Symmetries of the 4-matrices Schlesinger system

M.V.Babich[1]

January 11, 2000

Abstract

All types of symmetries, such as a permutation of parameters, Schlesinger and Okamoto transformations are considered. The new form of Schlesinger system and the action of modular group on the corresponding linear system are studied, the geometrical meaning of the Okamoto transformation is discussed.

0 Schlesinger system

Consider the following linear differential equation on $SL(2,\mathbb{C})$-valued function Ψ of the complex variable λ:

$$\frac{d}{d\lambda}\Psi = \sum_k A_k \frac{1}{\lambda-\lambda_k}\Psi, \quad \Psi \in SL(2,\mathbb{C}), A_k \in sl(2,\mathbb{C}), \lambda, \lambda_k \in \mathbb{C}. \tag{1}$$

The solution $\Psi(\lambda)$ is a multivalued function with branch points λ_k. When Ψ is prolonged along a closed path going around λ_k, Ψ is changed on some (λ-independent) matrix M_k. Matrices M_k are called the monodromy matrices. In general, monodromy matrices depend on the positions of branch points λ_k, if we consider them as parameters. How should we change matrices A_k for keeping constant all monodromy matrices M_k and eigenvalues of A_k?

L. Schlesinger [1] found that these conditions are equivalent to the following system of equations on matrices A_k, so called *Schlesinger system* (Ss):

$$\frac{\partial A_k}{\partial \lambda_j} = \frac{1}{\lambda_j-\lambda_k}[A_j, A_k] \;\; (j \neq k), \quad \frac{\partial A_k}{\partial \lambda_k} = \sum_{j\neq k} \frac{1}{\lambda_k-\lambda_j}[A_j, A_k], \tag{2}$$

function Ψ depends on parameters λ_k in according to the equations

$$\frac{\partial}{\partial \lambda_k}\Psi = \frac{1}{\lambda_k-\lambda}A_k\Psi \quad \forall k. \tag{3}$$

[1]Permanent address: St.Petersburg department of the Steklov Mathematical Institute, Fontanka 27, St.Petersburg, 191011 Russia; E-mail: mbabich@pdmi.ras.ru

In the present paper I am considering the simplest non-trivial case of four matrices $A_1, A_2, A_3, A_0 = -(A_1+A_2+A_3)$. By the fractional-linear transformation we move points $\{\lambda_1,\lambda_2,\lambda_3,\lambda_0\}$ into $\{0,x,1,\infty\}$. Function $\Psi(\lambda)$ depends on the complex parameter x and, as a function of λ, has four branch points on $\overline{\mathbb{C}}$: $0, 1, x$ and infinity.

Equations (1), (3) take the form

$$\begin{cases} \frac{\partial}{\partial\lambda}|_{x=const}\Psi = \left(\frac{1}{\lambda}A_1 + \frac{1}{\lambda-x}A_2 + \frac{1}{\lambda-1}A_3\right)\Psi \\ \frac{\partial}{\partial x}|_{\lambda=const}\Psi = \frac{1}{x-\lambda}A_2\Psi, \end{cases} \tag{4}$$

and Ss takes the form

$$\frac{d}{dx}A_1 = \frac{1}{x}\left[A_2, A_1\right], \quad \frac{d}{dx}A_2 = \frac{1}{x}\left[A_1,A_2\right] + \frac{1}{x-1}\left[A_3,A_2\right], \quad \frac{d}{dx}A_3 = \frac{1}{x-1}\left[A_2,A_3\right]. \tag{5}$$

The set (a_1, a_2, a_3, a_0), where $a_k/2$ is an eigenvalue of A_k is called the set of parameters of Ss.

The symmetries of Ss are under consideration. The concept of symmetry is treated as a correspondence between two Ss's with different parameters. In this connection I want to mention here the works of D. Arinkin and S. Lysenko [2, 3], where using methods of theory of stacks, they found all the isomorphisms between moduli spaces corresponding to different Schlesinger systems.

1 Permutation of parameters

The source of this symmetry is evident on the level of a setting up the problem of isomonodromy deformation. The normalization $d\Psi\Psi^{-1}\to 0$ when $\lambda\to\infty$ destroys this symmetry, and we make some gauge transformation for the clarifying the action of S_4-group on the Ψ-function, the solution of the linear system.

Consider the universal covering of $\overline{\mathbb{C}}_\lambda \setminus \{0,1,x,\infty\}$, it is the complex plane $\mathbb{C}_u$. We define the correspondence between λ and u in the following way:

$$u = \frac{1}{4}\int_0^\lambda \frac{dt}{\sqrt{t(1-t)(x-t)}} \Big/ \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}}, \quad \lambda = \frac{\theta_2^2(0)\ \theta_1^2(u)}{\theta_3^2(0)\ \theta_0^2(u)}$$

$$\tau = i\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-(1-x)t)}} \Big/ \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-xt^2)}}, \quad x = \theta_2^4(0|\tau)/\theta_3^4(0|\tau),$$

and the theta functions are calculated by the period τ:

$$\theta[p,q](u|\tau) = \sum_{m\in\mathbb{Z}} \exp\{\pi i(m+p)^2\tau + 2\pi i(u+q)(m+p)\}$$

$\theta_1(u){:=}-\theta[\frac{1}{2},\frac{1}{2}](u|\tau),\ \theta_2(u){:=}\theta[\frac{1}{2},0](u|\tau),\ \theta_3(u){:=}\theta[0,0](u|\tau),\ \theta_0(u){:=}\theta[0,\frac{1}{2}](u|\tau).$

Theorem 1. *Let $G = G(x)$ be a solution of the equation*

$$-2G^{-1}\frac{d}{dx}G = \frac{1}{x}A_1 + \frac{2x-1}{x(x-1)}A_2 + \frac{1}{x-1}A_3,$$

and Ψ be a solution of the system (4).

