\tilde{e}_i^c \tilde{f}_i^c(X_T) = X_T$ provided $\tilde{f}_i^c(X_T) \neq 0$, we see from Proposition 6.2 that T' must be obtained from T by changing c of the non-i-matched i entries to $i+1$. Furthermore, if these entries were not the earliest then some of the new $i+1$ entries of T' would be i-matched and so we would not have $\tilde{e}_i^c \tilde{f}_i^c(X_T) = X_T$. □

Theorem 6.4. *The map from $\mathcal{T}(\lambda(\mathbf{w}))$ to $B(\mathbf{w})$ given by $T \mapsto X_T$ is an isomorphism of crystals. That is,*

$$\mathrm{wt}(X_T) = \mathrm{wt}(T),\ \varepsilon_i(X_T) = \varepsilon_i(T),\ \varphi_i(X_T) = \varphi_i(T) \tag{6.1}$$

$$\tilde{e}_i(X_T) = X_{\tilde{e}_i(T)},\ \tilde{f}_i(X_T) = X_{\tilde{f}_i(T)}. \tag{6.2}$$

Proof. Since the equations of (6.1) hold for the highest weight element of the two crystals (the tableau with each entry equal to its row number and the unique element of $B(\mathbf{0}, \mathbf{w})$), the result will follow from the properties of a crystal if we show that $\tilde{f}_i(X_T) = X_{\tilde{f}_i(T)}$.

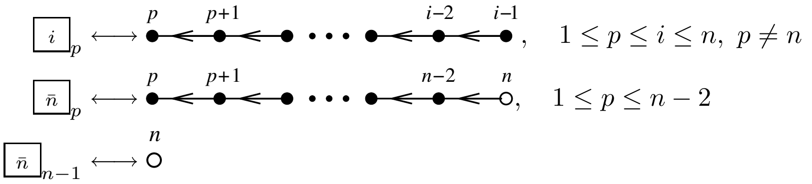
Let $\varepsilon_i(X_T) = c$, $\tilde{e}_i^c(X_T) = X_{T_1}$ and $\tilde{f}_i^{c+1}(X_{T_1}) = X_{T_2}$. Note that the $i+1$ entries which were changed to i to obtain T_1 from T must have been earlier than those i entries which are not i-matched. Otherwise, these $i+1$ entries would have been i-matched. Thus it follows from Proposition 6.2 and Corollary 6.3 that T_2 is obtained from T by switching the earliest non-i-matched i entry to an $i+1$. But this is precisely how $\tilde{f}_i(T)$ is obtained from T and so $\tilde{f}_i(T) = T_2$. □

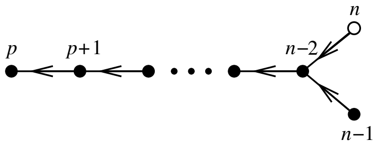
7. Coincidence of the crystal actions: Type D

The crystal graph of a finite dimensional representation of $\mathfrak{g} = \mathfrak{so}_{2n}$ is realized on Young tableaux and on the set of irreducible components of Nakajima's quiver varieties. In this section we define a crystal isomorphism between the two realizations. Our method of proof is somewhat different than that for type A. For type A, we were able to prove independently that the irreducible components are enumerated by the Young tableaux appearing in the combinatorial crystal basis using the results of [3]. However, for type D, the situation is different. While for the spin representations we have a natural enumeration of the irreducible components by certain Young diagrams (see [14]), which can be easily translated to the language of tableaux as was done in Section 5 for type A, this is not true for arbitrary integrable highest weight representations of type D. The more general enumeration will fall out from our comparison of the two presentations of the crystal graph.

We first associate a representation of our quiver to each possible entry in a D_n-tableau in $\mathcal{B}(Y)$. To simplify our presentation, we will describe our quiver representations with graphs. Each vertex represents a basis vector of the quiver representation and will be labelled with a number 1 through n. This number indicates the degree of the vector. If two vertices are labelled i and j (corresponding to the basis vectors v_i and v_j), a solid oriented edge from the vertex labelled i to the one labelled j will indicate that $x_{h_{ij}}(v_i) = v_j$ and a dotted oriented edge will indicate that $x_{h_{ij}}(v_i) = -v_j$. All other components of x are zero.

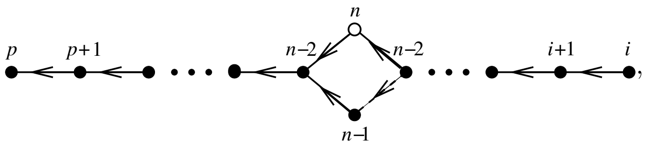
We first consider the entries of T_0. We make the following identifications.



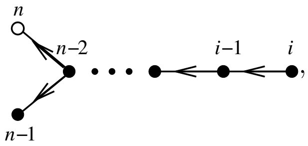
$\boxed{\overline{n-1}}_p \longleftrightarrow$ 

$, \quad 1 \le p \le n-2$

$\boxed{\overline{n-1}}_{n-1} \longleftrightarrow$ 

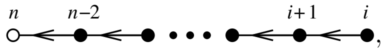
$\boxed{\bar{i}}_p \longleftrightarrow$ 

$, \quad 1 \le p \le n-2,\ i \le n-2$

$\boxed{\bar{i}}_{n-1} \longleftrightarrow$ 

$, \quad i \le n-2$

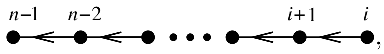
We use a superscript $+$ or $-$ when we want to emphasize that an entry belongs to T^+/T^+_{sp} or T^-/T^-_{sp} respectively. For the entries $\boxed{\alpha}^{\pm}_p$ with $1 \le p \le n-1$, the correspondence is as for entries of T_0. Note that the only possible $\boxed{i}^+_n$ is when $i = n$ (since $i \ge n$ by column strictness) and this corresponds to the empty string. Also, $\boxed{\bar{n}}^+_n$ is not allowed by condition (D2) of [4]. The remaining possible entries of T^+ correspond to strings as follows.

$\boxed{\bar{i}}^+_n \longleftrightarrow$ 

$, \quad 1 \le i \le n-2$

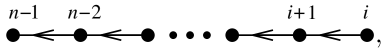
$\boxed{\overline{n-1}}^+_n \longleftrightarrow$ 

Similarly, $\boxed{i}^-_n$ is not possible for any $1 \le i \le n-1$ by column strictness nor for $i = n$ by condition (D3) of [4]. $\boxed{\bar{n}}^-_n$ corresponds to the empty string and the remaining possible entries of T^- correspond to strings as follows.

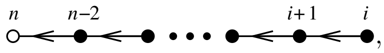
$\boxed{\bar{i}}^-_n \longleftrightarrow$ 

$, \quad 1 \le i \le n-1$

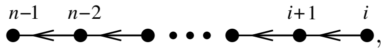
The entries in the half boxes $\boxed{\alpha}$ of $T^{\pm}_{sp}$ correspond to the empty string if $1 \le \alpha < \overline{n-1}$. Otherwise, the entries correspond to strings as follows.

$\boxed{\bar{i}}^+_p \longleftrightarrow$ 

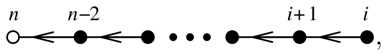
$, \quad 1 \le i \le n-2,\ n-p$ is odd

$\boxed{\bar{i}}^+_p \longleftrightarrow$ 

$, \quad 1 \le i \le n-2,\ n-p$ is even

$\boxed{\bar{i}}^-_p \longleftrightarrow$ 

$, \quad 1 \le i \le n-2,\ n-p$ is even

$\boxed{\bar{i}}^-_p \longleftrightarrow$ 

$, \quad 1 \le i \le n-2,\ n-p$ is odd

$$\begin{array}{|c|}\hline \ \\ \hline n\text{-}1 \\ \hline\end{array}^{+}_{p} \longleftrightarrow \overset{n-1}{\bullet}, \quad 1 \le i \le n-2,\ n-p \text{ is odd}$$

$$\begin{array}{|c|}\hline \ \\ \hline n\text{-}1 \\ \hline\end{array}^{+}_{p} \longleftrightarrow \overset{n}{\circ}, \quad 1 \le i \le n-2,\ n-p \text{ is even}$$

$$\begin{array}{|c|}\hline \ \\ \hline n\text{-}1 \\ \hline\end{array}^{-}_{p} \longleftrightarrow \overset{n-1}{\bullet}, \quad 1 \le i \le n-2,\ n-p \text{ is even}$$

$$\begin{array}{|c|}\hline \ \\ \hline n\text{-}1 \\ \hline\end{array}^{-}_{p} \longleftrightarrow \overset{n}{\circ}, \quad 1 \le i \le n-2,\ n-p \text{ is odd}$$

As before, we will often blur the distinction between entries, strings and their corresponding representations and between vertices and the vector corresponding to them. Thus we will refer to the strings of T and say that one string is earlier than another if its corresponding entry is, etc.

