Noncommutative spaces.

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Introduction

In commutative geometry, "spaces" are understood either as

— locally ringed topological spaces (otherwise called *geometric spaces*), or as

— sheaves of sets on the category of affine schemes endowed with an appropriate topology. Here *appropriate topology* varies from the Zariski topology ([DeG, I.3.11]) to a flat (**fpqc**, or **fppf**) topology. In the case of the **fpqc** topology, spaces can be identified with functors X from the category **CRings** of commutative unital rings to **Sets** which preserve finite products and such that the diagram $X(R) \longrightarrow X(T) \rightrightarrows X(T \otimes_R T)$ is exact for any faithfully flat ring morphism $R \longrightarrow T$. Respectively, commutative schemes are defined either as locally ringed spaces which are locally affine, or as sheaves of sets obtained by glueing affine schemes (i.e. representable functors) for Zariski topology.

Only comparatively few of the known examples of what might be regarded as noncommutative analogues of schemes, or formal schemes, can be realized as ringed spaces, like D-schemes of Beilinson-Bernstein [BB], [BD], "virtual" noncommutative formal spaces introduced by Kapranov [Ka], and affine noncommutative schemes of P. Cohn [C]. Usually noncommutative analogues of schemes appear as

— categories (regarded as categories of quasi-coherent sheaves) over a base category, like the **Proj** of a graded noncommutative ring (see [M1], [V1], [V2], [A2], [AZ], [OW], and a number of other works) and the flag variety of a quantized enveloping algebra (see [LR2], [R3]),

— or functors from the category Alg_k of associative unital k-algebras to **Sets**, like projective spaces introduced in [KR1] and different Grassmannians and flag varieties constructed in [KR2].

We study the categorical approach to spaces in [KR3]. The purpose of this paper is to provide a geometric background for a number of examples of noncommutative spaces of the second type, i.e. spaces defined as functors $Alg_k \longrightarrow$ **Sets**.

We define the category $\mathbf{Aff}_k$ of noncommutative affine k-schemes as the category of representable functors from Alg_k to **Sets**. Thus Yoneda imbedding $R \longmapsto Alg_k(R, -)$ induces an equivalence between $\mathbf{Aff}_k$ and the category Alg_k^{op} opposite to Alg_k. A flat cover on $\mathbf{Aff}_k$ is given by a finite set $\{A \longrightarrow A_i \mid i \in J\}$ of flat algebra morphisms such that the corresponding morphism $A \longrightarrow \prod_{i \in J} A_i$ is faithful. Flat covers play an important role in noncommutative setting due to the fact that the flat descent holds in the noncommutative case providing means for studying categories of quasi-coherent sheaves (this is used in [KR1]). But, in general, flat covers do not form a pretopology: the invariance under a base change fails. This leads to a weaker version of a (pre)topology based on the notion of a *Q-category* (here 'Q' stands for 'quotient'). By definition, a Q-category is a pair, $\bar{A} \underset{u^*}{\overset{u_*}{\rightleftarrows}} A$, of functors such that the functor u^* is fully faithful and left adjoint to u_*, which implies that A is a quotient category of $\bar{A}$ and u_* is a localization functor. Both Grothendieck sites and Grothendieck pretopologies can be viewed as Q-categories. For any category C, we have a Q-category $C^{\mathbb{A}} = (C^{\bar{A}} \rightleftarrows C^{A})$ of functors on $\mathbb{A}$ with values in C. In particular, we have the Q-category of presheaves on a Grothendieck site, or a pretopology.

The main basic notion in this formalism is the notion of a sheaf: given a Q-category $\mathbb{A} = (\bar{A} \underset{u^*}{\overset{u_*}{\rightleftarrows}} A)$, an object x of the category A is called an $\mathbb{A}$-*sheaf* if the canonical map

$$\bar{A}(\bar{y}, u^*(x)) \longrightarrow A(u_*(\bar{y}), x), \quad g \longmapsto \eta_x^{-1} \circ u_*(g), \tag{1}$$

is an isomorphism for all $\bar{y} \in Ob\bar{A}$. Here η_u is an adjunction isomorphism $Id_A \xrightarrow{\sim} u_*u^*$.

In the case of the Q-category of presheaves on a Grothendieck site, or a Grothendieck pretopology, we recover sheaves in the usual sense.

Flat covers on $\mathbf{Aff}_k$ give a rise to a Q-category, and noncommutative spaces represent sheaves of sets on this Q-category. Similarly to the commutative case, thus defined 'spaces' are identified with functors $Alg_k \xrightarrow{X}$ **Sets** which preserve finite products and such that the natural diagram

$$X(R) \longrightarrow X(T) \rightrightarrows X(T \star_R T) \tag{2}$$

is exact for any faithfully flat ring morphism $R \longrightarrow T$. Here $T \star_R T$ is a traditional notation for the fiber coproduct $T \coprod_R T$ in the category of associative unital rings. One can show that all representable functors are spaces, i.e. the category $\mathbf{Aff}_k$ of noncommutative affine k-schemes is a full subcategory of the category of 'spaces'.

As in commutative case, **fpqc** covers are not always the best choice, and there are noncommutative versions of other types of covers on the category of noncommutative affine schemes which are used to define other categories of 'spaces'. For instance, smooth covers seem to be a more sensible choice for a big part of examples we consider (in [KR] and [KR2]). These covers form a Q-category, but, in general, not a pretopology.

There is another interpretation of Q-categories illustrated by the following example: A is the category **CRings** of commutative unital rings, $\bar{A}$ the category of (commutative) ring epimorphisms with a nilpotent kernel, the functor $A \to \bar{A}$ maps any ring to the identical endomorphism of this ring, and $\bar{A} \to A$ maps a ring epimorphism $S \to R$ to its target, R. In this case, sheaves (resp. monopresheaves) on $\mathbb{A}$ turn out to be formally étale (resp. formally unramified) functors **CRings** $\longrightarrow$ **Sets**; and *epipresheaves* (defined in 3.1.4) are formally smooth functors. This example suggests that Q-categories might be regarded also as "categories of thickennings". An appropriate (not quite obvious) choice of a Q-category produces a noncommutative version of formally étale, formally unramified, and formally smooth functors.

The paper is organized as follows.

First three sections contain preliminaries on Q-categories. In Sections 1 and 2, we define a Q-category and morphisms of Q-categories and give a number of examples. In Section 3, we introduce the notions of a sheaf, a monopresheaf and an epipresheaf in a Q-category and illustrate these notions using examples of Section 2.

In Section 4, we interpret epipresheaves, monopresheaves and sheaves of sets on a Q-category as resp. *formally smooth, formally unramified*, and *formally étale* functors. In this case, the Q-category is thought as the Q-category of thickennings.

Most of Section 5 is dedicated to definition and basic properties, and some examples of *formally smooth, formally unramified* and *formally étale morphisms* of presheaves of sets on a Q-category. We introduce locally finitely presentable morphisms and define smooth (resp. unramified, resp. étale) morphisms as locally finitely presentable morphisms which are formally smooth (resp. formally unramified, resp. formally étale). We define *open immersions* as smooth monomorphisms and obtain general properties of open immersion as a consequence of those of smooth morphisms.

In Section 6, we introduce *formally* $\mathbb{A}$*-infinitesimal morphisms* (which might be regarded as formal thickenings of spaces) as the dual notion to that of formally $\mathbb{A}$-smooth morphisms. We make one more step giving a symmetric form to the duality between formally smooth and formally infinitesimal morphisms. Curiously, this leads to the interpretation of separated (resp. universally closed, resp. proper) morphisms of schemes as formally unramified (resp. formally smooth, resp. formally étale) morphisms for an appropriate choice of the class of formally infinitesimal morphisms of schemes.

In Section 7, we discuss closed immersions and separated morphisms.

The second part of the work is dedicated to noncommutative locally affine spaces and schemes. In Section 8, we introduce locally affine spaces and schemes. With any Q-category, we associate another Q-category of a 'topological nature' called a *quasi-cosite*. We define "$\mathbb{A}$-spaces" as sheaves of sets on the quasi-cosite associated with $\mathbb{A}$. For any quasi-topology on the category of $\mathbb{A}$-spaces, we define locally affine spaces and schemes in the most direct way. Starting with another Q-category, $\mathbb{A}_1 = (\bar{A}_1 \rightleftarrows A)$ (having the same underlying category A) regarded as a Q-category of thickennings, we define three natural quasi-topologies in which covers are sets of resp. $\mathbb{A}_1$-smooth morphisms, $\mathbb{A}_1$-étale morphisms, and $\mathbb{A}_1$-open immersions.

In Section 9, we consider basic applications of this formalism. We recover commututative schemes and algebraic spaces taking as $\mathbb{A}$ the Q-category of commutative rings with **fpqc** cocovers. This means that A is the category of commutative rings and objects of $\bar{A}$ are finite conservative families of flat ring morphisms. And $\bar{A}_1$ is the category of ring surjective morphisms with a nilpotent kernel. An appropriate noncommutative generalization of this setting produces (via the formalism of Section 8) the notions of noncommutative schemes and algebraic spaces.

Section 10 is devoted to the noncommutative Grassmannian which is one of important examples of a noncommutative locally affine space.

The paper has two appendices. The first appendix contains some complementary facts about Q-categories. In Appendix 2, we discuss finiteness conditions.

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I. Q-categories and sheaves.

Q-categories were introduced (initially in [R5]) as a milder version of Grothendieck sites. We define noncommutative spaces as sheaves on Q-categories. Another interpretation of Q-categories leads to (a generalization of) the notions of formally smooth, formally unramified, and formally étale morphisms which play a significant role in this work and its sequals. We establish, among other things, the following dictionary:

$$\begin{array}{rcl} \text{sheaves of sets (on Q-categories)} & \longleftrightarrow & \text{formally étale spaces} \\ \text{monopresheaves of sets} & \longleftrightarrow & \text{formally unramified spaces} \\ \text{epipresheaves of sets} & \longleftrightarrow & \text{formally smooth spaces} \end{array}$$

Thus, formally étale (resp. formally unramified, resp. formally smooth) morphisms might be regarded as relative versions of sheaves (resp. monopresheaves, resp. epipresheaves) on a Q-category.

1. Q-categories. A Q-*category* is a pair of functors $\bar{A} \xrightarrow{u_*} A \xrightarrow{u^*} \bar{A}$ such that the functor u^* is fully faithful and left adjoint to u_*. We shall regard functors u_*, u^* as resp. direct and inverse image functors of a morphism $\bar{A} \xrightarrow{u} A$ and write this data as $\bar{A} \overset{u}{\rightleftarrows} A$.

A *morphism* from a Q-category $\bar{A} \overset{u}{\rightleftarrows} A$ to a Q-category $\bar{A}' \overset{u'}{\rightleftarrows} A'$ is a triple $(\Phi, \bar{\Phi}, \phi)$, where $A \xrightarrow{\Phi} A'$ and $\bar{A} \xrightarrow{\bar{\Phi}} \bar{A}'$ are functors and ϕ is a functor isomorphism $\Phi u_* \longrightarrow u'_* \bar{\Phi}$. The composition of two morphisms, is defined by

$$(\Phi', \bar{\Phi}', \phi') \circ (\Phi, \bar{\Phi}, \phi) = (\Phi'\Phi, \bar{\Phi}'\bar{\Phi}, \bar{\Phi}'\phi \circ \phi'\Phi)$$

For a universum $\mathfrak{U}$, we denote by $QCat_{\mathfrak{U}}$, or simply by $QCat$, the category whose objects are Q-categories $\bar{A} \overset{u}{\rightleftarrows} A$ such that A and $\bar{A}$ belong to $Cat_{\mathfrak{U}}$.

1.1. Remark. Since the functor u^* is fully faithful, the direct image functor u_* is a localization. A morphism $(\Phi, \bar{\Phi}, \phi)$ from $\bar{A} \overset{u}{\rightleftarrows} A$ to $\bar{A}' \overset{u'}{\rightleftarrows} A'$ is defined uniquely up to isomorphism by the functor $\bar{A} \xrightarrow{\bar{\Phi}} \bar{A}'$ and the compatibility of $\bar{\Phi}$ with the corresponding localizations (expressed by ϕ).

In fact, the isomorphism $\Phi u_* \xrightarrow{\phi} u'_* \bar{\Phi}$ induces an isomorphism $\Phi \longrightarrow u'_* \bar{\Phi} u^*$. This follows from the fact that, since the functor u^* is fully faithful, an adjunction morphism $Id_A \xrightarrow{\eta_u} u_* u^*$ is an isomorphism. The isomorphism ϕ is equivalent (after replacing Φ by $u'_* \bar{\Phi} u^*$) to the canonical morphism

$$u'_* \bar{\Phi} u^* u_* \longrightarrow u'_* \bar{\Phi} \tag{1}$$

induced by an adjunction morphism $u^* u_* \xrightarrow{\epsilon_u} Id_{\bar{A}}$. The compatibility of $\bar{\Phi}$ with localizations means exactly that (1) is an isomorphism.

1.2. The 2-category of Q-categories. The category $QCat$ is the category of 1-morphisms of a 2-category, $QCat^2$: given a pair of 1-morphisms $(\Phi, \bar{\Phi}, \phi)$, $(\Psi, \bar{\Psi}, \psi)$

from $\bar{A} \overset{u}{\rightleftarrows} A$ to $\bar{A}' \overset{u'}{\rightleftarrows} A'$, a 2-arrow from $(\Phi, \bar{\Phi}, \phi) \longrightarrow (\Psi, \bar{\Psi}, \psi)$ is a pair of morphisms, $\Phi \xrightarrow{\alpha} \Psi$, $\bar{\Phi} \xrightarrow{\bar{\alpha}} \bar{\Psi}$ such that the diagram

$$\begin{tikzcd}
\Phi u_* \arrow[r, "\phi"] \arrow[d, "\alpha u_*"'] & u'_* \bar{\Phi} \arrow[d, "u'_* \bar{\alpha}"] \\
\Psi u_* \arrow[r, "\phi"] & u'_* \bar{\Psi}
\end{tikzcd}$$

commutes.

1.3. Induced Q-categories. Fix a Q-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$. Let $\Phi : B \longrightarrow A$ be a functor. Let $\bar{B}$ denote the fiber product of $B \xrightarrow{\Phi} A \xleftarrow{u_*} \bar{A}$. Recall that objects of $\bar{B}$ are triples $(x, \alpha, \bar{y})$, where $x \in ObB$, $\bar{y} \in Ob\bar{A}$, and α is an isomorphism $\Phi(x) \xrightarrow{\sim} u_*(\bar{y})$. Morphisms are defined in an obvious way. There are natural projection functors

$$\bar{B} \xrightarrow{\bar{\Phi}} \bar{A}, \quad (x, \alpha, \bar{y}) \longmapsto \bar{y}, \quad \text{and} \quad \bar{B} \xrightarrow{u_{\Phi *}} B, \quad (x, \alpha, \bar{y}) \longmapsto x.$$

We define a functor $B \xrightarrow{u_\Phi^*} \bar{B}$ by $x \longmapsto (x, \eta_u \Phi(x), u^* \Phi(x))$. It follows that $u_{\Phi *} \circ u_\Phi^* = Id_B$ and $u_\Phi^* \circ u_{\Phi *}(x, \alpha, \bar{y}) = (x, \eta_u \Phi(x), u^* \Phi(x))$. We have a canonical morphism

$$\epsilon_{u_\Phi}(x, \alpha, \bar{y}) = (id_x, \epsilon_u(\bar{y}) \circ u^*(\alpha) : u_\Phi^* \circ u_{\Phi *}(x, \alpha, \bar{y}) \longrightarrow (x, \alpha, \bar{y})$$

functorial in $(x, \alpha, \bar{y})$. Hence $\epsilon_{u_\Phi} = \{\epsilon_{u_\Phi}(x, \alpha, \bar{y}) | \ (x, \alpha, \bar{y}) \in Ob\bar{B}\}$ is a funcrtor morphism $u_\Phi^* \circ u_{\Phi *} \longrightarrow Id_{\bar{B}}$. One can see that ϵ_{u_Φ} and the identical functor $Id_B \longrightarrow Id_B = u_{\Phi *} u_\Phi^*$ are adjunction morphisms for the pair of functors $u_{\Phi *}$, u_Φ^*. In particular, the functor $B \xrightarrow{u_{\Phi *}} \bar{B}$ is fully faithful.

Notice that $u_* \circ \bar{\Phi}(x, \alpha, \bar{y}) = u_*(\bar{y})$, and $\Phi \circ u_{\Phi *}(x, \alpha, \bar{y}) = \Phi(x)$. Thus

$$\phi(x, \alpha, \bar{y}) = \alpha : \Phi \circ u_{\Phi *}(x, \alpha, \bar{y}) = \Phi(x) \longrightarrow \Phi(x) = u_* \circ \bar{\Phi}(x, \alpha, \bar{y}), \ (x, \alpha, \bar{y}) \in Ob\bar{B},$$

defines a functor isomorphism $\Phi \circ u_{\Phi *} \xrightarrow{\phi} u_* \circ \bar{\Phi}$.

Altogether, we have obtained a Q-category $\mathbb{A}_\Phi = \mathbb{B} = (\bar{B} \overset{u_\Phi}{\rightleftarrows} B)$ induced by the functor Φ and a canonical Q-category morphism $(\bar{\Phi}, \phi, \Phi) : \mathbb{B} \longrightarrow \mathbb{A}$.

1.3.1. Two special cases. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category. For any object x of the category A, we have the categories A/x and $x\backslash A$ and the canonical functors $A/x \longrightarrow A \longleftarrow x\backslash A$. We denote the corresponding induced Q-categories resp. by $\mathbb{A}/x$ and $x\backslash\mathbb{A}$.

1.4. Q^o-categories. A Q^o-*category* is a pair of functors $\bar{A} \xrightarrow{u_*} A \xrightarrow{u^*} \bar{A}$ such that the functor u^* is fully faithful and a right adjoint to u_*. In other words, the data $\bar{A} \xrightarrow{u_*} A \xrightarrow{u^*} \bar{A}$ is a Q^o-category iff the dual data, $\bar{A}^{op} \xrightarrow{u_*^{op}} A^{op} \xrightarrow{u^{*op}} \bar{A}^{op}$ is a Q-category.

All facts and constructions on Q-categories (resp. Q^o-categories) have their dual versions which will be used when needed.

2. Examples of Q-categories and $\mathbf{Q}^o$-categories. Given a category A, there are two families of basic examples of Q-categories (resp. Q^o-categories) having A as the underlying category:

– the Q-category of cosieves on A (cf. 2.1) and its Q-subcategories,

– dually, the Q^o-category of sieves on A and its Q^o-subcategories (among them Grothendieck sites with the base A);

– the Q-category (resp. the Q^o-category) of morphisms in A (see 2.5) and its Q-subcategories (resp. Q^o-subcategories).

Our main class of examples are Q-categories of functors on a Q-category (or Q-categories of presheaves on a Q^o-category) (see 2.6).

2.1. The Q-category of cosieves. Fix a category A. Let $\mathbf{S}A$ denote the *category of cosieves* on A defined as follows. Objects of $\mathbf{S}A$ are pairs (x, R), where $x \in ObA$ and R is a cosieve in $x\backslash A$. Morphisms from (x, R) to (x', R') are given by arrows $x \xrightarrow{f} x'$ such that $R'_f \subseteq R$. Here R'_f is a cosieve in $x\backslash A$ whose objects are all pairs $(v, \xi \circ f)$ such that $(v, \xi) \in ObR'$. There is a functor $A \xrightarrow{u^*} \mathbf{S}A$ which assigns to each object x of A the pair $(x, x\backslash A)$. The functor u^* is fully faithful and has a canonical right adjoint,

$$\mathbf{S}A \xrightarrow{\ u_*\ } A, \quad (x, R) \longmapsto x.$$

This defines a Q-category of cosieves, $\mathbf{S}A \rightleftarrows A$.

Cosieves in $x\backslash A$ are in a natural one-to-one correspondence with subfunctors of the functor $A(x, -)$. Thus the Q-category of cosieves is isomorphic to a Q-category, $\bar{A} \rightleftarrows A$, defined as follows. Objects of $\bar{A}$ are pairs (x, R), where $x \in ObA$, R is a subfunctor of $A(x, -)$. Morphisms from (x, R) to (y, S) are morphisms $x \xrightarrow{f} y$ such that the functor morphism $A(y, -) \xrightarrow{A(f,-)} A(x, -)$ induces a morphism $S \longrightarrow R$ of the subfunctors. The functor u_* maps a pair (x, R) to x. The functor u^* assigns to any object x of A the pair $(x, A(x, -))$.

2.2. Quasi-(co)sites and (co)sites. Let A be a category, and let $\mathfrak{T}$ be a map which assigns to every object x of A a set $\mathfrak{T}(x)$ of subfunctors of $A(x, -)$ which contains $A(x, -)$ itself. We shall identify the pair $(A, \mathfrak{T})$ with the full Q-subcategory $\bar{A}_{\mathfrak{T}} \rightleftarrows A$ of the Q-category $\mathbf{S}A \rightleftarrows A$ of cosieves objects of which are all pairs (x, R), where $x \in ObA$ and $R \in \mathfrak{T}(x)$.

We call the pair $(A, \mathfrak{T})$ a *quasi-cosite* if two conditions hold:

(a) for any pair $R,\ R' \in \mathfrak{T}(x)$, $R \cap R' \in \mathfrak{T}(x)$,

(b) if $R \in \mathfrak{T}(x)$ and R' is a subfunctor of $A(x, -)$ containing R, then $R' \in \mathfrak{T}(x)$.

Quasi-sites correspond to quasi-cosites on the dual category A^{op}.

Grothendieck sites are quasi-sites. Recall that a site is a pair $(A, \mathfrak{T})$, where $\mathfrak{T}$ is a *topology*, i.e. a map which assigns to each $x \in ObA$ a set $\mathfrak{T}(x)$ of subfunctors of $A(-, x)$ (called *refinements of* x) satisfying the conditions:

(i) for any $R \in \mathfrak{T}(x)$ and any arrow $f : y \longrightarrow x$, the subfunctor $R_f = R \times_{A(-,x)} A(-,y)$ of $A(-y)$ is a refinement of y (i.e. it belongs to $\mathfrak{T}(y)$.

(ii) If $R \in \mathfrak{T}(x)$ and $\widetilde{R}$ is a subfunctor of $A(-,x)$ such that $\widetilde{R}_f \in \mathfrak{T}(y)$ for any $f \in R(y),\ y \in ObA,$, then $\widetilde{R} \in \mathfrak{T}(x)$.

Q-categories dual to Grothendieck sites are called *cosites.* The Q-category of cosieves, $(\mathbf{S}A \rightleftarrows A)$ and its Q-subcategory $\mathbb{A}_{dis} = (\bar{A}_{dis} \rightleftarrows A)$, where $\bar{A}_{dis}$ is formed by all pairs $(x, x\backslash A),\ x \in ObA$, are two extreme examples of cosites.

Cosites might be regarded as a topology in terms of "closed sets". If $(\bar{A} \overset{u}{\rightleftarrows} A)$ is a cosite, then A might be viewed as the category of closed sets of a would-be space.

2.3. A quasi-cosite associated with a Q-category. Fix a Q-category $(\bar{A} \overset{u}{\rightleftarrows} A)$. To any $\bar{y} \in Ob\bar{A}$, we assign the category $\bar{y}\backslash u^*$ of pairs (f, x), where f is a morphism $\bar{y} \longrightarrow u^*(x)$. The functor u_* induces a morphism, $\Phi^\sim = (\Phi, id_{u_*}, Id_A)$, from $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ to the cosite $\mathbf{S}A \rightleftarrows A$. Here Φ is a functor $\bar{A} \longrightarrow \mathbf{S}A$ which assigns to any object $\bar{y}$ of $\bar{A}$ the pair $(u_*(\bar{y}), R_{\bar{y}})$, where $R_{\bar{y}}$ denotes the cosieve in $u_*(\bar{y})\backslash A$ formed by all $(v, u_*(\bar{y}) \overset{\xi}{\to} v)$ such that $\xi = \eta_u^{-1}(v) \circ \bar{\xi}$ for some $\bar{y} \overset{\bar{\xi}}{\longrightarrow} u^*(v)$. The *quasi-cosite,* $\mathfrak{T}\mathbb{A} = (\mathfrak{T}_{\mathbb{A}} \rightleftarrows A)$, *associated with* $\mathbb{A}$ is the smallest quasi-cosite containing the image of the functor Φ. The triple (Id_A, Φ, id) is a canonical morphism from $\mathbb{A}$ to the Q-category $(\mathbf{S}A \rightleftarrows A)$ of cosieves on A.

2.3.1. Note. If $\mathbb{A}$ is a quasi-cosite, then $\mathfrak{T}\mathbb{A}$ is naturally isomorphic to $\mathbb{A}$. Dually, with every Q^o-category $\mathbb{A}$, one can associate a quasi-site which is naturally isomorphic to $\mathbb{A}$ if $\mathbb{A}$ is a quasi-site.

2.3.2. Proposition. *Suppose* $\mathbb{A}$ *has the property:*

(*) for any $\bar{y} \in Ob\bar{A}$ *and any morphism* $x \overset{f}{\longrightarrow} u_*(\bar{y})$, *there exists a morphism* $\bar{x} \overset{\bar{f}}{\longrightarrow} \bar{y}$ *and an isomorphism* $u_*(\bar{x}) \overset{\alpha}{\longrightarrow} x$ *such that* $u_*(\bar{f}) = f \circ \alpha$.

Then the quasi-cosite associated with $\mathbb{A}$ *is a cosite.*

Proof is left to the reader. ■

2.4. Quasi-(co)sites and (co)covers. Let τ be a function which assigns to each object, x, of the category A a family, τ_x, of sets of arrows to x which contains $\{x \overset{id_x}{\to} x\}$. This data defines a category, A_τ, whose objects are all pairs $(x, \mathcal{U})$, where $x \in ObA$, $\mathcal{U} \in \tau_x$; we shall call them *covers.* Morphisms from $(x,\mathcal{U})$ to $(y, \mathcal{V})$ are morphisms $x \overset{f}{\longrightarrow} y$ such that for any arrow $x_u \overset{u}{\to} x$ in $\mathcal{U}$ there exists an arrow $y_v \overset{v}{\to} y$ in $\mathcal{V}$ and a morphism $x_u \overset{g_{uv}}{\longrightarrow} y_v$ such that $f \circ u = v \circ g_{uv}$. The functor $A_\tau \longrightarrow A$ which assigns to every pair $(x,\mathcal{U})$ the object x and to every morphism $(x,\mathcal{U}) \overset{f}{\longrightarrow} (y, \mathcal{V})$ the morphism $x \overset{f}{\longrightarrow} y$ is a right adjoint to the fully faithful functor $A \longrightarrow A_\tau$ which maps every object x of A to $(x, \{id_x\})$. This defines a Q^o-category $\mathbb{A}_\tau = (A_\tau \rightleftarrows A)$.

Consider the quasi-site $\mathfrak{T}\mathbb{A}_\tau$ associated with $\mathbb{A}_\tau$. The functor $A_\tau \overset{\Phi}{\longrightarrow} \mathfrak{T}A_\tau$ assigns to every cover $(x,\mathcal{U})$ the pair $(x, R_{\mathcal{U}})$, where $R_{\mathcal{U}}$ is the sieve associated with the set of arrows $\mathcal{U}$: it consists of all arrows to x which factor through some of the arrows of $\mathcal{U}$.

If for any morphism $y \xrightarrow{f} x$ and any $\mathcal{U} \in \tau_x$, there exists $\mathcal{V} \in \tau_y$ such that f is a morphism $(y, \mathcal{V}) \longrightarrow (x, \mathcal{U})$, then the quasi-site associated with $\mathbb{A}_\tau$ is a site. In particular, if τ is a Grothendieck pretopology, we obtain this way the site associated with τ.

2.4.1. Covers. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a quasi-site. A set of arrows $\mathcal{U} = \{x_i \to x|\ i \in J\}$ in A is called a *cover* (or an $\mathbb{A}$-cover) of x, if the pair $(x, R_{\mathcal{U}})$, where $R_{\mathcal{U}}$ is the sieve associated to $\mathcal{U}$, is an object of $\bar{A}$.

It follows from the definition of a quasi-site that

(i) every set of arrows to x which contains a cover is a cover;

(ii) if $\mathcal{U}$ and $\mathcal{U}'$ are covers of x, then $\mathcal{U} \times_x \mathcal{U}' = \{x_u \times_x x_v \longrightarrow x \mid u \in \mathcal{U},\ v \in \mathcal{U}'\}$ is a cover of x, provided the pull-backs $x_u \times_x x_v$ exist for all $u \in \mathcal{U},\ v \in \mathcal{U}'$.

2.5. The Q-category and the Q^o-category of morphisms of a category. Fix a category A. Consider the category A^2 objects of which are morphisms of the category A, and morphisms from $x \xrightarrow{f} y$ to $x' \xrightarrow{f'} y'$ are commutative squares

$$\begin{array}{ccc} x & \xrightarrow{g} & x' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ y & \xrightarrow{h} & y' \end{array} \qquad (1)$$

Denote by u^* the functor $A \longrightarrow A^2$ which assigns to any object x of A the object $x \xrightarrow{id_x} x$ and to any morphism f the corresponding commutative square. The functor u^* is fully faithful and has a right adjoint, u_*, which maps any object $x \xrightarrow{f} y$ of A^2 to x and any morphism (1) to $x \xrightarrow{g} x'$. In fact, $u_* u^* = Id_A$, and there is a natural morphism $u^* u_* \xrightarrow{\epsilon_u} Id_{A^2}$ which assigns to any object $x \xrightarrow{f} y$ of the category A^2 the morphism

$$\begin{array}{ccc} x & \xrightarrow{id_x} & x \\ {\scriptstyle id_x}\downarrow & & \downarrow{\scriptstyle f} \\ x & \xrightarrow{f} & y \end{array}$$

from $u^* u_*(x \xrightarrow{f} y)$ to $(x \xrightarrow{f} y)$. One can see that $Id_A \xrightarrow{id} u_* u^*$ and ϵ_u are adjunction morphisms.

Dually, the functor u^* has a natural left adjoint, $u_!$, which assigns to any object $x \xrightarrow{f} y$ of A^2 the object y and to any morphism (1) the morphism $y \xrightarrow{h} y'$.

2.5.1. Q-subcategories of $(A^2 \rightleftarrows A)$**.** Let $\bar{A}$ be a full subcategory of the category A^2 which contains all objects $x \xrightarrow{id_x} x$. Then the functor u^* takes values in the subcategory $\bar{A}$, hence it induces a structure of a Q-subcategory, $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$, of the Q-category $A^2 \rightleftarrows A$. The functor $u_! : A^2 \longrightarrow A$ induces a functor $\bar{A} \longrightarrow A$ left adjoint to u^*.

2.5.2. Q-categories with a functor $u_!$ **and the Q-category of morphisms.** Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category such that the functor u^* has a left adjoint, $u_!$. Then there is a canonical morphism $u_* \xrightarrow{\mathfrak{r}_u} u_!$ equal to the composition $u_*(\eta^!_u \circ \epsilon_u) \circ \eta_u u_*$. Here $Id_{\bar{A}} \xrightarrow{\eta^!_u} u^* u_!$, $Id_A \xrightarrow{\eta_u} u_* u^*$, and $u^* u_* \xrightarrow{\epsilon_u} Id_{\bar{A}}$ are adjunction morphisms. Thus we have a canonical morphism $(\Psi_{\mathbb{A}}, id_{u_*}, Id_A) : \mathbb{A} \longrightarrow (A^2 \rightleftarrows A)$, where the functor $\Psi_{\mathbb{A}}$ assigns to any object, $\bar{y}$, of $\bar{A}$ the canonical morphism $u_*(\bar{y}) \xrightarrow{\mathfrak{r}_u(\bar{y})} u_!(\bar{y})$.