Then the function $\hat{\Psi} = G\Psi$ satisfies the system

$$\begin{cases} \hat{\Psi}' = 2\sum_{k=0}^{3} \log' \theta_k(u|\tau)\hat{A}_k\hat{\Psi} \\ \\ \dot{\hat{\Psi}} = 2\sum_{k=0}^{3} \dot{\log}\, \theta_k(u|\tau)\hat{A}_k\hat{\Psi}, \end{cases} \tag{6}$$

where $...' = \partial_u|_{\tau=const}$, $\dot{...} = \partial_\tau|_{u=const}$, $\hat{A}_k = GA_kG^{-1}$.

Proof:

Consider (6); the change of variables $(u,\ \tau) \longrightarrow (\lambda,\ x)$ gives[2]

$$\begin{cases} \frac{\partial}{\partial\lambda}|_{x=const}\hat{\Psi}\hat{\Psi}^{-1} = \frac{1}{\lambda}\hat{A}_1 + \frac{1}{\lambda - x}\hat{A}_2 + \frac{1}{\lambda-1}\hat{A}_3 \\ \frac{\partial}{\partial x}|_{\lambda=const}\hat{\Psi}\hat{\Psi}^{-1} = \frac{1}{x-\lambda}\hat{A}_2 - \frac{1}{2x}\hat{A}_1 - \frac{2x-1}{2x(x-1)}\hat{A}_2 - \frac{1}{2(x-1)}\hat{A}_3. \end{cases}$$

If $G = G(x)$ is a solution of the equation

$$-2\frac{d}{dx}GG^{-1} = \frac{1}{x}\hat{A}_1 + \frac{2x-1}{x(x-1)}\hat{A}_2 + \frac{1}{x-1}\hat{A}_3,$$

one can see that $\Psi = G^{-1}\hat{\Psi}$ satisfy the system (4).

QED

It is worth to notice the puzzling resemblance of the linear representation (6) to the linear system arisen in description of isomonodromic deformations on the torus [7].

Remark 1. *Using the analytical properties of $\hat{\Psi}_u\hat{\Psi}^{-1}$ and $\hat{\Psi}_\tau\hat{\Psi}^{-1}$, we can write the following representation for the second equation of (6):*

$$4\pi i\ \dot{\hat{\Psi}} = (\hat{A}' + \hat{A}\log'\theta_1(2u|\tau))\hat{\Psi}\ ,\ \textit{where}\ \hat{A} = \sum_{k=0}^{3}\log'\theta_k(u|\tau)\hat{A}_k.$$

[2]For the details see the Appendix

For the representation (6) the compatibility conditions, i.e. the version of Ss (5) take the following symmetrical form

$$\begin{cases} \dot{\hat{A}}_1 = 2\left[\hat{A}_2 \,\dot{\log}\; \theta_2 + \hat{A}_3 \,\dot{\log}\; \theta_3 + \hat{A}_0 \,\dot{\log}\; \theta_0, \hat{A}_1\right] \\ \dot{\hat{A}}_2 = 2\left[\hat{A}_1 \,\dot{\log}\; \theta_2 + \hat{A}_0 \,\dot{\log}\; \theta_3 + \hat{A}_3 \,\dot{\log}\; \theta_0, \hat{A}_2\right] \\ \dot{\hat{A}}_3 = 2\left[\hat{A}_0 \,\dot{\log}\; \theta_2 + \hat{A}_1 \,\dot{\log}\; \theta_3 + \hat{A}_2 \,\dot{\log}\; \theta_0, \hat{A}_3\right] \\ \dot{\hat{A}}_0 = 2\left[\hat{A}_3 \,\dot{\log}\; \theta_2 + \hat{A}_2 \,\dot{\log}\; \theta_3 + \hat{A}_1 \,\dot{\log}\; \theta_0, \hat{A}_0\right] \end{cases} \tag{7}$$

with the standard notation $\theta_k = \theta_k(0|\tau)$.

Let us define the action of the modular group Σ on functions $\Psi(u|\tau)$ as follows:

$$\text{if } \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{Z}) \cong \Sigma, \text{ then } \sigma\Psi(u|\tau) = \Psi\left(\frac{u}{\gamma\tau+\delta}|\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right).$$

The system (6) is invariant with respect to the action of the modular group in the following sense

Theorem 2. *Let $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Sigma$ be an element of the modular group, and $\hat{\Psi}(u|\tau)$ be a solution of (6) for some set $\hat{A}_k(\tau)$, $k = 1,2,3,0$. Then the matrix $\tilde{\Psi} = \sigma\hat{\Psi}$ is a solution of*

$$\begin{cases} \tilde{\Psi}' = 2\sum_{k=0}^{3} \log' \theta_k(u|\tau)\tilde{A}_k\tilde{\Psi} \\ \dot{\tilde{\Psi}} = 2\sum_{k=0}^{3} \dot{\log}\; \theta_k(u|\tau)\tilde{A}_k\tilde{\Psi}, \end{cases} \tag{8}$$

where $\tilde{A}_k(\tau) = \hat{A}_{k_\sigma}(\sigma\tau)$; $\sigma\tau = (\alpha\tau+\beta)/(\gamma\tau+\delta)$; k_σ is the permutation of the indices 2,3 and 0 that corresponds to the permutation of half-periods $\{\frac{\gamma\tau+\delta}{2}; \frac{(\alpha+\gamma)\tau+\beta+\delta}{2}; \frac{\alpha\tau+\beta}{2}\} \longrightarrow \{\frac{1}{2}; \frac{1+\tau}{2}; \frac{\tau}{2}\} \pmod{1,\tau}$.

Proof:
It is well known that Σ is generated by two elements $S = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. It is sufficient to verify the statement for S and T, and to check that the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ produces the identical transformation $\tilde{A}_k(\tau) = \hat{A}_k(\tau)$.