We reserve the use of the word string to refer to those representations corresponding to a possible tableau entry. If we can remove a vertex of degree i from a string a and obtain another string, we call both the string and the vertex *i-removable*. When we do not wish to specify the degree of the vertex, we may simply say that it is *removable*. It is possible for a string to have two removable vertices. In this case the degree of the two vertices are necessarily $n-1$ and n. Note that $\boxed{\alpha}_p$ is i-removable iff it is non-empty and $\varepsilon_i(\boxed{\alpha}) = 1$. If we can add a vertex of degree i to a string $\boxed{\alpha}_p$ and obtain another string, we say that this string is *i-admissible*. Equivalently, $\boxed{\alpha}_p$ is i-admissible iff $\varphi_i(\boxed{\alpha}) = 1$. Note that if a pair of strings is i-matched then one is i-removable (or empty) and the other is i-admissible. If a string is i-admissible, we call the vertex to which the added degree i vertex would map a *free* vertex. In the case that the new vertex would map into two vertices (necessarily of degree $n-1$ and n), we call both vertices free. Note that the removable and free vertex/vertices of a string are nearly always the same. The only case in which these notions differ is in certain strings with a vertex of degree n or $n-1$ but not both in which this vertex is the removable one and a vertex of degree $n-2$ is the free vertex.

Note that the column of half boxes is completely determined by the strings to which its entries correspond and that the action of the Kashiwara operator $\tilde{f}_i$ simply corresponds to adding a vertex of degree i to a string with right endpoint of degree $i+1$ provided that no string with right endpoint of degree i already exists. In this case we say the string is i-admissible. Similarly, the action of $\tilde{e}_i$ corresponds to removing the degree i right endpoint of a string provided that no string with right endpoint $i+1$ (or $n, n-1$ if $i = n-2$) already exits. In this case we say the string is i-removable. So rather than saying that the half column itself is i-matched to some entries in the rest of the tableaux, we can say the individual strings are i-matched. Because of this, we are able to treat the entries of the half column in much the same way as the entries in the full columns. However, when doing so, we will often treat the entries of the half column as if they appeared in the nth row. This is because of the degree of their "terminal" vertex, which is not of degree given by the row of the entry as in the case of the entries of full boxes. We also say that any string corresponding to a half box is earlier/later than any other since in the Far-Eastern reading, the half column is considered all at once. For a more detailed examination of the geometric construction of the spin representations see [14].

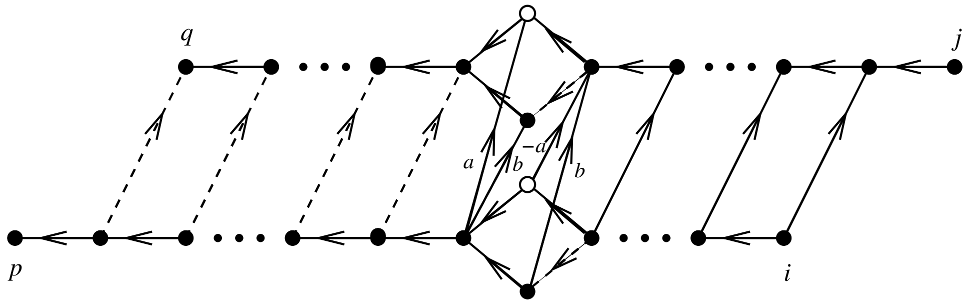


FIGURE 6. A map from the string corresponding to an entry $\bar{i}$ in the pth row of a tableau into the string corresponding to an entry $\bar{j}$ in the qth row for $i, j, p, q \leq n-2$. Unmarked solid and dotted lines indicate a coefficient of 1 and -1 respectively. Otherwise the lines are labelled by the value of the coefficient. These must satisfy $a + b = -1$.

When we refer to the components of a vector in $\mathbf{V}$, we will always be referring to the basis consisting of (the vectors corresponding to) the vertices of the strings involved. We say that a vertex v maps into a vertex w if the result of applying the map $x_h(v)$ has a non-zero w-component for some $h \in H$. We also say that the string containing v maps into the string containing w.

Lemma 7.1. *If $\alpha, \beta \in \mathbf{N}$, $\alpha < \beta$ (where we say that $n < \bar{n}$ and $\bar{n} < n$) and $p < q$, then $\boxed{\alpha}_p$ can map into $\boxed{\beta}_q$ in such a way that if $\boxed{\alpha}_p$ is i-admissible and $\boxed{\beta}_q$ is i-removable, then the free vertex/vertices of $\boxed{\alpha}_p$ map(s) into the i-removable vertex of $\boxed{\beta}_q$.*

Proof. See Figure 6 for the most complicated case. The other cases can be obtained by taking appropriate subdiagrams of this one. □

Lemma 7.1 is very similar to what happened in the type A case. However, we can also have a different type of mapping between strings in type D. This can be seen in the following proposition.

Proposition 7.2. *At least one free vertex of any i-admissible string can map into the i-removable vertex of any later i-removable string. The strings involved may correspond to either half or full boxes.*

Proof. We first deal with the case of strings corresponding to full boxes. Let the i-admissible string be $\boxed{\alpha}_p$ and the i-removable string be $\boxed{\beta}_q$. First suppose $1 \leq \alpha, \beta < \overline{n-1}$. If $i \leq n-2$ then the only possibility is that $\alpha = i$ and $\beta = i+1$. Then since $\boxed{\beta}_q$ is later than $\boxed{\alpha}_p$ and $\beta > \alpha$, we must have $p < q$ and the result follows from Lemma 7.1. If $i = n-1$ then $\alpha = n-1$ or $\bar{n}$ and $\beta = n$ or $\overline{n-1}$. In all cases, Lemma 7.1 again gives us the result. If $i = n$ then $\alpha = n-1$ or n and $\beta = \bar{n}$ or $\overline{n-1}$. Again, Lemma 7.1 provides the desired result in all cases. If $\overline{n-1} \leq \alpha, \beta \leq \bar{1}$, the result also follows from Lemma 7.1.

Now consider the case when $1 \leq \alpha < \overline{n-1}$ and $\overline{n-1} \leq \beta \leq \bar{1}$. Then if $i \leq n-2$, we must have $\alpha = i$ and $\beta = \bar{i}$. In this case $\boxed{\alpha}_p$ can map into $\boxed{\beta}_q$ as in Figure 7.

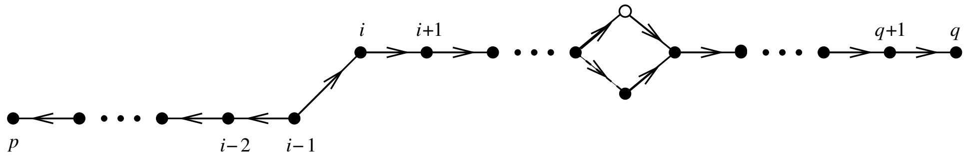


FIGURE 7. The string corresponding to an entry i in the pth row of a tableau mapping into the string corresponding to an entry $\bar{i}$ in the qth row for $i < n-2$. The second string has been reversed from its usual presentation.

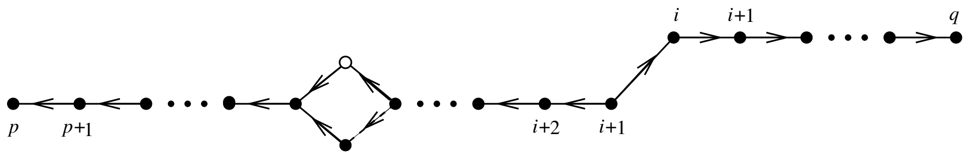


FIGURE 8. The string corresponding to an entry $\overline{i+1}$ in the pth row of a tableau mapping into the string corresponding to an entry $i+1$ in the qth row for $i \leq n-3$. The second string has been reversed from its usual presentation.