On the other hand, there is a canonical morphism from the Q-category $\mathbb{A}$ to the Q-category of cosieves $\mathbf{S}A \rightleftarrows A$ defined in 2.3. Notice that for any $\bar{y} \in \bar{A}$, the category $\bar{y}\backslash u^*$ is isomorphic to the category $u_!(\bar{y})\backslash A$. Two morphisms, $g_1, g_2 : u_!(\bar{y}) \longrightarrow x$ define the same object of the sieve $R_{\bar{y}}$ (cf. 2.3) iff the square

$$\begin{array}{ccc} u_*(\bar{y}) & \xrightarrow{\mathfrak{r}_u(\bar{y})} & u_!(\bar{y}) \\ {\scriptstyle \mathfrak{r}_u(\bar{y})}\big\downarrow & & \big\downarrow{\scriptstyle g_1} \\ u_!(\bar{y}) & \xrightarrow{g_2} & x \end{array} \qquad (1)$$

is commutative. Suppose $u_!(\bar{y}) \coprod_{u_*(\bar{y})} u_!(\bar{y})$ exists, and let $p_1,\ p_2$ be canonical coprojections $u_!(\bar{y}) \rightrightarrows u_!(\bar{y}) \coprod_{u_*(\bar{y})} u_!(\bar{y})$. Then the commutativity of (1) means that $g_1 = g \circ p_1$ and $g_2 = g \circ p_2$ for a uniquely determined morphism $u_!(\bar{y}) \coprod\limits_{u_*(\bar{y})} u_!(\bar{y}) \xrightarrow{\ g\ } x$.

The following is one of our main examples of a Q-category with the functor $u_!$.

2.6. The Q-category of infinitesimal algebra epimorphisms. Let A be the category Alg_k of associative unital k-algebras, and let $\bar{A}$ be the full subcategory of the category Alg_k^2 of k-algebra morphisms whose objects are epimorphisms with a nilpotent kernel.

2.6.1. Note. The commutative version of 2.6 (i.e. A is the category $CAlg_k$ of commutative algebras and $\bar{A}$ is the subcategory of $CAlg_k^2$ whose objects are commutative algebra epimorphisms with nilpotent kernel) can be interpreted as the category of infinitesimal extensions of affine schemes over k.

2.6.2. The Q-category of thickennings of a scheme. A non-affine version of the example 2.6.1 is the Q-category of thickennings of a scheme. Fix a scheme $\mathbf{X}$. Let A be the category of (Zariski) open subschemes of $\mathbf{X}$, and $\bar{A}$ the category of thickennings: objects of the category $\bar{A}$ are nilpotent scheme closed immersions $U \longrightarrow T$, where U is any open subscheme of $\mathbf{X}$. The fully faithful functor $A \xrightarrow{u^*} \bar{A}$, $U \longmapsto (U \xrightarrow{id_U} U)$, is left adjoint to the functor $\bar{A} \xrightarrow{u_*} A$ sending an immersion $U \longrightarrow T$ to U.

2.7. Q-categories of functors. Fix a category C. To any Q-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$, we assign a Q-category $C^{\mathbb{A}} = (C^{\bar{A}} \overset{C^u}{\rightleftarrows} C^A)$. Here C^A denotes the category of functors $A \longrightarrow C$ and C^u is a morphism with the inverse image functor

$$C^{u_*} : C^A \longrightarrow C^{\bar{A}}, \quad F \longmapsto F \circ u_*,$$

and the direct image functor $C^{u^*} : G \longmapsto G \circ u^*$. If C is the category **Sets**, we shall write $\mathbb{A}^\vee = (\bar{A}^\vee \overset{u^\vee}{\rightleftarrows} A^\vee)$ instead of $\mathbf{Sets}^{\mathbb{A}} = (\mathbf{Sets}^{\bar{A}} \overset{\mathbf{Sets}^u}{\rightleftarrows} \mathbf{Sets}^A)$.

3. Sheaves, monopresheaves and epipresheaves in a Q-category.

3.1. Definitions. Fix a Q-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$.

3.1.1. $\mathbb{A}$-sheaves. We call an object x of the category A an $\mathbb{A}$-*sheaf* if the canonical map

$$\bar{A}(\bar{y}, u^*(x)) \longrightarrow A(u_*(\bar{y}), x), \quad g \longmapsto \eta_x^{-1} \circ u_*(g), \tag{1}$$

is an isomorphism for any $\bar{y} \in Ob\bar{A}$. Here η_u is an adjunction (iso)morphism $Id_A \longrightarrow u_*u^*$.

We denote by $\mathfrak{F}\mathbb{A}$ the full subcategory of the category A generated by $\mathbb{A}$-sheaves.

3.1.2. $\mathbb{A}$-monopresheaves. We call an object x of the category A an $\mathbb{A}$-*monopresheaf*, or an $\mathbb{A}$-*separated presheaf*, if the canonical map (1) is injective for any $\bar{y} \in Ob\bar{A}$. We denote by $\mathfrak{M}\mathbb{A}$ the full subcategory of A formed by $\mathbb{A}$-monopresheaves.

3.1.3. The canonical morphism ρ_u. Let x be an object of the category A such that the functor $A(u_*(-), x)$ is representable, i.e. $A(u_*(-), x) \simeq \bar{A}(-, u^!(x))$ for some $u^!(x) \in Ob\bar{A}$. There is a canonical morphism $\rho_u(x) : u^*(x) \longrightarrow u^!(x)$ corresponding to the isomorphism $\eta_u^{-1}(x) : u_*u^*(x) \longrightarrow x$. It follows from the definitions that x is an $\mathbb{A}$-monopresheaf iff the morphism $\rho_u(x)$ is a monomorphism. Note, however, that x can be a monopresheaf without the functor $A(u_*(-), x)$ being representable.

3.1.4. $\mathbb{A}$-epipresheaves. We call an object x of A an $\mathbb{A}$-*epipresheaf* if the functor $A(u_*(-), x)$ is representable and the canonical morphism $\rho_u(x) : u^*(x) \longrightarrow u^!(x)$ (cf. 3.1.3) is a strict epimorphism.

We denote by $\mathfrak{E}\mathbb{A}$ the full subcategory of the category A formed by $\mathbb{A}$-epipresheaves.

It follows from 3.1.3 that *an object x of A is an $\mathbb{A}$-sheaf iff it is an $\mathbb{A}$-monopresheaf and an $\mathbb{A}$-epipresheaf.*

3.1.5. Dual notions. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q^o-category. We call an object x of A a *sheaf* (resp. a *monopresheaf*, resp. an *epipresheaf*) in $\mathbb{A}$ if x is a sheaf (resp. a monopresheaf, resp. an epipresheaf) in the dual Q-category $\mathbb{A}^{op}$.

3.2. Sheaves and monopresheaves in $\mathbb{A}^\wedge$. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category, $u = (u^*, u_*)$. And let $\mathbb{A}^\wedge = (\bar{A}^\wedge \overset{\widehat{u}}{\rightleftarrows} A^\wedge)$ be the corresponding Q-category of presheaves of sets. Note that since u_* is a right adjoint to u^*, the functor $A^\wedge \xrightarrow{\widehat{u}_*} \bar{A}^\wedge, X \longmapsto X \circ u_*$, is a right adjoint to $\widehat{u}^*$. Thus, a presheaf $A^{op} \xrightarrow{X} \mathbf{Sets}$ is a sheaf iff the canonical morphism $\widehat{u}^*(X) \longrightarrow \widehat{u}_*(X) = X \circ u_*$ is an isomorphism. That is for any $Y \in Ob\bar{A}$, the canonical morphism

$$\widehat{u}^*(X)(Y) = \bar{A}^\wedge(Y, \widehat{u}^*(X)) \simeq \operatorname{co}\lim_{(V,\xi)\in A/X} \bar{A}^\wedge(Y, u^*(V)) \longrightarrow Xu_*(Y) = A^\wedge(u_*(Y), X) \tag{1}$$

is an isomorphism.

Similarly, *a presheaf* $A^{op} \xrightarrow{X} \mathbf{Sets}$ *is a monopresheaf iff the canonical morphism* $\widehat{u}^*(X) \longrightarrow X \circ u_*$ *is a monomorphism. That is for any* $Y \in Ob\bar{A}$*, the canonical morphism (1) is a monomorphism.*

It follows that an object x of A is a sheaf (resp. a monopresheaf) in the Q-category $\mathbb{A}$ iff the presheaf of sets $A(-,x)$ is a sheaf (resp. a monopresheaf) in $\mathbb{A}^\wedge$.

3.3. Sheaves and monopresheaves in Q-categories of functors. Fix a Q-category $\mathbb{A} = (\bar{A} \overset{\bar{u}}{\rightleftarrows} A)$ and a category C. For any pair of functors, $\bar{A} \xrightarrow{\bar{G}} C$ and $A \xrightarrow{F} C$, we have the canonical map

$$C^{\bar{A}}(\bar{G}, F \circ u_*) \longrightarrow C^A(\bar{G} \circ u^*, F), \quad \phi \longmapsto F\eta_u^{-1} \circ \phi u^*, \tag{1}$$

(cf. 2.7). By definition, the functor F is a $C^{\mathbb{A}}$-sheaf (resp. a $C^{\mathbb{A}}$-monopresheaf) iff the morphism (1) is bijective (resp. injective) for any $\bar{G}$.

3.3.1. Note. We shall usually call $C^{\mathbb{A}}$-sheaves (resp. $C^{\mathbb{A}}$-monopresheaves) *sheaves* (resp. *monopresheaves*) *on* $\mathbb{A}$ *with values in* C.

3.4. Sheaves in $\mathbb{A}$ and sheaves on $\mathbb{A}$. The following proposition shows that these two notions are, in a sense, dual to each other.

3.4.1. Proposition. *Let* $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ *be a Q-category such that the inverse image functor* u^* *has a left adjoint,* $u_!$*. The functor* $A(x,-) : A \longrightarrow \mathbf{Sets}$ *is a sheaf (resp. a monopresheaf) in* $\mathbb{A}^\vee = \mathbf{Sets}^{\mathbb{A}}$ *if and only if* x *is a sheaf (resp. a monopresheaf) in the* Q^o*-category* $\mathbb{A}_! = (\bar{A} \overset{u^o}{\rightleftarrows} A)$.

Proof. We denote the functor $A(x,-)$ by F.

(i) Suppose $\bar{G} = \bar{A}(\bar{y},-) : \bar{A} \longrightarrow \mathbf{Sets}$ for some object $\bar{y}$ of $\bar{A}$. Then by Yoneda's Lemma,

$$\bar{A}^\vee(\bar{G}, F \circ u_*) \simeq F(u_*(\bar{y})) = A(x, u_*(\bar{y})) \simeq \bar{A}(u^*(x), \bar{y}).$$

Here $A^\vee = \mathbf{Sets}^A$.

(ii) Since the functor u^* has a left adjoint, $u_!$, we have $G \circ u^* = \bar{A}(\bar{y}, u^*(-)) \simeq A(u_!(\bar{y}),-)$, hence

$$A^\vee(\bar{G} \circ u^*, F) \simeq A^\vee(A(u_!(\bar{y}),-), A(x,-))$$

and by Yoneda's Lemma, $A^\vee(A(u_!(\bar{y}),-), A(x,-) \simeq A(x, u_!(\bar{y}))$. Thus we have a commutative diagram

$$\begin{array}{ccc} \bar{A}^\vee(\bar{G}, F \circ u_*) & \longrightarrow & A^\vee(\bar{G} \circ u^*, F) \\ \wr\big\uparrow & & \big\uparrow\wr \\ \bar{A}^{op}(\bar{y}, u^*(x)) & \longrightarrow & A^{op}(u_!(\bar{y}), x) \end{array} \tag{1}$$

in which vertical arrows are canonical isomorphisms. This shows that if the functor $A(x,-)$ is a sheaf (resp. a monopresheaf) of sets on the Q-category $\mathbb{A}$, the object x is a sheaf in the Q^o-category $\mathbb{A}_! = (\bar{A} \overset{u^o}{\rightleftarrows} A)$.

(iii) Any functor $\bar{A} \xrightarrow{G} \mathbf{Sets}$ is a colimit of the diagram $h^{\bar{A}}/G$, where $h^{\bar{A}}$ denotes the Yoneda's functor $\bar{A}^{op} \longrightarrow \bar{A}^{\vee}$, $\bar{y} \longmapsto \bar{A}(\bar{y}, -)$. Thus, the canonical morphism

$$A^{\vee}(G, F \circ u_*) \longrightarrow A^{\vee}(G \circ u^*, F) \tag{2}$$

is isomorphic to the limit

$$\lim_{\bar{y} \in Ob(h^{\bar{A}}/G)} \Big(\bar{A}^{op}(\bar{y}, u^*(x)) \xrightarrow{\alpha_{\bar{y},x}} A^{op}(u_!(\bar{y}), x) \Big). \tag{3}$$

If x is a sheaf in $\mathbb{A}_!$, then all morphisms $\alpha_{\bar{y},x}$ in (3) are isomorphisms, hence their limit, (2), is an isomorphism.

Similarly, if x is a monopresheaf in $\mathbb{A}_!$, then all morphisms $\alpha_{\bar{y},x}$ in (3) are monomorphisms, hence their limit, (2), is a monomorphism. ∎

3.5. The functors $u^!_C$ and $(u_C)_!$. Suppose the category C has small limits. Then the functor $C^{u^*} : C^{\bar{A}} \longrightarrow C^A$, $G \longmapsto G \circ u^*$, has a right adjoint, $u^!_C$, given for all $\bar{y} \in Ob\bar{A}$ and any $F : A \longrightarrow C$ by

$$u^!_C(F)(\bar{y}) = \lim(F \circ \mathfrak{G}_{\bar{y}}), \tag{1}$$

where $\mathfrak{G}_{\bar{y}}$ is the functor $\bar{y}\backslash u^* \longrightarrow A$, $(x, \bar{y} \longrightarrow u^*(x)) \longmapsto x$. If $C = \mathbf{Sets}$, we shall write $\widehat{u}^!$ instead of $u^!_{\mathbf{Sets}}$.

A functor F is a sheaf (resp. a monopresheaf, resp. an epipresheaf) iff the canonical morphism $F \circ u_* \longrightarrow u^!_C(F)$ is an isomorphism (resp. a monomorphism, resp. a strict epimorphism), i.e. for any $\bar{y} \in Ob\bar{A}$, the canonical morphism

$$F(u_*(\bar{y})) \longrightarrow \lim_{V \in Ob(\bar{y}\backslash u^*)} F(V) \tag{2}$$

is an isomorphism (resp. a monomorphism, resp. a strict epimorphism).

3.5.1. Lemma. *(a) A functor $A \xrightarrow{F} C$ is a sheaf (resp. a monopresheaf) on the Q-category $\mathbb{A}$ iff the functor*

$$C(z, F-) : A \longrightarrow \mathbf{Sets}$$

is a sheaf (resp. a monopresheaf) for any $z \in ObC$.

(b) A functor $A \xrightarrow{F} C$ is an epipresheaf on $\mathbb{A}$ iff $u^!_C(F)$ exists and for any $z \in ObC$, the functor $C(F-, z) : A \longrightarrow \mathbf{Sets}^{op}$ is an epipresheaf.

Proof. (a) (i) Suppose first that C has limits of small diagrams. Since for any $z \in ObC$, the functor $C(z, -)$ preserves small limits, the morphism 3.5(2) is an isomorphism (resp. a monomorphism) iff

$$C(z, F(u_*(\bar{y}))) \longrightarrow \lim_{V \in Ob(\bar{y}\backslash u^*)} C(z, F(V))$$

is an isomorphism for all $z \in ObC$.

(ii) In the general case, denote by $F^\wedge$ the composition of the functor $A \xrightarrow{F} C$ with the Yoneda embedding $C \xrightarrow{h} C^\wedge$. The functor F is a sheaf in $C^{\mathbb{A}}$ iff $F^\wedge$ is a sheaf in $(C^\wedge)^{\mathbb{A}}$. Since $C^\wedge$ is a category with limits, the assertion follows from (i).

(b) We leave the argument to the reader. ■

3.5.2. The functor $(u_C)_!$. Fix a Q-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ and a category C. Suppose the category C has small colimits. Then the functor

$$C^{u_*} : C^A \longrightarrow C^{\bar{A}}, \quad F \longmapsto F \circ u_*,$$

has a left adjoint, $(u_C)_!$, given by

$$(u_C)_!(G)(x) = \operatorname{colim}(G \circ \mathfrak{F}_{u_*/x}),$$

where $\mathfrak{F}_{u_*/x}$ is the canonical functor $u_*/x \longrightarrow \bar{A}$, $(\bar{y}, u_*(\bar{y}) \to x) \longmapsto \bar{y}$.

We shall write $\widehat{u}_!$ instead of $(u_{\mathbf{Sets}})_!$.

3.5.2.1. Special cases. If the functor u_* has a right adjoint, $u^!$, then $(u_C)_!$ is isomorphic to the functor $G \longmapsto G \circ u^!$.

If $G = \bar{A}(\bar{y}, -)$ for some $\bar{y} \in Ob\bar{A}$, then $u_!(G) \simeq A(u_*(\bar{y}), -)$.

3.6. Presheaves and sheaves on a Q-category with values in Sets. Let F be a functor $A \longrightarrow \mathbf{Sets}$. Then

$$\widehat{u}^!(F)(\bar{y}) = \lim_{V \in Ob(\bar{y} \backslash u^*)} F(V) \simeq A^\vee(\bar{A}(\bar{y}, u^*(-)), F)$$

and $\widehat{u}^*(F)(\bar{y}) = F \circ u_* \simeq A^\vee(u_*(\bar{y}), F)$. Here $A^\vee$ denotes the category of functors from A to **Sets**, and $u_*(\bar{y})$ (as any other object of A) is identified with the corresponding corepresentable functor, $A(u_*(\bar{y}), -)$. The canonical morphism $\bar{A}(\bar{y}, u^*(-)) \longrightarrow A(u_*(\bar{y}), -)$ induces a morphism

$$A^\vee(A(u_*(\bar{y}), -), F) \longrightarrow A^\vee(\bar{A}(\bar{y}, u^*(-)), F). \tag{1}$$

The presheaf F is a sheaf (resp. a monopresheaf, resp. an epipresheaf) iff the morphism (1) is an isomorphism (resp. a monomorphism, resp. an epimorphism).

3.6.1. Note. The observation above extends to the case of presheaves with values in a category C having small limits. In this case, the Q-category $C^{\mathbb{A}}$ is equivalent to the Q-subcategory $C^{\mathbb{A}^\vee}_{\mathfrak{cc}}$ of the category $C^{(\mathbb{A}^\vee)^{op}}$ formed by presheaves on resp. $\bar{A}^\vee$ and $A^\vee$ with values in C having a right adjoint (cf. 2.10.2). For any presheaf $F^\vee : (A^\vee)^{op} \longrightarrow C$, the canonical morphism $\bar{A}(\bar{y}, u^*(-)) \longrightarrow A(u_*(\bar{y}), -)$ induces a morphism

$$F^\vee(A(u_*(\bar{y}), -)) \longrightarrow F^\vee(\bar{A}(\bar{y}, u^*(-))). \tag{2}$$

The presheaf $F^\vee$ is a sheaf (resp. a monopresheaf, resp. an epipresheaf) iff the morphism (2) is an isomorphism (resp. a monomorphism, resp. an epimorphism).

3.7. Presheaves and sheaves on Q-categories with a functor $u_!$. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category such that the functor u^* has a left adjoint, $u_!$. For instance, $\mathbb{A}$ is a full Q-subcategory of the Q-category $(A^2 \rightleftarrows A)$ of morphisms (cf. 2.5 and 2.5.1). Denote by $\mathfrak{r}_u$ the canonical morphism $u_* \longrightarrow u_!$. For any category C, the functor $C^{u_!}$ is a right adjoint to C^{u^*}. Thus, a functor $A \xrightarrow{F} C$ is a sheaf (resp. a monopresheaf, resp. an epipresheaf) on $\mathbb{A}$ iff the morphism

$$F \circ u_* \xrightarrow{F(\mathfrak{r}_u)} F \circ u_! \qquad (1)$$

is an isomorphism (resp. a monomorphism, resp. a strict epimorphism).

Denote by Σ_u the family $\{\mathfrak{r}_u(\bar{y}) : u_*(\bar{y}) \longrightarrow u_!(\bar{y}) \mid \bar{y} \in Ob\bar{A}\}$ of morphisms of A. It follows that the category of sheaves on $\mathbb{A}$ with values in a category C is isomorphic to the category of functors $\Sigma_u^{-1}A \longrightarrow C$, where $\Sigma_u^{-1}A$ is the quotient category by Σ_u.

3.7.1. Sheaves on Q-categories and localizations. Let Σ be a family of morphisms of a category A containing all isomorphisms of A (or, at least, all identical morphisms). Denote by A_Σ the full subcategory of A^2 formed by morphisms of Σ. This defines a Q-subcategory, $\mathbb{A}_\Sigma = (A_\Sigma \rightleftarrows A)$ of $(A^2 \rightleftarrows A)$. Sheaves on $\mathbb{A}_\Sigma$ with values in a category C are functors $A \xrightarrow{F} C$ which transform morphisms of Σ into invertible morphisms. In other words, the category of sheaves on $\mathbb{A}_\Sigma$ with values in C is isomorphic to the category of functors from the quotient category $\Sigma^{-1}A$ to C.

3.8. Sheaves on Grothendieck sites. Below we show that (pre)sheaves on a site can be realized as (pre)sheaves on the Q-category of the form $\mathbb{A}_\Sigma$ (see 3.7.1) where Σ is the class of covers of a pretopology; in particular, it satisfies (left) Ore conditions.

3.8.1. A Grothendieck pretopology associated with a quasi-(co)site. Let $(A, \mathfrak{T})$ be a quasi-cosite. For any functor $A \xrightarrow{X} \mathbf{Sets}$, we denote by $\mathfrak{T}^\vee(X)$ the set of all subfunctors, R, of X such that for any object x of A and any morphism $A(x,-) \longrightarrow X$, the subfunctor $R \times_X A(x,-)$ of $A(x,-)$ belongs to $\mathfrak{T}(x)$. It follows that $X \xrightarrow{id_X} X$ belongs to $\mathfrak{T}^\vee(X)$ and the function $X \longmapsto \mathfrak{T}^\vee(X)$ is invariant under a base change: for any morphism $Y \longrightarrow X$ and any $R \in \mathfrak{T}^\vee(X)$, the subfunctor $R \times_X Y \longrightarrow Y$ belongs to $\mathfrak{T}^\vee(Y)$. Thus, the function $\mathfrak{T}^\vee$ is a pretopology on the category $A^\vee$ of functors $A \longrightarrow \mathbf{Sets}$ which we regard as a Q^o-category. Since any cover in this pretopology consists of one arrow, the Q^o-category $(\mathfrak{T}^\vee \rightleftarrows A^\vee)$ is a Q^o-subcategory of the Q^o-category $((A^\vee)^2 \rightleftarrows A^\vee)$ of morphisms of $A^\vee$.

If $(A, \mathfrak{T})$ is a cosite, then the Yoneda embedding, $A^{op} \xrightarrow{h^A} A^\vee$, determines a Q-category embedding $(\mathfrak{T} \rightleftarrows A) \longrightarrow (\mathfrak{T}^\vee \rightleftarrows A^\vee)^{op}$.

Dually, if $(A, \mathfrak{T})$ is a site, then the Yoneda embedding, $A \xrightarrow{h_A} A^\wedge$, induces a Q^o-category embedding $(\mathfrak{T} \rightleftarrows A) \longrightarrow (\mathfrak{T}^\wedge \rightleftarrows A^\wedge)$.

3.8.2. Proposition. *Let $(A, \mathfrak{T})$ be a site, and let C be a category with small limits. Then the Q-category of presheaves on $(A, \mathfrak{T})$ with values in C (which is, by definition, the Q-category of functors on the Q-category $(\mathfrak{T}^{op} \rightleftarrows A^{op})$ with values in C) is naturally equivalent to the full Q-subcategory of the Q-category of presheaves on $(\mathfrak{T}^\wedge \rightleftarrows A^\wedge)$ formed by those presheaves which preserve small limits.*

In particular, $A^{op} \xrightarrow{F} C$ is a sheaf (resp. monopresheaf, resp. epipresheaf) iff the corresponding presheaf on $(\mathfrak{T}^\wedge \rightleftarrows A^\wedge)$ is a sheaf (resp. monopresheaf, resp. epipresheaf).

Proof. The assertion follows from Proposition II.1.3 in [GZ] (cf. 2.10.2). Details are left to the reader. ■

3.9. Sheaves on a Q-category and sheaves on the associated quasi-cosite. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category, and let $\mathfrak{T}\mathbb{A}$ be the quasi-cosite associated with $\mathbb{A}$ (see 2.4). For any category C, the categories of sheaves (resp. categories of monopresheaves, resp. categories of epipresheaves) on $\mathbb{A}$ and on $\mathfrak{T}\mathbb{A}$ with values in a category C are, usually, quite different.

In fact, suppose the functor u^* has a left adjoint, $u_!$. Then sheaves (resp. monopresheaves, resp. epipresheaves) on $\mathbb{A}$ with values in the category C are precisely those functors $A \xrightarrow{F} C$ which map morphisms of $\Sigma_u = \{\mathfrak{r}_u(\bar{y}) : u_*(\bar{y}) \longrightarrow u_!(\bar{y}) \mid \bar{y} \in Ob\bar{A}\}$ to invertible morphisms (resp. to monomorphisms, resp. to strict epimorphisms) (cf. 3.7).

Suppose, in addition, that for any $\bar{y} \in Ob\bar{A}$, there exists a push-forward

$$\begin{array}{ccc} u_*(\bar{y}) & \xrightarrow{\mathfrak{r}_u(\bar{y})} & u_!(\bar{y}) \\ {\scriptstyle \mathfrak{r}_u(\bar{y})}\downarrow & & \downarrow \\ u_!(\bar{y}) & \longrightarrow & u_!(\bar{y}) \coprod_{u_*(\bar{y})} u_!(\bar{y}) \end{array}$$

Then a functor $A \xrightarrow{F} C$ is a sheaf on $\mathfrak{T}\mathbb{A}$ iff for any $\bar{y} \in Ob\bar{A}$, the diagram

$$F(u_*(\bar{y})) \xrightarrow{\mathfrak{r}_u(\bar{y})} F(u_!(\bar{y})) \rightrightarrows F(u_!(\bar{y}) \coprod_{u_*(\bar{y})} u_!(\bar{y})) \qquad (1)$$

is exact (cf. 2.4).

3.9.1. Sheaves on the associated quasi-cosite: the general case. Suppose that the Q-category $\mathbb{A}$ is arbitrary, but the category C has small limits. Then the category $C^{\mathbb{A}}$ can be realised as the Q-subcategory, $C_{\mathfrak{cc}}^{(\mathbb{A}^\vee)^{op}}$, of the category of presheaves on $\mathbb{A}^\vee$ with values in C (cf. 2.10.2). Let $(A^\vee)^{op} \xrightarrow{F^\vee} C$ denote a presheaf corresponding to a functor $A \xrightarrow{F} C$. By 3.6.1, F is a sheaf on $\mathbb{A}$ iff for any $\bar{y} \in Ob\bar{A}$, the morphism

$$F^\vee(A(u_*(\bar{y}), -)) \longrightarrow F^\vee(\bar{A}(\bar{y}, u^*(-))). \qquad (2)$$

induced by the canonical morphism $\bar{A}(\bar{y}, u^*(-)) \longrightarrow A(u_*(\bar{y}), -)$ is an isomorphism.

A functor $A \xrightarrow{F} C$ is a sheaf on $\mathfrak{T}\mathbb{A}$ iff for any $\bar{y} \in Ob\bar{A}$, the diagram

$$F^\vee(A(u_*(\bar{y}), -)) \longrightarrow F^\vee(\bar{A}(\bar{y}, u^*(-)) \rightrightarrows F^\vee(\bar{A}(\bar{y}, u^*(-)) \coprod_{A(u_*(\bar{y}),-))} \bar{A}(\bar{y}, u^*(-))) \qquad (3)$$

is exact. If follows from the definition of the presheaf $F^\vee$ that $F^\vee(A(u_*(\bar{y}),-)) \simeq F(u_*(\bar{y}))$. If the functors $A(\bar{y},u^*(-))$ and $\bar{A}(\bar{y},u^*(-)) \prod_{A(u_*(\bar{y}),-))} \bar{A}(\bar{y},u^*(-))$ are representable, the diagram (3) is equivalent to the diagram (1).

If $C = \mathbf{Sets}$, then $F^\vee(X) = Hom_{A^\vee}(X,F)$ for any $X \in ObA^\vee$.

3.9.2. Note. Observe that the category of sheaves on $\mathbb{A}$ is a (stictly full) subcategory of the category of sheaves on the associated quasi-cosite, $\mathfrak{T}\mathbb{A}$.

3.9.3. Sheaves in terms of covers. Let A be a category, and let τ be a function which assigns to every object x of A the family, τ_x of sets of arrows to x (called covers of x, cf. 2.4) which contains $\{id_x\}$. We call τ a *quasi-pretopology* if for any two covers $\mathcal{U}$, $\mathcal{V}$ and any two elements $(x_u \stackrel{u}{\to} x) \in \mathcal{U}$ and $(x_v \stackrel{v}{\to} x) \in \mathcal{V}$, the pull-back $x_u \times_x x_v$ exists. We shall identify the τ with the corresponding Q^o-category $\mathbb{A}_\tau = (A_\tau \rightleftarrows A)$ (cf. 2.4).

Suppose $\mathbb{A} = \mathbb{A}_\tau^{op} = (A_\tau \rightleftarrows A)^{op}$ for a quasi-pretopology τ. Let $\bar{y} = (y_i \stackrel{\phi_i}{\to} y|\ i \in I)$ be an object of A_τ (i.e. a cover; cf. 2.4.1) and x an object of A. Then $A^{op}(u_*(\bar{y}),x) = \mathcal{A}(x,y)$ and $A_\tau^{op}(\bar{y},u^*(x)) = A_\tau((x \stackrel{id_x}{\to} x),\bar{y}) = \coprod_{i\in I} A(x,y_i)$. Here coproduct is the coproduct in **Sets**, i.e. the disjoint union. The canonical morphism

$$A_\tau^{op}(\bar{y},u^*(x)) = \coprod_{i\in I} A(x,y_i) \longrightarrow A^{op}(u_*(\bar{y}),x) = A(x,y)$$

maps each morphism $x \xrightarrow{f_i} y_i$ to the composition of f_i and $y_i \xrightarrow{\phi_i} y$. The equivalence relation

$$\Big(\coprod_{i\in I} A(x,y_i)\Big) \prod_{A(x,y)} \Big(\coprod_{i\in I} A(x,y_i)\Big) \subseteq \Big(\coprod_{i\in I} A(x,y_i)\Big) \prod \Big(\coprod_{i\in I} A(x,y_i)\Big)$$

is defined correspondingly: a morphism $x \xrightarrow{f_i} y_i$ is equivalent to a morphism $x \xrightarrow{f_j} y_j$ iff $\phi_i \circ f_i = \phi_j \circ f_j$. It follows that

$$\Big(\coprod_{i\in I} A(x,y_i)\Big) \prod_{A(x,y)} \Big(\coprod_{i\in I} A(x,y_i)\Big) = \coprod_{i,j\in I} \Big(A(x,y_i) \prod_{A(x,y)} A(x,y_j)\Big) = \coprod_{i,j\in I} A(x, y_i \times_y y_j).$$

Thus the diagram (2) in the case of a quasi-pretopology is (isomorphic to)

$$\coprod_{i,j\in I} A(-,y_i \times_y y_j) \rightrightarrows \coprod_{i\in I} A(-,y_i) \longrightarrow A(-,y) \qquad (4)$$

Let F be any presheaf of sets on A. Then the functor $A^\wedge(-,F)$ maps the sequence (4) to the diagram

$$F(y) \longrightarrow \prod_{i\in I} F(y_i) \rightrightarrows \prod_{i,j\in I} F(y_i \times_y y_j). \qquad (5)$$

Thus a presheaf F is a sheaf on the quasi-site $\mathfrak{T}\mathbb{A}_\tau$ associated with $\mathbb{A}_\tau$ iff the functor $A^\wedge(-,F)$ maps the diagram (4) to an exact diagram.