Consider S:

$$S\hat{\Psi}(u|\tau) = \hat{\Psi}(u|\tau+1)$$

Equalities

$$\theta_1(u|\tau) = e^{-i\pi/4}\theta_1(u|\tau+1), \quad \theta_2(u|\tau) = e^{-i\pi/4}\theta_2(u|\tau+1),$$

and

$$\theta_3(u|\tau) = \theta_0(u|\tau+1), \quad \theta_0(u|\tau) = \theta_3(u|\tau+1),$$

imply that $\tilde\Psi = S\hat\Psi$ satisfies (8) with

$$\tilde A_1(\tau) = \hat A_1(\tau+1),\ \tilde A_2(\tau) = \hat A_2(\tau+1),\ \tilde A_3(\tau) = \hat A_0(\tau+1),\ \tilde A_0(\tau) = \hat A_3(\tau+1).$$

Consider T:

$$T\hat\Psi(u|\tau) = \tilde\Psi(u|\tau) = \hat\Psi(-u/\tau|-1/\tau) = \hat\Psi(v|\tau_T),$$

where $\tau_T = -1/\tau$, $v = \tau_T u$.

$$\begin{aligned}&\tfrac{1}{2}d\tilde\Psi(u|\tau) = \sum_{k=0}^{3}\log'\theta_k(v|\tau_T)\hat A_k(\tau_T)\tilde\Psi dv + \sum_{k=0}^{3}\overset{\bullet}{\log}\,\theta_k(v|\tau_T)\hat A_k(\tau_T)\tilde\Psi d\tau_T = \\ &\sum_{k=0}^{3}\log'\theta_k(v|\tau_T)\hat A_k(\tau_T)\tau_T\tilde\Psi du + \sum_{k=0}^{3}\left(\overset{\bullet}{\log}\,\theta_k(v|\tau_T) + \log'\theta_k(v|\tau_T)u\right)\hat A_k(\tau_T)(\tau_T)^2\tilde\Psi d\tau\end{aligned} \tag{9}$$

From the equalities

$$\theta_1(u|\tau) = -i\mu\theta_1(v|\tau_T),\ \theta_2(u|\tau) = \mu\theta_0(v|\tau_T),\ \theta_3(u|\tau) = \mu\theta_3(v|\tau_T),$$

and

$$\theta_0(u|\tau) = \mu\theta_2(v|\tau_T),\ \text{with}\ \mu = \sqrt{-i\tau}\exp\{\tau_T\pi i u^2\},$$

we get

$$\begin{aligned}d\log\theta_k(u|\tau) &= d\log\mu + d\log\theta_{k_T}(v|\tau_T) = d\log\mu + \log'\theta_{k_T}(v|\tau_T)dv + \overset{\bullet}{\log}\,\theta_{k_T}(v|\tau_T)d\tau_T = \\ &\quad d\log\mu + \log'\theta_{k_T}(v|\tau_T)du + \left(\overset{\bullet}{\log}\,\theta_{k_T}(v|\tau_T) + \log'\theta_{k_T}(v|\tau_T)u\right)d\tau_T,\end{aligned}$$

in which the index k_T takes values $1,0,3,2$ when k takes values $1,2,3,0$.

$$\begin{aligned}\log'\theta_{k_T}(v|\tau_T)\tau_T &= \log'\theta_k(u|\tau) - \frac{\partial}{\partial u}\log\mu \\ \overset{\bullet}{\log}\,\theta_{k_T}(v|\tau_T)(\tau_T)^2 &= \overset{\bullet}{\log}\,\theta_k(u|\tau) - u\tau_T\left(\log'\theta_k(u|\tau) - \frac{\partial}{\partial u}\log\mu\right) - \frac{\partial}{\partial\tau}\log\mu.\end{aligned}$$

Substituting the last two equalities into (9) produces

$$\frac{1}{2}d\tilde{\Psi}(u|\tau) = \sum_{k=0}^{3} \log' \theta_k(u|\tau)\hat{A}_{k_T}(\tau_T)du + \sum_{k=0}^{3} \overset{\bullet}{\log}\, \theta_k(u|\tau)\hat{A}_{k_T}(\tau_T)d\tau. \qquad (10)$$

Here we have used the identity $\sum_k \hat{A}_k = 0$ and the fact that μ does not depend on the index "k" of theta-functions. Equality (10) means that $\tilde{\Psi}$ satisfies (8) with

$$\tilde{A}_1(\tau) = \hat{A}_1(-1/\tau), \tilde{A}_2(\tau) = \hat{A}_0(-1/\tau), \tilde{A}_3(\tau) = \hat{A}_3(-1/\tau), \tilde{A}_0(\tau) = \hat{A}_2(-1/\tau).$$

To conclude the proof we must show that actions of "σ" and "$-\sigma$" coincide, but that is clear since $\overset{\bullet}{\log}\ \theta_k(u) = \overset{\bullet}{\log}\ \theta_k(-u)$ and $\log' \theta_k(u) = \log' \theta_k(-u)$.

We have defined the action of the modular group on functions of two variables and denoted this group of transformations by Σ. Let us add one generator more: $\sigma'\Psi(u|\tau) = \Psi(u+1|\tau)$ and denote the corresponding group of transformations by Σ'.

From Theorem 2 and the trivial observation that the function $\tilde{\Psi} = \sigma'\Psi(u|\tau) = \Psi(u+1|\tau)$ satisfies (8) with $\tilde{A}_{1,2}(\tau) = \hat{A}_{2,1}(\tau)$, $\tilde{A}_{3,0}(\tau) = \hat{A}_{0,3}(\tau)$, there follows

Proposition 1.1. *Let $\Psi(u|\tau)$ be a solution of (6) with parameters (a_1, a_2, a_3, a_0), and let $(1,2,3,0) \longrightarrow (j_1, j_2, j_3, j_0)$ be any permutation of indices. Then there is transformation $\sigma \in \Sigma'$ such that $\tilde{\Psi} = \sigma\Psi$ satisfies (8) with parameters $(a_{j_1}, a_{j_2}, a_{j_3}, a_{j_0})$.*

We say that system (6) is invariant under an arbitrary permutation of parameters.