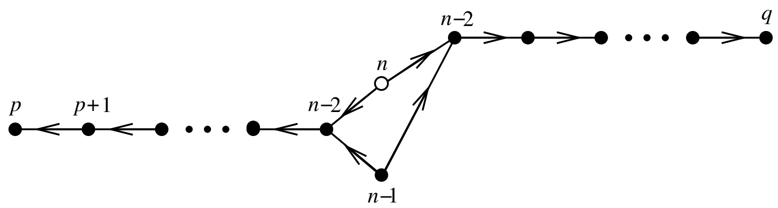


FIGURE 9. The string corresponding to an entry $\overline{n-1}$ in the pth row in a tableau mapping into the string corresponding to an entry $n-1$ in the qth row. The second string has been reversed from its usual presentation.

If $i = n-1$, then $\alpha = n-1$ or $\bar{n}$ and $\beta = \overline{n-1}$. In both cases $\alpha < \beta$, so $p < q$ and thus Lemma 7.1 applies. If $i = n$, then $\alpha = n-1$ or n and $\beta = \overline{n-1}$. Again, Lemma 7.1 applies in either case.

Finally, consider the case when $\overline{n-1} \leq \alpha \leq \bar{1}$ and $1 \leq \beta < \overline{n-1}$. The only possibility is $i \leq n-2$. Then $\alpha = \overline{i+1}$ and $\beta = i+1$ and $\boxed{\alpha}_p$ can map into $\boxed{\beta}_q$ as in Figure 8 (for $i \leq n-3$) or Figure 9 (for $i = n-2$).

Now consider the case when one or both of the strings correspond to a half box. If both do, then the i-admissible string must be $\boxed{\overline{i+1}}^{\pm}$ and the i-removable string must be $\boxed{\bar{i}}^{\pm}$. Thus the two entries must be adjacent in the column and so one has left endpoint $n-1$ and the other n. If $\boxed{\bar{i}}^{\pm}$ has left endpoint n then the map is as in Figure 10. Otherwise, we simply interchange the labelling of the leftmost vertex of each string.

If only one of the strings corresponds to a half box, this must necessarily be the i-removable string $\boxed{i}^{\pm}$ since the half entries occur later than all the full entries.

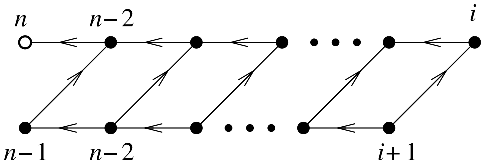


FIGURE 10. The string corresponding to the entry $\overline{i+1}$ in a half box mapping into the string corresponding to the entry $\bar{i}$ in a half box.

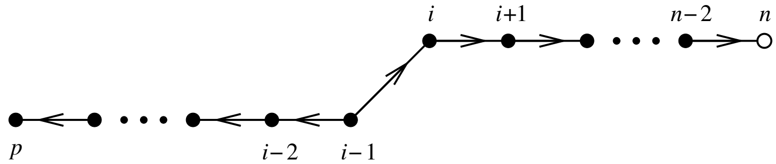


FIGURE 11. The string corresponding to the entry i in the the pth row of tableau mapping into the string corresponding to the entry $\bar{i}$ in a half box.

Let the i-admissible string be $\boxed{\alpha}_p$. If $\alpha \geq \overline{n-1}$ we can take the appropriate subdiagram of Figure 6 setting either a or b equal to zero if the i-removable string has left endpoint n or $n-1$ respectively. If $\alpha < \overline{n-1}$ then the map can be as in Figure 11 (we picture the case where the endpoint of the string corresponding to the half box is of degree n and $\alpha \neq \bar{n}$ – the other cases are similar). $\square$

Now, for a given tableau T, let $\mathbf{v}^T$ be the sum of the graded dimensions of the strings corresponding to the entries of T and let $\mathbf{V}^T$ be a vector space with graded dimension $\mathbf{v}^T$. That is, for $i \in I$, $\mathbf{v}_i^T$ is equal to the number of vertices of degree i occurring in the strings of T. Let $x' \in \Lambda_{\mathbf{V}^T}$ be the direct sum of the representations corresponding to the strings of T. We then define A_T to be the set of all $x \in \Lambda_{\mathbf{V}^T}$ such that for any vertex v and $h \in H$, $x_h(v)$ has the same w-component as $x'_h(v)$ for any w in the same string as v and all the other components of $x_h(v)$ lie in later strings. Roughly speaking, we permit strings to map only into later strings.

Let $\mathcal{C}_T$ be the union of the $G_{\mathbf{V}^T}$-orbits of the points of A_T and let $\bar{\mathcal{C}}_T$ be its closure. Then define

$$X_T \stackrel{\text{def}}{=} \left(\left(\bar{\mathcal{C}}_T \times \sum_{i \in I} \operatorname{Hom}(\mathbf{V}_i^T, \mathbf{W}_i) \right) \cap \Lambda(\mathbf{v}^T, \mathbf{w})^{\text{st}} \right) / G_{\mathbf{V}^T}.$$

Note that a priori some X_T may be empty.

Lemma 7.3. *An i-admissible empty string corresponding to a full box can only be i-matched to another empty string immediately to the south of it.*

Proof. Unless $i = n$, an i-admissible empty string a corresponding to a full box must be of type $\boxed{i}_i$. Assume a is of this type. Suppose it is i-matched to another empty string b, necessarily of type $\boxed{i{+}1}_{i+1}$. By column strictness, the entry directly north of b must be of type $\boxed{i}_i$. Then b is i-matched to this entry which must therefore be the string a.

Now suppose a is i-matched to the string b of type $\boxed{\bar{\imath}}_q$. Since b is later than a, it is weakly west of a and $q > i$. Then the entry in the ith row of the column containing b must be equal to i since the tableau in semistandard. But then by condition (D1) of [4], we must have that q is greater than the length of the column containing b which is a contradiction.

The only remaining possibility is that $i = n$, a is of type $\boxed{n-1}_{n-1}$ and is n-matched to a string b of type $\boxed{\bar{n}}_q$. Since b is later than a and the tableau is semistandard, we must have $q = n$ and thus b corresponds to the empty string.

Note that the case where $i = n$, a is of type $\boxed{n}_n$ and b is of type $\boxed{\overline{n-1}}_q$ is not possible since we would have to have $q = n$ and then b cannot be later than a by the fact that the tableau must be semistandard. Similarly the case where $i = n-1$, a is of type $\boxed{\bar{n}}_n$ and b is of type $\boxed{n}_q$ or $\boxed{\overline{n-1}}_q$ is not possible by the fact that the tableau must be semistandard. $\square$

Proposition 7.4. *Let $W \subset \mathbf{V}_i$ be the space spanned by the i-removable vertices of the i-matched, i-removable entries of a tableau T. If $i = n-2$ then for each i-matched, i-admissible string with a degree $n-2$ vertex, we extend W by the span of this vertex as well. Then a generic point of A_T has a representative such that the images under $\{x_h \mid \operatorname{in}(h) = i\}$ of the free vertices of the i-admissible strings to which these entries are i-matched (in the case $i = n-2$, for each i-admissible string with two free vertices that does not have a degree $n-2$ vertex we consider only the degree n free vertex), projected to W, form a basis of W.*

Proof. By the definition of i-matching, there are always more i-removable strings later than an i-matched i-admissible string than other i-admissible strings. Thus the result follows from Proposition 7.2. We need the slight modification for the case $i = n-2$ so that the images of the free vertices are linearly independent. $\square$

Corollary 7.5. *Let a be an i-admissible, i-matched string in a tableau T and take $x \in A_T$. Suppose we extend x to a vector v of degree i in such a way that the resulting point is in $\Lambda_{\mathbf{V}}$. Then if v maps into the free vertex of a, either it must also map into some string earlier than a or there is an i-admissible empty string earlier than a.*

Proof. This follows immediately from Proposition 7.4 and the moment map condition except for the case when $i = n-2$ and one or more of the i-admissible, i-matched strings of T is of type $\boxed{\overline{n-1}}_{n-1}$ which is the only string with two free vertices that does not have a degree $n-2$ vertex. Let us say that b is such a string and it does not have an empty string immediately north of it (which would necessarily be i-admissible). Then the string immediately north has a vector v' of degree $i = n-2$ mapping into the degree $n-1$ and n vertices of a. Thus we can replace v by a linear combination of v and v' and assume that it only maps into the degree n vertex (we could just have easily have used the degree $n-1$ vertices). We can do this for each such string b. Thus the result again follows from Proposition 7.4. $\square$