4. Formally étale, formally unramified, and formally smooth functors.

4.1. Example: (pre)sheaves on the Q-category of infinitesimal epimorphisms and formally étale functors. Let A be the category $CAlg_k$ of commutative associative unital k-algebras, $\bar{A}$ a full subcategory of A^2 whose objects are k-algebra epimorphisms with a nilpotent kernel. We denote by $C\mathbb{A}lg_k^{inf}$ the corresponding full Q-subcategory of $(CAlg_k^2 \rightleftarrows CAlg_k)$. We call $C\mathbb{A}lg_k^{inf}$ the Q-category of commutative *infinitesimal epimorphisms.*

A functor $CAlg_k \xrightarrow{F}$ **Sets** is *formally étale* (resp. *formally unramified*, resp. formally smooth) if the canonical morphism $F \circ u_* \longrightarrow F \circ u_!$ is an isomorphism (resp. a monomorphism, resp. a strict epimorphism). Comparing with 3.7, we obtain that F is formally étale (resp. formally unramified, resp. formally smooth) iff it is a sheaf (resp. a monopresheaf, resp. an epipresheaf) of sets on the Q-category $C\mathbb{A}lg_k^{inf}$.

4.2. A noncommutative version. Let A be the category Alg_k of associative unital algebras over a commutative ring k, $\bar{A}$ a full subcategory of A^2 whose objects are algebra epimorphisms with a nilpotent kernel. We denote by $\mathbb{A}lg_k^{inf}$ the corresponding full Q-subcategory of $(Alg_k^2 \rightleftarrows Alg_k)$. We call $\mathbb{A}lg_k^{inf}$ the *Q-category of infinitesimal epimorphisms.*

4.3. Proposition. *Let a functor $Alg_k \xrightarrow{F}$* **Sets** *be corepresentable by a k-algebra R.*

(a) The functor F is an epipresheaf on $\mathbb{A}$ iff the algebra R is quasi-free in the sense of Quillen and Cuntz [CQ1]. The latter is equivalent to the condition: the $R \otimes_k R^{op}$-module $\Omega^1_{R|k}$ of Keller differentials of R (which is the kernel of the multiplication $R \otimes_k R \longrightarrow R$) is projective.

(b) The functor F is a monopresheaf on $\mathbb{A}$ iff $\Omega^1_{R|k} = 0$.

Proof. A standard argument shows that F is an epipresheaf (resp. a monopresheaf) iff for any k-algebra epimorphism $S \xrightarrow{\phi} R$ such that $Ker(\phi)^2 = 0$, there exists a splitting (resp. at most one splitting), that is a k-algebra morphism $R \xrightarrow{\psi} S$ such that $\phi \circ \psi = id_R$.

(a) Thus F is an epipresheaf iff $Ext^2_{R^e}(R, M) = 0$ for any R^e-module M. Here R^e denote the k-algebra $R \otimes_k R^{op}$. Consider the long exact sequence

$$... \longrightarrow Ext^i_{R^e}(R, M) \longrightarrow Ext^i_{R^e}(R^e, M) \longrightarrow Ext^i_{R^e}(\Omega^1_{R|k}, M) \longrightarrow Ext^{i+1}_{R^e}(R, M) \longrightarrow ...$$

corresponding to the short exact sequence $0 \longrightarrow \Omega^1_{R|k} \longrightarrow R^e \longrightarrow R \longrightarrow 0$. Since $Ext^i_{R^e}(R^e, M) = 0$ for all $i \geq 1$ and all R^e-modules M, $Ext^i_{R^e}(\Omega^1_{R|k}, M) \simeq Ext^{i+1}_{R^e}(R, M)$ for all $i \geq 1$ and all R^e-modules M. In particular, $Ext^2_{R^e}(R, M) = 0$ for all M iff $Ext^1_{R^e}(\Omega^1_{R|k}, M) = 0$ for all M. The latter means precisely that $\Omega^1_{R|k}$ is a projective R^e-module.

(b) Let $R \xrightarrow{\psi} S$ be a k-algebra morphism such that $\phi \circ \psi = id_R$. It gives a decomposition of S into a semidirect product of R and an R-bimodule, M, with multiplication defined by $(r, m)(r', m') = (rr', r \cdot m' + m \cdot r')$. Any other splitting, $R \xrightarrow{\psi'} S$, is (id_R, d), where $R \xrightarrow{d} M$ is a derivation sending k to zero. Thus, the set of splittings of ϕ is in one-to-one correspondence with $Der_{R|k}(M)$. But $Der_{R|k}(M) \simeq Hom_{R^e}(\Omega^1_{R|k}, M)$. Hence ϕ is unramified iff $\Omega^1_{R|k} = 0$. ■

4.4. Formally $\mathbb{A}$-smooth functors. Examples 4.1 and 4.2 suggest the following interpretation of epipresheaves, monopresheaves and sheaves on a Q-category:

4.4.1. Definition. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category. We say that a functor $A \xrightarrow{F} \mathbf{Sets}$ is *formally* $\mathbb{A}$*-smooth* (resp. *formally* $\mathbb{A}$*-unramified*, resp. *formally* $\mathbb{A}$*-étale*) if it is an epipresheaf (resp. a monopresheaf, resp. a sheaf) on $\mathbb{A}$; i.e. the canonical morphism

$$\widehat{u}^*(F) = F \circ u_* \longrightarrow \widehat{u}^!(F) = \lim_{(V,\xi) \in Ob(\bar{y} \backslash u^*)} F(V) \qquad (1)$$

is a strict epimorphism (resp. a monomorphism, resp. an isomorphism).

4.4.2. A reformulation. One can, using isomorphisms

$$\widehat{u}^!(F)(\bar{y}) \simeq A^\vee(\bar{A}(\bar{y}, u^*(-)), F) \quad \text{and} \quad \widehat{u}^*(F)(\bar{y}) = F \circ u_* \simeq A^\vee(u_*(\bar{y}), F)$$

(see 3.6), reformulate the notion of the formal $\mathbb{A}$-smoothness as follows.

A functor $A \xrightarrow{F} \mathbf{Sets}$ is formally $\mathbb{A}$-smooth iff the canonical map

$$A^\vee(u_*(\bar{y}), F) \longrightarrow A^\vee(\bar{A}(\bar{y}, u^*(-)), F), \quad g \longmapsto g \circ \alpha_{\bar{y}}, \qquad (2)$$

is surjective for all $\bar{y} \in Ob\bar{A}$. Here $u_*(\bar{y})$ is identified with its image in $A^\vee$, i.e. the functor $A(u_*(\bar{y}), -) : A \longrightarrow \mathbf{Sets}$, and $\alpha_{\bar{y}}$ denotes the canonical morphism

$$\bar{A}(\bar{y}, u^*(-)) \longrightarrow A(u_*(\bar{y}), -), \quad h \longmapsto \eta_u(-)^{-1} \circ u_*(h). \qquad (3)$$

A functor $A \xrightarrow{F} \mathbf{Sets}$ is formally $\mathbb{A}$-unramified (resp. formally $\mathbb{A}$-étale) iff the map (2) is injective (resp. bijective) for all $\bar{y} \in Ob\bar{A}$.

4.5. Formally $\mathbb{A}$-smooth, formally $\mathbb{A}$-unramified, and formally $\mathbb{A}$-étale objects. We say that an object x of the category A is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified, resp. formally $\mathbb{A}$-étale) if the functor $A(x, -) : A \longrightarrow \mathbf{Sets}$ is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified, resp. formally $\mathbb{A}$-smooth).

4.5.1. $\mathbb{A}$-smooth, $\mathbb{A}$-unramified, and $\mathbb{A}$-étale objects. We say that an object x of A is $\mathbb{A}$*-smooth* (resp. $\mathbb{A}$-unramified) if it is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified) and finitely presented. The latter means that the functor $A(x, -)$ preserves colimits of filtered diagrams. We call an object x $\mathbb{A}$*-étale* if it is both $\mathbb{A}$-smooth and $\mathbb{A}$-unramified.

4.5.2. Example. Let A be the category of associative unital algebras over a commutative ring k, $\bar{A}$ a full subcategory of A^2 formed by all algebra epimorphisms. Then for any projective k-module V, the tensor algebra, $T_k(V)$, of V is formally $\mathbb{A}$-smooth. It is $\mathbb{A}$-smooth iff the projective k-module V is of finite type.

Note that there is only one (up to isomorphism) $\mathbb{A}$-étale algebra: the ring k.

4.6. Quasi-free and separable algebras. Let A be the category $\mathfrak{A}ss_k$ whose objects are associative k-algebras. Morphisms from a k-algebra R to a k-algebra S are equivalence classes of algebra morphisms $R \longrightarrow S$ by the following equivalence relation: two algebra morphisms, $R \underset{g}{\overset{f}{\rightrightarrows}} S$, are equivalent if they are conjugated, i.e. $g(-) = tf(-)t^{-1}$ for an invertible element t of S. Let $\bar{A}$ be the full subcategory of A^2 formed by the equivalence classes of algebra epimorphisms with a nilpotent kernel. We denote by $\mathfrak{A}ss_k^{inf}$ the corresponding full Q-subcategory of $(A^2 \rightleftarrows A)$.

Note that the restriction of the natural functor $Ass_k \longrightarrow \mathfrak{A}ss_k$ to the subcategory of commutative algebras is a strict fully faithful functor ('strict' means that it is injective on objects) which induces a strict fully faithful morphism $C\mathbb{A}lg_k^{inf} \hookrightarrow \mathfrak{A}ss_k^{inf}$ of Q-categories (cf. 4.1).

Recall that a k-algebra R is called *separable* if R is a projective left R^e-module, $R^e = R \otimes_k R^o$. It follows from the exact sequence of R^e-modules

$$0 \longrightarrow \Omega^1_{R|k} \longrightarrow R^e \longrightarrow R \longrightarrow 0$$

that if R is separable, then $\Omega^1_{R|k}$ is a projective R^e-module, i.e. R is quasi-free [CQ1].

4.6.1. Proposition. *Let R be an associative k-algebra.*
(a) The following conditions are equivalent:
(i) The algebra R is formally $\mathfrak{A}ss_k^{inf}$-smooth.
(ii) The left R^e-module of Keller differentials, $\Omega^1_{R/k} = Ker(R^e \to R)$, is projective.
(b) The following conditions are equivalent:
(iii) R is formally $\mathfrak{A}ss_k^{inf}$-étale.
(iv) R is formally $\mathfrak{A}ss_k^{inf}$-unramified.
(v) R is separable.

Proof. (a) Let $S \xrightarrow{\phi} R$ be a k-algebra morphism such that there exists a k-algebra morphism $R \xrightarrow{\psi} S$ right inverse to ϕ in the category $\mathfrak{A}ss_k$. The latter means, in particular, that $\phi \circ \psi$ is conjugate to id_R; i.e. there exists an invertible element t of R such that for any $r \in R$, $\phi \circ \psi(r) = trt^{-1}$. The composition, ψ_t, of ψ with the inner automorphism $r \longmapsto t^{-1}rt$ is a right inverse to ϕ in the category Alg_k. This shows that R is formally $\mathfrak{A}ss_k^{inf}$-smooth iff it is formally $\mathbb{A}lg_k^{inf}$-smooth (cf. 4.2). The assertion follows from 4.3 (or [CQ1]).

(b) The implication (iii)⇒(iv) is true by definition.

(iv)⇒(v). Let M be an R^e-module, S a semiproduct of R and M, $S \xrightarrow{\phi} R$ the canonical epimorphism. It follows from (a) that any right inverse to ϕ in the sense of $\mathfrak{A}ss_k$ is conjugate to a right inverse, $R \xrightarrow{\psi} S$ to ϕ in the sense of Alg_k. The morphism ψ is of the form $r \longmapsto r + D(r)$ for some (any) derivation $R \xrightarrow{D} M$ which sends k to zero. If R is $\mathfrak{A}ss_k^{inf}$-unramified, the morphism ψ is equivalent to the morphism $R \longrightarrow S$, $r \longmapsto r$. This means that there exists an invertible element u of S such that $\psi(r) = uru^{-1}$ for all $r \in R$. The element u can be written as $t(1_R + z)$, where 1_R is the unit of R, t is an invertible element of R, and $z \in M$. Then

$$uru^{-1} = trt^{-1} + (tzt^{-1})(trt^{-1}) - (trt^{-1})(tzt^{-1}) \tag{1}$$

In particular, $\phi \circ \psi(r) = trt^{-1}$ for all $r \in R$. But $\phi \circ \psi = id_R$, hence the element t is central. Thus $\psi(r) = r + z_t r - r z_t$, where $z_t = tzt^{-1}$, i.e. D is an inner derivation. It is known [CQ1] (and easy to prove) that *R is a separable k-algebra iff any derivation of R in any R^e-module M is inner*, hence the implication.

(v)⇒(iii). Let R be a separable k-algebra. Let $T \xrightarrow{\phi} S$ be a k-algebra morphism with a nilpotent kernel and $R \xrightarrow{f} S$ an arbitrary algebra morphism. It follows from the argument in [CQ1] that any two liftings of f to a morphism $R \longrightarrow T$ are conjugate by an element t of T such that $1 - t$ belongs to the kernel of ϕ, in particular it is nilpotent. Conversely, such a lifting property implies that R is separable. ■

5. Formally $\mathbb{A}$-smooth, formally $\mathbb{A}$-unramified, and formally $\mathbb{A}$-étale morphisms. Fix a Q-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$. Let X, Y be functors $A \longrightarrow$ **Sets**.

5.1. Definition. We call a morphism $X \xrightarrow{f} Y$ *formally $\mathbb{A}$-smooth* if for any $\bar{y} \in Ob\bar{A}$ and for any pair of morphisms $A(u_*(\bar{y}), -) \xrightarrow{g} Y$, $\bar{A}(\bar{y}, u^*(-)) \xrightarrow{g'} X$ such that the diagram

$$\begin{array}{ccc} \bar{A}(\bar{y}, u^*(-)) & \xrightarrow{g'} & X \\ \alpha_{\bar{y}} \downarrow & & \downarrow f \\ A(u_*(\bar{y}), -) & \xrightarrow{g} & Y \end{array} \tag{1}$$

commutes, there exists a morphism $A(u_*(\bar{y}), -) \xrightarrow{\gamma} X$ such that $\gamma \circ \alpha_{\bar{y}} = g'$ and $f \circ \gamma = g$. In other words, the diagram

$$\begin{array}{ccc} \bar{A}(\bar{y}, u^*(-)) & \xrightarrow{g'} & X \\ \alpha_{\bar{y}} \downarrow & \nearrow_{\gamma} & \downarrow f \\ A(u_*(\bar{y}), -) & \xrightarrow{g} & Y \end{array} \tag{2}$$

commutes.

5.2. Definition. We call a morphism $X \xrightarrow{f} Y$ *formally $\mathbb{A}$-unramified* if for any $\bar{y} \in Ob\bar{A}$ and any pair of morphisms, $A(u_*(\bar{y}), -) \xrightarrow{g} Y$, $\bar{A}(\bar{y}, u^*(-)) \xrightarrow{g'} X$, making the diagram 5.1(1) commute, there exists at most one morphism, $A(u_*(\bar{y}), -) \xrightarrow{\gamma} X$, such that the diagram 5.1(2) commute.

5.2.1. Note. Any monomorphism $X \xrightarrow{f} Y$ is formally $\mathbb{A}$-unramified.

5.3. Definition. We call a morphism $X \xrightarrow{f} Y$ *formally $\mathbb{A}$-étale* if it is both formally $\mathbb{A}$-smooth and formally $\mathbb{A}$-unramified.

5.3.1. A reformulation. Consider the diagram 5.1(1). We have canonical, functorial in $\bar{y}$ and X isomorphisms

$$A^\vee(\bar{A}(\bar{y}, u^*(-)), X) = A^\vee(\widehat{u}_*(\bar{A}(\bar{y}, -)), X) \simeq \bar{A}^\vee(\bar{A}(\bar{y}, -), \widehat{u}^!(X) \simeq \widehat{u}^!(X)(\bar{y})$$

and

$$A^\vee(u_*(\bar{y}), -), Y) \simeq Y(u_*(\bar{y})) = \widehat{u}^*(Y)(\bar{y}).$$

The commutative diagram

$$\begin{array}{ccc} \widehat{u}^*(X) & \xrightarrow{\widehat{u}^*(f)} & \widehat{u}^*(Y) \\ {\scriptstyle \alpha_{\widehat{u}}(X)}\downarrow & & \downarrow{\scriptstyle \alpha_{\widehat{u}}(Y)} \\ \widehat{u}^!(X) & \xrightarrow{\widehat{u}^!(f)} & \widehat{u}^!(Y) \end{array} \qquad (1)$$

induces a morphism

$$\widehat{u}^*(X) \longrightarrow \widehat{u}^*(Y) \times_{\widehat{u}^!(Y)} \widehat{u}^!(X) \qquad (2)$$

5.3.1.1. Proposition. *(a) A morphism $X \xrightarrow{f} Y$ is formally $\mathbb{A}$-unramified (resp. formally $\mathbb{A}$-étale) iff (2) is a monomorphism (resp. an isomorphism).*

(b) A morphism $X \xrightarrow{f} Y$ is formally $\mathbb{A}$-smooth iff for any $\bar{y} \in Ob\bar{A}$, the map

$$\widehat{u}^*(X)(\bar{y}) \longrightarrow (\widehat{u}^*(Y) \times_{\widehat{u}^!(Y)} \widehat{u}^!(X))(\bar{y})$$

is surjective. In particular, (2) is an epimorphism.

Proof is left to the reader. ■

5.3.1.2. Corollary. *Suppose $Y \in ObA^\vee$ is formally $\mathbb{A}$-étale. Then a morphism $X \xrightarrow{f} Y$ is formally $\mathbb{A}$-étale iff X is formally $\mathbb{A}$-étale.*

Proof. The morphism f is formally $\mathbb{A}$-étale iff the square (1) is cartesian. In particular, if the right vertical arrow of (1) is an isomorphism (which means exactly that Y is formally $\mathbb{A}$-étale), then the left vertical arrow of (1) is an isomorphism too, i.e. X is formally $\mathbb{A}$-étale.

Conversely, if both vertical arrows of the square (1) are isomorphisms (i.e. both X and Y are formally $\mathbb{A}$-étale), then the square (1) is cartesian. ■

5.3.2. Formally étale morphisms and $\mathbb{A}$-sheaves. Let $*$ denote the functor which assigns to all objects of A a one element set – a final object of the category $A^\vee$. For any $X \in ObA^\vee$, denote by π_X the unique map $X \longrightarrow *$. It follows that X is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified, resp. formally $\mathbb{A}$-étale) iff the map $X \xrightarrow{\pi_X} *$ is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified, resp. formally $\mathbb{A}$-étale).

Thus, formally $\mathbb{A}$-étale morphisms might be viewed as relative versions of sheaves of sets on $\mathbb{A}$.

5.4. Proposition. *(a) The composition of formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified, resp. formally $\mathbb{A}$-étale) morphisms is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified, resp. formally $\mathbb{A}$-étale).*

(b) Let X, Y, Z be functors $A \longrightarrow$ **Sets**, *and let $X \xrightarrow{f} Y$ and $Y \xrightarrow{h} Z$ be functor morphisms.*

(i) Suppose $h \circ f$ is formally $\mathbb{A}$-unramified. Then f is formally $\mathbb{A}$-unramified.

(ii) Suppose h is formally $\mathbb{A}$-unramified. If $X \xrightarrow{h\circ f} Z$ is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-étale, then f is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-étale).

(c) Let $S \in ObA^{\vee}$, and let $(X,\xi) \xrightarrow{f} (Y,\mu)$, $(X',\xi') \xrightarrow{f'} (Y',\mu')$ be morphisms of objects over S. The morphisms f, f' are formally $\mathbb{A}$-unramified (resp. formally $\mathbb{A}$-smooth, resp. formally $\mathbb{A}$-étale) iff the morphism $f \times_S f' : X \times_S X' \longrightarrow Y \times_S Y'$ has the respective property.

Proof. (a) Suppose $X \xrightarrow{h\circ f} Z$ is formally $\mathbb{A}$-smooth. Let

$$\begin{array}{ccc} \bar{A}(\bar{y}, u^*(-)) & \xrightarrow{\;\;g'\;\;} & X \\ \alpha_{\bar{y}} \big\downarrow & & \big\downarrow h \circ f \\ A(u_*(\bar{y}), -) & \xrightarrow{\;\;g\;\;} & Z \end{array} \tag{1}$$

be a commutative diagram. Since the morphism h is formally $\mathbb{A}$-smooth, there exists a morphism $A(u_*(\bar{y}), -) \xrightarrow{\gamma'} Y$ such that the diagram

$$\begin{array}{ccc} \bar{A}(\bar{y}, u^*(-)) & \xrightarrow{f\circ g'} & Y \\ \alpha_{\bar{y}} \big\downarrow & \nearrow_{\gamma'} & \big\downarrow f \\ A(u_*(\bar{y}), -) & \xrightarrow{g} & Z \end{array} \tag{2}$$

commutes. In particular, $\gamma' \circ \alpha_{\bar{y}} = f \circ g'$, i.e. the diagram

$$\begin{array}{ccc} \bar{A}(\bar{y}, u^*(-)) & \xrightarrow{\;\;g'\;\;} & X \\ \alpha_{\bar{y}} \big\downarrow & & \big\downarrow f \\ A(u_*(\bar{y}), -) & \xrightarrow{\;\;\gamma'\;\;} & Y \end{array} \tag{3}$$

commutes. Since f is formally $\mathbb{A}$-smooth, there exists a morphism $\gamma : A(u_*(\bar{y}), -) \longrightarrow X$ such that the diagram

$$\begin{array}{ccc} \bar{A}(\bar{y}, u^*(-)) & \xrightarrow{g'} & X \\ \alpha_{\bar{y}} \big\downarrow & \nearrow_{\gamma} & \big\downarrow f \\ A(u_*(\bar{y}), -) & \xrightarrow{\gamma'} & Y \end{array} \tag{4}$$

commutes. Combining (3) and (4), we obtain the commutative diagram

$$\begin{array}{ccc} \bar{A}(\bar{y}, u^*(-)) & \xrightarrow{g'} & X \\ \alpha_{\bar{y}} \big\downarrow & \nearrow_{\gamma} & \big\downarrow h \circ f \\ A(u_*(\bar{y}), -) & \xrightarrow{g} & Z \end{array}$$

Hence $h \circ f$ is formally $\mathbb{A}$-smooth.

We leave to the reader the checking that if f and h are formally $\mathbb{A}$-unramified (resp. formally $\mathbb{A}$-étale), then $h \circ f$ is formally $\mathbb{A}$-unramified (resp. formally $\mathbb{A}$-étale).

(b) (i) Suppose $h \circ f$ is formally $\mathbb{A}$-unramified, and let

$$\begin{array}{ccc} \bar{A}(\bar{y}, u^*(-)) & \xrightarrow{g'} & X \\ \alpha_{\bar{y}} \downarrow & \nearrow_{\gamma} & \downarrow f \\ A(u_*(\bar{y}), -) & \xrightarrow{g} & Y \end{array}$$

be a commutative diagram. Then the diagram

$$\begin{array}{ccc} \bar{A}(\bar{y}, u^*(-)) & \xrightarrow{g'} & X \\ \alpha_{\bar{y}} \downarrow & \nearrow_{\gamma} & \downarrow h \circ f \\ A(u_*(\bar{y}), -) & \xrightarrow{h \circ g} & Z \end{array}$$

commutes. Since $h \circ f$ is formally $\mathbb{A}$-unramified, the morphism γ is uniquely defined.

(ii) Suppose h is formally $\mathbb{A}$-unramified and $h \circ f$ is formally $\mathbb{A}$-smooth. Let

$$\begin{array}{ccc} \bar{A}(\bar{y}, u^*(-)) & \xrightarrow{g'} & X \\ \alpha_{\bar{y}} \downarrow & & \downarrow f \\ A(u_*(\bar{y}), -) & \xrightarrow{g} & Z \end{array}$$

be a commutative diagram. Since $h \circ f$ is formally $\mathbb{A}$-smooth, there exists a commutative diagram

$$\begin{array}{ccc} \bar{A}(\bar{y}, u^*(-)) & \xrightarrow{g'} & X \\ \alpha_{\bar{y}} \downarrow & \nearrow_{\gamma} & \downarrow h \circ f \\ A(u_*(\bar{y}), -) & \xrightarrow{h \circ g} & Z \end{array}$$

Since h is formally $\mathbb{A}$-unramified, $g = f \circ \gamma$.

(c) The proof of the assertion (c) is left to the reader. ■

5.5. Corollary. *Let X, Y, Z be functors $A \longrightarrow$* **Sets**, *and let $X \xrightarrow{f} Y$, $Y \xrightarrow{h} Z$ be functor morphisms. Suppose h is formally $\mathbb{A}$-étale. Then $h \circ f$ is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified, resp. formally $\mathbb{A}$-étale) iff f belongs to the same class.*

5.6. Proposition. *Let $\mathbb{A}$ be a Q-category. Let X, Y, Y' be functors $A \longrightarrow$* **Sets** *and $Y' \xrightarrow{h} Y \xleftarrow{f} X$ functor morphisms. If $X \xrightarrow{f} Y$ is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-nonramified, resp. formally $\mathbb{A}$-étale), then the canonical projection $X \times_Y Y' \xrightarrow{f'} Y'$ is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified, resp. formally $\mathbb{A}$-étale).*

Proof. Set $X' = X \times_Y Y'$ and denote by h' the canonical projection $X' \longrightarrow X$. Consider a commutative diagram

$$\begin{tikzcd} \bar{A}(\bar{y}, u^*(-)) \arrow[r, "g'"] \arrow[d, "\alpha_{\bar{y}}"'] & X' \arrow[r, "h'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ A(u_*(\bar{y}), -) \arrow[r, "g"] & Y' \arrow[r, "h"] & Y \end{tikzcd} \qquad (1)$$

(a) Suppose f is formally $\mathbb{A}$-smooth. Then there exists a morphism $A(u_*(\bar{y}), -) \xrightarrow{\gamma} X$ such that the diagram

$$\begin{tikzcd} \bar{A}(\bar{y}, u^*(-)) \arrow[r, "h' \circ g'"] \arrow[d, "\alpha_{\bar{y}}"'] & X \arrow[d, "f"] \\ A(u_*(\bar{y}), -) \arrow[r, "h \circ g"] \arrow[ur, "\gamma"'] & Y \end{tikzcd} \qquad (2)$$

commutes. By the universal property of the fiber product, there exists a unique morphism $\bar{A}(\bar{y}, u^*(-)) \xrightarrow{\gamma'} X'$ such that the diagram

$$\begin{tikzcd} \bar{A}(\bar{y}, u^*(-)) \arrow[r, "g'"] \arrow[d, "\alpha_{\bar{y}}"'] & X' \arrow[d, "f'"] \\ A(u_*(\bar{y}), -) \arrow[r, "g"] \arrow[ur, "\gamma'"'] & Y' \end{tikzcd} \qquad (3)$$

commutes. This shows that the projection f' is formally smooth.

(b) Suppose the morphism $X \xrightarrow{f} Y$ is formally $\mathbb{A}$-unramified. And suppose that there exists a morphism $\bar{A}(u_*(\bar{y}), -) \xrightarrow{\gamma'} X'$ such that the diagram (3) commutes. This implies that the diagram (2), with $\gamma = h' \circ \gamma'$, is commutative. Since f is formally $\mathbb{A}$-unramified, the commutativity of (2) determines γ uniquely. By the universal property of fiber product, the morphisms γ' is uniquely determined by the morphisms γ and h, hence f' is formally $\mathbb{A}$-unramified.

(c) It follows from (a) and (b) that if f is formally $\mathbb{A}$-étale, then f' is formally $\mathbb{A}$-étale too. ∎

5.7. Definition. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category, and let R, S be objects of A. We call a morphism $R \xrightarrow{\phi} S$ *formally* $\mathbb{A}$*-smooth* (resp. *formally* $\mathbb{A}$*-unramified*, resp. *formally* $\mathbb{A}$*-étale*) iff the corresponding functor morphism, $A(\phi, -) : A(S, -) \longrightarrow A(R, -)$, has the respective property.

5.7.1. A special case. If the functor u^* has a left adjoint, $u_!$, one can define formally $\mathbb{A}$-smooth, formally $\mathbb{A}$-unramified and formally $\mathbb{A}$-étale morphisms of A in terms of the category A itself:

A morphism $R \xrightarrow{f} S$ of A is formally $\mathbb{A}$-smooth iff any $\bar{y} \in Ob\bar{A}$ and a commutative diagram of the form

$$\begin{tikzcd} u_!(\bar{y}) & S \arrow[l, "g'"'] \\ u_*(\bar{y}) \arrow[u, "\alpha'_{\bar{y}}"] & R \arrow[l, "g"'] \arrow[u, "f"'] \end{tikzcd} \qquad (1)$$

(where $\alpha'_{\bar{y}}$ is the canonical functorial morphism) extends to a commutative diagram

$$\begin{array}{ccc} u_!(\bar{y})) & \xleftarrow{g'} & S \\ {\scriptstyle \alpha'_{\bar{y}}} \uparrow & \swarrow_{\gamma} & \uparrow f \\ u_*(\bar{y}) & \xleftarrow{g} & R \end{array} \tag{2}$$

A morphism $R \xrightarrow{f} S$ is formally $\mathbb{A}$-unramified iff for any commutative diagram of the form (1) there exists at most one morphism $S \xrightarrow{\gamma} u_*(\bar{y})$ such that the diagram (2) commutes.

5.8. Formally smooth and formally étale morphisms of algebras. If $\mathbb{A}$ is the Q-category Alg_k^{inf} of 4.2, we call formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified, resp. formally $\mathbb{A}$-étale) morphisms simply *formally smooth* (resp. *formally unramified*, resp. *formally étale*). We have the following relative analogue of Proposition 4.6.2:

5.8.1. Proposition. *Let R, S be associative k-algebras, and let $R \xrightarrow{\phi} S$ be a k-algebra morphism.*

(a) The morphism ϕ is formally unramified iff the morphism $S \otimes_R S \longrightarrow S$, $s \otimes t \longmapsto st$, is an isomorphism, or, equivalently, $\Omega^1_{S|R} = Ker(S \otimes_R S \longrightarrow S) = 0$.