2 Schlesinger transformations

The idea of the Schlesinger transformation belong to L. Schlesinger himself [1]. Different authors wrote down different formulae for such transformations designed for their own purposes (see [5, 9, 10, 11]). My intention is to explain the conception of the Schlesinger transformation and give such formulae that might be easily modified, as desired.

Consider function Ψ, which is a solution of (4) or (6). From the local considerations near the singular points (see [4, 5, 8])

$$\Psi(u+\omega_J) = (G_J + O(u^2)) \begin{pmatrix} u^{a_J} & 0 \\ 0 & u^{-a_J} \end{pmatrix} (I + j_J^{\pm}\sigma_{\pm} \log u)C_J, \qquad (11)$$

where $G_J + O(u^2)$ is some power series in u^2, $G_J, C_J \in SL(2,\mathbb{C})$, $\sigma_+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\sigma_- = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and $j_J^{\pm}$ vanish for any $\pm a_J \notin \mathbb{Z}^+ = \{0,1,2,3\ldots\}$. Matrices C_J and values j_J are constants. By ω_J we denoted a half-period, zero of the corresponding theta-function: $\theta_J(\omega_J) = 0$.

It follows from (11), that $\hat{A}_J = G_J\, æ_J\, (G_J)^{-1}$, where $2æ_J = a_J \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ if $a_J \neq 0$ and $2æ_J = j_J^{\mp}\sigma_{\pm}$ if $a_J = 0$.

The (constant) monodromy matrices are

$$M_J = C_J^{-1} \begin{pmatrix} e^{\pi i a_J} & 0 \\ 0 & e^{-\pi i a_J} \end{pmatrix} \begin{pmatrix} 1 & \pi i j_J^{\pm} \\ 0 & 1 \end{pmatrix} C_J$$

Let F be the analytical and single-valued outside the singularities of Ψ matrix-function. Suppose, we found such F, that in the singular points ω_J it has such a representation that the product $F\Psi = \tilde{\Psi}$ has the representation (11) again, maybe with some new $\tilde{G}_J$, $\tilde{a}_J$. If so, the function $\tilde{\Psi}$ has the same (consequently constant!) monodromy matrices as Ψ. After the normalization "det $\tilde{\Psi} = 1$" and proper gauge transformation, $\tilde{\Psi}$ satisfies systems (6) or (4).

Note that from the uniqueness theorem for the corresponding Riemann-Hilbert problem follows that essentially (up to normalizations and gauge transformations) function Ψ is determined by parameters a_k and the monodromy matrices M_k. Schlesinger transformation does not change the monodromy matrices, consequently different Schlesinger transformations commute (up to a gauge), and any Schlesinger transformation can be constructed as a sequence of so-called "elementary" Schlesinger transformations, which are constructed below.

We present such a matrix F which adds one to two parameters (for example a_I and a_J):

$$F_{I+,J+} = \begin{pmatrix} \theta_I^2(u) & 0 \\ 0 & \theta_J^2(u) \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix},$$

where vector-rows (f_{11}, f_{12}) and (f_{21}, f_{22}) are arbitrary vectors orthogonal to the vector-columns of the leading term of (11) $((G_J)_{11}, (G_J)_{21})^T$ and $((G_I)_{11}, (G_I)_{21})^T$ respectively; for example:

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} -(G_J)_{21} & (G_J)_{11} \\ -(G_I)_{21} & (G_I)_{11} \end{pmatrix}$$

One can easily verify that such a construction of the matrix (f_{ij}) preserves form (11) for the product $F_{I+,J+}\Psi = \tilde{\Psi}$.

In the same manner we can write down matrix-functions $F_{I-,J+}$, $F_{I+,J-}$ and $F_{I-,J-}$ that give all possible changes of the values a_I, a_J by ± 1:

$$F_{I+,J-} = \begin{pmatrix} \theta_I^2(u) & 0 \\ 0 & \theta_J^2(u) \end{pmatrix} \begin{pmatrix} -(G_J)_{22} & (G_J)_{12} \\ -(G_I)_{21} & (G_I)_{11} \end{pmatrix}$$

$$F_{I^-,J^+} = \begin{pmatrix} \theta_I^2(u) & 0 \\ 0 & \theta_J^2(u) \end{pmatrix} \begin{pmatrix} -(G_J)_{21} & (G_J)_{11} \\ -(G_I)_{22} & (G_I)_{12} \end{pmatrix}$$

$$F_{I^-,J^-} = \begin{pmatrix} \theta_I^2(u) & 0 \\ 0 & \theta_J^2(u) \end{pmatrix} \begin{pmatrix} -(G_J)_{22} & (G_J)_{12} \\ -(G_I)_{22} & (G_I)_{12} \end{pmatrix}$$

Note that it is impossible to make the trivial generalization of this construction and get a half-integer changes of a_J, using some function F with simple zeros.

3 Okamoto transformation

In this section we consider the transformation that was found by K. Okamoto [6] for the Painlevé VI equation, which is equivalent to Ss. This transformation connects Schlesinger systems with parameters (a_1, a_2, a_3, a_0) and $(a_1, a_2, a_3, a_0)J$, where

$$J = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

To make this subject more visual we change the "$sl(2,\mathbb{C})$-language" to the equivalent "$\mathbb{C}^3$-language".

Let us introduce the scalar and vector products on the $sl(2,\mathbb{C})$:

$$\langle A, B\rangle = 2tr\ AB, \quad [A,B] = AB - BA.$$

We need the following versions[3] of the well known formulae from linear algebra

$$\begin{aligned} -<[A,B]C>^2 &= \det\begin{pmatrix} <A,A> & <A,B> & <A,C> \\ <B,A> & <B,B> & <B,C> \\ <C,A> & <C,B> & <C,C> \end{pmatrix} \quad (12) \\ -[A,[B,C]] &= B<A,C> - C<A,B>. \end{aligned}$$

In this interpretation equations (5) govern the evaluation of tetrahedron A_1, A_2, A_3, A_0, where A_k is the radius-vector of the same name vertex; the origin is in the masses center. A gauge transformation of the linear system (4) leads to the turn $A_k \longrightarrow GA_kG^{-1}$ (where $G = G(x) \in SL(2,\mathbb{C})$) of the tetrahedron as a solid body.