Theorem 7.6. (1) *The set*

$$\{X_T \mid T \in \mathcal{B}(Y)\}$$

is precisely the set of irreducible components of $\cup_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \mathbf{w})$, and

(2) *$T \mapsto X_T$ is an isomorphism of crystals. That is,*

$$\operatorname{wt}(X_T) = \operatorname{wt}(T),\ \varepsilon_i(X_T) = \varepsilon_i(T),\ \varphi_i(X_T) = \varphi_i(T), \tag{7.1}$$

$$\tilde{e}_i(X_T) = X_{\tilde{e}_i(T)}, \tag{7.2}$$

$$\tilde{f}_i(X_T) = X_{\tilde{f}_i(T)}. \tag{7.3}$$

Proof. The variety $\mathcal{L}(0, \mathbf{w})$ corresponding to the empty tableau is a single point and equations (7.1) hold in this case. Also for $|\mathbf{v}| = \sum_i \mathbf{v}_i = 1$, (1) is true because our varieties are again points. Furthermore, it is easy to see that (7.2) and (7.3) hold for the highest weight tableau. Thus, we can prove the theorem by induction on the height $|\mathbf{v}|$ of $\mathbf{v}$. So fix $\mathbf{v}$ and assume that (1), (7.1), (7.2) and (7.3) hold for all $\mathbf{v}'$ with $|\mathbf{v}'| < |\mathbf{v}|$. Note that this implies that (1), (7.1) and (7.2) hold for all $\mathbf{v}'$ with $|\mathbf{v}'| \leq |\mathbf{v}|$ by the properties of a crystal (e.g. $\tilde{e}_i \tilde{f}_i b = b$ provided $\tilde{f}_i b \neq 0$). Thus it is enough to show that for an irreducible component X_T of $\mathcal{L}(\mathbf{v}, \mathbf{w})$, (7.3) holds. Also, since we already know that both crystals are isomorphic to that of the irreducible integrable highest weight module of highest weight $\mathbf{w}$, it follows that $\tilde{f}_i(X_T) = 0$ if and only if $\tilde{f}_i(T) = 0$ for any irreducible component X_T of $\mathcal{L}(\mathbf{v}, \mathbf{w})$. Thus we neglect this case.

By the induction hypothesis, we have that $\tilde{e}_i^c(X_T) = X_{T'}$ where $T' = \tilde{e}_i^c(T)$. Thus it suffices to show that $\tilde{f}_i^{c+1}(X_{T'}) = X_{\tilde{f}_i^{c+1} T'}$. Let $(\mathbf{V}, x)$ be a generic point of $A_{T'}$, hence the x-part of a representative of a point of $X_{T'}$. We need to describe the set of representations $(\mathbf{V}', x')$ such that

$$\mathbf{V}'_j = \mathbf{V}_j \ \forall\ i \neq j, \quad \mathbf{V}'_i = \mathbf{V}_i \oplus \mathbb{C}^{c+1},$$
$$\mathbf{V} \text{ is } x'\text{-stable}, \quad x'|_{\mathbf{V}} = x.$$

Thus we need only describe how x' can act on the additional space $\mathbb{C}^{c+1}$ appearing at degree i in $\mathbf{V}'$.

Pick a basis for $\mathbb{C}^{c+1}$, let v be the first element of the basis, and extend x generically to v. So v maps into all the vertices it possibly can. We claim that if $T'' = \tilde{f}_i(T')$, then the extension of x to $V \oplus \mathbb{C}v$ lies in $A_{T''}$. Let a be the earliest string that v maps into and let it map into the vertex w of this string. Let the degree of w be j. Consider the following cases

(1) The string a has a vertex w' of degree i mapping into w.
(2) The vertex w maps into a vertex w' in a of degree i and there is no degree i vertex in a mapping into w.
(3) Neither (1) nor (2) holds. That is, w does not map into any degree i vertex in a and no degree i vertex in a maps into w.

First consider case (1). If w' does not map into any other vertex of a besides w then we can, by choosing a different representative of our orbit, replace v by a linear combination of v and w' which does not map into a. So we may assume that this case does not occur. If w' maps into another vertex w'' in a then we must have that the degree of w' is $n-2$ and the two vertices of a it maps into are of degree $n-1$ and n. Then a must be of type $\boxed{\alpha}_p$ for $\alpha \geq \overline{n-2}$ and $p \leq n-1$. Now, if $p < n-1$ then the moment map condition implies (see Figure 12) that either

- v maps into some vertex of a string earlier than a which in turn maps into the degree $n-2$ vertex of a that w maps into, or
- v maps into w and w'' in such a way that, considering only a-components, $x_{h_{n,n-2}} x_{h_{n-2,n}}(v) + x_{h_{n-1,n-2}} x_{h_{n-2,n-1}}(v) = 0$.

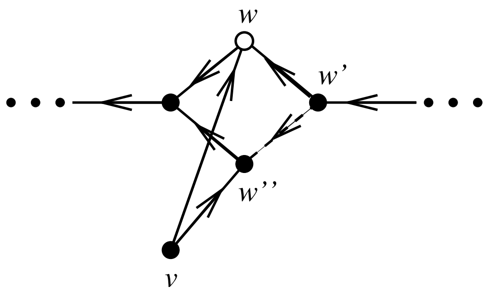


FIGURE 12. The vector v mapping into the degree n vertex of a string corresponding to an entry α, $\alpha \geq \overline{n-2}$, in row p, $p < n-1$.

The first option violates our choice of a and thus we can neglect it. Consider the second option. Since w' satisfies the same equation, we can again replace v by a linear combination of v and w' and assume that v does not map into a. So we can assume that $p = n-1$. If the string directly north of a is not empty, it has a degree $n-2$ vertex v' mapping into w and w'. We can thus replace v by a linear combination of v, v' and w' (since generically the images of v' and w' are linearly independent) and assume that v does not map into a. If the string directly north of a is empty then we consider v to be added to this string.

Next consider case (2). We have that $x_{h_{ji}} x_{h_{ij}}(v)$ has a non-zero w'-component if we consider only a and v. Thus, by the moment map condition, v must map into some other vertex w'' mapping into w'. w'' cannot be in another string because we picked a to be the earliest string into which v maps. Thus, w'' must be in the string a. The only possibility is that $i = n-2$, the degree of w is $n-1$ or n and the degree of w'' is n or $n-1$ respectively. Then, because we are assuming that there is no degree i vertex mapping into w, the string a must be of type $\boxed{\overline{n-1}}$ and adding v yields the string of type $\boxed{\overline{n-2}}$.

Finally consider case (3). The only possibilities for w are that

- a is an i-admissible string, w is its free vertex, and v maps into it in the obvious way yielding the unique string obtained by adding a vertex of degree i to a,
- w is not a free vertex and is the terminal vertex of a string (that is, w maps into no other vertex in a), or
- w is the degree $n-2$ vertex of a string of type $\boxed{\bar{i}}^{\pm}$.

Consider the second possibility. Assume that $j \neq n-1, n$. It is not possible for a to be in the northernmost row in the tableau since then $j = 1$ and we must have $i = 2$ (since v maps into w). But since w is not a free vertex, there must already be a vertex of degree 2 mapping into it which contradicts (3). If the string directly north of a in the tableau is not empty then there is another vertex v' in this string of degree i mapping into w in the manner of Lemma 7.1. Then we could replace v by a suitable linear combination of v and v' and assume that v does not map into w after all. So there must be an empty string above a and we consider v to be added to that string. If $j = n$ (the case $j = n-1$ is exactly analogous) then $i = n-2$ and a is either of type $\boxed{\ \bar{i}\ }_{n-1}$ or $\boxed{\bar{i}}^{\pm}$. But since (3) assumes that w has no vertex of degree i in a mapping into it and w is not free, a is actually of type $\boxed{\bar{n}}_{n-1}$, $\boxed{\overline{n-1}}^{\pm}_{n}$ or $\boxed{\overline{n\text{-}1}}^{\pm}$. If a is type $\boxed{\bar{n}}_{n-1}$ then the argument used above applies and there is an empty string north of a, which we consider v to be added to. If a is of type $\boxed{\overline{n-1}}^{\pm}_{n}$ or $\boxed{\overline{n\text{-}1}}^{\pm}$, then v maps into the string in such a way as to extend it to the string $\boxed{\overline{n-2}}^{\pm}_{n}$ or $\boxed{\overline{n\text{-}2}}^{\pm}$ respectively.