(b) Suppose the k-algebra R is separable. Then the morphism $R \xrightarrow{\phi} S$ is formally smooth iff $\Omega^1_{S|R}$ is a projective left S^e-module.

Proof. A standard argument shows that $R \xrightarrow{\phi} S$ is formally smooth (resp. formally unramified) iff for any R-ring epimorphism $T \xrightarrow{\alpha} S$ such that $Ker(\alpha)^2 = 0$, there exists an R-ring morphism (resp. at most one R-ring morphism) $S \xrightarrow{\beta} T$ such that $\alpha \circ \beta = id_S$.

(a) Let $S \xrightarrow{\beta} T$ be an R-ring morphism such that $\alpha \circ \beta = id_S$. It gives a decomposition of T into a semidirect product of S and an S-bimodule, M, with multiplication defined by $(s, m)(s', m') = (ss', s \cdot m' + m \cdot s')$. Any other right inverse to α, is of the form (id_S, D), where $S \xrightarrow{D} M$ is a derivation sending R to zero. The latter means precisely that D is an R^e-module morphism, $R^e = R \otimes_k R^o$. Thus, the set of splittings of α is in one-to-one correspondence with the set $Der_{S|R}(M)$ of derivations $S \xrightarrow{D} M$ which are R^e-module morphisms. But $Der_{S|R}(M)$ is naturally isomorphic to $Hom_{S^e}(\Omega^1_{S|R}, M)$. Hence ϕ is unramified iff $\Omega^1_{S|R} = 0$.

(b) Suppose the k-algebra R is separable, i.e. R is a projective R^e-module. Then the S^e-bimodule $S \otimes_R S$ is projective.

In fact, for any S^e-module M, there is a functorial isomorphism $Hom_{S^e}(S \otimes_R S, M) \simeq Hom_{R^e}(R, \phi_*(M))$. Here ϕ_* is the pull-back functor $S^e - mod \longrightarrow R^e - mod$ induced by the morphism ϕ. Since R is a projective R^e-module and the functor ϕ_* is exact, the functor $M \longmapsto Hom_{R^e}(R, \phi_*(M))$ is exact. Therefore the functor $M \longmapsto Hom_{S^e}(S \otimes_R S, M)$ is exact, i.e. $S \otimes_R S$ is a projective S^e-module.

By 4.6.1, the algebra R is separable iff it is $\mathfrak{A}ss^{inf}$-étale. The latter means that the morphism $k \longrightarrow R$ is formally $\mathfrak{A}ss^{inf}$-étale. If follows from 5.4(ii), that the morphism

$R \xrightarrow{\phi} S$ is formally $\mathfrak{Ass}^{inf}$-smooth iff the composition of $k \longrightarrow R$ and ϕ is formally $\mathfrak{Ass}^{inf}$-smooth, i.e. the k-algebra S is formally $\mathfrak{Ass}^{inf}$-smooth. By 4.3 and 4.6.1, the k-algebra S is formally $\mathfrak{Ass}^{inf}$-smooth iff it is formally smooth (i.e. Alg_k^{inf}-smooth). On the other hand, the algebra S is formally smooth iff $Ext^2_{S^e}(S, M) = 0$ for any S^e-module M. Consider the long exact sequence

$$... \longrightarrow Ext^i_{S^e}(S, M) \longrightarrow Ext^i_{S^e}(S \otimes_R S, M) \longrightarrow Ext^i_{S^e}(\Omega^1_{S|R}, M) \longrightarrow Ext^{i+1}_{S^e}(S, M) \longrightarrow ... \quad (1)$$

corresponding to the short exact sequence $0 \longrightarrow \Omega^1_{S|R} \longrightarrow S \otimes_R S \longrightarrow S \longrightarrow 0$. Since $S \otimes_R S$ is a projective S^e-module, $Ext^i_{S^e}(S \otimes_R S, M) = 0$ for all $i \geq 1$ and all S^e-modules M. Therefore $Ext^i_{S^e}(\Omega^1_{S|R}, M) \simeq Ext^{i+1}_{S^e}(S, M)$ for all $i \geq 1$ and all S^e-modules M. In particular, $Ext^2_{S^e}(S, M) = 0$ for all M iff $Ext^1_{S^e}(\Omega^1_{S|R}, M) = 0$ for all M. The latter means precisely that $\Omega^1_{S|R}$ is a projective S^e-module. ■

5.8.1.1. Corollary. *Suppose R is a separable k-algebra. Then a k-algebra morphism $R \xrightarrow{\phi} S$ is formally unramified iff it is formally étale.*

Proof. By 5.8.1(a), $R \xrightarrow{\phi} S$ is unramified iff $\Omega^1_{S|R} = 0$. By 5.8.1(b), ϕ is formally smooth iff $\Omega^1_{S|R}$ is a projective S^e-module. In particular, ϕ is formally smooth (hence étale), if $\Omega^1_{S|R} = 0$. ■

5.8.2. Proposition. *Let R, S be associative k-algebras, and let $R \xrightarrow{\phi} S$ be a k-algebra morphism. The following conditions are equivalent:*

(i) ϕ is formally unramified and flat.

(ii) ϕ is a flat monomorphism.

(iii) ϕ^* is an exact localization.

If the conditions above hold, then

(iv) ϕ is formally étale.

Proof. (ii)⇒(i), because every monomorphism is formally unramified.

(i)⇒(iii). By 5.8.1(a), the canonical morphism $S \otimes_R S \longrightarrow S$, $s \otimes t \longmapsto st$, is an isomorphism. Since $\phi^*\phi_* \simeq (S \otimes_R S) \otimes_S -$ and $Id_{S-mod} \simeq S \otimes_S -$, this means precisely that the adjunction morphism $\phi^*\phi_* \longrightarrow Id_{S-mod}$ is an isomorphism. The latter is equivalent to the full faithfulness of the direct image functor ϕ_*. By [GZ], Proposition I.1.3, ϕ^* is a localization.

(iii)⇒(ii) follows from the fact that any morphisms $R \xrightarrow{\phi} S$ such that its inverse image functor, ϕ^*, is a localization, is an algebra epimorphism.

In fact, let $S \overset{f_1}{\underset{f_2}{\rightrightarrows}} T$ be a pair of algebra morphisms such that $f_1 \circ \psi = f_2 \circ \phi$, i.e. we have the diagram of algebra morphisms over R:

$$\begin{tikzcd} S \arrow[rr, shift left, "f_1"] \arrow[rr, shift right, "f_2"'] & & T \\ & R \arrow[ul, "\phi"] \arrow[ur, "\gamma"'] & \end{tikzcd} \quad (1)$$

Here $\gamma = f_1 \circ \phi$. Applying to (1) first scalar restriction functor and then the functor $\phi_*\phi^*$, we obtain the diagram $\phi_*\phi^*(R) \longrightarrow \phi_*\phi^*\phi_*(S) \rightrightarrows \phi_*\phi^*\gamma_*(T)$ which is isomorphic to the diagram

$$\phi_*\phi^*(R) \longrightarrow \phi_*(S) \rightrightarrows \gamma_*(T), \tag{2}$$

because, due to the fact that ϕ^* is a localization, ϕ_* is a fully faithful functor, or, equivalently, $\phi^*\phi_* \simeq Id_{S-mod}$. Notice that the morphism $\phi_*\phi^*(R) \longrightarrow \phi_*(S)$ in (2) is an isomorphism. Since it equalizes the pair $\phi_*(S) \rightrightarrows \gamma_*(T)$, this pair is trivial. Hence the initial pair of morphisms is trivial: $f_1 = f_2$.

$\{$(iii),(i)$\}$ $\Rightarrow$(iv). It suffices to show that if $R \xrightarrow{\phi} S$ is an exact localization, then ϕ is formally smooth. A standard argument shows that a morphism $R \xrightarrow{\phi} S$ is smooth iff any R-ring strict epimorphism (i.e. a surjection) $T \xrightarrow{g} S$ such that the square of the kernel of g is zero, has right inverse. Denote the kernel of g by J. Thus we have an exact sequence of R-bimodules

$$0 \longrightarrow J \longrightarrow T \longrightarrow S \longrightarrow 0. \tag{3}$$

Denote by Φ^* the functor

$$R^e - mod \longrightarrow S^e - mod, \quad M \longmapsto S \otimes_R M \otimes_R S.$$

Notice that this functor is an exact localization having a (necessarily fully faithful) right adjoint, Φ_*. In particular, it maps the exact sequence (3) into exact sequence. Applying the functor Φ^* to the diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & T \arrow[rr] & & S \arrow[r] & 0 \\ & & & R \arrow[ul] \arrow[ur] & & \end{tikzcd} \tag{4}$$

we obtain the diagram

$$\begin{tikzcd} 0 \arrow[r] & \Phi^*(J) \arrow[r] & \Phi^*(T) \arrow[rr] & & \Phi^*(S) \arrow[r] & 0 \\ & & & \Phi^*(R) \arrow[ul] \arrow[ur] & & \end{tikzcd} \tag{5}$$

Since Φ^* is a localization, the natural morphism $S \longrightarrow \Phi_*\Phi^*(S)$ is an isomorphism, $\Phi^*(R) = S \otimes_R S \simeq S$, and the k-algebra morphism $\Phi^*(\phi) : \Phi^*(R) \longrightarrow \Phi^*(S)$ is an isomorphism.

Note that J is an S-bimodule. This implies that $\Phi_*\Phi^*(J) \simeq J$. Thus we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d] & T \arrow[rr] \arrow[d] & & S \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Phi_*\Phi^*(J) \arrow[r] & \Phi_*\Phi^*(T) \arrow[rr] & & \Phi_*\Phi^*(S) \arrow[r] & 0 \\ & & & S \arrow[ul] \arrow[ur] & & \end{tikzcd} \tag{6}$$

whose left and right vertical arrows are isomorphisms. Since both rows are exact sequences, it follows that the adjunction morphism $T \longrightarrow \Phi_*\Phi^*(T)$ is an isomorphism too, hence the assertion. ∎

5.9. Formally $\mathfrak{Ass}_k^{inf}$-unramified and formally $\mathfrak{Ass}_k^{inf}$-étale morphisms. The following assertion is a relative version of 4.6.1.

5.9.1. Proposition. *Let R, S be associative k-algebras, and let $R \xrightarrow{\phi} S$ be a k-algebra morphism.*

1) The following conditions are equivalent:

(i) The morphism ϕ is formally $\mathfrak{Ass}_k^{inf}$-unramified.

(ii) Any derivation $S \xrightarrow{D} M$ which is an R^e-module morphism is inner.

(iii) The canonical S^e-module epimorphism $S \otimes_R S \longrightarrow S$ has a right inverse.

2) Suppose that the k-algebra R is separable. Then

(a) The morphism $R \xrightarrow{\phi} S$ is formally $\mathfrak{Ass}_k^{inf}$-smooth iff $\Omega^1_{S|R}$ is a projective S-bimodule.

(b) The following conditions are equivalent:

(iv) The morphism ϕ is formally $\mathfrak{Ass}_k^{inf}$-unramified.

(v) ϕ is formally $\mathfrak{Ass}_k^{inf}$-étale.

(vi) S is a separable k-algebra (i.e. S is a projective S^e-module).

Proof. 1) (i)⇔(ii). Let T be a semidirect product of S and an S^e-bimodule M, and let $T \xrightarrow{\alpha} S$ the natural projection, $(s,z) \longmapsto s$. Any k-algebra morphism $S \longrightarrow T$ which is right inverse to α in category $\mathfrak{Ass}_k$ is conjugate to a k-algebra morphism, $S \xrightarrow{\beta} T$, which is right inverse to α in Alg_k. Any such morphism β is of the form $s \longmapsto (s, D(s))$, where $S \xrightarrow{D} M$ is an $S|R$-derivation. If $R \xrightarrow{\phi} S$ is $\mathfrak{Ass}_k^{inf}$-unramified, β is of the form $s \longmapsto usu^{-1}$. The argument of 4.6.1 shows that this (together with the equality $\alpha\circ\beta = id_S$) implies that D is an inner derivation.

Conversely, if the morphism $S \xrightarrow{\beta} T$ is given by $s \longmapsto (s, D(s))$, where D is an inner derivation, i.e. $D(s) = s\cdot z - z\cdot s$ for some element z of M and all $s \in S$, then $\beta(s) = usu^{-1}$, where $u = 1_S - z$).

(ii)⇒(iii). The functor $Der_{S|R} : S^e - mod \longrightarrow \mathbf{Sets}$, $M \longmapsto Der_{S|R}(M)$, is representable by the S^e-module $\Omega^1_{S|R} = Ker(S \otimes_R S \longrightarrow S)$. The canonical monomorphism $\Omega^1_{S|R} \xrightarrow{i_\phi} S \otimes_R S$ induces a map

$$Hom_{S^e}(S \otimes_R S, M) \longrightarrow Hom_{S^e}(\Omega^1_{S|R}, M) \tag{1}$$

Notice that $Hom_{S^e}(S \otimes_R S, M) \simeq Hom_{R^e}(R, \phi_*(M))$, and $Hom_{R^e}(R, \phi_*(M))$ is naturally isomorphic to the center, $\mathfrak{z}(\phi_*(M)) = \{v \in M |\ r\cdot v = v\cdot r \text{ for all } r \in R\}$, of the R^e-module $\phi_*(M)$. The composition of the bijection $\mathfrak{z}(\phi_*(M)) \longrightarrow Hom_{S^e}(S \otimes_R S, M)$ and the map (1) assigns to each central element, z, of $\phi_*(M)$ the corresponding inner derivation, $s \longmapsto s\cdot z - z\cdot s$. Thus, each derivation of $Der_{S|R}(M)$ is inner iff the map (1) is surjective. In the case $M = \Omega^1_{S|R}$, this implies the existence of an S^e-module morphism

$S\otimes_R S \xrightarrow{p} \Omega^1_{S|R}$ such that $p\circ i_\phi = id$. Or, equivalently, the canonical S^e-module morphism $S \otimes_R S \longrightarrow S$ has a right inverse.

The implication (iii)⇒(ii) follows from the argument above.

2) (a) A k-algebra morphism $R \xrightarrow{\phi} S$ is formally $\mathfrak{Ass}_k^{inf}$-smooth iff it is formally Alg_k^{inf}-smooth. By 5.8.1, if R is a separable k-algebra, then ϕ is formally Alg_k^{inf}-smooth iff $\Omega^1_{S|R}$ is a projective S^e-module.

(b) By the argument of 5.8.1, if R is a separable k-algebra, then the S^e-module $S \otimes_R S$ is projective. By 1), the morphism $R \xrightarrow{\phi} S$ is $\mathfrak{Ass}_k^{inf}$-unramified iff the S^e-module morphism $S \otimes_R S \longrightarrow S$ has a right inverse. Since $S \otimes_R S$ is projective, the latter implies that S is a projective S^e-module, hence (equivalently) $\Omega^1_{S|R}$ is a projective S^e-module, i.e. the morphism ϕ is formally $\mathfrak{Ass}_k^{inf}$-smooth. This proves the implications (iv)⇒(v)⇒(vi)⇒(v). The implication (v)⇒(iv) is true by definition. ∎

5.9.2. Corollary. *The following conditions on a k-algebra morphism $R \xrightarrow{\phi} S$ are equivalent:*

(a) ϕ is formally $\mathfrak{Ass}_k^{inf}$-étale.

(b) The adjunction morphism $\phi^*\phi_* \xrightarrow{\epsilon_\phi} Id_{S-mod}$ has a right inverse.

Proof. (a)⇒(b). By 5.9.1, the canonical S^e-module epimorphism $S \otimes_R S \xrightarrow{\mu} S$ has a right inverse, $S \xrightarrow{tau'} S \otimes_R S$. The morphism τ' defines a morphism, $Id_{S-mod} \xrightarrow{\tau} \phi^*\phi_*$. The equality $\mu\circ\tau = id_S$ implies that the composition of τ with the adjunction morphism, $\phi^*\phi_* \xrightarrow{\epsilon_\phi} Id_{S-mod}$ is the identity morphism.

(b)⇒(a). Conversely, any morphism, $Id_{S-mod} \xrightarrow{\tau'} \phi^*\phi_*$, is induced by an S^e-module morphism, $S \xrightarrow{\tau} S \otimes_R S$. The morphism τ' is a right inverse to the adjunction morphism $\phi^*\phi_* \xrightarrow{\epsilon_\phi} Id_{S-mod}$ iff the composition of the bimodule morphism τ with the canonical morphism $S \otimes_R S \longrightarrow S$ equals to id_S. ∎

5.10. Another description of the category $\mathfrak{Ass}_k$.

5.10.1. Proposition. *Two k-algebra morphisms, $R \overset{\phi}{\underset{\psi}{\rightrightarrows}} S$, are conjugate iff the corresponding inverse image functors, $R - mod \overset{\phi^*}{\underset{\psi^*}{\rightrightarrows}} S - mod$, are isomorphic.*

Proof. (a) Suppose that ψ and ϕ are conjugate, i.e. there exists an invertible element, t, of S such that $\psi(r) = t\phi(r)t^{-1}$ for all $r \in R$. For any R-module $\mathcal{M} = (M, m)$, we have a commutative diagram

$$\begin{array}{ccc} S\otimes_k M & \xrightarrow{\cdot t} & S\otimes_k M \\ {\scriptstyle \gamma_\psi}\downarrow & & \downarrow{\scriptstyle \gamma_\phi} \\ S\otimes_{R,\psi} M & \xrightarrow{\lambda_t} & S\otimes_{R,\phi} M \end{array} \tag{1}$$

Here $\cdot t$ denotes the S-module morphism $s\otimes z \longmapsto st\otimes z$ for all $s\in S,\ z\in M$; $\gamma_\psi,\ \gamma_\phi$ are canonical epimorphisms.

In fact, for any $s \in S,\ r \in R,\ z \in M$, $\gamma_\psi(s\psi(r) \otimes z) = \gamma_\psi(s \otimes r \cdot z)$, and $\cdot t(s \otimes r \cdot z) = st \otimes r \cdot z$.

On the other hand, $\cdot t(s\psi(r) \otimes z) = s\psi(r)t \otimes z = st\phi(r) \otimes z$, and $\gamma_\phi(st\phi(r) \otimes z) = \gamma_\phi(st \otimes r \cdot z)$. Since γ_ψ is by definition the cokernel of two maps

$$S \otimes_k R \otimes_k M \underset{\psi_r}{\overset{\psi_l}{\rightrightarrows}} S \otimes_k M, \quad s \otimes r \otimes z \overset{\psi_l}{\longmapsto} s\psi(r) \otimes z, \text{ and } s \otimes r \otimes z \overset{\psi_r}{\longmapsto} s \otimes r \cdot z,$$

it follows the existence of a (necessarily unique) morphism $S \otimes_{R,\psi} M \xrightarrow{\lambda_t} S \otimes_{R,\phi} M$ such that the diagram (1) commutes; i.e. λ_t is given by $\gamma_\psi(s \otimes z) \longmapsto \gamma_\phi(st \otimes z)$.

(b) Conversely, suppose $\phi,\ \psi$ are k-algebra morphisms such that there is a functorial isomorphism $u : \psi^* \xrightarrow{\sim} \phi^*$. Identifying both $\phi^*(R)$ and $\psi^*(R)$ with the left S-module S, we obtain, in particular, an S-module morphism $u(R) : S \longrightarrow S$. Since S is a unital ring, $u(R)$ equals to $s \overset{\cdot t}{\longmapsto} st$ for some $t \in S$. Since u is a functor morphism, for any $r \in R$, $u(R) \circ \psi^*(\cdot r) = \phi^*(\cdot r) \circ u(R)$. This means that $s\psi(r)t = st\phi(r)$ for any $s \in S$, hence $\psi(r) = t\phi(r)t^{-1}$. ∎

5.10.2. Corollary. *The category $\mathfrak{Ass}_k$ is isomorphic to the category whose objects are associative k-algebras; morphisms are equivalence classes of k-algebra morphisms with respect to the following relation: two k-algebra morphisms $\phi, \psi : R \longrightarrow S$ are equivalent iff the inverse image functors $\phi^*, \psi^* : R - mod \longrightarrow S - mod$ are isomorphic.*

5.11. Formally $\mathbb{A}$-open immersions. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category, $X,\ Y$ functors $A \longrightarrow$ **Sets**. We call a morphism $X \xrightarrow{f} Y$ a *formally $\mathbb{A}$-open immersion* if it is a formally $\mathbb{A}$-smooth monomorphism, or, equivalently, a formally $\mathbb{A}$-étale monomorphism.

5.11.1. Proposition. *(a) The composition of formally $\mathbb{A}$-open immersions is a formally $\mathbb{A}$-open immersion.*

(b) Let $X \longrightarrow Y$ be a formally $\mathbb{A}$-open immersion. For any morphism $T \longrightarrow Y$, the canonical projection $T \times_Y X \longrightarrow T$ is a formally $\mathbb{A}$-open immersion.

(c) Let $S \in ObA^\vee$, and let $(X, \xi) \xrightarrow{f} (Y, \mu),\ (X', \xi') \xrightarrow{f'} (Y', \mu')$ be morphisms of objects over S. The morphisms $f,\ f'$ are formally $\mathbb{A}$-open immersions iff the morphism $f \times_S f' : X \times_S X' \longrightarrow Y \times_S Y'$ is a formally $\mathbb{A}$-open immersion.

Proof. The assertions (a), (b), (c) follow from the fact that both monomorphisms and formally $\mathbb{A}$-étale morphisms are stable under composition, base change, and fiber product (see 5.4, 5.6). ∎

5.11.2. Proposition. *Let $X \underset{f_2}{\overset{f_1}{\rightrightarrows}} Y \xrightarrow{h} Z$ be a diagram of presheaves of sets on A^{op} (i.e. functors $A \longrightarrow$ **Sets**) such that $h \circ f_1 = h \circ f_2$. If the morphism h is $\mathbb{A}$-unramified, then the morphism $Ker(f_1, f_2) \xrightarrow{i} X$ is a formally $\mathbb{A}$-open immersion.*

Proof. Consider the diagram

$$\begin{array}{ccccccc}
\bar{A}(\bar{y}, u^*(-)) & \xrightarrow{g'} & Ker(f_1, f_2) & & & & \\
\alpha_{\bar{y}} \big\downarrow & & \big\downarrow i & & & & \\
A(u_*(\bar{y}), -) & \xrightarrow{g} & X & \overset{f_1}{\underset{f_2}{\rightrightarrows}} & Y & \xrightarrow{h} & Z
\end{array} \qquad (1)$$

with commutative square. The diagram

$$\begin{array}{ccc}
\bar{A}(\bar{y}, u^*(-)) & \xrightarrow{f_1 \circ i \circ g'} & Y \\
\alpha_{\bar{y}} \big\downarrow & & \big\downarrow h \\
A(u_*(\bar{y}), -) & \xrightarrow{h \circ f_1 \circ g} & Z
\end{array}$$

commutes; and since h is unramified, there is at most one morphism $A(u_*(\bar{y}), -) \xrightarrow{\gamma} Y$ such that $h \circ \gamma = h \circ f_1 \circ g$. Since $h \circ f_1 = h \circ f_2$, this uniqueness implies that $f_1 \circ g = f_2 \circ g$. Therefore, there exists a unique morphism $A(u_*(\bar{y}), -) \xrightarrow{\lambda} X$ such that $g = i \circ \lambda$. This shows that the monomorphism i is formally $\mathbb{A}$-smooth, hence the assertion. ■

5.11.3. Corollary. *Any section of a formally $\mathbb{A}$-unramified morphism is a formally $\mathbb{A}$-open immersion.*

Proof. Let $Y \xrightarrow{s} X$ be a section of a formally $\mathbb{A}$-unramified morphism $X \xrightarrow{f} Y$. Then the morphism s induces an isomorphism $Y \longrightarrow Ker(sf, id_X)$. The assertion follows now from 5.11.2. ■

5.11.4. Formally open immersions of affine schemes.

5.11.4.1. Proposition. *Let $\mathbb{A}$ be the Q-category Alg_k^{inf} of 4.2. Then the following conditions on a flat k-algebra morphism $R \xrightarrow{\phi} S$ are equivalent:*

(i) ϕ is formally $\mathbb{A}$-unramified.

(ii) ϕ is formally $\mathbb{A}$-étale.

(iii) ϕ^* is a localization.

(iv) ϕ is a formally open $\mathbb{A}$-immersion.

Proof. The equivalence of the first three conditions is the content of 5.8.2. By the definition of a formally $\mathbb{A}$-open immersion, $(iv) \Rightarrow (ii)$.

$\{(iii), (ii)\} \Rightarrow (iv)$. By definition, $R \xrightarrow{\phi} S$ is a formally open $\mathbb{A}$-immersion iff the morphism of functors $Alg_k(\phi, -) : Alg_k(S, -) \longrightarrow Alg_k(R, -)$ is a formally étale monomorphism, or, equivalently, ϕ is a formally étale algebra epimorphism. It is étale by (ii). And $\phi^* \simeq S \otimes_R -$ being a localization, implies that ϕ is an epimorphism (see the argument of 5.8.2, $(iii) \Rightarrow (i)$). ■

5.11.4.2. Proposition. *A flat k-algebra morphism $R \xrightarrow{\phi} S$ is a formally $\mathfrak{Ass}_k^{inf}$-open immersion iff the functor ϕ^* is a localization.*

Proof. Suppose ϕ^* is a localization. Then the adjunction morphism $\phi^*\phi_* \longrightarrow Id_{S-mod}$ is an isomorphism. Therefore, by 5.9.2, ϕ is formally $\mathfrak{A}ss_k^{inf}$-étale. Let $S \underset{f_2}{\overset{f_1}{\rightrightarrows}} T$ be a pair of algebra morphisms such that $f_1 \circ \phi$ is equivalent to $f_2 \circ \phi$ (i.e. both define the same morphism in $\mathfrak{A}ss_k$). Then, by 5.10.1, $(f_1 \circ \phi)^* \simeq (f_2 \circ \phi)^*$, or, equivalently, $\phi_* \circ f_{1*} \simeq \phi_* \circ f_{2*}$. Since ϕ_* is a fully faithful functor, this implies that $f_{1*} \simeq f_{2*}$, hence f_1 and f_2 define the same morphism in $\mathfrak{A}ss_k$.

Conversely, suppose ϕ is a formally $\mathfrak{A}ss_k^{inf}$-open immersion. In particular, it is a formally $\mathfrak{A}ss_k^{inf}$-smooth morphism. It follows from 4.3 and 4.6.1 that formal $\mathfrak{A}ss_k^{inf}$-smoothness is the same as formal Ass_k^{inf}-smoothness. ∎

5.12. Smooth, étale, and unramified morphisms. Open immersions.

5.12.1. Finitely presentable and locally finitely presentable morphism. Let C be a category, $C^\wedge$ the category of presheaves of sets on C (i.e. functors $C^{op} \longrightarrow$ **Sets**). Fix an object, Y, of $C^\wedge$. We call a morphism $X \xrightarrow{f} Y$ *locally finitely presentable* (resp. *locally of finite type*) if for any filtered projective system $D \xrightarrow{\mathfrak{D}} C/Y$, the canonical map

$$\operatorname{colim} C^\wedge/Y(\mathfrak{D}, (X, f)) \longrightarrow C^\wedge/Y(\lim \mathfrak{D}, (X, f)) \qquad (1)$$

is bijective (resp. injective). Here C/Y denotes the full subcategory of the category $C^\wedge/Y$ whose objects are pairs $(V, V \to Y)$ with representable V.

A morphism $X \xrightarrow{f} Y$ is called *finitely presentable* (resp. *of finite type*) if for any filtered projective system $D \xrightarrow{\mathfrak{D}} C^\wedge/Y$, the canonical map

$$\operatorname{colim} C^\wedge/Y(\mathfrak{D}, (X, f)) \longrightarrow C^\wedge/Y(\lim \mathfrak{D}, (X, f)) \qquad (2)$$

is bijective (resp. injective).

In order to avoid repetitions, we introduce intermediate notions. Fix a full subcategory $\mathcal{E}$ of the category $C^\wedge$ containing all representable functors. Let $X,\ Y$ be objects of $C^\wedge$. We call a morphism $X \xrightarrow{f} Y$ of $\mathcal{E}$-finite type (resp. $\mathcal{E}$-finitely presentable) if for any filtered projective system $D \xrightarrow{\mathfrak{D}} \mathcal{E}/Y$, the canonical map

$$\operatorname{colim} C^\wedge/Y(\mathfrak{D}, (X, f)) \longrightarrow C^\wedge/Y(\lim \mathfrak{D}, (X, f))$$

is injective (resp. bijective).

5.12.2. Proposition. *Let $\Sigma^1_{\mathcal{E}}$ (resp. $\Sigma^0_{\mathcal{E}}$) denote the class of all $\mathcal{E}$-finitely presentable (resp. of $\mathcal{E}$-finite type) morphisms of the category $C^\wedge$.*

(a) Both $\Sigma^0_{\mathcal{E}}$ and $\Sigma^1_{\mathcal{E}}$ are closed under composition and contain all isomorphisms.

(b) If the morphism f in the cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

belongs to $\Sigma^i_{\mathcal{E}}$, then f' belongs to $\Sigma^i_{\mathcal{E}}$, $i = 0, 1$.

(c) Suppose that $X \xrightarrow{f} Y$ and $Z \xrightarrow{h} W$ are morphisms over S which belong to $\Sigma^i_{\mathcal{E}}$. Then $X \times_S Z \xrightarrow{f \times_S h} Y \times_S W$ belongs to $\Sigma^i_{\mathcal{E}}$, $i = 0, 1$.

(d) If the composition $g \circ f$ of two morphisms is $\mathcal{E}$-finitely presentable and g is of $\mathcal{E}$-finite type, then f is $\mathcal{E}$-finitely presentable.

Proof. (a) Let morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ belong to $\Sigma^i_{\mathcal{E}}$. We claim that the composition $X \xrightarrow{gf} Z$ belongs to the same class; i.e. for any filtered projective system $D \xrightarrow{\mathfrak{D}} \mathcal{E}/Z$, the canonical map

$$\operatorname{colim} C^\wedge/Z(\mathfrak{D}, (X, gf)) \longrightarrow C^\wedge/Z(\lim \mathfrak{D}, (X, gf))$$

is injective if $i = 0$ and bijective if $i = 1$. First, consider the case $i = 0$.

Let (u_ν) and (u'_ν) be two inductive systems of arrows $\mathfrak{D}(\nu) \longrightarrow (X, gf)$, $\nu \in ObD$, (i.e. $gfu_\nu = gfu'_\nu$ for all ν) such that the compositions of u_ν and u'_ν with the canonical morphism $\lim \mathfrak{D} \xrightarrow{p_\nu} \mathfrak{D}(\nu)$ are equal. With more reason, $(fu_\nu)p_\nu = (fu'_\nu)p_\nu$. Since $Y \xrightarrow{g} Z$ is of $\mathcal{E}$-finite type, $fu_\mu = fu'_\mu$ for an appropriate μ. Replacing $\mathfrak{D}$ by the composition, $\mathfrak{D}_\mu$, of $\mathfrak{D}$ with the canonical functor $\mu\backslash D \longrightarrow D$, we can regard (u_ν) and (u'_ν) as inductive systems of arrows $\mathfrak{D}_\mu(\nu) \longrightarrow (X, f)$, $\nu \in Ob\mu\backslash D$ which equalize the canonical morphism $\lim \mathfrak{D} = \lim \mathfrak{D}_\mu \xrightarrow{p_\nu} \mathfrak{D}_\mu(\nu)$. Since $X \xrightarrow{f} Y$ belongs to $\Sigma^0_{\mathcal{E}}$, there exists λ such that $u_\lambda = u'_\lambda$; i.e. the systems (u_ν) and (u'_ν) define the same element of $\operatorname{colim} C^\wedge/Z(\mathfrak{D}, (X, gf))$.