[3] $sl(2,\mathbb{R}) \cong \mathbb{R}^{2,1}$, not to $\mathbb{R}^3$, that is the reason for the minus sign in the left-hand sides

Consider "the Gram-matrix", i.e. the pair-wise scalar products of radius-vectors of vertices. Note, that the scalar product does not depend on the gauge transformations, and $< A_i, A_k >=< \hat{A}_i, \hat{A}_k >= a_{ik}$, so we consider (5) and (7) simultaneously:

$$a_{ik} :=< A_i, A_k >= 2tr\ (A_i A_k), \quad a_{kk} = 2tr\ A_k^2 = -4\det A_k = (a_k)^2.$$

Matrix entries a_{ik} satisfy the relations

$$\begin{gathered} a_{ik} = a_{ki} \\ \sum_{i=0}^{3} a_{ik} = 0\ \forall k \\ a_{ik} + (a_{ii} + a_{kk})/2 = a_{jl} + (a_{jj} + a_{ll})/2 \text{ for mutually different } i,j,k,l. \end{gathered} \tag{13}$$

The second and the third rows follow from the equalities $< \sum A_i, A_k >= 0$ and $< A_i + A_k, A_i + A_k >=< -A_j - A_l, -A_j - A_l >$ which are hold since $\sum A_i = 0$.

The equations which determine the evaluation of the matrix (a_{ik}) are

$$\begin{cases} \dfrac{d}{d\tau} a_{12} = \dfrac{d}{d\tau} a_{03} = 4 \left(\dfrac{d}{d\tau} \log\ \dfrac{\theta_0}{\theta_3} \right) a_{230} \\ \\ \dfrac{d}{d\tau} a_{13} = \dfrac{d}{d\tau} a_{20} = 4 \left(\dfrac{d}{d\tau} \log\ \dfrac{\theta_2}{\theta_0} \right) a_{230} \\ \\ \dfrac{d}{d\tau} a_{10} = \dfrac{d}{d\tau} a_{32} = 4 \left(\dfrac{d}{d\tau} \log\ \dfrac{\theta_3}{\theta_2} \right) a_{230} \\ \\ \quad a_{kk} = const \qquad \forall k, \end{cases} \tag{14}$$

or

$$\frac{d}{d\tau} a_{1k} = \frac{d}{d\tau} a_{ij} = -4 \left(\frac{d}{d\tau} \log\ \frac{\theta_i}{\theta_j} \right) a_{ijk}, \quad i,j,k \in \{2,3,0\},$$

where $\theta_k = \theta_k(0|\tau)$ are the theta-constants, $a_{ijk} =< [A_i, A_j], A_k >$ is a one and a half of the volume of the tetrahedron A_1, A_2, A_3, A_0 (with some sign).

Note that the sum of equations (14) is zero, so there are only two independent equations of the first order on two unknown functions (for example a_{12} and a_{13}); all other a_{ik} are linear combinations of them and constant parameters a_{kk} (equalities (13)). We can exclude one unknown and get an ordinary differential equation of the second order. After some (non-trivial!) change of variables it can be brought to the Painlevé VI equation.

If we know a_{ik}, the tetrahedron can be restored up to a position, and linear system (4) up to a gauge transformation. Let us set as a goal the following natural question:

What different Schlesinger systems are controlled by the same system of equations (14)?

The answer is:

They are Schlesinger systems, connected by Okamoto transformation.

Let us comment this statement. We say that Ss's $A_1,\dots,A_0$ and $\tilde{A}_1,\dots,\tilde{A}_0$ have the same systems (14) if $a_{ik}=\tilde{a}_{ik}+const_{ik}$ and $a_{230}=\tilde{a}_{230}$, where $const_{ik}$ are independent on the variable τ.

From now on we will assign our tetrahedron A_1,A_2,A_3,A_0 by three vectors $\vec{X},\vec{Y},\vec{Z}$:

$$\vec{X}:=A_1+A_2,\ \vec{Y}:=A_1+A_3,\ \vec{Z}:=A_1+A_0$$

$$2A_1=\vec{X}+\vec{Y}+\vec{Z},\ 2A_2=\vec{X}-\vec{Y}-\vec{Z},\ 2A_3=-\vec{X}+\vec{Y}-\vec{Z},\ 2A_0=-\vec{X}-\vec{Y}+\vec{Z}.$$

Vectors $\vec{X},\vec{Y},\vec{Z}$ connect the antipodal (having no common vertexes) edges of the tetrahedron and form a basis of the $\mathbb{C}^3$ space. The calculation of its Gram-matrix gives

$$\begin{pmatrix} X^2 & a_{10/23} & a_{13/20}\\ a_{10/23} & Y^2 & a_{12/30}\\ a_{13/20} & a_{12/30} & Z^2\end{pmatrix} := \begin{pmatrix} <\vec{X},\vec{X}> & <\vec{X},\vec{Y}> & <\vec{X},\vec{Z}>\\ <\vec{Y},\vec{X}> & <\vec{Y},\vec{Y}> & <\vec{Y},\vec{Z}>\\ <\vec{Z},\vec{X}> & <\vec{Z},\vec{Y}> & <\vec{Z},\vec{Z}>\end{pmatrix} =$$

$$\begin{pmatrix} <\vec{X},\vec{X}> & \frac{1}{2}(a_{11}+a_{00}-a_{22}-a_{33}) & \frac{1}{2}(a_{11}+a_{33}-a_{22}-a_{00})\\ \frac{1}{2}(a_{11}+a_{00}-a_{22}-a_{33}) & <\vec{Y},\vec{Y}> & \frac{1}{2}(a_{11}+a_{22}-a_{33}-a_{00})\\ \frac{1}{2}(a_{11}+a_{33}-a_{22}-a_{00}) & \frac{1}{2}(a_{11}+a_{22}-a_{33}-a_{00}) & <\vec{Z},\vec{Z}>\end{pmatrix}$$