If w is the degree $n-2$ vertex of a string of type $\boxed{\bar{i}}^{\pm}$, then the terminal vertex of this string must be of different degree ($n-1$ or n) than v. By the argument used above, there must be an i-admissible empty string in the half column and we can assume that v is added to this string.

Thus we have shown that earliest string that v maps into (taking the empty string in the relevant cases) is an i-admissible string and it maps into the free vertex in the case the string is non-empty. Suppose this string is non-empty and i-matched. Then by Corollary 7.5, v also maps into some earlier string. But this contradicts our choice of a. Thus v is added to the earliest i-admissible string which is not i-matched. Note that this applies to empty strings by Lemma 7.3 since the empty strings that appeared in the argument above were all directly north of a non-empty string and since we can assume that v is added to the earliest non-i-matched empty string.

Repeating the above process for the other basis vectors of $\mathbb{C}^{c+1}$, we see that after extending x to all of $\mathbb{C}^{c+1}$, we have changed the earliest $c+1$ non-i-matched i-admissible entries (i.e $x' \in A_{\tilde{f}_i^{c+1}}(T')$). One should note that for the entries of the column of half boxes, changing an entry really corresponds to changing two entries and shuffling the order of the entries. So we have shown that $x' \in A_{\tilde{f}_i^{c+1}T'}$. However, to show that we obtain all of $A_{\tilde{f}_i^{c+1}T'}$, we must show that the definition of $A_{\tilde{f}_i^{c+1}T'}$ does not allow for any maps into $v \in \mathbb{C}^{c+1}$ since V must be x'-stable.

By definition, we need only consider strings earlier than the string to which v was added. Suppose the vertex w of an earlier string maps into v. If w maps into another vertex v' of degree i in a string d, then if d is earlier than a, we can replace v' by a linear combination of v' and v and assume that w maps into v' but not v. If a is earlier than d, we can replace v by a multiple of v and v' and assume that w does not map into v'. Continuing this process we can assume that the only degree i vertex that w maps into is v. But then v could have mapped into w. Thus by our choice of the string to which we add v, we must have replaced the original (generically mapping) v by a linear combination of itself and other vectors $v_1, \ldots, v_l$ so that it does not map into w. Reversing this process, we can replace v by a linear combination of itself and the vectors $v_1, \ldots, v_l$ so that it maps into w. But then by nilpotency, w cannot map into v. So w must have mapped into some v_k, $1 \leq k \leq l$. This is a contradiction. $\square$

Remark 7.7. It can be shown that we could have used the same construction to define the irreducible components in finite type A. However, in type A we also have the description in terms of conormal bundles since the strings only contain edges in Ω. This description does not work in type D because our strings contain edges in both Ω and $\bar{\Omega}$.

Remark 7.8. Note that in both type A and D, our strings look like subgraphs of the crystal graph of the vector representation. Furthermore, the existence of

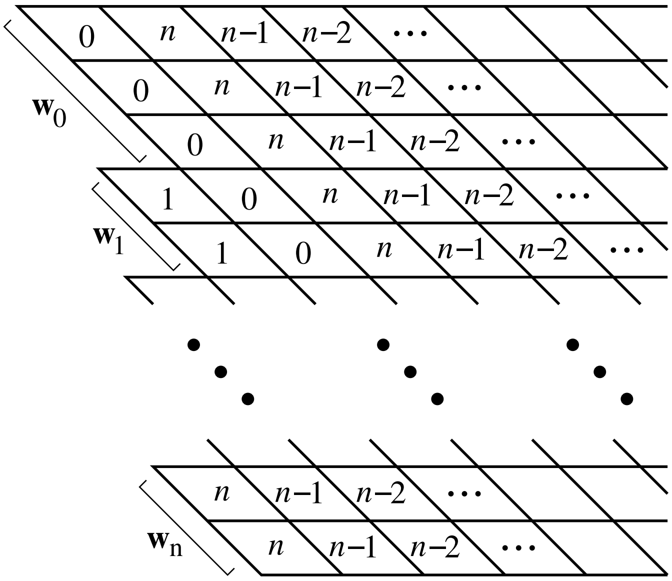


FIGURE 13. The ground state pyramid P_λ for $\lambda = \sum_{i=0}^{n} \mathbf{w}_i\omega_i$. The labelling of the squares indicates the color block that must be placed on that square.

certain "maximal" maps between strings (see Lemma 7.1 for type D) gives us a geometric interpretation of the ordering of tableaux entries.

8. Young pyramids and the geometric crystal structure for type $A_n^{(1)}$

There exists a realization of the crystal graph of the basic representations of the classical affine Lie algebras $A_n^{(1)}$ and $D_n^{(1)}$ (among others) in terms of combinatorial objects known as Young walls (see [4, 5]). It turns out that there is a natural enumeration of the irreducible components of Nakajima's quiver variety by Young walls. In fact, for type $A_n^{(1)}$, we use the geometry to suggest a generalization of the combinatorial construction to higher level. Thus we define a new combinatorial object which we call a *Young pyramid*. On these objects, which in level one reduce to Young walls, we will realize the crystal of an arbitrary irreducible integrable highest weight representation of the Lie algebra $\mathfrak{g} = \widehat{\mathfrak{sl}}_{n+1}$ of type A_n.

Let $\lambda = \sum_{i=0}^{n} \mathbf{w}_i\omega_i$ be a dominant integral weight. We define P_λ to be the *ground state pyramid* of weight λ as shown in Figure 13. We then build on the ground state by placing colored blocks (with color 0 through n). We call a series of blocks, one on top of the other, a *stack*. The words row and column will refer to the rows and columns of the ground state pyramid and the blocks placed on them. We will use the compass directions to refer to the relative positions of stacks (or slots in the ground state) and the words above and below or up and down to refer to the relative position of blocks within a stack. The rules for building the pyramids are as follows.

(1) The pyramid must be built on top of the ground state pyramid.

(2) A block placed in an empty slot of the ground state wall must be of the same color as the square on which it is placed.
(3) A block placed on a block of color i must be of the unique color j such that $j \equiv i+1 \mod n+1$.
(4) Any stack must be weakly taller than a stack east or south of it. That is, the height of the stacks must weakly decrease as we move south and east.

We say a Young pyramid is *proper* if the height of a stack in the northernmost row is weakly shorter than any stack in the southernmost row $n+1$ columns to the west. Pictorially, a Young pyramid is proper if when we move the first row to the southern edge of the Young pyramid and shift it west by $n+1$ boxes, the condition that the heights of columns must weakly decrease as we move south and east is still satisfied. We say that a Young pyramid is *n-reduced* if for any given height, it does not contain $n+1$ columns of that height with each of the $n+1$ colors for the top block. Note that in the case that λ is a fundamental weight, our Young pyramids are simply the Young walls defined in [4] and all pyramids are proper. However being n-reduced is different than being reduced as defined in [4]. Let $\mathcal{F}(\lambda)$ be the set of all proper Young pyramids built on the ground state pyramid P_λ and let $\mathcal{P}(\lambda)$ be the set of all n-reduced proper Young pyramids in $\mathcal{F}(\lambda)$.

For two integers $k' \leq k$, define $\mathbf{V}(k',k) \in \mathcal{V}$ to be the vector space with basis $\{e_r \mid k' \leq r \leq k\}$. We require that e_r has degree $i \in \{0,1,\dots,n\}$, where $r \equiv i \mod n+1$. Let $x(k',k) \in \mathbf{E}_{\mathbf{V}(k',k),\Omega}$ be defined by $x(k',k) : e_r \mapsto e_{r-1}$ for $k' \leq r \leq k$, where $e_{k'-1} = 0$. Note that the isomorphism class of this representation does not change when k' and k are simultaneously translated by a multiple of $n+1$.

Let y be a non-empty stack of some $P \in \mathcal{P}(\lambda)$ with bottom block of color k' and height (number of boxes) l. Define $k = k'+l-1$. Then let $\mathbf{V}^y = \mathbf{V}(k',k)$ and $x^y = x(k',k)$. Equivalently, $\mathbf{V}^y$ is the vector space with basis given by the blocks of the stack y with the degree of each block given by its color and x_y is defined by mapping each block to the block immediately below it (and mapping the bottom block to zero). As for tableaux, we will use the words stack and representation interchangeably.