Suppose now that the morphisms f and g belong to $\Sigma^1_{\mathcal{E}}$. Let $D \xrightarrow{\mathfrak{D}} \mathcal{E}/Z$ be a filtered projective system, and let $\lim \mathfrak{D} \xrightarrow{h} (X, gf)$ be an arbitrary morphism. Consider the morphism $\lim \mathfrak{D} \xrightarrow{fh} (Y, g)$. Since $Y \xrightarrow{g} Z$ belongs to $\Sigma^1_{\mathcal{E}}$, there exists a unique element, $\mathfrak{u}$, of $\operatorname{colim} C^\wedge/Z(\mathfrak{D}, (Y, g))$ whose image in $C^\wedge/Z(\lim \mathfrak{D}, (Y, g))$ coincides with fh. Let (u_ν) be an inductive system of arrows $\{\mathfrak{D}_\mu(\nu) \longrightarrow (Y, g)\}$ representing the element $\mathfrak{u}$; i.e. the diagrams

$$\begin{array}{ccc} \lim \mathfrak{D}_\mu & \xrightarrow{h} & X \\ {\scriptstyle p_\nu}\downarrow & & \downarrow{\scriptstyle f} \\ \mathfrak{D}_\mu(\nu) & \xrightarrow{u_\nu} & Y \end{array}$$

commute (here $\mathfrak{D}_\mu$ has the same sense as above). Since $X \xrightarrow{f} Y$ belongs to $\Sigma^1_{\mathcal{E}}$, there is a unique element of $\operatorname{colim} C^\wedge/Y(\mathfrak{D}_\mu, (X, f))$ whose image in $C^\wedge/Y(\lim \mathfrak{D}_\mu, (X, f))$ is given by h. Here we use that $\lim \mathfrak{D}_\mu = \lim \mathfrak{D}$.

(b) Suppose that $X \xrightarrow{f} Y$ belongs to $\Sigma^1_{\mathcal{E}}$. Let $D \xrightarrow{\mathfrak{D}} \mathcal{E}/Y'$ be a filtered projective system, and let $\lim \mathfrak{D} = (V, V \xrightarrow{v} Y')$. Fix a morphism $(V, v) \xrightarrow{h} (X', f')$. Since f belongs to $\Sigma^1_{\mathcal{E}}$, the morphism $(V, gv) \xrightarrow{g'h} (X, f)$ is the image of a unique element, $\mathfrak{u}$, of $\operatorname{colim} C^\wedge/Y(g_*\mathfrak{D}, (X, f))$. Here $g_*\mathfrak{D} : D \longrightarrow C^\wedge/Y$ is the diagram obtained by composition $\mathfrak{D}$ with g. Let (u_ν) be an inductive system of arrows $\{g_*\mathfrak{D}(\nu) \longrightarrow (X, f)\}$ representing the element $\mathfrak{u}$. Then the diagrams

$$\begin{array}{ccc}
(V, gv) & \xrightarrow{h} & (X', fg') \\
p_\nu \downarrow & & \downarrow g' \\
g_*\mathfrak{D}(\nu) & \xrightarrow{u_\nu} & (X, f)
\end{array}$$

commute. By the universal property of cartesian squares, there exists a unique morphism $\mathfrak{D}(\nu) \xrightarrow{u'_\nu} (X', f')$ such that $u_\nu = g'u'_\nu$.

The proof of remaining assertions follows a similar routine. We leave the arguments to the reader. ■

5.12.3. Definitions. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category. Let X, Y be presheaves of sets on A^{op} (i.e. functors $A \longrightarrow$ **Sets**). We call a morphism $X \xrightarrow{f} Y$ *smooth* (resp. *étale*, resp. *unramified*) if it is locally finitely representable and formally smooth (resp. formally étale, resp. formally unramified).

We call an $\mathbb{A}$-smooth monomorphism an $\mathbb{A}$-*open immersion.*

6. Formally smooth and formally infinitesimal morphisms. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category. We call a morphism $U \xrightarrow{\phi} T$ of presheaves of sets on A^{op} *formally* $\mathbb{A}$-*infinitesimal* if for any formally $\mathbb{A}$-smooth morphism $X \xrightarrow{f} Y$ and for any commutative diagram

$$\begin{array}{ccc}
U & \xrightarrow{g'} & X \\
\phi \downarrow & & \downarrow f \\
T & \xrightarrow{g} & Y
\end{array} \qquad (1)$$

there exists a morphism $T \xrightarrow{\gamma} X$ such that the diagram

$$\begin{array}{ccc}
U & \xrightarrow{g'} & X \\
\phi \downarrow & \nearrow_{\gamma} & \downarrow f \\
T & \xrightarrow{g} & Y
\end{array} \qquad (2)$$

commutes.

6.1. Proposition. *(a) Any split monomorphism (in particular, any isomorphism) is formally* $\mathbb{A}$*-infinitesimal.*

(b) The composition of formally $\mathbb{A}$*-infinitesimal morphisms is formally* $\mathbb{A}$*-infinitesimal.*

(c) Let $U \xrightarrow{\phi} T$ *be a formally* $\mathbb{A}$*-infinitesimal morphism and* $U \xrightarrow{\psi} V$ *any morphism. Then the canonical morphism* $V \longrightarrow V \coprod_U V$ *is formally* $\mathbb{A}$*-infinitesimal.*

Proof. (a) Obvious.

(b) Let $U \xrightarrow{\phi} T$ and $T \xrightarrow{\psi} S$ be formally $\mathbb{A}$-infinitesimal morphisms. And let

$$\begin{tikzcd} U \arrow[r, "g'"] \arrow[d, "\psi\phi"'] & X \arrow[d, "f"] \\ S \arrow[r, "g"] & Y \end{tikzcd}$$

be a commutative diagram with f formally $\mathbb{A}$-smooth. Since ϕ is formally $\mathbb{A}$-infinitesimal, there exists a morphism $T \xrightarrow{\gamma} X$ such that the diagram

$$\begin{tikzcd} U \arrow[r, "g'"] \arrow[d, "\phi"'] & X \arrow[d, "f"] \\ T \arrow[r, "g\psi"] \arrow[ur, "\gamma"'] & Y \end{tikzcd}$$

commutes. Since ψ is formally $\mathbb{A}$-infinitesimal, there exists a morphism $S \xrightarrow{\gamma'} X$ such that the diagram

$$\begin{tikzcd} T \arrow[r, "\gamma"] \arrow[d, "\psi"'] & X \arrow[d, "f"] \\ S \arrow[r, "g"] \arrow[ur, "\gamma'"'] & Y \end{tikzcd}$$

commutes.

(c) Consider a commutative diagram

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d, "\phi"'] & V \arrow[r, "g'"] \arrow[d, "p_1"] & X \arrow[d, "f"] \\ T \arrow[r, "p_2"] & T\coprod_U V \arrow[r, "g"] & Y \end{tikzcd}$$

in which $p_1,\ p_2$ are canonical projections. Since ϕ is formally $\mathbb{A}$-infinitesimal, there exists a morphism $T \xrightarrow{\gamma} X$ such that the diagram

$$\begin{tikzcd} U \arrow[r, "g'\psi"] \arrow[d, "\phi"'] & X \arrow[d, "f"] \\ T \arrow[r, "gp_2"] \arrow[ur, "\gamma"'] & Y \end{tikzcd}$$

commutes. Due to the universal properties of fiber coproducts, the commutative diagram

$$\begin{tikzcd} U \arrow[r, "\psi"] \arrow[d, "\phi"'] & V \arrow[d, "g'"] \\ T \arrow[r, "\gamma"] & X \end{tikzcd}$$

determines uniquely a morphism $T\coprod_U V \xrightarrow{\gamma'} X$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{g'} & X \\ \phi \downarrow & & \downarrow f \\ T\coprod_U V & \xrightarrow{g} & Y \end{array}$$

commutes. ∎

6.2. The Q-category $\mathbb{A}^\vee_{\mathbb{A}}$. We denote by $\mathbb{A}^\vee_{\mathbb{A}}$ the full Q-subcategory, $(A^\vee_{\mathbb{A}} \rightleftarrows A^\vee)$, of the Q-category $((A^\vee)^2 \rightleftarrows A^\vee)$ of morphisms of $A^\vee$ (cf. 2.5), where $A^\vee_{\mathbb{A}}$ denotes the full subcategory of $(A^\vee)^2$ whose objects are formally $\mathbb{A}$-infinitesimal morphisms. It follows from 6.1 that $\mathbb{A}^\vee_{\mathbb{A}}$ is a Q-category corresponding to a pretopology.

6.3. General pattern. Fix a category C and a family, $\mathfrak{M}$, of morphisms of C containing all identical morphisms.

(i) We call a morphism $X \xrightarrow{f} Y$ in C *formally $\mathfrak{M}$-smooth* if any commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{g} & X \\ \phi \downarrow & & \downarrow f \\ S & \xrightarrow{g'} & Y \end{array} \qquad (1)$$

such that $\phi \in \mathfrak{M}$ extends to a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{g'} & X \\ \phi \downarrow & \nearrow_{\gamma} & \downarrow f \\ S & \xrightarrow{g} & Y \end{array} \qquad (2)$$

(ii) We call $X \xrightarrow{f} Y$ *formally $\mathfrak{M}$-unramified* if for any commutative diagram (1) such that $\phi \in \mathfrak{M}$, there exists at most one morphism $S \xrightarrow{\gamma} X$ such that the diagram (2) commutes.

(iii) We call $X \xrightarrow{f} Y$ *formally $\mathfrak{M}$-étale* if it is both formally $\mathfrak{M}$-smooth and formally $\mathfrak{M}$-unramified.

We denote by $\mathfrak{M}_{fsm}$ (resp. $\mathfrak{M}_{fnr}$, resp. $\mathfrak{M}_{fet}$) the class of all formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale) morphisms.

6.4. $\mathfrak{N}$-infinitesimal morphisms. On the other hand, given a class $\mathfrak{N}$ of morphisms of C, denote by $\mathfrak{N}_{inf}$ the class of all morphisms $T \xrightarrow{\phi} S$ of C such that any commutative diagram (1) such that $X \xrightarrow{f} Y$ belongs to $\mathfrak{N}$ extends to a diagram (2). Morphisms of $\mathfrak{N}_{inf}$ will be called *$\mathfrak{N}$-infinitesimal morphisms.*

6.5. Remarks. (a) Given a Q-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$, take as C the category $A^\vee$ of functors $A \longrightarrow$ **Sets**, and set $\mathfrak{M} = \mathfrak{M}_{\mathbb{A}}$ to be the family of morphisms $\{\bar{A}(\bar{y}, u^*(-)) \xrightarrow{\alpha_{\bar{y}}}$

$A(u_*(\bar{y}), -) \mid \bar{y} \in Ob\bar{A}\}$. It follows from definitions that a morphism $X \longrightarrow Y$ is formally $\mathbb{A}$-smooth (resp. formally $\mathbb{A}$-unramified) iff it is formally $\mathfrak{M}_{\mathbb{A}}$-smooth (resp. formally $\mathfrak{M}_{\mathbb{A}}$-unramified).

Similarly, the notion of an $\mathbb{A}$-infinitesimal morphism is a special case of a $\mathfrak{N}$-infinitesimal morphism for an obvious choice of the category C and the family of morphisms $\mathfrak{N}$; namely, $C = A^\vee$, and $\mathfrak{N}$ is the class of formally $\mathbb{A}$-smooth morphisms.

(b) The main reason to introduce this setting here is the natural "duality"

$$\mathfrak{M} \longmapsto \mathfrak{M}_{fsm}, \quad \mathfrak{N} \longmapsto \mathfrak{N}_{inf}.$$

If follows from definitions that $\mathfrak{M} \subseteq \mathfrak{N}_{inf}$ iff $\mathfrak{N} \subseteq \mathfrak{M}_{fsm}$. In a more symmetric way, the latter relations can be expressed as follows:

Any commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{g} & X \\ \phi \big\downarrow & & \big\downarrow f \\ S & \xrightarrow{g'} & Y \end{array}$$

such that $\phi \in \mathfrak{M}$ and $f \in \mathfrak{N}$ extends to a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{g'} & X \\ \phi \big\downarrow & \nearrow_{\gamma} & \big\downarrow f \\ S & \xrightarrow{g} & Y \end{array}$$

6.6. Proposition. *Let $\mathfrak{M}$ be a family of arrows of a category C.*

(a) The class $\mathfrak{M}_{fsm}$ (resp. $\mathfrak{M}_{fnr}$, resp. $\mathfrak{M}_{fet}$) of formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale) morphisms is closed under composition and contains all isomorphisms of the category C.

(b) Let $X \xrightarrow{f} Y$, $Y \xrightarrow{h} Z$ be morphisms of C.

(i) If $h \circ f$ is formally $\mathfrak{M}$-unramified, then f is formally $\mathfrak{M}$-unramified.

(ii) Suppose h is formally $\mathfrak{M}$-unramified. If $X \xrightarrow{h \circ f} Z$ is formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-étale), then f is formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-étale).

(c) Let $X \xrightarrow{\xi} S \xleftarrow{\xi'} X'$ and $Y \xrightarrow{\nu} S \xleftarrow{\nu'} Y'$ be morphisms such that there exist $X \times_S X'$ and $Y \times_S Y'$. Let $(X, \xi) \xrightarrow{f} (Y, \nu)$ and $(X', \xi') \xrightarrow{f'} (Y', \nu)$ be morphisms of objects over S. The morphisms f, f' are formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale) iff the morphism $f \times_S f' : X \times_S X' \longrightarrow Y \times_S Y'$ has the respective property.

(d) Let $X \xrightarrow{f} Y \xleftarrow{h} S$ be such a diagram that there exists a fiber product $X \times_S Y$. If f is formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally étale), then the canonical projection $X \times_S T \xrightarrow{f'} T$ is formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale).

Proof. The argument follows the lines of the proofs of 5.4 and 5.6. ∎

6.6.1. Corollary. *Let $X \xrightarrow{f} Y$, $Y \xrightarrow{h} Z$ be morphisms of C. Suppose h is formally $\mathfrak{M}$-étale. Then $h \circ f$ is formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale) iff f belongs to the same class.*

6.7. Proposition. *Let $\mathfrak{N}$ be a family of arrows of C.*
(a) Any split monomorphism (in particular, any isomorphism) belongs to $\mathfrak{N}_{inf}$.
(b) The class $\mathfrak{N}_{inf}$ of $\mathfrak{N}$-infinitesimal morphisms is closed under composition.

(c) Let $T \xleftarrow{\phi} U \xrightarrow{\psi} S$ be morphisms such that there exists $T \coprod_U S$. If ϕ belongs to $\mathfrak{N}_{inf}$, then the canonical morphism $S \longrightarrow T \coprod_U S$ belongs to $\mathfrak{N}_{inf}$.

Proof. The argument is similar to the proof of 6.1. ∎

6.8. Example: separated, universally closed, and proper morphisms. Let A be the category $CAlg_k$ of commutative k-algebras. Let $\bar{A}$ be the category of faithfully flat k-algebra morphisms, $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ the corresponding full Q-subcategory of $(A^2 \rightleftarrows A)$. *Spaces* in the sense of Grothendieck, in particular schemes and algebraic spaces, are sheaves of sets on the Q-category $\mathbb{A}$.

Let $\mathfrak{M}'_v$ be the family of canonical injections of valuation rings to their fields of fractions; and let $\mathfrak{M}_v$ denote the image of $\mathfrak{M}'_v$ in the category $A^\vee$ of functors $A \longrightarrow$ **Sets**.

6.8.1. Proposition. *Let $X \xrightarrow{f} Y$ be a quasi-separated scheme morphism. Then*
(a) The morphism f is separated iff it is formally $\mathfrak{M}_v$-unramified.
(b) The morphism f is universally closed iff it is formally $\mathfrak{M}_v$-smooth.
(c) The morphism f is proper iff it is formally $\mathfrak{M}_v$-étale.

Proof. The assertions (a) and (c) are equivalent resp. to the Grothendieck's criterion of separatedness and properness (see EGA, Ch.II, 7.2.3 and 7.2.8). A proof of the assertion (b) can be extracted from the argument of Theorem 7.2.8, EGA, Ch.II. ∎

Standard properties of separated and proper morphisms become special cases of assertions on formally $\mathfrak{M}$-unramified and formally $\mathfrak{M}$-étale morphisms (cf. 5.4 and 5.6):

6.8.2. Corollary. *(a) Any monomorphism is a separated morphism.*
(b) A composition of two separated (resp. proper) morphisms is separated (resp. proper).
(c) Separated (resp. proper) morphisms are stable under base change.

(d) If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are two morphisms such that $g \circ f$ is separated, then f is separated.

(e) If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are two morphisms such that g is separated and $g \circ f$ is proper, then f is proper.

(f) If $X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y'$ are separated (resp. proper) morphisms over S, then their product, $f \times_S f' : X \times_S X' \longrightarrow Y \times_S Y'$, is also separated (resp. proper).

6.8.3. Remarks. (a) One can introduce the notions of *formally separated* and *formally proper* morphisms by omitting the condition that the morphism in question is quasi-compact. In terms of the family $\mathfrak{M}_v$, a morphism is formally separated (resp. formally proper) iff they are formally $\mathfrak{M}_v$-unramified (resp. formally $\mathfrak{M}_v$-étale). It follows that the assertions obtained from 6.8.1 and 6.8.2 by dropping the quasi-compactness condition and inserting 'formally' at appropriate places, are corollaries of 6.6.

(b) The notions of a (formally) proper morphism and a (formally) separated morphism make sense for morphisms of arbitrary presheaves of sets on the category A^{op}, not only for scheme morphisms, because the notions of a (formally) $\mathfrak{M}$-smooth and (formally) $\mathfrak{M}$-unramified morphisms make sense for morphisms of presheaves of sets on A^{op}.

(c) At the moment, it is not clear what might be an adequate noncommutative version of the family $\mathfrak{M}_v$.

7. Affine morphisms. Closed immersions. Separated morphisms and separated presheaves.

7.1. Representable morphisms. Let $\mathcal{P}$ be a class of morphisms of the category A having the following properties:

(a) A composition of a morphism from $\mathcal{P}$ with any isomorphism belongs to $\mathcal{P}$.

(b) If $X \xrightarrow{f} Y$ is a morphism from $\mathcal{P}$, then for any $Z \xrightarrow{g} Y$, there exists a fiber product $X \times_Y Z$ and the projection $X \times_Y Z \longrightarrow Z$ belongs to $\mathcal{P}$.

Let F, G be presheaves of sets. A morphism $F \longrightarrow G$ is called *representable by a morphism of* $\mathcal{P}$ if for any $h_X \longrightarrow G$, the projection $F \times_G h_X \longrightarrow h_X$ is of the form h_u for a morphism $u \in \mathcal{P}$. In particular, the functor $F \times_G h_X$ is representable.

Denote by $\mathcal{P}^\wedge$ the class of all morphisms of $A^\wedge$ representable by morphisms of $\mathcal{P}$. Clearly a morphism $h_X \longrightarrow h_Y$ belongs to $\mathcal{P}^\wedge$ iff it is of the form h_w with $w \in \mathcal{P}$.

7.1.1. Lemma. *The class $\mathcal{P}^\wedge$ is invariant under the base change: if $F \longrightarrow G$ belongs to $\mathcal{P}^\wedge$ and $H \longrightarrow G$ is an arbitrary morphism, then the projection $H \times_G F \longrightarrow H$ belongs to $\mathcal{P}^\wedge$.*

Proof is left to the reader. ∎

7.1.2. Lemma. *Let $\mathcal{P}$ and $\mathcal{Q}$ be classes of morphisms of the category A satisfying the conditions (a), (b). Then*

(i) The intersection $\mathcal{P} \cap \mathcal{Q}$ has the properties (a) and (b).

(ii) If $\mathcal{P}$ is closed under the composition, then $\mathcal{P}^\wedge$ has the same property.

Proof is left to the reader. ∎

7.1.3. Standard examples. 1) The class $\mathfrak{M} = \mathfrak{M}(A)$ of all monomorphisms has the property 7.1(b) and is closed under the composition.

2) Same holds for the class $\mathcal{E}^u = \mathcal{E}^u(A)$ of universal epimorphisms. Recall that a morphism $X \xrightarrow{f} Y$ is called a *universal epimorphism* if for any morphism $V \longrightarrow Y$, there exists a fiber product $X \times_Y V$ and the canonical projection $X \times_Y V \longrightarrow V$ is an epimorphism.

7.2. Affine morphisms. Let A be a category with fiber products, and let $\mathcal{P}$ be the class of all morphisms of A. In this case we shall call $\mathcal{P}$-representable morphisms *affine.*

It follows that *a presheaf morphism $F \longrightarrow G$ is affine iff for any object X of A and any morphism $h_X \longrightarrow G$, the presheaf $F \times_G h_X$ is representable.*

7.2.1. Lemma. *Let A be a category with finite limits, and let G be a presheaf of sets on a category A. The following conditions are equivalent:*

(a) For any object X of A, any morphism $h_X \longrightarrow G$ is affine.

(b) The diagonal morphism $G \longrightarrow G \times G$ is affine.

Proof. (a)⇒(b). Let X be an object of A and $h_X \xrightarrow{f} G \times G$ an arbitrary morphism. Taking compositions of f with projections $G \times G \rightrightarrows G$, we obtain a pair of morphisms $h_X \overset{f_1}{\underset{f_2}{\rightrightarrows}} G$. Their fiber product, $h_X \times_G h_X$, is a part of the cartesian square

$$\begin{array}{ccc} G & \xrightarrow{\Delta_G} & G \times G \\ \uparrow & & \uparrow f_1 \times f_2 \\ h_X \times_G h_X & \longrightarrow & h_X \times h_X \end{array}$$

The condition (a) implies that $h_X \times_G h_X$ is affine. Since $h_X \longrightarrow G \times G$ is arbitrary, the diagonal morphism $\Delta_G : G \longrightarrow G \times G$ is affine.

(b)⇒(a). Let X, Y be objects of A, and let $h_X \longrightarrow G \longleftarrow h_Y$ be arbitrary morphisms. Consider the cartesian square

$$\begin{array}{ccc} G & \xrightarrow{\Delta_G} & G \times G \\ \uparrow & & \uparrow \\ h_X \times_G h_Y & \longrightarrow & h_X \times h_Y \end{array}$$

Since A has finite products, $h_X \times h_Y$ is representable, $h_X \times h_Y \simeq h_{X \times Y}$. Since by hypothesis (b) the diagonal morphism Δ_G is affine, the presheaf $h_X \times_G h_Y$ is representable too, hence the assertion. ■

7.3. Strict monomorphisms and closed immersions. For a morphism $Y \xrightarrow{f} X$ of a category A, denote by Λ_f the class of all pairs of morphisms $u_1, u_2 : X \rightrightarrows V$ equalizing f. A morphism $Y \xrightarrow{f} X$ is called a *strict monomorphism* if any morphism $g : Z \longrightarrow X$ such that $\Lambda_f \subseteq \Lambda_g$ has a unique decomposition $g = f \circ g'$. It follows that any strict monomorphism is a monomorphism. We denote the class of strict monomorphisms of the category A by $\mathfrak{M}_s(A)$, or by $\mathfrak{M}_s$. The class $\mathfrak{E}_s = \mathfrak{E}_s(A)$ of strict epimorphisms is defined dually.

Clearly the composition of a strict monomorphism with an isomorphism is a strict monomorphism. If the category A has fiber products, then the class $\mathfrak{M}_s = \mathfrak{M}_s(A)$ of strict monomorphisms of the category A satisfies the condition 7.1(b) too.

In fact, consider the diagram

$$\begin{array}{ccccc} X \times_Y V & \xrightarrow{p_2} & V & & \\ p_1 \downarrow & & \downarrow g & & \\ X & \xrightarrow{f} & Y & \rightrightarrows & Z \end{array} \qquad (1)$$

where $Y \rightrightarrows Z$ is an arbitrary pair of arrows from the class Λ_f of arrows equalizing f. But then p_2 is a universal arrow equalizing all pairs $\Lambda_f \circ g = \{(u_1 g, u_2 g) | \ (u_1, u_2) \in \Lambda_f\}$.

7.3.1. Note. Suppose a morphism $Y \xrightarrow{f} X$ is such that there exists a fiber coproduct $X \coprod_Y X$. Then f is a strict monomorphism iff it is a kernel of the coprojections $Y \rightrightarrows Y \coprod_X Y$. In particular, if the category A has fiber coproducts, then strict monomorphisms can be defined as morphisms $X \longrightarrow Y$ such that the diagram $X \longrightarrow Y \rightrightarrows Y \coprod_X Y$ is exact.

7.3.2. Lemma. *(a) If the composition, gf, of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a strict monomorphism, then f is a strict monomorphism.*

(b) Any retraction is a strict monomorphism.

Proof. (a) If gf is a universal morphism with respect to the class of arrows

$$\Lambda_{gf} = \{Z \overset{u_1}{\underset{u_2}{\rightrightarrows}} V \mid u_1 gf = u_2 gf\},$$

then f is universal for the class of arrows $\Lambda_{gf} \circ g = \{(u_1 g, u_2 g) | \ (u_1, u_2) \in \Lambda_{gf}\}$.

(b) Let $X \xrightarrow{p} Y$ is a retraction, i.e. there exists a morphism $Y \xrightarrow{e} X$ such that $ep = id_X$. Then p is a kernel of the pair $X \overset{id_X}{\underset{pe}{\rightrightarrows}} X$.

In fact, if $Y \xrightarrow{f} X$ is a morphism equalizing the pair (id_X, pe), then $f = p \circ (ef)$; and this decomposition is unique because p is a monomorphism. ■

7.3.3. Closed immersions of presheaves of sets. Let F, G be presheaves of sets on A. We call a morphism $F \longrightarrow G$ a *closed immersion* if it belongs to $\mathfrak{M}_s^\wedge$, i.e. if it is representable by a strict monomorphism. In particular, a closed immersion $h_X \longrightarrow h_Y$ of representable functors is of the form h_u, where u is a strict monomorphism.

7.3.4. Example. Let A be the category $CAff/k$ of commutative affine schemes over $Spec(k)$. Then strict monomorphisms are exactly closed immersions of affine schemes. Let X and Y be arbitrary schemes identified with the corresponding sheaves of sets on the category $C = CAff/k$. Then a morphism $X \longrightarrow Y$ is a closed immersion in the sense of the definition 7.3.1 iff it is a closed immersion of schemes in the conventional sense.

This example shows in particular that a strict monomorphism of (pre)sheaves is not necessarily a closed immersion. For instance, if $X \xrightarrow{f} Y$ is a scheme morphism, the diagonal morphism $X \xrightarrow{\Delta_f} X \times_Y X$ is a kernel of the natural pair of arrows $X \times_Y X \rightrightarrows X$, hence it is a strict monomorphism of sheaves of sets. But Δ_f is a closed immersion (in the sense of 7.3.2) only if the scheme morphism f is separated. Note that, in general, Δ_f is not even affine.

7.3.5. Example. Let R be an associative k-algebra and A the category $(R\backslash Alg_k)^{op}$ of noncommutative affine schemes over R. Let $(S, R \xrightarrow{s} S)$ and $(T, R \xrightarrow{t} T)$ be R-rings and $(S, s) \xrightarrow{f} (T, t)$ and R-ring morphism. The corresponding morphism of affine schemes is a strict monomorphism iff the diagram $T\prod_S T \rightrightarrows T \xrightarrow{f} S$ is exact. The latter means that S is the quotient of T by the two-sided ideal $Ker(f)$.

7.4. Separated morphisms and separated presheaves. Let X, Y be presheaves of sets on a category A. We call a morphism $X \xrightarrow{f} Y$ *separated* if the canonical morphism $X \xrightarrow{\Delta_f} X \times_Y X$ is a closed immersion. We say that a presheaf X on A is *separated* if the diagonal morphism $X \longrightarrow X \times X$ is a closed immersion.

Let $\bullet$ denote the constant presheaf $A^{op} \longrightarrow \mathbf{Sets}$ with values in a one point set. Since $\bullet$ is a final object of the category of presheaves of sets on A, a presheaf X is separated iff the (unique) morphism $X \longrightarrow \bullet$ is separated.

II. Locally affine spaces and schemes. Grassmannians.

8. Locally affine spaces.

8.1. Spaces and covers. Fix a Q-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$. Let $\mathfrak{T}\mathbb{A}$ be the associated quasi-cosite (cf. 2.4). We call (the category of) sheaves of sets on $\mathfrak{T}\mathbb{A}$ (the category of) $\mathbb{A}$-*spaces*, or simply *spaces* if it is clear what is $\mathbb{A}$. We denote by $\mathcal{E}sp_{\mathbb{A}}$ the full subcategory of $A^\vee$ formed by $\mathbb{A}$-spaces, and by $\widehat{u}^+$ the sheafification functor $A^\vee \longrightarrow \mathcal{E}sp_{\mathbb{A}}$ (a left adjoint to the inclusion functor $\mathcal{E}sp_{\mathbb{A}} \longrightarrow A^\vee$).

If $\mathfrak{T}\mathbb{A}$ is a cosite, then (and only then) the sheafification functor is exact. Otherwise, it is only right exact, as any functor having a left adjoint.

We call a space X *affine*, if X is corepresentable, i.e. $X \simeq A(x,-)$ for some $x \in ObA$. We are particularly interested in the case when the Q-category $\mathbb{A}$ is subcanonical, i.e. all corepresentable functors are $\mathbb{A}$-spaces.

8.2. Remark. In what follows, the Q-category $\mathbb{A}$, or ruther the associated quasi-cosite $\mathfrak{T}\mathbb{A}$, is just a device serving to define the subcategory of spaces, $\mathcal{E}sp_{\mathbb{A}}$. We might start with choosing somehow a strictly full subcategory, $\mathfrak{E}$, of the category $A^\vee$ such that the inclusion functor $\mathfrak{E} \hookrightarrow A^\vee$ has a right adjoint, and declare $\mathfrak{E}$ the category of Spaces (cf. A1.10). The requirement that $\mathbb{A}$ should be subcanonical means that $Ob\mathfrak{E}$ contains all corepresentable functors. Note, however, that this setting is not more general than the one we started with: if we take as $\mathbb{A}$ the Q-category $A^\vee \rightleftarrows \mathfrak{E}$, the category $\mathcal{E}sp_{\mathbb{A}}$ of $\mathbb{A}$-spaces coincides with $\mathfrak{E}$.

8.3. Affine τ-covers, locally affine τ-spaces, and τ-schemes. Let τ be a quasi-topology on the category $\mathcal{E}sp_{\mathbb{A}}$ of $\mathbb{A}$-spaces.

We call a τ-cover $\{X_u \xrightarrow{u} X \mid u \in \mathcal{U}\}$ *affine*, if all X_u are affine (i.e. corepresentable).

We call an $\mathbb{A}$-space X τ-*locally affine* if it has an affine τ-cover.

We call the cover $\{X_u \xrightarrow{u} X \mid u \in \mathcal{U}\}$ *2-affine* if it is affine and for any $u, v \in \mathcal{U}$, the space $X_u \times_X X_v$ has an affine τ-cover.