We use the notation $a_{1i/jk}=(a_{11}+a_{ii}-a_{jj}-a_{kk})/2=const$, $a_\Sigma=X^2+Y^2+Z^2=a_{11}+a_{22}+a_{33}+a_{00}=const$. You can see that for any Ss all the pair-wise products of the vectors of this basis are constants, except the "squares of lengths" of the vectors $\vec{X},\vec{Y},\vec{Z}$ which are connected by one relation: $X^2+Y^2+Z^2=a_\Sigma=const$. We see that the state of Ss can be naturally parameterized by a point of a sphere with radius $\sqrt{a_\Sigma}$. The squares of the Cartesian coordinates of this point satisfy equations (14) ($a_{12}=X^2/2+const,\ a_{13}=Y^2/2+const,\ a_{10}=Z^2/2+const$). For the value a_{230} the following expression, using (12), can be written down:

$$-4a_{230}^2=\det\begin{pmatrix} X^2 & a_{10/23} & a_{13/20}\\ a_{10/23} & Y^2 & a_{12/30}\\ a_{13/20} & a_{12/30} & Z^2\end{pmatrix} = X^2Y^2Z^2-\frac{1}{3}\left(X^2(2a_{12/30}^2-a_{10/23}^2-a_{13/20}^2)+\right.$$

$$\left.Y^2(2a_{13/20}^2-a_{12/30}^2-a_{10/23}^2)+Z^2(2a_{10/23}^2-a_{12/30}^2-a_{13/20}^2)\right)+2a_{12/30}a_{13/20}a_{10/23}-$$

$$\frac{1}{3}a_\Sigma(a_{12/30}^2+a_{13/20}^2+a_{10/23}^2)=X^2Y^2Z^2+\frac{2}{3}(4\sigma_2^2-\sigma_1^2)X^2-\frac{1}{3}(4\sigma_2^2-\sigma_1^2+24\sqrt{\sigma_4})Y^2-$$

$$\frac{1}{3}(4\sigma_2^2-\sigma_1^2-24\sqrt{\sigma_4})Z^2-16\sigma_3+\frac{8}{3}\sigma_1\sigma_2-\sigma_1^3,$$

where, following Okamoto [6], we denoted

$$a_1=b_1+b_2,\ a_2=b_1-b_2,\ a_3=b_3+b_4,\ a_0=b_3-b_4,$$

$$\sigma_1=\sum_i b_i^2,\ \sigma_2=\sum_{i<j} b_i^2 b_j^2,\ \sigma_3=\sum_{i<j<k} b_i^2 b_j^2 b_k^2,\ \sqrt{\sigma_4}=b_1b_2b_3b_4.$$

We can see that a_{230} is invariant under a permutation of values b_i. From the other hand we can see that the equality

$$\det\begin{pmatrix} X^2+const_X & \tilde a_{10/23} & \tilde a_{13/20} \\ \tilde a_{10/23} & Y^2+const_Y & \tilde a_{12/30} \\ \tilde a_{13/20} & \tilde a_{12/30} & Z^2+const_Z \end{pmatrix} = \det\begin{pmatrix} X^2 & a_{10/23} & a_{13/20} \\ a_{10/23} & Y^2 & a_{12/30} \\ a_{13/20} & a_{12/30} & Z^2 \end{pmatrix}$$

is equivalent to the following set of equalities:

$$const_X=const_Y=const_Z=0,\ \sigma_1=\tilde\sigma_1,\ \sigma_2=\tilde\sigma_2,\ \sigma_3=\tilde\sigma_3,\ b_1b_2b_3b_4=\tilde b_1\tilde b_2\tilde b_3\tilde b_4,$$

where values with tilde are computed for $\tilde a_{kk}$.

These equalities imply that the only possible transformations are permutations of b_k and simultaneous changes of signs of some pair b_i, b_k. Generators of this group act on the set of parameters of Ss (a_1,a_2,a_3,a_0) as a change of the sign of a_k (which is a trivial transformation since our equations contain only squares of a_k's) and one non-trivial generator

$$(\tilde a_1,\tilde a_2,\tilde a_3,\tilde a_0)=(a_1,a_2,a_3,a_0)\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

that was the goal of our consideration. Note that it is difficult to imagine how this transformation of tetrahedron looks "geometrically", because of its non-reality. In the case of orthogonal vectors $\vec X,\vec Y,\vec Z$ (corresponding to equal parameters $a_k = a_i\ \forall k,i$; let for example $a_k=1$):

$$\det\begin{pmatrix} <\vec X,\vec X> & <\vec X,\vec Y> & <\vec X,\vec Z> \\ <\vec Y,\vec X> & <\vec Y,\vec Y> & <\vec Y,\vec Z> \\ <\vec Z,\vec X> & <\vec Z,\vec Y> & <\vec Z,\vec Z> \end{pmatrix} = \det\begin{pmatrix} X^2 & 0 & 0 \\ 0 & Y^2 & 0 \\ 0 & 0 & 1-X^2-Y^2 \end{pmatrix} =$$

$$\det\begin{pmatrix} X^2 & 1/2 & 1/2 \\ 1/2 & Y^2 & 1/2 \\ 1/2 & 1/2 & 1-X^2-Y^2 \end{pmatrix} = \det\begin{pmatrix} <\vec{\tilde X},\vec{\tilde X}> & <\vec{\tilde X},\vec{\tilde Y}> & <\vec{\tilde X},\vec{\tilde Z}> \\ <\vec{\tilde Y},\vec{\tilde X}> & <\vec{\tilde Y},\vec{\tilde Y}> & <\vec{\tilde Y},\vec{\tilde Z}> \\ <\vec{\tilde Z},\vec{\tilde X}> & <\vec{\tilde Z},\vec{\tilde Y}> & <\vec{\tilde Z},\vec{\tilde Z}> \end{pmatrix},$$

we can see that $<\vec{\tilde X}+\vec{\tilde Y}-\vec{\tilde Z},\vec{\tilde X}+\vec{\tilde Y}-\vec{\tilde Z}>=0$. For real vectors it implies that $\vec{\tilde Z}=\vec{\tilde X}+\vec{\tilde Y}$, the determinant is identically zero, and all the values a_{ij} are constant.