Define $\mathbf{V}^P = \bigoplus_y \mathbf{V}^y$ where the sum is over all stacks of P and let $\mathbf{v}^P$ be its graded dimension. Then define $x^P_\Omega \in \mathbf{E}_{\mathbf{V}^P,\Omega}$ to be the direct sum of the representations x^y over all the stacks of P. Denote by $\mathcal{O}_P$ the $G_{\mathbf{V}^P}$-orbit through x^P_Ω. Let $\mathcal{C}_P$ be the conormal bundle of $\mathcal{O}_P$ and let $\bar{\mathcal{C}}_P$ be its closure. Let $\mathbf{W}$ be a vector space of dimension $\mathbf{w}$. Then define

$$X_P = \left(\left(\bar{\mathcal{C}}_P \times \sum_{j\in I} \mathrm{Hom}((\mathbf{V}^P)_j, \mathbf{W}_j)\right) \cap \Lambda(\mathbf{v}^P, \mathbf{w})^{st}\right) / G_{\mathbf{V}^P}.$$

Proposition 8.1. *We have that $P \leftrightarrow X_P$ is a one-to-one correspondence between the set $\mathcal{P}(\lambda)$ and the set of irreducible components of $\cup_{\mathbf{v}} \mathcal{L}(\mathbf{v}, \mathbf{w})$.*

Proof. This follows immediately from Theorem 6.3 of [3]. Each row of a Young pyramid corresponds to a Maya diagram (Y, γ). The stacks correspond to the rows of the Young diagram Y and the charge γ is given by the color of the slot in the ground state pyramid of the westernmost place in the row. Then the conditions on the heights of the columns is precisely the condition on the ordering of the Maya diagrams and the n-reduced conditions coincide. □

We call a block of color i in a proper Young pyramid *i-removable* if the pyramid remains a proper Young pyramid after removing this block. We say that a stack is i-removable if its top block is. Similarly, we call a place where we may add a block of color i and obtain another proper Young pyramid, an *i-admissible* slot. We say that a stack is i-admissible if the top of that stack is an i-admissible slot.

We number the stacks of P, starting at zero, from north to south along columns and from west to east. When we say that one stack occurs earlier or later than another, we are referring to this ordering. Let y_i be the ith stack of P. We assign to y_i a $+$ if the stack is i-admissible and a $-$ if the stack is i-removable. From the sequence of $+$'s and $-$'s, cancel out every $(+,-)$ pair to obtain a sequence of $-$'s followed by $+$'s. This is called the *i-signature* of P. If two stacks correspond to a $(+,-)$ pair, we say they are i-matched. We then define $\tilde{e}_i P$ to be the Young pyramid obtained from P by removing the i-block from the stack of P corresponding to the rightmost $-$ in the i-signature of P. If no $-$ exists in the i-signature of P then $\tilde{e}_i P = 0$. We define $\tilde{f}_i P$ to be the Young wall obtained from P by adding an i-block to the stack corresponding to the leftmost $+$ in the i-signature of P. If there is no $+$ in the i-signature of P then $\tilde{f}_i P = 0$.

Note that for the case of a fundamental weight, when the Young pyramids are Young walls, the i-signature just defined is different from the one defined in [4] because our Young wall is reversed compared to those in [4] and thus the cancellation of $(+,-)$ pairs occurs in the opposite direction.

Define the maps

$$\mathrm{wt} : \mathcal{F}(\omega_i) \to P,\ \varepsilon_i : \mathcal{F}(\omega_i) \to \mathbb{Z},\ \varphi_i : \mathcal{F}(\omega_i) \to \mathbb{Z}$$

by

$$\begin{aligned} \mathrm{wt}(P) &= \omega_i - \sum_{j \in I} k_j \alpha_j \\ \varepsilon_i(P) &= \text{the number of } - \text{ in the } i\text{-signature of } P \\ \varphi_i(P) &= \text{the number of } + \text{ in the } i\text{-signature of } P, \end{aligned}$$

where k_i is the number of i-blocks in P that have been added to the ground-state pyramid P_λ.

Proposition 8.2. *The maps* $\mathrm{wt} : \mathcal{F}(\lambda) \to P$, $\tilde{e}_i, \tilde{f}_i : \mathcal{F}(\lambda) \to \mathcal{F}(\lambda) \cup \{0\}$, $\varepsilon_i, \varphi_i : \mathcal{F}(\lambda) \to \mathbb{Z}$ *define a $U_q(\mathfrak{g})$-crystal structure on the set $\mathcal{F}(\omega_i)$ of all proper Young pyramids.*

Proof. This is an straightforward verification. ☐

Lemma 8.3. *If two stacks y_j and y_k of a proper n-reduced Young pyramid $P \in \mathcal{P}(\lambda)$ are of equal height and have top blocks of color i and $i-1$ respectively, then y_j is in the column immediately west of y_k and y_k is not i-admissible.*

Proof. Since y_j and y_k have equal height, all stacks placed on the squares of the ground state lying with the box with opposite corners defined by the stacks y_j and y_k have this height. Then the fact that P is n-reduced implies that y_j is in the column immediately west of the column containing y_k. This implies that the stack immediately west of y_k is of equal height. Thus y_k is not i-admissible. ☐

Lemma 8.4. *For a Young pyramid $P \in \mathcal{P}(\lambda)$, a generic point of the irreducible component $X_P \in B(\mathbf{w})$ has a representative (x,t) such that*

(1) *The top $(i-1)$-colored block of an i-admissible stack maps into the i-removable block of another stack if and only if it is i-matched to it.*
(2) *If there is a stack with top block of color $(i-1)$ such that there are stacks of equal height in the column immediately to its left then its top block only maps into the i-colored top block of the southernmost stack of this height.*

Proof. By the moment map condition, an $(i-1)$-colored top block of a stack can only map into the i-colored top block of a weakly shorter stack. By Lemma 8.3, if there is a stack of equal height, then it must occur in the column immediately to the left. By a suitable change of basis (i.e. changing our orbit representative), we may assume it only maps into the southernmost of these. If there is no stack of equal height with a top block of color i then we need only consider strictly shorter stacks, which must occur to the east. Then the result follows by the argument used in the proof of Lemma 6.1. □

Theorem 8.5. *The map $P \mapsto X_P$ from $\mathcal{P}(\lambda)$ to the set of irreducible components of $\cup_{\mathbf{v}}\mathcal{L}(\mathbf{v},\mathbf{w})$ is an isomorphism of crystals.*

Proof. The proof is almost exactly analogous to that of Theorem 6.4 where we use Lemma 8.4 instead of Lemma 6.1. □

Remark 8.6. Note that for the case when λ is a fundamental weight and our Young pyramids are Young walls, the crystal action we obtain is different from the one in [4] because our $(+,-)$ matching occurs in the opposite direction. However, it is also possible to recover the crystal action of [4] from the geometry of quiver varieties for type $A_n^{(1)}$. One simply has to reverse the orientation of the quiver and repeat the above process. Then by rotating the Young walls counterclockwise by 90 degrees, we obtain the crystal graph as presented in [4]. However, the n-reduced condition is slightly more difficult to describe in this setting and the connection to [3] is a little less transparent.

Remark 8.7. Note that the theory of Young pyramids could easily be developed for type A_∞ and $\widehat{\mathfrak{gl}}_{n+1}$ using the corresponding results of [3] instead of the $A_n^{(1)}$ results used here. For both cases we would no longer have the n-reduced condition and for type A_∞ our set of colors would be the integers rather than the set $\{0,1,\dots,n\}$. The question of determining the geometric crystal structure for $\widehat{\mathfrak{gl}}_{n+1}$ would require an extension of the results of [13] to the varieties defined in [3].

9. Young walls and the geometric crystal structure for type $D_n^{(1)}$

Let $\mathfrak{g} = \widehat{\mathfrak{so}}_{2n}$, $n \geq 4$, be the affine Lie algebra of type $D_n^{(1)}$. For a level one fundamental weight ω_k of $\mathfrak{g}$ (i.e. $k \in \{0,1,n-1,n\}$), define $\mathcal{F}(\omega_k)$ as in [4] to be the set of all proper Young walls built on the ground state wall Y_{ω_k}. Recall that a Young wall is proper if none of its full columns have the same height. The null root of $\mathfrak{g}$ is given by

$$\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n.$$

A part of a column consisting of one 0-block, one 1-block, two i-blocks for $2 \leq i \leq n-2$, one $(n-1)$-block and one n-block in some cyclic order is called a *δ-column.* A column in a proper Young wall Y is said to contain a *removable δ* if we may remove a δ-column from Y and still obtain a proper Young wall. A proper Young wall is said to be *reduced* if none of its columns contain a removable δ. Let $\mathcal{Y}(\omega_k)$ denote

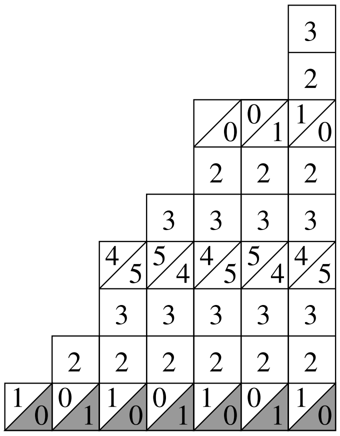


FIGURE 14. A Young wall $Y \in \mathcal{Y}(\omega_1)$ for $n = 5$.