A τ-cover $\{X_u \xrightarrow{u} X \mid u \in \mathcal{U}\}$ will be called a *Zariski affine τ-cover* if it is affine and consists of monomorphisms.

We call an $\mathbb{A}$-space X a τ-*scheme* if it has a Zariski affine τ-cover.

A τ-cover $\{X_u \xrightarrow{u} X \mid u \in \mathcal{U}\}$ will be called *Zariski 2-affine* if it is Zariski affine and for any $u, v \in \mathcal{U}$, the space $X_u \times_X X_v$ has a Zariski affine τ-cover.

We denote by $\mathcal{LA}_{\mathbb{A},\tau}$ the full subcategory of $\mathcal{E}sp_{\mathbb{A}}$ whose objects are τ-locally affine $\mathbb{A}$-spaces, and by $Sch_{\mathbb{A},\tau}$ the full subcategory of $\mathcal{E}sp_{\mathbb{A}}$ whose objects are τ-schemes.

8.3.1. Note. Let $\{X_u \xrightarrow{u} X \mid u \in \mathcal{U}\}$ be a 2-cover, and let $\{X^{\nu}_{u,v} \xrightarrow{\nu_{u,v}} X_u \times_X X_v \mid u,v \in \mathcal{U},\ \nu \in \mathcal{J}_{u,v}\}$ be an affine cover of $X_u \times_X X_v,\ u,v \in \mathcal{U}$. We can associate with this data the following diagram formed by affine spaces and their morphisms:

$$\begin{array}{ccc} & X^{\nu}_{u,v} & \\ \swarrow & & \searrow \\ X_u & & X_v \end{array} \qquad u,v \in \mathcal{U},\ \nu \in \mathcal{J}_{u,v} \tag{1}$$

Suppose the quasi-pretopology τ is such that all covers $\{X_u \xrightarrow{u} X \mid u \in \mathcal{U}\}$ are strictly epimorphic families of arrows, i.e. $\coprod_{u \in \mathcal{U}} X_u \longrightarrow X$ is a strict epimorphism. Then the space X is a colimit of the diagram (1). In particular, the space X can be reconstructed from the local affine data given by the diagram (1).

8.4. Semiseparated and weakly separated covers and spaces. We call a cover $\{X_u \xrightarrow{u} X \mid u \in \mathcal{U}\}$

– *weakly separated* if X_u and the pull-back $X_u \times_X X_v$ are affine (i.e. corepresentable) for all $u,v \in \mathcal{U}$,

– *semiseparated* if the space X_u and the morphism $X_u \xrightarrow{u} X$ are affine for all u.

Clearly semiseparated covers are weakly separated, and weakly separated covers are 2-covers; i.e. a space which has a weakly separated cover is locally affine. We call a space which has a semiseparated (resp. weakly separated) affine cover *semiseparated* (resp. *weakly separated*).

8.4.1. Proposition. *Suppose that* $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ *is subcanonical (i.e. all corepresentable functors* $A \longrightarrow$ **Sets** *are* $\mathbb{A}$*-spaces), and* A *has products. Then every separated locally affine* $\mathbb{A}$*-space is* τ*-semiseparated.*

Proof. Let X be any separated $\mathbb{A}$-space, not necessarily locally affine. By definition, X is separated iff the diagonal morphism $\Delta_X : X \longrightarrow X \times X$ is a closed immersion, i.e. it is representable by strict monomorphisms. The latter means that for any pair of morphisms $T \overset{p_1}{\underset{p_2}{\rightrightarrows}} X$ with T affine, the canonical morphism $Ker(p_1,p_2) \longrightarrow T$ is a strict monomorphism of affine spaces (i.e. corepresentable functors; cf. 7.4). In particular, the diagonal morphism Δ_X is affine. Let $T,\ V$ be affine $\mathbb{A}$-spaces and $T \xrightarrow{f} X \xleftarrow{g} V$ arbitrary morphisms. Consider the cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \times X \\ \uparrow & & \uparrow f \times g \\ T \times_X V & \xrightarrow{\mathrm{j}} & T \times V \end{array} \tag{2}$$

Since A has products, the product of corepresentable functors T and V is a corepresentable functor. Since $\mathbb{A}$ is subcanonical, this product is an affine $\mathbb{A}$-space. In particular, it coincides with the product $T \times V$ in the diagram (2) (taken in the category of $\mathbb{A}$-spaces). Since $T \times V$ is affine, the morphism $T \times_X V \xrightarrow{\ \mathfrak{j}\ } T \times V$ is a strict monomorphism of affine $\mathbb{A}$-spaces. In particular, $T \times_X V$ is affine. This shows that any morphism from an affine $\mathbb{A}$-space to X is affine, hence the assertion. ∎

8.4.2. Example: semiseparated schemes and algebraic spaces. Let $\mathcal{X}$ be a (commutative) scheme, or an algebraic space. An affine cover $\{U_i \stackrel{u_i}{\to} \mathcal{X} \mid i \in J\}$ is called *semiseparated* if each morphism $U_i \stackrel{u_i}{\to} \mathcal{X}$ is affine. A scheme (or algebraic space) is called *semiseparated* if it has a semiseparated cover.

8.5. Natural quasi-topologies on $\mathcal{E}sp_{\mathbb{A}}$. Let $\mathcal{P}$ be a class of morphisms in $Esp_{\mathbb{A}}$ which contains all identical arrows and is closed under the composition (that is $\mathcal{P}$ is a subcategory of $\mathcal{E}sp_{\mathbb{A}}$ having same objects as $\mathcal{E}sp_{\mathbb{A}}$). We call a set of arrows $\{X_u \stackrel{u}{\to} X \mid u \in \mathcal{U}\}$ from a *$\mathcal{P}$-cover* of X if

(i) it is strictly epimorphic;

(ii) all arrows of $\mathcal{U}$ belong to $\mathcal{P}$.

'Strictly epimorphic' means that the corresponding morphism $\coprod_{u\in\mathcal{U}} X_u \longrightarrow X$ is a strict epimorphism. This defines a quasi-topology, $\tau^{\mathcal{P}}$, on $\mathcal{E}sp_{\mathbb{A}}$. It remains to choose the class $\mathcal{P}$.

Let $\mathbb{A}_1 = (\bar{A}_1 \rightleftarrows A)$ be another Q-category with the same underlying category A (thought as the Q-category of thickennings). Then we have the following choices:

– the class $\mathcal{P}_{\text{ét}}$ (resp. $\mathcal{P}_{f\text{ét}}$) of (resp. formally) $\mathbb{A}_1$-étale morphisms,

– the class $\mathcal{P}_{sm}$ (resp. $\mathcal{P}_{fsm}$) of (resp. formally) $\mathbb{A}_1$-smooth morphisms,

– the class $\mathcal{P}_{zar}$ (resp. $\mathcal{P}_{fzar}$) of (resp. formally) $\mathbb{A}_1$-open immersions (cf. 5.12.3).

We denote the corresponding quasi-topologies resp. by $\tau_{\text{ét}}$, $\tau_{f\text{ét}}$, τ_{sm}, τ_{fsm}, τ_{zar}, and τ_{fzar} and call them resp. *étale, formally étale, smooth, formally smooth, Zariski* and *formally Zariski* quasi-topology.

We call τ_{zar}-locally affine $\mathbb{A}$-spaces *$\mathbb{A}_1$-schemes.*

8.6. Remark. Each of the classes, $\mathcal{P}_{\text{ét}}$, $\mathcal{P}_{sm}$, and $\mathcal{P}_{zar}$ is stable under a base change. But, strict epimorphisms fail, in general, to be invariant under a base change, hence $\tau_{\text{ét}}$, τ_{sm}, and τ_{zar} are not topologies usually. Same holds for formal versions of these quasi-pretopologies.

Let $\mathcal{P}$ be closed under base change. We define a $\tau_c^{\mathcal{P}}$-cover as a *universally strictly epimorphic* set of arrows $\{X_u \stackrel{u}{\to} X \mid u \in \mathcal{U}\}$ contained in $\mathcal{P}$. This means that for any morphism $Y \longrightarrow X$, the set of arrows $\{X_u \times_X Y \to Y \mid u \in \mathcal{U}\}$ is a $\tau^{\mathcal{P}}$-cover (i.e. is strictly epimorphic). It follows that $\tau_c^{\mathcal{P}}$-covers form a pretopology, and the corresponding topology is the topology coinduced by τ, i.e. it is the strongest among those topologies which are coarser than τ.

9. Commutative and noncommutative schemes and algebraic spaces.

9.1. Commutative schemes and algebraic spaces. Let A be the category $CAlg_k$ of commutative k-algebras, $\bar{A}_1$ the full subcategory of A^2 formed by k-algebra epimorphisms with nilpotent kernels, $\bar{A}$ the full subcategory of A^2 formed by faithfully flat morphisms. Then $\mathbb{A}$-spaces are sheaves of sets on affine schemes endowed with the flat topology, i.e. *spaces* in the Grothendieck's sense. And $\mathbb{A}_1$-schemes are usual commutative schemes. If $\tau = \tau_{\text{ét}}$, then locally affine (resp. separated) spaces are Artin's (resp. separated) algebraic spaces.

9.2. Noncommutative schemes. Let A be the category Alg_k of associative k-algebras, $\bar{A}_1$ the full subcategory of A^2 formed by k-algebra epimorphisms with nilpotent kernels, $\bar{A}$ the full subcategory of A^2 formed by faithfully flat morphisms. The notion of an $\mathbb{A}_1$-scheme is a natural noncommutative version of a scheme. If $\tau = \tau_{\text{ét}}$ (cf. 8.5), then, by 5.11.4.1, locally affine $\mathbb{A}$-spaces are, precisely, schemes (in the sense of 8.5).

9.3. Noncommutative algebraic spaces. Let A be the category $\mathfrak{Ass}_k$ of associative k-algebras with morphisms defined up to conjugation (cf. 4.6). Let $\bar{A}$ be the full subcategory of A^2 formed by equivalence classes of faithfully flat algebra morphisms, and let $\bar{A}_1$ be the full subcategory of A^2 formed by equivalence classes (with respect to conjugation, see 4.6) of k-algebra epimorphisms with nilpotent kernels. If τ is the étale quasi-topology (cf. 8.5), then locally affine $\mathbb{A}$-spaces in this setting seem to be an adequate noncommutative version of Artin's algebraic spaces.

Schemes and locally affine spaces with respect to the smooth quasi-topology are same in the settings of 9.2 and 9.3.

9.4. Remark. The definition of Artin's algebraic space obtained in 9.1 is more general than the ones usually used. Usually some finiteness restrictions are imposed. For instance, Knutson considers only quasi-compact quasi-separated algebraic spaces. And in [A1], algebraic spaces are separated.

Similarly, in the noncommutative case, one can impose finiteness conditions, for example consider quasi-compact and quasi-separated locally affine spaces and schemes (see definitions in 8.2 and Appendix 2).

10. Vector fibers and Grassmannians.

Fix an associative unital k-algebra R. Let A be the category $R\backslash Alg_k$ of associative *k-algebras over R* (i.e. pairs $(S, R \to S)$, where S is a k-algebra and $R \to S$ a k-algebra morphism) which we call for convenience R-rings. We denote by R^e the k-algebra $R \otimes_k R^o$.

10.1. Vector fiber associated with a bimodule. Let $\mathcal{M}$ be a left R^e-module. We denote by $\mathbb{V}_R(\mathcal{M})$ the 'spectrum' of the tensor algebra $T_R(\mathcal{M}) = \oplus_{n\geq 0}\mathcal{M}^{\otimes n}$ of the R^e-module $\mathcal{M}$. Here $\mathcal{M}^{\otimes 0} = R$ and $\mathcal{M}^{\otimes(n+1)} = \mathcal{M} \otimes_R \mathcal{M}^{\otimes n}$ for $n \geq 0$.

10.1.1. Lemma. *For any unital ring morphism $R \xrightarrow{\varphi} S$, there is a natural isomorphism*

$$\mathbf{Spec}S \prod_{\mathbf{Spec}R} \mathbb{V}_R(\mathcal{M}) \xrightarrow{\sim} \mathbb{V}_S(\bar{\varphi}^*(\mathcal{M}))$$

over $\mathbf{Spec}S$. *Here* $\bar{\varphi}^*(\mathcal{M})) = S \otimes_R \mathcal{M} \otimes_R S$.

Proof. Consider arbitrary commutative square

$$\begin{array}{ccc}
\mathbf{Spec}A & \longrightarrow & \mathbb{V}_R(\mathcal{M}) \\
\big\downarrow & & \big\downarrow \\
\mathbf{Spec}S & \longrightarrow & \mathbf{Spec}R
\end{array},$$

or, the corresponding commutative square of k-algebra morphisms

$$\begin{array}{ccc} T_R(\mathcal{M}) & \longrightarrow & A \\ \Big\uparrow & & \Big\uparrow \\ R & \xrightarrow{\varphi} & S \end{array} \qquad (1)$$

The algebra morphism $T_R(\mathcal{M}) \longrightarrow A$ is uniquely determined by an R^e-module morphism $\mathcal{M} \longrightarrow A$. The commutativity of the diagram (1) implies that the pair of morphisms $\mathcal{M} \longrightarrow A \longleftarrow S$ defines an S^e-module morphism $\bar{\varphi}^*(\mathcal{M}) = S^e \otimes_{R^e} \mathcal{M} \longrightarrow A$ which, in turn, uniquely determines a k-algebra morphism $T_S(\bar{\varphi}^*(\mathcal{M})) \longrightarrow A$. Therefore $T_S(\bar{\varphi}^*(\mathcal{M})) \simeq T_R(\mathcal{M}) \star_R S$ as algebras over S; hence the assertion. ■

10.1.2. Proposition. *Let $\mathcal{M}$ be a left R^e-module. The space $\mathbb{V}_R(\mathcal{M})$ is locally of finite type (resp. locally finitely presentable) over R iff the R^e-module $\mathcal{M}$ is of finite type (resp. locally finitely presentable).*

Proof. Let $A = R\backslash Alg_k$, and let $D \xrightarrow{\mathfrak{D}} A$ be a filtered inductive system. Then we have a commutative diagram of canonical morphisms

$$\begin{array}{ccc} \text{colim } Hom_A(T_R(\mathcal{M}), \mathfrak{D}) & \xrightarrow{\lambda} & Hom_A(T_R(\mathcal{M}), \text{colim } \mathfrak{D}) \\ \wr\Big\downarrow & & \Big\downarrow\wr \\ \text{colim } Hom_{R^e}(\mathcal{M}, \Phi_R\mathfrak{D}) & \xrightarrow{\widetilde{\lambda}} & Hom_{R^e}(\mathcal{M}, \Phi_R(\text{colim } \mathfrak{D})) \end{array} \qquad (2)$$

Here $A \xrightarrow{\Phi_R} R^e{-}mod$ is the functor which maps a R-ring $(S, R \longrightarrow S)$ to the left R^e-module S. The functor Φ_R is a right adjoint to the functor

$$R^e - mod \xrightarrow{T_R} A, \quad \mathcal{M} \longmapsto (T_R(\mathcal{M}), R \to T_R(\mathcal{M}).$$

The functor Φ_R preserves colimits of filtered inductive systems, i.e. the canonical morphism colim $\Phi_R\mathfrak{D} \longrightarrow \Phi_R(\text{colim } \mathfrak{D})$ is an isomorphism. Thus, the map $\widetilde{\lambda}$ in (2) is the composition of a canonical map

$$\text{colim } Hom_{R^e}(\mathcal{M}, \Phi_R\mathfrak{D}) \xrightarrow{\widehat{\lambda}} Hom_{R^e}(\mathcal{M}, \text{colim } \Phi_R\mathfrak{D})$$

and an isomorphism $Hom_{R^e}(\mathcal{M}, \text{colim } \Phi_R\mathfrak{D}) \xrightarrow{\sim} Hom_{R^e}(\mathcal{M}, \Phi_R(\text{colim } \mathfrak{D}))$. Together with the commutativity of (2), this means that λ is injective (resp. bijective) iff $\widehat{\lambda}$ is injective (resp. bijective). Therefore, the morphism $\mathbb{V}_R(\mathcal{M}) \longrightarrow \mathbf{Spec}(R)$ is locally of finite type (resp. locally finitely presentable) if the R^e-module $\mathcal{M}$ is of finite type (resp. locally finitely presentable).

Let now $\widetilde{D} \xrightarrow{\widetilde{\mathfrak{D}}} R^e - mod$ be a filtered inductive system. Let $\mathcal{E}x_R$ denote the functor $R^e - mod \longrightarrow A = R\backslash Alg_k$ denote the functor which assigns to every left R^e-module $\mathcal{L}$ the pair $(\mathcal{L}_R, R \to \mathcal{L}_R)$, where $\mathcal{L}_R$ is the extension R by $\mathcal{L}$. Then for every R^e-module L, the composition $\Phi_R \circ \mathcal{E}x_R$ transfers $\mathcal{L}$ into $R \oplus \mathcal{L}$. Taking in (2) $\mathfrak{D} = \mathcal{E}x_R \circ \widetilde{\mathfrak{D}}$, we obtain (from the following (2) discussion) a commutative diagram

$$\begin{array}{ccc} \mathrm{colim}\ Hom_A(T_R(\mathcal{M}), \mathfrak{D}) & \xrightarrow{\ \lambda\ } & Hom_A(T_R(\mathcal{M}), \mathrm{colim}\ \mathfrak{D}) \\ \wr\downarrow & & \downarrow\wr \\ \mathrm{colim}\ Hom_{R^e}(\mathcal{M}, \Phi_R\mathfrak{D}) & \xrightarrow{\ \widehat{\lambda}\ } & Hom_{R^e}(\mathcal{M}, \mathrm{colim}\ \Phi_R\mathfrak{D}) \end{array} \tag{3}$$

If $\mathbb{V}_R(\mathcal{M})$ is locally of finite type (resp. locally finitely presentable) over R, then the map λ in (3) is injective (resp. bijective). And λ is injective (resp. bijective) iff $\widehat{\lambda}$ has the same property. Since there are natural isomorphisms

$$\begin{aligned} \mathrm{colim}\ Hom_{R^e}(\mathcal{M}, \Phi_R\mathfrak{D}) &\simeq \mathrm{colim}\ Hom_{R^e}(\mathcal{M}, R \oplus \widetilde{\mathfrak{D}}) \\ &\simeq Hom_{R^e}(\mathcal{M}, R) \oplus \mathrm{colim}\ Hom_{R^e}(\mathcal{M}, \widetilde{\mathfrak{D}}) \end{aligned}$$

and

$$\begin{aligned} Hom_{R^e}(\mathcal{M}, \mathrm{colim}\ \Phi_R\mathfrak{D}) &\simeq Hom_{R^e}(\mathcal{M}, \mathrm{colim}\ (R \oplus \widetilde{\mathfrak{D}}) \\ &\simeq Hom_{R^e}(\mathcal{M}, R) \oplus Hom_{R^e}(\mathcal{M}, \mathrm{colim}\ \widetilde{\mathfrak{D}}) \end{aligned}$$

compatible with the map $\widehat{\lambda}$ in (3), $\widehat{\lambda}$ is injective (resp. bijective) iff the canonical map $\mathrm{colim}\ Hom_{R^e}(\mathcal{M}, \widetilde{\mathfrak{D}}) \longrightarrow Hom_{R^e}(\mathcal{M}, \mathrm{colim}\ \widetilde{\mathfrak{D}})$ is injective (resp. bijective). This shows that if $\mathbb{V}_R(\mathcal{M})$ is locally of finite type (resp. locally finitely presentable) over R, then the R^e-module $\mathcal{M}$ is of finite type (resp. locally finitely presentable). ■

10.2. A note on a base change. Fix an object S of a category $\mathcal{E}$. Let $\mathcal{E}/S \xrightarrow{f_S} \mathcal{E}$ and $\mathcal{E}^\wedge/S \xrightarrow{\widehat{f}_S^*} \mathcal{E}^\wedge$ be the forgetful functors. The functor f_S induces a functor

$$f_S^* : \mathcal{E}^\wedge \longrightarrow (\mathcal{E}/S)^\wedge, \quad X \longmapsto X \circ f_S, \tag{1}$$

The functor $\widehat{f}_S^*$ has a right adjoint

$$\widehat{f}_{S_*} : \mathcal{E}^\wedge \longrightarrow \mathcal{E}^\wedge/S, \quad X \longmapsto (X \times S, X \times S \to S). \tag{2}$$

For every presheaf X on $\mathcal{E}$, there is a natural isomorphism

$$f_S^*(X) \simeq Hom_{\mathcal{E}^\wedge/S}(h^S(-), \widehat{f}_{S_*}(X)), \tag{3}$$

where the functor $\mathcal{E}/S \xrightarrow{h^S} \mathcal{E}^\wedge/S$ is induced by the Yoneda embedding. In other words, the functor $f_S^*(X)$ is the restriction to $\mathcal{E}/S$ of a functor representable in $\mathcal{E}^\wedge/S$.

If a presheaf X on $\mathcal{E}$ is representable by an object $\bar{X}$, and there exists a product $\bar{X} \times S$ in $\mathcal{E}$, then the presheaf $f_S^*(X)$ on $\mathcal{E}/S$ is representable by the object $(\bar{X} \times S, \bar{X} \times S \to S)$. This situation is illustrated by 10.1.1. In the general case, we shall identify $\mathcal{E}/S$ with a full subcategory of $\mathcal{E}^\wedge/S$ and omit the embedding h^S.

10.3. Inner hom. Let M, V be left R-modules. Consider the functor

$$\mathcal{H}_{_R}(M,V) : R\backslash Alg_k \longrightarrow Sets$$

which assigns to each algebra $R \xrightarrow{\phi} S$ over R the set $Hom_{_S}(\phi^*(M), \phi^*(V))$.

10.3.1. Proposition. *Let M and V be left R-modules. For any unital ring morphism $R \xrightarrow{\varphi} S$, there is a natural isomorphism*

$$\mathbf{Spec}S \prod_{\mathbf{Spec}R} \mathcal{H}_{_R}(M,V) \xrightarrow{\ \sim\ } \mathcal{H}_{_S}(\varphi^*(M), \varphi^*(V))$$

over $\mathbf{Spec}S$ *(cf. 10.2(5)).*

Proof. Consider a commutative square

$$\begin{array}{ccc} \mathbf{Spec}A & \xrightarrow{\ \xi\ } & \mathcal{H}_{_{M,V}} \\ \gamma \downarrow & & \downarrow \\ \mathbf{Spec}S & \xrightarrow{\ \phi\ } & \mathbf{Spec}R \end{array}$$

The morphism ξ corresponds to an element of $\mathcal{H}_{_{M,V}}(R \to A)$, i.e. to an A-module epimorphism $A \otimes_R M \longrightarrow A \otimes_R V$. Since $A \otimes_R - \simeq A \otimes_S (S \otimes_R -)$, this epimorphism defines an element of $\mathcal{H}_{_{\phi^*(M),\phi^*(V)}}(S \to A)$ which uniquely determines a morphism $\mathbf{Spec}A \xrightarrow{\bar{\xi}} \mathcal{H}_{_{\phi^*(M),\phi^*(V)}}$ over $\mathbf{Spec}S$. This implies the assertion. ∎

10.3.2. Lemma. *If V is a projective R-module of finite type, then the functor $\mathcal{H}_{_R}(M,V)$ is representable.*

Proof. In fact, for any algebra $R \xrightarrow{\phi} S$ over R, we have:

$$Hom_{_S}(\phi^*(M), \phi^*(V) \simeq Hom_{_R}(M, \phi^*(V)) = Hom_{_R}(M, S \otimes_R V).$$

If V is a projective R-module of finite type, then $S \otimes_R V \simeq Hom^R(V^\vee, S)$, where $V^\vee$ is the right R-module dual to V, i.e. $V^\vee = Hom_{_R}(V, R)$; and $Hom^R(-,-)$ denotes the functor of right R-module morphisms. Thus,

$$Hom_{_R}(M, S \otimes_R V) \simeq Hom_{_R}(M, Hom^R(V^\vee, S)) \simeq Hom_{_{R^e}}(M \otimes_k V^\vee, S)$$

and

$$Hom_{_{R^e}}(M \otimes_k V^\vee, S) \simeq R\backslash Alg_k(\mathbf{T}_R(M \otimes_k V^\vee), S),$$

hence the assertion. ∎

10.3.3. Corollary. *Let M be a left R-module and V a projective left R-module of finite type. Then, for any unital ring morphism $R \xrightarrow{\varphi} S$, there is a natural isomorphism*

$$S \star_R \mathbf{T}_{_R}(M \otimes_k V^\vee) \xrightarrow{\ \sim\ } \mathbf{T}_{_S}(\varphi^*(M) \otimes_k \varphi^*(V)^\vee)$$

over S. Here $\varphi^*(V)^\vee = Hom_S(\varphi^*(V), S) \simeq Hom_R(V, S)$.

Proof. By 10.3.2, the functor $\mathcal{H}_R(M, V)$ is representable by the tensor algebra $T_R(M\otimes_k V^\vee)$ of the R^e-module $M \otimes_k V^\vee$. In particular, the functor $\mathcal{H}_S(\varphi^*(M), \varphi^*(V))$ is represented by the tensor algebra of S^e-module

$$\varphi^*(M) \otimes_k \varphi^*(V^\vee) = S \otimes_R M \otimes_k V^\vee \otimes_R S \simeq S^e \otimes_{R^e} (M \otimes_k V^\vee).$$

The assertion follows now from 10.1.1 (see also 10.2.1). ∎

10.3.4. Corollary. *Let M be a left R-module and V a projective left R-module of finite type. If the R-module M is locally of finite type (resp. locally finitely presentable), then the functor $\mathcal{H}_R(M, V)$ is locally of finite type (resp. locally finitely presentable) over R. If V is a generator of the category $R-mod$, then the converse holds; i.e. the functor $\mathcal{H}_R(M, V)$ is of finite type (resp. finitely presentable) over R iff the R-module M is of finite type (resp. finitely presentable).*

Proof. By 10.3.2, the functor $\mathcal{H}_R(M, V)$ is representable by the tensor algebra $T_R(M\otimes_k V^\vee)$ of the R^e-module $M \otimes_k V^\vee$, i.e. $\mathcal{H}_R(M, V) \simeq \mathbb{V}_R(M \otimes_k V^\vee)$. If the R-module M is of finite type (resp. finitely presentable), then the R^e-module $M \otimes_k V^\vee$ is of finite type (resp. finitely presentable). If V is a generator of the category $R-mod$ (that is $Hom_R(V, -)$ is a faithful functor), then the R^e-module $M \otimes_k V^\vee$ is of finite type (resp. finitely presentable) iff the R-module M has this property. The assertion follows now from 10.1.2. ∎

10.4. The functor $Gr_{M,V}$. Let $A = R\backslash Alg_k$. Let M be a left R-module and V a projective left R-module. Consider the functor $Gr_{M,V} : A \longrightarrow$ **Sets** which assigns to any R-ring $(S, R \xrightarrow{s} S)$ (an object of A) the set of isomorphism classes of S-module epimorphisms $s^*(M) \longrightarrow s^*(V)$ (here $s^*(M) = S \otimes_R M$) and to any R-ring morphism $(S, R \xrightarrow{s} S) \xrightarrow{\phi} (T, R \xrightarrow{t} T)$ the map

$$Gr_{M,V}(S, s) \longrightarrow Gr_{M,V}(T, t)$$

induced by the inverse image functor $S - mod \xrightarrow{\phi^*} T - mod$, $\mathcal{N} \longmapsto T \otimes_S \mathcal{N}$.

10.4.1. The functor $G_{M,V}$. Denote by $G_{M,V}$ the functor $A \longrightarrow$ **Sets** which assigns to any R-ring $(S, R \xrightarrow{s} S)$ the set of pairs of morphisms $s^*(V) \xrightarrow{v} s^*(M) \xrightarrow{u} s^*(V)$ such that $u \circ v = id_{s^*(V)}$ and acts naturally on morphisms. Since V is a projective module, the map

$$\pi = \pi_{M,V} : G_{M,V}(S, s) \longrightarrow Gr_{M,V}(S, s), \quad (v, u) \longmapsto [u], \tag{1}$$

is a (strict) functor epimorphism.

10.4.2. Relations. Denote by $\mathfrak{R}_{M,V}$ the "functor of relations" $G_{M,V} \prod_{Gr_{M,V}} G_{M,V}$. By definition, $\mathfrak{R}_{M,V}$ is a subfunctor of $G_{M,V} \times G_{M,V}$ which assigns to each R-ring, $(S, R \xrightarrow{s} S)$, the set of all 4-tuples $(u_1, v_1; u_2, v_2) \in G_{M,V} \times G_{M,V}$ such that the epimorphisms $u_1, u_2 : s^*(M) \longrightarrow s^*(V)$ are equivalent. The latter means that there exists an isomorphism

$\varphi : s^*(V) \longrightarrow s^*(V)$ such that $u_2 = \varphi \circ u_1$, or, equivalently, $\varphi^{-1} \circ u_2 = u_1$. Since $u_i \circ v_i = id$, $i = 1, 2$, these equalities imply that $\varphi = u_2 \circ v_1$ and $\varphi^{-1} = u_1 \circ v_2$. Thus $\mathfrak{R}_{M,V}(S, s)$ is a subset of all $(u_1, v_1; u_2, v_2) \in G_{M,V}(S, s) \prod G_{M,V}(S, s)$ satisfying the following relations:

$$u_2 = (u_2 \circ v_1) \circ u_1, \quad u_1 = (u_1 \circ v_2) \circ u_2 \tag{2}$$

in addition to the relations describing $G_{M,V}(S, s) \prod G_{M,V}(S, s)$:

$$u_1 \circ v_1 = id_{S \otimes_R V} = u_2 \circ v_2 \tag{3}$$

Denote by p_1, p_2 the canonical projections $\mathfrak{R}_{M,V} \rightrightarrows G_{M,V}$. It follows from the surjectivity of $G_{M,V} \longrightarrow Gr_{M,V}$ that the diagram

$$\mathfrak{R}_{M,V} \overset{p_1}{\underset{p_2}{\rightrightarrows}} G_{M,V} \xrightarrow{\ \pi\ } Gr_{M,V} \tag{4}$$

is exact.

10.4.3. Proposition. *If both M and V are projective modules of a finite type, then the functors $G_{M,V}$ and $\mathfrak{R}_{M,V}$ are corepresentable.*

Proof. (a) Suppose the R-module V is finite. For any algebra morphism $\phi : R \longrightarrow S$, we have the following functorial isomorphisms:

$$Hom_S(\phi^*(M), \phi^*(V)) \simeq Hom_R(M, \phi_*\phi^*(V)) = Hom_R(M, S \otimes_R V) \simeq$$

$$Hom_R(M, Hom^R(V^\vee, S)) \simeq Hom_{R^e}(M \otimes_k V^\vee, S) \simeq R\backslash Alg_k(\mathbf{T}_R(M \otimes_k V^\vee), S)$$

Here $Hom^R(V^\vee, S)$ is the (left) R-module of right R-module morphisms from $V^\vee$ to S, $R^e = R \otimes_k R^{op}$, and $\mathbf{T}_R(M \otimes_k V^\vee)$ is the tensor algebra of the R-bimodule $M \otimes_k V^\vee$.