4 Appendix

I want to change variables $(\tau, u) \longrightarrow (x, \lambda)$ in the system

$$\begin{cases} \hat{\Psi}' = 2\sum_{k=0}^{3} \log' \theta_k(u|\tau)\hat{A}_k\hat{\Psi} \\ \overset{\bullet}{\hat{\Psi}} = 2\sum_{k=0}^{3} \overset{\bullet}{\log}\, \theta_k(u|\tau)\hat{A}_k\hat{\Psi}, \end{cases}$$

where

$$\begin{aligned} u = \frac{1}{4K}\int_0^{\lambda} \frac{dt}{\sqrt{t(1-t)(x-t)}} &= \frac{1}{4K}\int_0^{\lambda/x} \frac{x\,dt}{\sqrt{xt_1(1-xt_1)x(1-t_1)}} = \\ & \frac{1}{2K}\int_0^{\sqrt{\lambda/x}} \frac{dt_2}{\sqrt{(1-t_2^2)(1-xt_2^2)}}; \end{aligned}$$

$$\begin{aligned} \lambda = \sqrt{x}\ \theta_1^2(u)/\theta_0^2(u) = x\ \mathrm{sn}^2 v,\ 1-\lambda &= \sqrt{1-x}\ \theta_3^2(u)/\theta_0^2(u) = \mathrm{dn}^2 v, \\ x - \lambda &= \sqrt{x(1-x)}\ \theta_2^2(u)/\theta_0^2(u) = x\ \mathrm{cn}^2 v \end{aligned}$$

$$K = \frac{1}{2}\int_0^x \frac{dt}{\sqrt{t(1-t)(x-t)}} = \int_0^1 \frac{dt_2}{\sqrt{(1-t_2^2)(1-xt_2^2)}}, \quad v = 2Ku,$$

the theta functions are calculated by the period τ:

$$\tau = \int_1^{1/x} \frac{dt_1}{\sqrt{t_1(1-t_1)(1-xt_1)}} \Big/ \int_0^1 \frac{dt_1}{\sqrt{t_1(1-t_1)(1-xt_1)}}, \quad x = \theta_2^4(0|\tau)/\theta_3^4(0|\tau).$$

Consider the first equation.

$$2\sum_k \hat{A}_k \frac{\partial}{\partial\lambda}|_{x=const} \log\theta_k(u) = 2\sum_{k\neq 0} \hat{A}_k \frac{\partial}{\partial\lambda}|_{x=const} \log\frac{\theta_k(u)}{\theta_0(u)} = \frac{1}{\lambda}\hat{A}_1 + \frac{1}{\lambda-x}\hat{A}_2 + \frac{1}{\lambda-1}\hat{A}_3$$

For the transformation of the second equation we derive some formulae:

$$\begin{aligned} \frac{\theta_k''(u)}{\theta_k(u)} - \frac{\theta_0''(u)}{\theta_0(u)} &= \log''\frac{\theta_k(u)}{\theta_0(u)} + \left(\frac{\theta_k'(u)}{\theta_k(u)}\right)^2 - \left(\frac{\theta_0'(u)}{\theta_0(u)}\right)^2 = \log''\frac{\theta_k(u)}{\theta_0(u)} + \left(\log'\frac{\theta_k(u)}{\theta_0(u)}\right)^2 + \\ &2\log'\frac{\theta_k(u)}{\theta_0(u)}\log'\theta_0(u) = \frac{(\theta_k(u)/\theta_0(u))''}{\theta_k(u)/\theta_0(u)} + 2\left(\log'\frac{\theta_k(u)}{\theta_0(u)}\right)\log'\theta_0(u). \end{aligned}$$

Using the heat equation $4\pi i \overset{\bullet}{\theta}_k (u) = \theta_k^{''}(u)$ we get

$$\overset{\bullet}{\log} \frac{\theta_1(u)}{\theta_0(u)} = \frac{1}{4\pi i}\left(\frac{\theta_1^{''}(u)}{\theta_1(u)} - \frac{\theta_0^{''}(u)}{\theta_0(u)}\right) = \frac{K^2}{\pi i}\left(\frac{1}{\mathrm{sn}\ v}\frac{\partial^2}{\partial v^2}\mathrm{sn}\ v + 2\frac{\partial}{\partial v}\log \mathrm{sn}\ v \quad Z\right) =$$
$$\frac{K^2}{\pi i}\left(-\mathrm{dn}^2\ v - x\mathrm{cn}^2\ v + 2\frac{\mathrm{cn}\ v\mathrm{dn}\ v}{\mathrm{sn}\ v}Z\right),$$
$$\overset{\bullet}{\log} \frac{\theta_2(u)}{\theta_0(u)} = \frac{K^2}{\pi i}\left(-\mathrm{dn}^2\ v + x\mathrm{sn}^2\ v - 2\frac{\mathrm{sn}\ v\mathrm{dn}\ v}{\mathrm{cn}\ v}Z\right),$$
$$\overset{\bullet}{\log} \frac{\theta_3(u)}{\theta_0(u)} = \frac{K^2}{\pi i}x\left(-\mathrm{cn}^2\ v + \mathrm{sn}^2\ v - 2\frac{\mathrm{sn}\ v\mathrm{cn}\ v}{\mathrm{sn}\ v}Z\right).$$