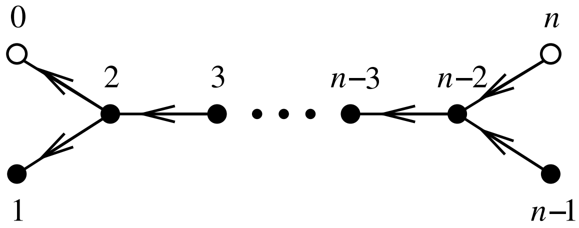


FIGURE 15. The quiver of type $D_n^{(1)}$

the set of all reduced proper Young walls in $\mathcal{F}(\omega_k)$. An example of an element of $\mathcal{Y}(\omega_k)$ is given in Figure 14.

Let $I = \{0, 1, \dots, n\}$ be the set of vertices of the Dynkin graph of $\mathfrak{g}$. We label and orient the quiver as in Figure 15 and we let h_{ij} denote the edge from vertex i to vertex j. Let y be a non-empty column of some $Y \in \mathcal{Y}(\omega_k)$. Let $\mathbf{V}^y$ be the vector space with basis given by the blocks of the column y with the degree of each block given by its color and let x^y be the map that sends each box to the box immediately below it (and mapping the bottom box to zero). In the case that a box v has two boxes w_1 and w_2 below it with the degree of w_1 less than the degree of w_2, then v maps into w_1 with coefficient -1 and into w_2 with coefficient 1. Define $\mathbf{V}^Y = \bigoplus_y \mathbf{V}^y$ where the sum is over all columns of Y and let $\mathbf{v}^Y$ be its graded dimension.

For $Y \in \mathcal{Y}(\omega_k)$, let $x' \in \Lambda_{\mathbf{V}^Y}$ be the direct sum of the representations corresponding to the columns of Y. We then define A_T to be the set of all $x \in \Lambda_{\mathbf{V}^T}$ such that for any vertex v in a column y and $h \in H$, $x_h(v)$ has the same w-component as $x'_h(v)$ for any w in y and all the other components of $x_h(v)$ lie in columns to the right of y. Roughly speaking, we permit columns to map into other columns to their right. It is informative to picture how one column can map into another. Two examples are given in Figures 16 and 17.

Let $\mathcal{C}_Y$ be the union of the $G_{\mathbf{V}^Y}$-orbits of the points of A_Y and let $\bar{\mathcal{C}}_Y$ be its closure. Corresponding to the weight ω_k, we have a dimension vector $\mathbf{w}^k$. It has

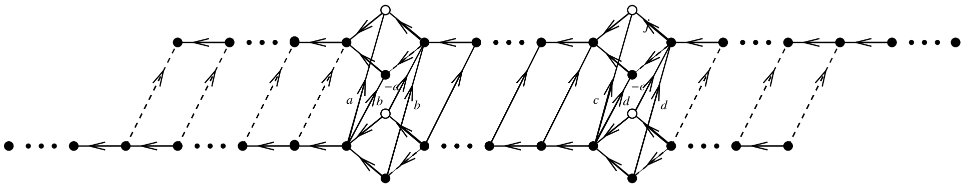


FIGURE 16. One column mapping into another. We depict the blocks by vertices. Unmarked solid and dotted lines indicate a coefficient of 1 and −1 respectively. Otherwise the lines are labelled by the value of the coefficient. These must satisfy $a + b = -1$ and $c + d = 1$.

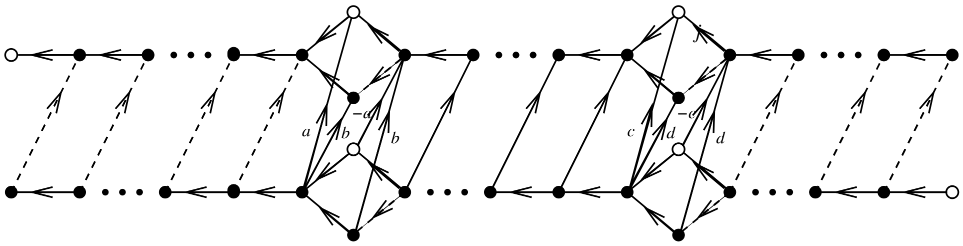


FIGURE 17. One column mapping into another. We depict the blocks by vertices. Unmarked solid and dotted lines indicate a coefficient of 1 and −1 respectively. Otherwise the lines are labelled by the value of the coefficient. These must satisfy $a + b = -1$ and $c + d = 1$.

kth component equal to one and all other components equal to zero. Let $\mathbf{W}$ be a vector space of dimension $\mathbf{w}^k$ Then define

$$X_Y \stackrel{\text{def}}{=} \left(\left(\bar{\mathcal{C}}_Y \times \sum_{i \in I} \mathrm{Hom}(\mathbf{V}_i^Y, \mathbf{W}_i) \right) \cap \Lambda(\mathbf{v}^Y, \mathbf{w}^k)^{\mathrm{st}} \right) / G_{\mathbf{V}^Y}.$$

We define the crystal action on the set $\mathcal{F}(\omega_k)$ as in [4]. We call a block of color i in a proper Young wall *i-removable* if the wall remains a proper Young wall after removing this block. We say that a column is i-removable if its top block is. Similarly, we call a place where we may add a block of color i and obtain another proper Young wall, an *i-admissible* slot. We say that a column is i-admissible if the top of that column is an i-admissible slot.

Let y_i be the ith column of Y, starting at zero and numbering from right to left. To a column y_i of Y we assign a $+$ if the column is i-admissible and a $-$ if the column is i-removable. Combining these $+$'s and $-$'s in the same order that the columns appear, we obtain a sequence of $+$'s and $-$'s. From this sequence, cancel out every $(+,-)$ pair to obtain a sequence of $-$'s followed by $+$'s. This is called the *i-signature* of Y. If two columns correspond to a $(+,-)$ pair, we say they are i-matched. We then define $\tilde{e}_i Y$ to be the Young wall obtained from Y by removing the i-block from the column of Y corresponding to the rightmost $-$ in the i-signature of Y. If no $-$ exists in the i-signature of Y then $\tilde{e}_i Y = 0$. We define $\tilde{f}_i Y$ to be the Young wall obtained from Y by adding an i-block to the column corresponding to the leftmost $+$ in the i-signature of Y. If there is no $+$ in the i-signature of Y then $\tilde{f}_i Y = 0$.

We also define the maps

$$\mathrm{wt} : \mathcal{F}(\omega_k) \to P,\ \varepsilon_i : \mathcal{F}(\omega_k) \to \mathbb{Z},\ \varphi_i : \mathcal{F}(\omega_k) \to \mathbb{Z}$$

by

$$\begin{aligned}
\mathrm{wt}(Y) &= \omega_k - \sum_{j \in I} k_j \alpha_j \\
\varepsilon_i(Y) &= \text{the number of } - \text{ in the } i\text{-signature of } Y \\
\varphi_i(Y) &= \text{the number of } + \text{ in the } i\text{-signature of } Y,
\end{aligned}$$

where k_i is the number of i-blocks in Y that have been added to the ground-state wall Y_{ω_k}.