(b) The set $G_{M,V}(S)$ is the kernel of the pair of morphisms

$$Hom_S(\phi^*(M), \phi^*(V)) \times Hom_S(\phi^*(V), \phi^*(M)) \rightrightarrows Hom_S(\phi^*(V), \phi^*(V)) \tag{5}$$

where one arrow assigns to each pair (u, v) the composition, $u \circ v$, of morphisms u and v, and the other one maps each pair (u, v) to the identity morphism, $id_{\phi^*(V)}$. Since the modules M and V are finite, we have canonical functorial isomorphisms:

$$Hom_S(\phi^*(M), \phi^*(V)) \times Hom_S(\phi^*(V), \phi^*(M)) \simeq$$

$$Hom_{R^e}(M \otimes_k V^\vee, S) \times Hom_{R^e}(V \otimes_k M^\vee, S) \simeq$$

$$Hom_{R^e}(M \otimes_k V^\vee \oplus V \otimes_k M^\vee, S) \simeq R\backslash Alg_k(\mathbf{T}_R(M \otimes_k V^\vee \oplus V \otimes_k M^\vee), S)$$

and

$$Hom_S(\phi^*(V), \phi^*(V)) \simeq R\backslash Alg_k(\mathbf{T}_R(V \otimes_k V^\vee), S)$$

Thus, to the diagram (1), there corresponds a diagram

$$\mathbf{T}_R(V \otimes_k V^\vee) \rightrightarrows \mathbf{T}_R(M \otimes_k V^\vee \oplus V \otimes_k M^\vee) \tag{6}$$

of algebra morphisms. The cokernel, $\mathcal{G}_{M,V}$, of the pair of morphisms (6) corepresents the kernel of the pair of morphisms (5). This proves the corepresentability of $G_{M,V}$.

(c) A similar argument proves the corepresentability of $\mathfrak{R}_{M,V}$. Details are left to the reader. ∎

10.4.3.1. Proposition. *If M and V are projective R-modules of a finite type, then the functors $G_{M,V}$, $\mathfrak{R}_{M,V}$, and $Gr_{M,V}$ are locally finitely presentable over R.*

Proof. It follows from the argument of 10.4.3 that $G_{M,V}$ is isomorphic to the kernel of a pair of arrows

$$\mathbb{V}_R(M \otimes_k V^\vee \oplus V \otimes_k M^\vee) \rightrightarrows \mathbb{V}_R(V \otimes_k V^\vee) \tag{7}$$

(see (6)). Since the R-modules M and V are projective of finite type, the R^e-modules $V \otimes_k V^\vee$ and $M \otimes_k V^\vee \oplus V \otimes_k M^\vee$ are projective of finite type; in particular, they are finitely presentable. Therefore, by 10.1.2, both functors in (7) are locally finitely presentable over R. The kernel of a pair of arrows between locally finitely presentable over R functors is locally finitely presentable over R; hence $G_{M,V}$ is locally finitely presentable over R. By a similar reason the functor of relations $\mathfrak{R}_{M,V}$ is locally finitely presentable over R. Since $Gr_{M,V}$ is a cokernel of a pair of arrows between locally finitely presentable over R functors (see 10.4.2(4)), it is locally finitely presentable too. ∎

10.4.4. Universality with respect to the base change.

10.4.4.1. Proposition. *Let M, V be R-modules. For any unital k-algebra morphism $R \xrightarrow{\phi} S$, there is a natural isomorphism between the diagram*

$$\mathbf{Spec}S \prod_{\mathbf{Spec}R} \Big(\mathfrak{R}_{M,V} \overset{p_1}{\underset{p_2}{\rightrightarrows}} G_{M,V} \xrightarrow{\ \pi\ } Gr_{M,V}\Big) \tag{1}$$

and the diagram

$$\mathfrak{R}_{\phi^*(M),\phi^*(V)} \overset{p_1}{\underset{p_2}{\rightrightarrows}} G_{\phi^*(M),\phi^*(V)} \xrightarrow{\ \pi\ } Gr_{\phi^*(M),\phi^*(V)} \tag{2}$$

In particular, $\mathbf{Spec}S \prod_{\mathbf{Spec}R} Gr_{M,V}$ *is isomorphic to* $Gr_{\phi^*(M),\phi^*(V)}$.

Proof. Consider a commutative square

$$\begin{array}{ccc} \mathbf{Spec}A & \xrightarrow{\ \xi\ } & Gr_{M,V} \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow \\ \mathbf{Spec}S & \xrightarrow{\ \phi\ } & \mathbf{Spec}R \end{array}$$

The morphism ξ corresponds to an element of $Gr_{M,V}(R \to A)$, i.e. to the equivalence class of an A-module epimorphism $A \otimes_R M \longrightarrow A \otimes_R V$. Since $A \otimes_R - \simeq A \otimes_S (S \otimes_R -)$, this epimorphism defines an element of $Gr_{\phi^*(M),\phi^*(V)}(S \to A)$ which corresponds to a morphism $\mathbf{Spec}A \xrightarrow{\bar{\xi}} Gr_{\phi^*(M),\phi^*(V)}$ over $\mathbf{Spec}S$. The latter means that the diagram

$$\begin{tikzcd} \mathbf{Spec}A \arrow[rr, "\bar{\xi}"] \arrow[dr, "\gamma"'] & & Gr_{\phi^*(M),\phi^*(V)} \arrow[dl] \\ & \mathbf{Spec}S & \end{tikzcd}$$

commutes. This implies that $\mathbf{Spec}S \prod\limits_{\mathbf{Spec}R} Gr_{M,V}$ is isomorphic to $Gr_{\phi^*(M),\phi^*(V)}$. Similarly, one can show that $\mathbf{Spec}S \prod\limits_{\mathbf{Spec}R} G_{M,V}$ is isomorphic to $G_{\phi^*(M),\phi^*(V)}$. It follows from the universality of these constructions that the isomorphisms can be chosen in such a way that the diagram

$$\begin{tikzcd} \mathbf{Spec}S \prod\limits_{\mathbf{Spec}R} G_{M,V} \arrow[r] \arrow[d] & \mathbf{Spec}S \prod\limits_{\mathbf{Spec}R} Gr_{M,V} \arrow[d] \\ G_{\phi^*(M),\phi^*(V)} \arrow[r, "\pi"] & Gr_{\phi^*(M),\phi^*(V)} \end{tikzcd} \qquad (3)$$

commutes. Notice that the functor $\mathbf{Spec}S \prod_{\mathbf{Spec}R} -$ preserves fibered products. Since $\mathfrak{R}_{M,V} = G_{M,V} \prod\limits_{Gr_{M,V}} G_{M,V}$, the diagram (3) induces an isomorphism

$$\mathbf{Spec}S \prod_{\mathbf{Spec}R} \mathfrak{R}_{M,V} \longrightarrow \mathfrak{R}_{\phi^*(M),\phi^*(V)}.$$

Hence the assertion. ■

10.4.4.2. Proposition. *Let M and V be projective left R-modules of finite type. And let $\mathcal{G}_{M,V}$ be a k-algebra representing the functor $G_{M,V}$ and $\mathcal{R}_{M,V}$ a k-algebra representing the functor $\mathfrak{R}_{M,V}$. Then, for any unital k-algebra morphism $R \xrightarrow{\phi} S$, there is a natural isomorphism between the k-algebras*

$$S \star_R \mathcal{G}_{M,V} \longrightarrow \mathcal{G}_{\phi^*(M),\phi^*(V)} \quad \textit{and} \quad S \star_R \mathcal{R}_{M,V} \longrightarrow \mathcal{R}_{\phi^*(M),\phi^*(V)}. \qquad (4)$$

Proof. By the part (b) of the argument of 10.4.3, the functor $G_{M,V}$ is the kernel of a pair of arrows

$$\mathcal{H}_R(M,V) \times \mathcal{H}_R(V,M) \rightrightarrows \mathcal{H}_S(V,V). \qquad (5)$$

By 10.4.4.3.2 and 10.4.4.3.3, all functors in this diagram are representable and satisfy the desired property with respect to a base change. This implies the first isomorphism (4). We leave to the reader establishing the second isomorphism. ■

10.5. Smoothness.

10.5.1. Proposition. *Let M, V be projective R-modules of finite type. Then all functors and all morphisms of the canonical diagram*

$$\mathfrak{R}_{M,V} \overset{p_1}{\underset{p_2}{\rightrightarrows}} G_{M,V} \xrightarrow{\ \pi\ } Gr_{M,V} \tag{1}$$

are smooth.

Proof. By 10.4.3.1, all functors in the diagram (1) are locally finitely presentable over R. It follows from 5.12.2(d) that all morphisms of the diagram (1) are locally finitely presentable. It remains to show that all functors and all morphisms of the diagram (1) are formally smooth.

Fix an R-ring epimorphism $T \longrightarrow S$ with a nilpotent kernel.

(a) By Yoneda's lemma, a morphism $\mathbf{Spec}(S) \longrightarrow G_{M,V}$ is uniquely defined by an element of $G_{M,V}(S)$, i.e. by a pair of S-module morphisms

$$S \otimes_R V \xrightarrow{g} S \otimes_R M \xrightarrow{h} S \otimes_R V \tag{2}$$

such that $h \circ g = id$. Since M and V are projective modules and the algebra morphism $T \longrightarrow S$ is an epimorphism, the diagram (2) can be lifted to a commutative diagram

$$\begin{array}{ccccc} T \otimes_R V & \xrightarrow{g'} & T \otimes_R M & \xrightarrow{h'} & T \otimes_R V \\ \downarrow & & \downarrow & & \downarrow \\ S \otimes_R V & \xrightarrow{g} & S \otimes_R M & \xrightarrow{h} & S \otimes_R V \end{array}$$

Since V is a module of finite type and the kernel of the morphism $T \longrightarrow S$ is nilpotent, in particular it is contained in the Jacobson's radical of T, the fact that the composition $h \circ g$ is an isomorphism implies (by Nakayama's Lemma) that $h' \circ g'$ is an isomorphism. Set $\bar{g} = g'$ and $\bar{h} = (h' \circ g')^{-1} \circ h'$. It follows that $\bar{h} \circ \bar{g} = id_{T \otimes_R V}$. Hence $G_{M,V}$ is formally smooth.

(b) A morphism $\mathbf{Spec}(S) \longrightarrow Gr_{M,V}$ is given by an element, ξ, of $Gr_{M,V}(S)$. Since the map $G_{M,V}(S) \longrightarrow Gr_{M,V}(S)$ is surjective, the element ξ is the image of an element, ξ', of $G_{M,V}(S)$. By (a), the element ξ' can be lifted to an element, ξ'_T, of $G_{M,V}(T)$. The image of ξ'_T in $Gr_{M,V}(T)$ is a preimage of ξ.

(c) A morphism $\mathbf{Spec}(S) \longrightarrow \mathfrak{R}_{M,V}$ is given by a pair of elements, (u_1, v_1), (u_2, v_2) of $G_{M,V}(S)$ satisfying the following relations:

$$u_2 = (u_2 \circ v_1) \circ u_1, \quad u_1 = (u_1 \circ v_2) \circ u_2 \tag{3}$$

in addition to the relations describing $G_{M,V}(S)$:

$$u_1 \circ v_1 = id = u_2 \circ v_2 \tag{4}$$

(cf. 10.4.2). By (a), each of the pairs can be lifted to an element of $G_{M,V}(T)$, resp. (u_1', v_1') and (u_2', v_2'). Set $\phi' = u_2' \circ v_1' : T \otimes_R V \longrightarrow T \otimes_R V$. It follows that $u_2' = \phi' \circ u_1'$. Since $S \otimes_T \phi' = u_2 \circ v_1$ is invertible and the kernel of $T \longrightarrow S$ is nilpotent, ϕ' is invertible too. This shows that the functor of relations, $\mathfrak{R}_{M,V}$, is formally smooth.

(d) Consider the commutative diagram

$$\begin{array}{ccc} G_{M,V} & \xrightarrow{\pi} & Gr_{M,V} \\ g \uparrow & & \uparrow g_1 \\ \mathbf{Spec} S & \xrightarrow{\bar{\phi}} & \mathbf{Spec} T \end{array} \tag{5}$$

in which $\bar{\phi}$ is the morphism corresponding to a ring epimorphism $\phi : T \longrightarrow S$ with a nilpotent kernel. The morphism g_1 in (5) is uniquely defined by an element of $Gr_{M,V}(R \to T)$, i.e. by a T-module epimorphism $T \otimes_R M \xrightarrow{u} T \otimes_R V$. By the same Yoneda's lemma, the morphism g in (5) is uniquely determined by an element of $G_{M,V}(S, s)$, i.e. a pair of S-module morphisms

$$S \otimes_R V \xrightarrow{v'} S \otimes_R M \xrightarrow{u'} S \otimes_R V$$

such that $u' \circ v' = id$. The commutativity of the diagram (5) is equivalent to the commutativity of the diagram

$$\begin{array}{ccccc} T \otimes_R V & & T \otimes_R M & \xrightarrow{u} & T \otimes_R V \\ \phi_V \downarrow & & \phi_M \downarrow & & \downarrow \phi_V \\ S \otimes_R V & \xrightarrow{v'} & S \otimes_R M & \xrightarrow{u'} & S \otimes_R V \end{array} \tag{6}$$

in which the vertical arrows correspond to the ring epimorphism $T \xrightarrow{\phi} S$, hence they are epimorphisms. Since $T \otimes_R V$ is a projective T-module and ϕ_M is a T-module epimorphism, there exists a T-module morphism $T \otimes_R V \xrightarrow{w} T \otimes_R M$ such that the diagram

$$\begin{array}{ccccc} T \otimes_R V & \xrightarrow{w} & T \otimes_R M & \xrightarrow{u} & T \otimes_R V \\ \phi_V \downarrow & & \phi_M \downarrow & & \downarrow \phi_V \\ S \otimes_R V & \xrightarrow{v'} & S \otimes_R M & \xrightarrow{u'} & S \otimes_R V \end{array} \tag{6$'$}$$

commutes. Since $T \otimes_R V$ is a projective T module of finite type, it follows from Nakayama's Lemma that $u \circ w$ is an isomorphism. Set $v = w \circ (u \circ w)^{-1}$. Then $u \circ v = id$ and the diagram

$$\begin{array}{ccccc} T \otimes_R V & \xrightarrow{v} & T \otimes_R M & \xrightarrow{u} & T \otimes_R V \\ \phi_V \downarrow & & \phi_M \downarrow & & \downarrow \phi_V \\ S \otimes_R V & \xrightarrow{v'} & S \otimes_R M & \xrightarrow{u'} & S \otimes_R V \end{array} \tag{6$''$}$$

commutes. The pair of arrows

$$T \otimes_R V \xrightarrow{v} T \otimes_R M \xrightarrow{u} T \otimes_R V \tag{7}$$

is an element of $G_{M,V}(R \to T)$ which corresponds to a morphism $\mathbf{Spec}T \xrightarrow{\gamma} G_{M,V}$.

Since the pair (7) is a preimage of the element $T \otimes_R M \xrightarrow{u} T \otimes_R V$ of $Gr_{M,V}(R \to T)$ corresponding to the morphism $\mathbf{Spec}T \xrightarrow{g_1} Gr_{M,V}$ (by definition of the morphism π), we have the equality: $\pi \circ \gamma = g_1$. The commutativity of the diagram (6") means exactly that the diagram

$$\begin{tikzcd} & G_{M,V} & \\ \mathbf{Spec}S \arrow[ur, "g"] \arrow[rr, "\bar{\phi}"] & & \mathbf{Spec}T \arrow[ul, "\gamma"'] \end{tikzcd}$$

commutes.

(e) Since the morphism π in the cartesian square

$$\begin{tikzcd} \mathfrak{R}_{M,V} \arrow[r, "p_1"] \arrow[d, "p_2"'] & G_{M,V} \arrow[d, "\pi"] \\ G_{M,V} \arrow[r, "\pi"] & Gr_{M,V} \end{tikzcd}$$

is formally smooth, the morphisms p_1 and p_2 are formally smooth (see 5.6). ∎

We need a slightly stronger version of a part of Proposition 10.5.1:

10.5.2. Proposition. *All morphisms of the canonical diagram (1) are coverings for the smooth topology.*

Proof. Let $T \xrightarrow{\xi} Gr_{M,V}$ be a morphism with affine T. Since the presheaf morphism $G_{M,V} \xrightarrow{\pi} Gr_{M,V}$ is surjective, there exists a morphism $T \xrightarrow{\xi'} Gr_{M,V}$ such that $\pi \circ \xi' = \xi$. This implies that the canonical projection $T \times_{Gr_{M,V}} G_{M,V} \xrightarrow{\pi'} T$ has a splitting; in particular, it is surjective. Since by 10.5.1, π is a smooth morphism, the projection π' is smooth too, hence the assertion. ∎

10.6. $\mathbb{A}$-Grassmannians. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category, where A is the category $R\backslash Alg_k$ of R-rings. We denote by $Gr^{\mathbb{A}}_{M,V}$ the $\mathbb{A}$-space associated with $Gr_{M,V}$. We call the functor $Gr^{\mathbb{A}}_{M,V}$ an $\mathbb{A}$-*Grassmannian of the type* (M,V). In particular, $Gr_{M,V}$ is the $\mathbb{A}$-Grassmannian, where $\mathbb{A}$ corresponds to the discrete cotopology.

Suppose M and V are projective modules of finite type. Let $\mathcal{G}_{M,V}$ and $\mathcal{R}_{M,V}$ be k-algebras corepresenting the functors resp. $G_{M,V}$ and $\mathfrak{R}_{M,V}$. And let $\mathfrak{p}_i$, $i = 1, 2$, be the morphisms $\mathcal{G}_{M,V} \rightrightarrows \mathcal{R}_{M,V}$ corresponding to the projections $\mathfrak{R}_{M,V} \rightrightarrows G_{M,V}$.

If $\mathbb{A}$ is subcanonical, then the exact diagram (4) induces an exact diagram of $\mathbb{A}$-spaces

$$\mathfrak{R}_{M,V} \rightrightarrows G_{M,V} \longrightarrow Gr^{\mathbb{A}}_{M,V}. \tag{7}$$

10.6.1. Note. If $\mathbb{A}$ is a Q-category corresponding to a Grothendieck (pre)topology, $Gr^{\mathbb{A}}_{M,V}(S,s)$ might be described as the set of equivalence classes of quotients of $S \otimes_R M$ which are *locally* isomorphic to $S \otimes_R V$. Here equivalence is also understood as a local isomorphism. If $\mathbb{A}$ does not correspond to a (pre)topology (more precisely, it is not stable under an arbitrary base change), a naive interpretation of this description produces a map from R-rings to **Sets** which, in general, is not functorial.

10.6.2. The Grassmannians $Gr^{+}_{M,V}$. Let M, V be projective left R-modules of finite type. Let A be the category $R\backslash Alg_k$; and suppose that $\bar{A} \subseteq A^2$ consists of all faithfully flat algebra morphisms. We shall write in this case $Gr^{+}_{M,V}$ instead of $Gr^{\mathbb{A}}_{M,V}$. The canonical morphisms $\mathfrak{p}_i : \mathcal{G}_{M,V} \longrightarrow \mathcal{R}_{M,V}$, $i = 1,2$, are faithfully flat. One can show that the $\mathbb{A}$-Grassmannian $Gr^{+}_{M,V}$ is a locally affine $\mathbb{A}$-space.

10.6.3. Noncommutative projective spaces. We shall write $N\mathbb{P}_M$ instead of Gr^{+}_{M,R^1} and call it the *noncommutative projective space of the R-module M*. Here R^1 denotes the free R-module of the rank one. This space was introduced and described in [KR1] in the case when R is a field, hence M is a finite-dimensional vector space over R.

10.7. Smooth topology and Grassmannians. We denote by $Gr^{\mathrm{sm}}_{M,V}$ the associated with $Gr_{M,V}$ sheaf of sets for the smooth topology (cf. 8.6). It follows from 10.5.1 and 10.5.2 that the exact diagram (1) in 10.5.1 induces an exact diagram of sheaves

$$\mathfrak{R}_{M,V} \overset{p_1}{\underset{p_2}{\rightrightarrows}} G_{M,V} \xrightarrow{\ \tilde{\pi}\ } Gr^{\mathrm{sm}}_{M,V} \tag{1}$$

whose arrows are covers in the smooth topology (see 10.5.2). In particular, $Gr^{\mathrm{sm}}_{M,V}$ is a locally affine space with respect to the smooth pretopology.

10.8. Affine Zariski subschemes of a Grassmannian. Noncommutative Grassmannians are not schemes. But they have affine Zariski subschemes (constructed below) which being restricted to commutative algebras produce a Zariski affine cover of the corresponding commutative Grassmannian (when it is not empty).

Fix an R-module morphism $V \xrightarrow{\phi} M$. For any R-ring $(S, R \xrightarrow{s} S)$, consider the set $F_{\phi;M,V}(S,s)$ of equivalence classes of all morphisms $s^*(M) \xrightarrow{v} V'$ such that $v \circ s^*(\phi)$ is an isomorphism. Here two morphisms, $s^*(M) \xrightarrow{v} V'$ and $s^*(M) \xrightarrow{v'} V''$, are equivalent iff $v' = \psi \circ v$ for some S-module isomorphism $V' \xrightarrow{\psi} V''$.

10.8.1. Proposition. *(a) The map $(S,s) \longmapsto F_{\phi;M,V}(S,s)$ is naturally extended to a subfunctor, $F_{\phi;M,V} : R\backslash Alg_k \longrightarrow$ **Sets**, of the functor $Gr_{M,V}$.*

(b) Suppose that the R-module V is projective of finite type. Then the functor $F_{\phi;M,V}$ is representable by an affine scheme and the morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ is affine.

Proof. (a) (i) Fix an object $(S, R \xrightarrow{s} S)$ of $R\backslash Alg_k$. If $s^*(M) \xrightarrow{v} V'$ belongs to $F_{\phi;M,V}(S,s)$, i.e. $v \circ s^*(\phi)$ is an isomorphism, then for any morphism $(S,s) \xrightarrow{h} (T,t)$, the composition $h^*(v) \circ h^*s^*(\phi)$ is an isomorphism, and $h^*s^*(\phi) \simeq t^*(\phi)$. There is a natural morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$.

(ii) Note that one can identify $F_{\phi;M,V}(S,s)$ with the set of all S-module epimorphisms $s^*(M) \xrightarrow{v} s^*(V)$ such that $v \circ s^*(\phi) = id_{s^*(V)}$. In fact, if $s^*(M) \xrightarrow{v'} V'$ is such that $w = v' \circ s^*(\phi) : s^*(V) \longrightarrow V'$ is an isomorphism, then $v = w^{-1} \circ v'$ has the required property.

(iii) One of the consequences of the observation (ii) is that the canonical morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ is a monomorphism.

(b) There are two maps,

$$Hom_S(s^*(M), s^*(V)) \underset{\beta_S}{\overset{\alpha_S}{\rightrightarrows}} Hom_S(s^*(V), s^*(V)),$$

defined by $v \overset{\alpha_S}{\longmapsto} v \circ s^*(\phi)$, $\quad v \overset{\beta_S}{\longmapsto} id_{s^*(V)}$. The maps α_S and β_S are functorial in (S,s), hence they define morphisms, resp. α and β, from the functor

$$S \longmapsto Hom_S(s^*(M), s^*(V)) \simeq Hom_R(M, s_*s^*(V)) = Hom_R(M, S \otimes_R V)$$

to the functor

$$S \longmapsto Hom_S(s^*(V), s^*(V)) \simeq Hom_R(V, s_*s^*(V)) = Hom_R(V, S \otimes_R V).$$

(iv) Suppose now that V is a projective R-module of finite type. Then, by 10.3.2, the first functor is representable by the vector fiber $\mathbb{V}_R(M \otimes_k V^\vee)$ of the left R^e-module $M \otimes_k V^\vee$, and the second one is representable by the vector fiber $\mathbb{V}_R(V \otimes_k V^\vee)$ of the projective R^e-module of finite type $V \otimes_k V^\vee$. Let α' and β' be morphisms from $\mathbb{V}_R(M \otimes_k V^\vee)$ to $\mathbb{V}_R(V \otimes_k V^\vee)$ corresponding to resp. α and β. The functor $F_{\phi;M,V}$ is the kernel of the pair (α, β), hence it is representable by the kernel, $\mathbf{F}_{\phi;M,V}$, of the pair (α', β').

(v) *The functor morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ is representable by an affine morphism*; i.e. for any R-ring $(S, R \xrightarrow{s} S)$ and any functor morphism $h_{(S,s)} \longrightarrow Gr_{M,V}$, the functor

$$R\backslash Alg_k \longrightarrow \mathbf{Sets}, \quad (T,t) \longmapsto F_{\phi;M,V}(T,t) \prod_{Gr_{M,V}(T,t)} h_{(S,s)}(T,t)$$

is representable by an affine subscheme of $\mathbf{Spec}S$.

In fact, by the Yoneda's lemma, any morphism $h_{(S,s)} \longrightarrow Gr_{M,V}$ is uniquely determined by an element of $Gr_{M,V}(S,s)$, i.e. by the equivalence class, $[v]$, of a locally split epimorphism $s^*(M) \xrightarrow{v} V'$. The corresponding map $h_{(S,s)}(T,t) \longrightarrow Gr_{M,V}(T,t)$ sends any morphism $(S,s) \xrightarrow{f} (T,t)$ to the equivalence class $[f^*(v)]$. The fiber product $F_{\phi;M,V}(T,t) \prod_{Gr_{M,V}(T,t)} h_{(S,s)}(T,t)$ consists of all pairs (w,γ), where $\gamma \in h_{(S,s)}(T,t)$ and $[T \otimes_k M \xrightarrow{w} T \otimes_k V]$ are such that $w \circ (T \otimes_k \phi) = id_{T \otimes_k V}$ and $w = \gamma^*(v)$. Since v and ϕ here are fixed, the fiber product $F_{\phi;M,V}(T,t) \prod_{Gr_{M,V}(T,t)} h_{(S,s)}(T,t)$ can be identified with the set of all morphisms $(S,s) \xrightarrow{\gamma} (T,t)$ of $R\backslash Alg_k$ (i.e. k-algebra morphisms $S \xrightarrow{\gamma} T$ satisfying $t = \gamma \circ s$) such that $\gamma^*(v \circ (T \otimes_k \phi)) = id_{T \otimes_k V}$. In other words, this fiber product is identified with the kernel of the pair of morphisms

$$h_{(S,s)}(T,t) \overset{\alpha_{(T,t)}}{\underset{\beta_{(T,t)}}{\rightrightarrows}} Hom_T(T \otimes_R V, T \otimes_R V)$$

defined by

$$\beta_{(T,t)} : \gamma \longmapsto id_{t^*(V)} = id_{T \otimes_k V}, \quad \alpha_{(T,t)} : \gamma \longmapsto \gamma^*(v) \circ (T \otimes_k \phi).$$

The morphisms $\beta_{(T,t)}$, $\alpha_{(T,t)}$ are functorial in (T,t), and $Hom_T(T \otimes_k V, T \otimes_k V) \simeq \mathbb{V}_R(V \otimes_k V^\vee)(T,t)$. Hence the morphisms $\beta = (\beta_{(T,t)})$, $\alpha = (\alpha_{(T,t)})$ define a pair of arrows $\mathbf{Spec}S \overset{\alpha'}{\underset{\beta'}{\rightrightarrows}} \mathbb{V}_R(V \otimes_k V^\vee)$, and the functor $(T,t) \longmapsto F_{\phi;M,V}(T,t) \prod_{Gr_{M,V}(T,t)} h_{(S,s)}(T,t)$ is representable by the kernel of the pair (α', β'). ∎

10.8.2. Proposition. *Let $M \xrightarrow{\phi} V$ be an R-module morphism, and let V be a projective R-module of finite type. If M is a finitely presentable R-module (resp. an R-module of finite type), then $F_{\phi;M,V}$ is locally finitely presentable (resp. locally of finite type) over R.*

Proof. By the part (iv) of the argument of 10.8.1, the functor $F_{\phi;M,V}$ is isomorphic to the kernel of a pair of arrows $\mathbb{V}_R(M \otimes_k V^\vee) \rightrightarrows \mathbb{V}_R(V \otimes_k V^\vee)$ over R. By 10.3.4, $\mathbb{V}_R(V \otimes_k V^\vee)$ is locally finitely presentable over R, and $\mathbb{V}_R(M \otimes_k V^\vee)$ is locally finitely presentable (resp. locally of finite type), if the R-module M is finitely presentable (resp. of finite type). The kernel of a pair of morphisms between locally finitely presentable functors (resp. functors locally of finite type) is locally finitely presentable (resp. locally of finite type); hence the assertion. ∎

10.8.2.1. Corollary. *Suppose M and V are projective R-modules of finite type. Then the canonical morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ is locally finitely presentable.*

Proof. By 10.4.3.1, $Gr_{M,V}$ is locally finitely presentable over R, and by 10.8.2, $F_{\phi;M,V}$ has the same property. By 5.12.2(d), the morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ is locally finitely presentable. ∎

10.8.3. Proposition. *Let M be a projective R-module and V a projective R-module of finite type. Then $F_{\phi;M,V}$ is formally smooth over R and the canonical morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ is a formally open immersion.*

If M is a projective module of finite type, then $F_{\phi;M,V}$ is smooth over R and the morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ is an open immersion.

Proof. (a) Let M be a projective R-module. Since by 10.5.1, $Gr_{M,V}$ is formally smooth over R and the composition of formally smooth morphisms is formally smooth, the formal smoothness of $F_{\phi;M,V}$ over R is a consequence of the formal smoothness of the canonical morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$.

Fix an R-ring epimorphism $(T,t) \xrightarrow{\alpha} (S,s)$ with a nilpotent kernel which is a part of a commutative diagram

$$\begin{array}{ccc} h_{(S,s)} & \longrightarrow & F_{\phi;M,V} \\ \alpha \big\downarrow & & \big\downarrow \\ h_{(T,t)} & \longrightarrow & Gr_{_{M,V}} \end{array} \tag{1}$$

By Yoneda's lemma, the morphism $h_{(S,s)} \longrightarrow F_{\phi;M,V}$ in (1) is uniquely defined by an element of $F_{\phi;M,V}$, i.e. by an S-module morphism $s^*(M) \xrightarrow{u'} s^*(V)$ such that $u' \circ s^*(\phi) = id_{s^*(V)}$ and morphism, and the morphism $h_{(T,t)} \longrightarrow Gr_{_{M,V}}$ is uniquely defined by an element of $Gr_{_{M,V}}(T,t)$. The commutativity of (1) is equivalent to the commutativity of the diagram

$$\begin{array}{ccccc} T \otimes_R V & \xrightarrow{t^*(\phi)} & T \otimes_R M & \xrightarrow{u} & T \otimes_R V \\ \alpha_V \big\downarrow & & \alpha_M \big\downarrow & & \big\downarrow \alpha_V \\ S \otimes_R V & \xrightarrow{s^*(\phi)} & S \otimes_R M & \xrightarrow{u'} & S \otimes_R V \end{array} \tag{2}$$

whose vertical arrows are induced by the ring epimorphism $T \xrightarrow{\alpha} S$, hence they are epimorphisms. Since $T \otimes_R V$ is a projective T module of finite type and the kernel of $T \xrightarrow{\alpha} S$ is nilpotent, it follows from Nakayama's lemma that $u \circ t^*(\phi)$ is an isomorphism. Set $\widetilde{u} = (u \circ t^*(\phi))^{-1} \circ u$. Then $\widetilde{u} \circ t^*(\phi) = id_{t^*(V)}$ and the diagram

$$\begin{array}{ccccc} T \otimes_R V & \xrightarrow{t^*(\phi)} & T \otimes_R M & \xrightarrow{\widetilde{u}} & T \otimes_R V \\ \alpha_V \big\downarrow & & \alpha_M \big\downarrow & & \big\downarrow \alpha_V \\ S \otimes_R V & \xrightarrow{s^*(\phi)} & S \otimes_R M & \xrightarrow{u'} & S \otimes_R V \end{array} \tag{3}$$

commutes. The pair of arrows

$$T \otimes_R V \xrightarrow{t^*(\phi)} T \otimes_R M \xrightarrow{\widetilde{u}} T \otimes_R V \tag{4}$$

is an element of $F_{\phi;M,V}(T,t)$ which corresponds to a morphism $h_{(T,t)} \xrightarrow{\gamma} F_{\phi;M,V}$. It follows from the construction that adjoining the morphism γ to the diagram (1) makes a commutative diagram. This shows that the canonical monomorphism $F_{\phi;M,V} \longrightarrow Gr_{_{M,V}}$ is formally smooth, hence a formally open immersion.