We denote $Z := \partial/\partial v \log \theta_0(v/(2K))$. Using

$$\frac{d\tau}{dx} = \frac{i\pi}{4K^2x(x-1)}$$

we rewrite these equalities:

$$\frac{\partial}{\partial x}|_{u=const} \log \frac{\theta_1(u)}{\theta_0(u)} = \frac{1}{4x(x-1)}\left(2\lambda - 1 - x + 2\frac{\mathrm{cn}\ v\mathrm{dn}\ v}{\mathrm{sn}\ v}Z\right)$$
$$\frac{\partial}{\partial x}|_{u=const} \log \frac{\theta_2(u)}{\theta_0(u)} = \frac{1}{4x(x-1)}\left(2\lambda - 1 - 2\frac{\mathrm{sn}\ v\mathrm{dn}\ v}{\mathrm{cn}\ v}Z\right)$$
$$\frac{\partial}{\partial x}|_{u=const} \log \frac{\theta_3(u)}{\theta_0(u)} = \frac{1}{4x(x-1)}\left(2\lambda - x - 2x\frac{\mathrm{sn}\ v\mathrm{cn}\ v}{\mathrm{dn}\ v}Z\right).$$

We pass on to the second equation:

$$\frac{\partial}{\partial x}|_{\lambda=const}\hat{\Psi}\hat{\Psi}^{-1} = \frac{\partial}{\partial x}|_{u=const}\hat{\Psi}\hat{\Psi}^{-1} - \left(\frac{\partial\lambda}{\partial x}|_{u=const}\right)\frac{\partial}{\partial\lambda}|_{x=const}\hat{\Psi}\hat{\Psi}^{-1}.$$

Let us calculate $\partial\lambda/\partial x$:

$$\frac{\partial\lambda}{\partial x}|_{u=const} = \lambda\frac{\partial}{\partial x}|_{u=const}\log\sqrt{x}\ \frac{\theta_1^2(u)}{\theta_0^2(u)} = \lambda\left(\frac{1}{2x} + 2\frac{\partial}{\partial x}|_{u=const}\log\frac{\theta_1(u)}{\theta_0(u)}\right) =$$

$$\lambda\left(\frac{1}{2x} + \frac{1}{2x(x-1)}(2\lambda - 1 - x + 2\frac{\mathrm{cn}\ v\ \mathrm{dn}\ v}{\mathrm{sn}\ v}Z)\right) = \frac{\lambda}{x(x-1)}\left(\lambda - 1 + \frac{\mathrm{cn}\ v\ \mathrm{dn}\ v}{\mathrm{sn}\ v}Z\right).$$

We get lastly

$$\frac{\partial}{\partial x}|_{\lambda=const}\hat{\Psi}\hat{\Psi}^{-1} = \frac{\hat{A}_1}{2x(x-1)}\left(2\lambda-1-x+2\frac{\text{cn }v\text{ dn }v}{\text{sn }v}Z\right)+\frac{\hat{A}_2}{2x(x-1)}\Bigg(2\lambda-1-$$

$$2\frac{\text{sn }v\text{ dn }v}{\text{cn }v}Z\Bigg)+\frac{\hat{A}_3}{2x(x-1)}\left(2\lambda-x-2x\frac{\text{cn }v\text{ sn }v}{\text{dn }v}Z\right)-\frac{\lambda}{x(x-1)}\Bigg(\lambda-1+\frac{\text{cn }v\text{ dn }v}{\text{sn }v}\cdot$$

$$Z\Bigg)\left(\frac{1}{\lambda}\hat{A}_1+\frac{1}{\lambda-x}\hat{A}_2+\frac{1}{\lambda-1}\hat{A}_3\right)=\frac{1}{x-\lambda}\hat{A}_2-\frac{1}{2x}\hat{A}_1-\frac{2x-1}{2x(x-1)}\hat{A}_2-\frac{1}{2(x-1)}\hat{A}_3$$

QED

Acknowledgments

I am pleased to thank D. Korotkin for the interest in this work and valuable discussions. I am grateful to MPIM-Bonn for support and hospitality during my stay there.

This work is partially supported by the grant RFFI 99-01-00687 of Russian Academy of Sciences.

References

[1] L. Schlesinger, *Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten* , J. Reine Math., **141**, 96–145 (1912).

[2] D. Arinkin, S. Lysenko, *On the Moduli of $SL(2)$-bundles with Connections on* $P^1 \setminus \{x_1, x_2, x_3, x_4\}$, International Mathematics Research Notices, **19**, 983–999 (1997).

[3] D. Arinkin, S. Lysenko, *Isomorphisms between Moduli Spaces of $SL(2)$-bundles with Connections on* $P^1 \setminus \{x_1, x_2, x_3, x_4\}$, Mathematical Research Letters, **4**, 181–190 (1997).

[4] M. Jimbo, T. Miwa, K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients I*, Phisica 2D, 306–352 (1981).

[5] M. Jimbo, T. Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II, III*, Phisica 2D, 407–448 (1981), Phisica 4D, 26-46, (1981).

[6] K. Okamoto, *Studies in the Painlevé equations I. Sixth Painlevé equation PVI*, Annali Mat. Pura Appl., **146**, 337–381 (1987).

[7] D. A. Korotkin and H. Samtleben, *On the quantization of isomonodromic deformations on the torus*, International Journal of Modern Physics A, v. 12, No. 11, 2013–2029 (1997).

[8] A.S.Fokas, A.R.Its, A.A.Kapaev and V.Yu.Novokshenov,*Painlevé transcendents as Nonlinear Special Functions*, Kluwer Ac.Pub.

[9] Mugan, U. and Sakka, A. *Schlesinger transformations for Painlevé VI equation*, J. Math. Phys. **36** (3), 1284–1298, (1995).

[10] A.Kitaev *Special functions of the isomonodromy type*, Preprint SFB-288-272 (Berlin), Jun 1997. 27pp.

[11] D. A. Korotkin, N. Manojlović and H. Samtleben, *Schlesinger transformations for elliptic isomonodromic deformations*, In preparation, (1999).