Proposition 9.1 ([4]). *The maps* $\mathrm{wt} : \mathcal{F}(\omega_k) \to P$, $\tilde{e}_i, \tilde{f}_i : \mathcal{F}(\omega_k) \to \mathcal{F}(\omega_k) \cup \{0\}$, $\varepsilon_i, \varphi_i : \mathcal{F}(\omega_k) \to \mathbb{Z}$ *define a* $U_q(\mathfrak{g})$*-crystal structure on the set* $\mathcal{F}(\omega_k)$ *of all proper Young walls.*

Proposition 9.2 ([4]). *For any* $Y \in \mathcal{Y}(\omega_k)$*, we have*

$$\tilde{e}_i Y \in \mathcal{Y}(\omega_k) \cup \{0\}, \quad \tilde{f}_i Y \in \mathcal{Y}(\omega_k) \cup \{0\}.$$

Hence the set $\mathcal{Y}(\omega_k)$ *has an affine crystal structure for the quantum affine algebra* $U_q(\mathfrak{g})$*. In fact,* $\mathcal{Y}(\omega)$ *is isomorphic to the crystal of the irreducible integrable (basic) representation of highest weight* ω_k*.*

Suppose that column y_k, $k > 0$, is i-admissible and column y_j to the right of y_k is i-removable. Since y_k is i-admissible, the column y_{k-1} immediately to its right must be at least two units taller than y_k. Consider the top block(s) of y_k and the block(s) of y_{k-1} two rows above the top block(s) of y_k. For the sake of argument, let us assume that we are only dealing with single, full-sized blocks (the general argument is analogous). One of these blocks is the same color as the second block from the top in y_j. Generically, the other block maps into the top block in y_j. That is, generically, either the top block of y_k or the block two rows above in y_{k-1} maps into the top block of y_j. Let us call the block that does the y_j*-mapping* block of y_k. In the case that our block is actually composed of two half-blocks, we call both the mapping blocks of y_k. If y_k is i-matched to y_j we simply call its y_j-mapping block the mapping block (dropping the y_j).

If the pattern for constructing Young walls dictates that a block of color i could be added to the top of a column y but the resulting wall would not be a proper Young wall, we say that y is *almost* i*-admissible.*

Proposition 9.3. *For a Young wall* $Y \in \mathcal{Y}(\omega_k)$*, a generic point of* A_Y *has a representative* (x, t) *such that*

(1) *The mapping block of an* i*-admissible column maps into the* i*-removable block of the column to which it is* i*-matched, and*
(2) *The top block(s) of an almost* i*-admissible column map(s) into the* i*-colored block one row above in the column immediately to its right.*

Furthermore, let $W \subset \mathbf{V}_i$ *be the space spanned by the* i*-removable blocks of the* i*-matched,* i*-removable columns of* Y*. If* $i = n-2$ *then for each* i*-matched,* i*-admissible column with a degree* $n-2$ *block one row below the top block, we extend*

W by the span of this block as well. Then a generic point of A_Y has a representative such that the images under $\{x_h \mid \mathrm{in}(h) = i\}$ of the mapping blocks of the i-admissible columns to which these columns are i-matched (in the case $i = n-2$, for each i-admissible column with two mapping blocks that does not have a degree $n-2$ block one row below, we consider only the degree n mapping block), projected to W, form a basis of W.

Proof. (1) follows from the above comments and (2) follows from the existence of the type of mappings between columns depicted in Figures 16 and 17. Then the second statement follows from the fact that by the definition of i-matching, there are always more i-removable strings later than an i-matched i-admissible string than other i-admissible strings. We need the slight modification for the case $i = n-2$ so that the images of the mapping blocks are linearly independent. □

Corollary 9.4. *Let y be an i-admissible, i-matched column in a Young wall Y and take $x \in A_Y$. Suppose we extend x to a vector v of degree i in such a way that the resulting point is in $\Lambda_{\mathbf{V}}$. Then if v maps into the top block(s) of y (or second block from the top if the top block has half unit thickness), either it must also map into some column to the left of y or the column to the left of y is i-admissible and empty.*

Proof. This follows from Proposition 9.3 just like Corollary 7.5 followed from Proposition 7.4. We use the fact that if we add a block v to a column y (thus, it maps into the top block(s) of y or the second block from the top if the top of y is of half unit thickness), then since each column generically maps into the column immediately to its right in the manner of Figures 16 and 17, the moment map condition implies that v must also map into the block(s) one row above it in the column immediately to the right. Thus v maps into the mapping block of y. □

Theorem 9.5. *The set $\{X_Y \mid Y \in \mathcal{Y}(\omega_k)\}$ is precisely the set of irreducible components of $\cup_{\mathbf{v}}\mathcal{L}(\mathbf{v}, \mathbf{w}^k)$ and the map $Y \mapsto X_Y$ from $\mathcal{Y}(\omega_k)$ to the set of irreducible components of $\cup_{\mathbf{v}}\mathcal{L}(\mathbf{v}, \mathbf{w}^k)$ is an isomorphism of crystals.*

Proof. The proof is almost exactly analogous to that of Theorem 7.6 where instead of Corollary 7.5, we use Corollary 9.4. The details will be omitted. □

Remark 9.6. Note that extending the above results to higher level requires a different technique than that used in type $A_n^{(1)}$. This is because for type $A_n^{(1)}$ every fundamental representation is of level 1 which is not the case for type $D_n^{(1)}$.

10. A connection to the path space realization

In [10] and [11], the full structure of the representations (rather than just the crystal structure) is defined on the homology of or a space of constructible functions on the quiver varieties. The irreducible components then correspond to certain elements in these spaces, yielding a basis for the representation. Thus, through the identification of irreducible components with combinatorial objects given in this paper, we have an action of the corresponding Lie algebra $\mathfrak{g}$ on the vector space spanned by the combinatorial objects. Furthermore, there is a natural correspondence between Young walls and the paths of the Kyoto model (see [4]). One simply reads off the top entries of columns to produce the corresponding path. This procedure can easily be extended to Young pyramids, yielding a nice correspondence between Young pyramids and paths. Then the geometric action translates into an action in the basis given by paths. For type $A_n^{(1)}$, an action in this basis was given in [1, 2] (the action for the level one case is given explicitly while the higher level case is less explicit). The geometric action seems to be different although the exact connection between the two is not known. For the case of $D_n^{(1)}$, the definition of an action on the space of paths or Young walls appears to be new.

References

[1] E. Date, M. Jimbo, A. Kuniba, T. Miwa, and M. Okado. Path space realization of the basic representation of $A_n^{(1)}$. In V. G. Kac, editor, *Infinite Dimensional Lie Algebras and Groups*, Advanced Series in Mathematical Physics 7, pages 108–123. World Scientific, 1989.

[2] E. Date, M. Jimbo, A. Kuniba, T. Miwa, and M. Okado. Paths, maya diagrams and representations of $\widehat{\mathfrak{sl}}(r, \mathbb{C})$. *Advanced Studies in Pure Mathematics*, (19):149–191, 1989.

[3] I. B. Frenkel and A. Savage. Bases of representations of type A affine Lie algebras via quiver varieties and statistical mechanics. *International Mathematics Research Notices*, (28):1521–1548, 2003.

[4] J. Hong and S.-J. Kang. *Introduction to quantum groups and crystal bases*. American Mathematical Society, 2000.

[5] S.-J. Kang. Crystal bases for quantum affine algebras and combinatorics of Young walls. *Proc. London Math. Soc.*, 86:26–69, 2003.

[6] M. Kashiwara and T. Nakashima. Crystal graphs for representations of the q-analogue of classical Lie algebras. *Journal of Algebra*, 165:295–345, 1994.

[7] M. Kashiwara and Y. Saito. Geometric construction of crystal bases. *Duke Mathematical Journal*, 89(1):9–36, 1997.

[8] G. Lusztig. Quivers, perverse sheaves, and quantized enveloping algebras. *Journal of the American Mathematical Society*, (4):365–421, 1991.

[9] H. Nakajima. Homology of moduli spaces of instantons on ALE spaces. *Journal of Differential Geometry*, 40:105–127, 1994.

[10] H. Nakajima. Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras. *Duke Mathematical Journal*, 76(2):365–416, 1994.

[11] H. Nakajima. Quiver varieties and Kac-Moody algebras. *Duke Mathematical Journal*, 91(3):515–560, 1998.

[12] H. Nakajima. t-analogs of q-characters of quantum affine algebras of type A_n, D_n. *Contemporary Mathematics*, 325:141–160, 2003.

[13] Y. Saito. Crystal bases and quiver varieties. *Mathematische Annalen*, 324(4):675–688, 2002.

[14] A. Savage. A geometric realization of spin representations and Young diagrams from quiver varieties. arXiv:math.AG/0307018.

Max-Planck-Institut für Mathematik, P.O. Box 7280, D-53072 Bonn, Germany

E-mail address: `alistair.savage@aya.yale.edu`