(b) Suppose now that M is a projective R-module of a finite type. Then by 10.8.2, the functor $F_{\phi;M,V}$ is locally finitely presentable over R, and by 10.8.2.1, the morphism $F_{\phi;M,V} \longrightarrow Gr_{_{M,V}}$ is locally finitely presentable. Therefore, $F_{\phi;M,V}$ is smooth over R and $F_{\phi;M,V} \longrightarrow Gr_{_{M,V}}$ is an open immersion. ■

10.8.4. Projective completion of a vector bundle. Let $M' = M \oplus V$, and let $V \xrightarrow{j_V} M'$ be the canonical morphism. The functor $F_{j_V;M',V}$ is isomorphic to the functor which assigns to any R-ring $(S, R \xrightarrow{s} S)$ the set $Hom_S(s^*(M), s^*(V)$ (cf. (ii) and (b) in the argument of 10.8.1). The latter functor is representable by the vector bundle

$\mathbb{V}_R(M \otimes_k V^\vee)$. If the R-modules M and V (hence M') are projective of finite type, then, by 10.8.3, we have a canonical affine open immersion $\mathbb{V}_R(M \otimes_k V^\vee) \longrightarrow Gr_{M',V}$.

In particular, if $R = k$, than taking $V = R^1 = k^1$, we obtain a canonical immersion $\mathbb{V}(M) \longrightarrow \mathbb{P}_{M \oplus R^1}$. The projective space $\mathbb{P}_{M \oplus R^1}$ can be regarded (as in the commutative case) as the *projective completion* of the vector bundle $\mathbb{V}_R(M)$.

10.8.5. Zero section and the hyperplane at infinity. Let M, V be R-modules. Set $M' = M \oplus V$, and let $V \xleftarrow{p_V} M \oplus V \xrightarrow{p_M} M$ be canonical projections. The projection p_V determines a canonical section $\mathbf{Spec}R \longrightarrow Gr_{M',V}$ which (following the commutative tradition) will be called the *zero section.* The projection $M' \xrightarrow{p_M} M$ induces a closed embedding $Gr_{M,V} \longrightarrow Gr_{M',V}$ called the *hyperplane at infinity.*

10.9. Grassmannians are separated. Recall that a presheaf of sets X on a category C is *separated* if the canonical morphism $\Delta_X : X \longrightarrow X \times X$ is a closed immersion (cf. 7.3, 7.4). Here C is the category $\mathbf{Aff}_k/\mathbf{Spec}R$ of *affine k-schemes over R* for some associative k-algebra R. In other words, $C = (R\backslash Alg_k)^{op}$.

10.9.1. Proposition. *For any pair M, V of projective R-modules of finite type, the presheaf $Gr_{M,V} : (\mathbf{Aff}_k/\mathbf{Spec}R)^{op} = R\backslash Alg_k \longrightarrow$ **Sets** is separated.*

Proof. Let $(S, R \xrightarrow{s} S)$ be an R-ring, and let $h_{(S,s)} \overset{u_1}{\underset{u_2}{\rightrightarrows}} Gr_{M,V}$ be a pair of morphisms over R. The claim is that the kernel of the pair (u_1, u_2) is representable by a closed immersion of affine schemes.

Let $s^*(M) \xrightarrow{\xi_i} s^*(V)$ be an epimorphism corresponding to u_i, $i = 1, 2$. Since $s^*(V)$ is a projective S-module, there exists an S-module morphism $s^*(V) \xrightarrow{\nu_i} s^*(M)$ such that $\xi_i \nu_i = id_{s^*(V)}$. Set $p_i = \nu_i \xi_i$. Then the diagram

$$s^*(M) \overset{id}{\underset{p_i}{\rightrightarrows}} s^*(M) \xrightarrow{\xi_i} s^*(V)$$

is exact. Consider the pairs of morphisms

$$s^*(M) \overset{\xi_1}{\underset{\xi_1 p_2}{\rightrightarrows}} s^*(V) \quad \text{and} \quad s^*(M) \overset{\xi_2}{\underset{\xi_2 p_1}{\rightrightarrows}} s^*(V). \tag{1}$$

There exists a universal R-ring morphism $(S, s) \xrightarrow{\psi} (T, t)$ such that the image by ψ^* of each of the pairs (1) belongs to the diagonal. We leave to the reader arguing that the morphism ψ is a closed immersion. ■

Appendix 1: Complements on Q-categories.

A1.1. A Q-category and a Q^o-category associated with a fully faithful functor. Let $G : A \longrightarrow B$ be a fully faithful functor.

A1.1.1. Denote by $\bar{A}$ the full subcategory of B objects of which are all $y \in ObB$ such that the functor $B(G(-), y) : A^{op} \longrightarrow$ **Sets** is representable. Denote by u^* the corestriction of G to the subcategory $\bar{A}$. Assigning to each $y \in Ob\bar{A}$ an object, $u_*(y)$, which represents the functor $B(G(-), y)$, determines a functor $u_* : \bar{A} \longrightarrow A$ right adjoint to u^*.

A1.1.2. Dually, let $\underline{A}$ be the full subcategory of B whose objects are all $z \in ObB$ such that the functor $B(z, G(-)) : A \longrightarrow$ **Sets** is corepresentable. Let $\underline{u}^*$ denote the corestriction of G to $\underline{A}$. The functor $\underline{u}^*$ has a left adjoint, $\underline{u}_!$. The pair of functors $\underline{u}^*$, $\underline{u}_!$ defines a Q^o-category $A \rightleftarrows \underline{A}$.

A1.1.3. Example. Fix a small category $\mathcal{D}$. Let B be the category $\mathcal{D}^A$ of functors $\mathcal{D} \longrightarrow A$. For any $x \in ObA$, denote by $i_x^{\mathcal{D}}$ the constant functor $\mathcal{D} \longrightarrow A$, $i_x^{\mathcal{D}}(\alpha) = id_x$ for all morphisms α of $\mathcal{D}$. The map $x \longmapsto i_x^{\mathcal{D}}$ defines a fully faithful functor, $i^{\mathcal{D}}$, from A to $\mathcal{D}^A$. The category $\bar{A}$ of A1.1.1 is the full subcategory of $\mathcal{D}^A$ formed by all functors $D : \mathcal{D} \longrightarrow A$ such that $\lim(D)$ exists. The functor $u_* : \bar{A} \longrightarrow A$, a right adjoint to $u^* = i^{\mathcal{D}}|^{\bar{A}}$, assigns to each functor D of $\bar{A}$ its limit.

Dually, the category $\underline{A}$ of A1.1.2 is the full subcategory of $\mathcal{D}^A$ formed by all functors $D' : \mathcal{D} \xrightarrow{D'} A$ such that $\mathrm{colim}(D')$ exists. The functor $\underline{u}_!$, a left adjoint to $\underline{u}^* = i^{\mathcal{D}}|^{\underline{A}}$, assigns to each functor D' of $\underline{A}$ its colimit.

If the category A has limits and colimits of functors $\mathcal{D} \longrightarrow A$, then $\bar{A} = \mathcal{D}^A = \underline{A}$. In this case, we write $u_!$ instead of $\underline{u}_!$. The canonical morphism $\mathfrak{r}_u : u_* \longrightarrow u_!$ (cf. 2.5) assigns to any functor $D : \mathcal{D} \longrightarrow A$ the natural morphism $\lim D \longrightarrow \mathrm{colim}\ D$.

The simplest example of such situation is $\mathcal{D} = (\cdot \to \cdot)$ and A an arbitrary category. We recover the Q-category $(A^2 \rightleftarrows A)$ of morphisms of A (cf. 2.5).

A1.1.4. Example. Let A be a category. Denote by $A^{\cdot \rightrightarrows \cdot}$ the category of functors $(\cdot \rightrightarrows \cdot) \longrightarrow A$. Let u^* denote the functor $A \longrightarrow A^{\cdot \rightrightarrows \cdot}$, $x \longmapsto (x \rightrightarrows x)$, where both arrows are identical. Suppose that the category A has kernels of pairs of morphisms. Then the functor u^* has a right adjoint, $u_* : (x \rightrightarrows y) \longmapsto Ker(x \rightrightarrows y)$. If the category A has cokernels of pairs of arrows, u^* has a left adjoint, $u_! : (x \rightrightarrows y) \longmapsto Cok(x \rightrightarrows y)$.

A1.2. Fully faithful functors and presheaves of sets. For a category A, let $A^\wedge$ denote the category of presheaves of sets on A, i.e. the category of contravariant functors from A to **Sets**. Let $g^* : A \longrightarrow B$ be a functor. The functor $g^{*\wedge} : B^\wedge \longrightarrow A^\wedge$, $X \longmapsto X \circ g^*$, admits a right and a left adjoint, resp. $\widehat{g}_*$ and $\widehat{g}^*$.

(i) The composition of $\widehat{g}^*$ with canonical embedding $A \hookrightarrow A^\wedge$ is isomorphic to g^*. For any presheaf $X \in ObA^\wedge$, we have:

$$\widehat{g}^*(X)(Y) \simeq \mathrm{colim}_{(V,\xi) \in A/X} B(Y, g^*(V)). \tag{1}$$

If the functor g^* has a left adjoint, $g_!$, then $\widehat{g}^* \simeq g_!^\wedge : Y \longmapsto Y \circ g_!$.

(ii) The functor $\widehat{g}^*$ is fully faithful iff the functor g^* is fully faithful.

Thus, if (and only if) g^* is fully faithful, the pair of functors $A^\wedge \xrightarrow{\widehat{g}^*} B^\wedge \xrightarrow{\widehat{g}_*} A^\wedge$ forms a Q-category.

(iii) Note that if the functor g^* is fully faithful and has a left adjoint, $g_!$, we recover a special case of A1.1, when $C = \mathbf{Sets}$.

A1.2.1. The dual picture. Let $g^* : A \longrightarrow B$ be a fully faithful functor. Applying to the dual functor, $g^{*op} : A^{op} \longrightarrow B^{op}$ the considerations of A1.1, we obtain a Q^o-category $(\bar{A}^\vee \rightleftarrows A^\vee) = (\bar{A}^\vee \rightleftarrows A^\vee)$.

A1.3. The Q-category of presheaves of sets. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category, $u = (u^*, u_*)$. It follows from A1.2 that the functor u^* determines a Q-category $\mathbb{A}^\wedge = (\bar{A}^\wedge \overset{\widehat{u}}{\rightleftarrows} A^\wedge)$, where $\widehat{u} = (\widehat{u}^*, \widehat{u}_*)$. We call $\mathbb{A}^\wedge$ the *Q-category of presheaves of sets on* $\mathbb{A}$. The canonical (Yoneda) embeddings $A \hookrightarrow A^\wedge$, $\bar{A} \hookrightarrow \bar{A}^\wedge$ define a Q-category embedding, $h_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathbb{A}^\wedge$, which we call Yoneda embedding too.

A1.3.1. The Q-category of presheaves of sets and the Q-category of functors. Let C be a category with small colimits. For any category, B, the composition with the Yoneda embedding, $h_B : B \longrightarrow B^\wedge$, induces an equivalence between the category C^B of functors $B \longrightarrow C$ and the category $C_{\mathfrak{c}}^{B^\wedge}$ of functors $B^\wedge \longrightarrow C$ having a right adjoint (see [GZ], Proposition II.1.3). In particular, for any Q-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$, the Yoneda embedding $h_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathbb{A}^\wedge$ induces an equivalence between the Q-category of functors $C^{\mathbb{A}}$ and the Q-category $C_{\mathfrak{c}}^{\mathbb{A}^\wedge} = (C_{\mathfrak{c}}^{\bar{A}^\wedge} \rightleftarrows C_{\mathfrak{c}}^{A^\wedge})$ formed by functors resp. $\bar{A}^\wedge \longrightarrow C$ and $A^\wedge \longrightarrow C$ having a right adjoint.

A1.3.2. The dual realization of the Q-category of functors. Suppose C is a category with small limits. The category C^B of functors $B \longrightarrow C$ is isomorphic to the category $(C^{op})^{B^{op}}$. By (the dual version of) Proposition II.1.3 in [GZ], the composition with the Yoneda embedding $h^B : B^{op} \longrightarrow (B^{op})^\wedge = B^\vee$ induces an equivalence between the category C^B and the category $(C^{op})_{\mathfrak{c}}^{B^\vee}$ of functors $(B^\vee) \longrightarrow C^{op}$ having a right adjoint. The category $(C^{op})_{\mathfrak{c}}^{B^\vee}$ is naturally isomorphic to the category $C_{\mathfrak{cc}}^{(B^\vee)^{op}}$ of functors $(B^\vee)^{op} \longrightarrow C$ (i.e. *presheaves on* $B^\vee$ *with values in the category* C) having a left adjoint.

It follows that for any Q-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$, the Yoneda embedding $h^{\mathbb{A}} : \mathbb{A}^{op} \longrightarrow \mathbb{A}^\vee$ induces an equivalence between the Q-category of functors $C^{\mathbb{A}}$ and the Q-category $C_{\mathfrak{cc}}^{(\mathbb{A}^\vee)^{op}} = (C_{\mathfrak{cc}}^{(\bar{A}^\vee)^{op}} \rightleftarrows C_{\mathfrak{cc}}^{(A^\vee)^{op}})$ formed by presheaves on resp. $\bar{A}^\vee$ and $A^\vee$ with values in C having a left adjoint.

A1.4. The functor $u^!$ and the Q-category $\mathbb{A}^!$. Denote by $A^!$ the full subcategory of the category A whose objects are all $x \in ObA$ such that the category u_*/x has a final object, $u^!(x)$. The map $x \longmapsto u^!(x)$ is extended to a functor, $u^! : A^! \to \bar{A}$, defined uniquely up to isomorphism. If $A^! = A$, then $u^!$ is a right adjoint to the functor u_*.

In the general case, let $\bar{A}^!$ denote the full subcategory of $\bar{A}$ whose objects are all $\bar{x} \in \bar{A}$ such that $u_*(\bar{x}) \in ObA^!$. Since the adjunction arrow $\eta_u : Id_A \to u_* u^*$ is an isomorphism, $u^*(A^!) \subseteq \bar{A}^!$. Thus we have a Q-subcategory, $\mathbb{A}^! = (\bar{A}^! \overset{\mathfrak{u}}{\rightleftarrows} A^!)$, of the Q-category $\mathbb{A}$ such that the functor $\mathfrak{u}^*$ has a right adjoint, $\mathfrak{u}^! = u^!$.

A1.4.1. Proposition. *The functor $u^!$ is fully faithful, or, equivalently, a canonical morphism $\epsilon_u^! : u_*u^! \to J_{A^!}$ is an isomorphism. Here $J_{A^!}$ denotes the inclusion functor $A^! \to A$.*

Proof. Passing to the Q-subcategory $\mathbb{A}^!$, we can assume that $A^! = A$, hence $J_{A^!}$ is the identical functor $A \to A$. The functor u_* is a localization (since it has a fully faithful left adjoint). Therefore the functor $u^!$, being a right adjoint to a localization, is fully faithful ([GZ], I.4). ■

A1.5. Sheaves in $\mathbb{A}^!$. There is a canonical morphism $\rho_u : u^*|_{A^!} \to u^!$ corresponding to the isomorphism $\eta_u^{-1} : u_*u^* \to Id_A$. By definition, $\mathfrak{F}\mathbb{A}$ is the full subcategory of $A^!$ whose objects are all $x \in ObA^!$ such that the canonical morphism $u^*(x) \to u^!(x)$ is an isomorphism. In particular, the category $\mathfrak{F}\mathbb{A}$ coincides with the category $\mathfrak{F}\mathbb{A}^!$ of sheaves in the Q-subcategory $\mathbb{A}^!$ of $\mathbb{A}$.

A1.5.1. Lemma. *The following conditions are equivalent:*

(i) $\mathfrak{F}\mathbb{A} = A$.

(ii) The functor u_* has a right adjoint, $u^!$, and the canonical morphism $u^* \to u^!$ is an isomorphism.

(iii) The functor u^* has a left adjoint, $u_!$, which is isomorphic to u_*.

Proof. The equivalence of (i) and (ii) follows from definitions.

$(ii) \Leftrightarrow (iii)$. The isomorphism $u^* \to u^!$ induces an isomorphism of the corresponding left adjoint functors, $u_* \to u_!$ (in particular, it induces the existance of a left adjoint, $u_!$, of u^*). Conversely, if u^* has a left adjoint, $u_!$, and the latter is isomorphic to u_*, then, obviously, u^* is a right adjoint to u_*, i.e. $u^* \simeq u^!$. ■

A1.6. Note. Set $(\bar{\mathfrak{F}}\mathbb{A} = u_*^{-1}(\mathfrak{F}\mathbb{A})$. The Q-subcategory, $(\bar{\mathfrak{F}}\mathbb{A} \rightleftarrows \mathfrak{F}\mathbb{A})$, of $\mathbb{A}$ induced by the inclusion functor $\mathfrak{F}\mathbb{A} \to A$ enjoys the equivalent properties of A1.5.1.

A1.7. The functor H_u. Fix a Q-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ such that the functor u^* has a left adjoint, $u_!$, and the functor u_* has a right adjoint, $u^!$. Denote by H_u the functor $u_!u^! : A \longrightarrow A$. Let $\tau_u : Id_A \longrightarrow H_u$ be the composition of the isomorphism $Id_A \longrightarrow u_!u^*$, the inverse to the adjunction isomorphism, $\eta_!^u : u_!u^* \longrightarrow Id_A$, and the morphism $u_!(\rho_u) : u_!u^* \longrightarrow u_!u^! = H_u$, where $\rho_u : u^* \longrightarrow u^!$ is the canonical morphism (cf. A1.5).

A1.7.1. Proposition. *The following conditions on $x \in ObA$ are equivalent:*

(a) x is an $\mathbb{A}$-sheaf.

(b) The canonical morphism $\rho_u(x) : u^*(x) \longrightarrow u^!(x)$ is an isomorphism.

(c) $u^!(x) \simeq u^*(y)$ for some $y \in ObA$.

(d) The morphism $\tau(x) : x \longrightarrow H_u(x)$ is an isomorphism.

Proof. (a)$\Leftrightarrow$(b) follow from definitions.

(b)$\Rightarrow$(c) is obviously true.

(c)$\Rightarrow$(b). If $u^!(x) \simeq u^*(y)$, then $u_*u^!(x) \simeq u_*u^*(y) \simeq y$. By A1.4.1, the adjunction morphism $Id_A \longrightarrow u_*u^!$ is an isomorphism. In particular, $x \simeq u_*u^!(x)$, hence $x \simeq y$.

(b)$\Rightarrow$(d) by definition of the morphism τ_u.

(d)⇒(b). One can check that $u^*(\tau_u) = \lambda_u \circ \rho_u$ and $u^!\tau_u = \rho_u H_u \circ \lambda_u u^!$. Here $\lambda_u : Id_{\bar{A}} \to u^! u_*$ is an adjunction morphism. Since $\tau_u(x)$ is an isomorphism, it follows from the first equality that $\rho_u(x)$ is a strict monomorphism, and from the second one, that $\rho_u H_u(x)$, hence $\rho_u(x)$, is a strict epimorphism. Thus $\rho_u(x)$ is an isomorphism. ∎

A1.7.2. Proposition. *1) Suppose u_* has a right adjoint, $u^!$. Then the following conditions on $x \in ObA$ are equivalent:*

(a) x is an $\mathbb{A}$-monopresheaf.

(b) The canonical morphism $\rho_u(x) : u^*(x) \longrightarrow u^!(x)$ is a monomorphism.

(c) There exists a monomorphism, $\xi : u^*(x) \longrightarrow u^!(y)$, for some $y \in ObA$.

2) Suppose both functors, $u^!$ and $u_!$, exist. If the morphism $\tau(x) : x \longrightarrow H_u(x)$ is a monomorphism, then x is an $\mathbb{A}$-monopresheaf. The converse is true under the condition that the functor $u_!$ maps monomorphisms to monomorphisms.

Proof. 1) (a)⇔(b)⇒(c) follow from definitions.

(c)⇒(b). The morphism ξ decomposes uniquely as the canonical morphism $\rho_u(x) : u^*(x) \longrightarrow x$ and a morphism $g : x \longrightarrow y$. Since the composition $g \circ \rho_u(x)$ is a monomorphism, $\rho_u(x)$ is a monomorphism.

2) Since u^* is left exact (as a functor having a left adjoint), $u^*(\tau_u(x))$ is a monomorphism, if $\tau_u(x)$ is a monomorphism. But $u^*(\tau_u(x)) = \lambda_u(x) \circ \rho_u(x)$, hence $\rho_u(x)$ is a monomorphism.

By definition, $\tau_u = (\eta^u_!)^{-1} \circ u_!(\rho_u)$ (cf. A1.7). Thus, if $\rho_u(x)$ is a monomorphism and $u_!$ maps monoarrows to monoarrows, $\tau_u(x)$ is a monomorphism too. ∎

A1.7.3. Lemma. *If x is an $\mathbb{A}$-epipresheaf (i.e. $\rho_u(x) : u^*(x) \longrightarrow u^!(x)$ is a strict epimorphism, cf. 3.1.4), then $\tau_u(x) : x \longrightarrow H_u(x)$ is a strict epimorphism.*

Proof. The functor $u_!$ maps strict epimorphisms to strict epimorphisms (as any functor having a right adjoint). In particular, if $\rho_u(x)$ is a strict epimorphism, then $\tau_u(x) = \eta^u_!(x)^{-1} \circ u_!(\rho_u(x))$ is a strict epimorphism. ∎

A1.8. Sheafification functors. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category, $\mathfrak{F}\mathbb{A}$ the category of $\mathbb{A}$-sheaves. A *sheafification functor* is a left adjoint to the inclusion functor $\mathfrak{F}\mathbb{A} \hookrightarrow A$ (if any). We are interested here in the case of Q-categories $C^{\mathbb{A}}$ and mostly for $C = \mathbf{Sets}$. We shall denote by $\mathfrak{F}(\mathbb{A}, C)$ the category of sheaves on a Q-category $\mathbb{A}$ with values in C. And we shall write $\mathfrak{F}\mathbb{A}^\vee$ in the case $C = \mathbf{Sets}$.

A1.8.1. Existence.

A1.8.1.1. Lemma. *Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category such that u^* preserves limits of certain type. Then the subcategory $\mathfrak{F}\mathbb{A}$ of sheaves in $\mathbb{A}$ is closed under limits of this type (taken in A).*

Proof. This follows from the characterization of objects of $\mathfrak{F}\mathbb{A}$ as those $x \in ObA$ for which the canonical morphism $u^*(x) \longrightarrow u^!(x)$ is an isomorphism and the fact that the functor $u^!$ (defined on a full subcategory, $A^!$, of A) preserves all limits (taken in A) of all small diagrams $\mathcal{D} \longrightarrow A^!$. ∎

A1.8.1.2. Lemma. *Let* $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ *be a Q-category such that* A *has small limits and the functor* u^* *preserves small limits. Suppose, in addition that for any object* x *of* A*, there exists an* $\mathbb{A}$*-sheaf,* y*, and a morphism* $x \to y$*. Then the inclusion functor* $\mathfrak{F}\mathbb{A} \hookrightarrow A$ *has a left adjoint.*

Proof. The assertion follows from A1.8.1.1 and the Freyd's criterion of the existence of a left adjoint functor to an inclusion functor (cf. [GZ]). ■

A1.8.1.3. Corollary. *Let* $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ *be a Q-category such that the functor* u^* *has a left adjoint,* $u_!$*, and* A *has small limits and a final object (the latter follows from the existence of small limits if* A *is equivalent to a small category). Then there exists a left adjoint to the inclusion functor* $\mathfrak{F}\mathbb{A} \hookrightarrow A$*.*

Proof. The existence of a left adjoint, $u_!$ to the functor u^* garantees that u^* preserves limits. Let $\bullet$ denote the final object to the category A. Notice that any functor having a left adjoint maps a final object to a final object. In particular, $u^*(\bullet)$ is a final objects of $\bar{A}$. This implies that $u^*(\bullet), \eta_u(\bullet)$ is a final object of the category $u_*/\bullet$, i.e. $\bullet$ is an $\mathbb{A}$-sheaf. Since, by definition, any object of A has a morphism to $\bullet$, the conditions of A1.8.1.2 are fulfilled, hence the assertion. ■

A1.8.1.4. Proposition. *Let* C *be a category with small limits and a final object (for instance,* $C =$ **Sets***, or* $C = k - mod$ *for some ring* k*). Then for any Q-category* $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$*, the inclusion functor* $\mathfrak{F}(\mathbb{A}, C) \hookrightarrow C^{\mathbb{A}}$ *has a left adjoint.*

Proof. Let $\bullet$ denote the final object of the category C. For any category B, the constant functor with value in $\bullet$ is a final object of the category B^C of functors from B to C. The functor $\widehat{u}^* : F \longmapsto F \circ u_*$ preserves small limits and maps the final object of the category C^A to the final object of $C^{\bar{A}}$. This implies, in particular, that the final object of C^A is a sheaf on $\mathbb{A}$. The assertion follows from A1.8.1.2 and (the argument of) A1.8.1.3. ■

A1.8.2. Construction. The functor H_u is used to construct a sheafification functor. We need the following technical fact.

A1.8.2.1. Proposition. *Let* $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ *be a Q-category, such that both functors,* $u_!$ *and* $u^!$ *exist. Let* $\tau_u : Id_A \longrightarrow H_u = u_! u^!$ *be the canonical morphism (cf. A1.7.1). Then* $H_u \tau_u = \tau_u H_u$*.*

Proof. There is a commutative diagram of canonical morphisms

$$\begin{array}{ccc}
u_! u^* & \xrightarrow{u_! \eta_u^* u^*} & H_u u_* u^* \\
{\scriptstyle \eta_u^!} \downarrow & & \uparrow {\scriptstyle H_u \eta_u} \\
Id_A & \xrightarrow{\tau_u} & H_u \\
{\scriptstyle \epsilon_u^*} \uparrow & & \downarrow {\scriptstyle \eta_u H_u} \\
u_* u^! & \xrightarrow{u_* \mathfrak{r}_u u^!} & u_* u^* H_u
\end{array} \qquad (1)$$

The upper diagram follows from the definition of τ_u (in A1.7.1), $\tau_u = u_! \rho_u \circ (\eta_u^!)^{-1}$, and the equality $\rho_u = u^! \eta_u^{-1} \circ \eta_u^* u^*$. The diagram (1) provides two formulas for τ_u:

(a) $\tau_u = \eta_u^{-1} H_u \circ u_* \lambda_u u^! \circ (\epsilon_u^*)^{-1}$

(b) $\tau_u = H_u \eta_u^{-1} \circ u_! \eta_u^* u^* \circ (\eta_u^!)^{-1}$.

Applying H_u from the left to the first expression and from the right to the second one, we obtain respectively:

(a') $H_u \tau_u = H_u \eta_u^{-1} H_u \circ u_!(u^! u_* \mathfrak{r}_u u^! \circ u^! \epsilon_u^{*-1}) = H_u \eta_u^{-1} H_u \circ u_!(u^! u_* \lambda_u u^! \circ \eta_u^* u^! = H_u \eta_u^{-1} H_u \circ u_!(u^! u_* \lambda_u \circ \eta_u^*) u^!$ since $u^! \epsilon_u^{*-1} = \eta_u^* u^!$.

(b') $\tau_u H_u = H_u \eta_u^{-1} H_u \circ (u_! \eta_u^* u^* u_! \circ \eta_u^{!-1} u_!) u^! = H_u \eta_u^{-1} H_u \circ (u_! \eta_u^* u^* u_! \circ u_! \mathfrak{r}_u) u^! = H_u \eta_u^{-1} H_u \circ u_!(\eta_u^* u^* u_! \circ \lambda_u) u^!$ since $\eta_u^{!-1} u_! = u_! \mathfrak{r}_u$. Thus the right hand sides of (a') and (b') coincide as claimed. ∎

A1.8.2.2. Corollary. *Under conditions of A1.8.2.1, $H_u^n \tau_u H_u^m = \tau_u H_u^{n+m}$ for any nonnegative integers m, n.*

We call functors which map monomorphisms to monomorphisms *monofunctors.*

A1.8.2.3. Corollary. *(a) Suppose H_u is a monofunctor and $x \in ObA$ is such that $\tau_u(x)$ is a monomorphism. Then $\tau_u(H_u(x))$ is a monomorphism.*

(b) Let $u_!$ be a monofunctor. Then the subcategory $\mathfrak{M}\mathbb{A} \hookrightarrow A$ of $\mathbb{A}$-monopresheaves is $H_u(x)$-invariant.

Proof. The assertion (a) is a consequence of the equality $H_u \tau_u = \tau_u H_u$.

(b) Since $u_!$ is a monofunctor, $H_u = u_! u^!$ is a monofunctor too. By A1.7.2, an object x is an $\mathbb{A}$-monopresheaf iff $\tau_u(x)$ is a monomorphism, hence the assertion. ∎

We set $\tau_u^1 = \tau_u$, and define morphisms $\tau_u^n : Id_A \longrightarrow H_u^n$, $n \geq 2$, by $\tau_u^n = H_u \tau_u^{n-1}$. Let $H_u^\sim$ denote the functor $\mathbb{N} \longrightarrow A$, $n \longmapsto H_u^n$, $(n \to m) \longmapsto H_u^n \tau_u^{m-n}$. The colimit of $H_u^\sim(x)$ (when it exists) will be denoted by H_u^∞.

A1.8.2.4. Proposition. *Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category such that there exist functors $u_!$ and $u^!$.*

(a) Suppose $u^!$ commutes with colimits of functors $\mathbb{N} \to A$. Then object x of A has an associate $\mathbb{A}$-sheaf iff the object $H_u^\infty(x) = \mathrm{colim} H_u^\sim$ exists. In the latter case, H_u^∞ is an $\mathbb{A}$-sheaf associated to x.

(b) Let $u^!$ commute with colimits of monofunctors $\mathbb{N} \to A$. Then an $\mathbb{A}$-monopresheaf x has an associate sheaf iff the $H_u^\infty(x)$ exists. In this case, $H_u^\infty(x)$ is an $\mathbb{A}$-sheaf associated to x.

Proof. (a) Suppose $H_u^\infty(x)$ exists. By hypothesis, the functor $u^!$ preserves the colimit of functors $\mathbb{N} \longrightarrow A$, in particular $u^!$ preserves the colimit of the functor $H_u^\sim(x)$. But then, since $u_!$ preserves all colimits, the functor H_u preserves colimit of $H_u^\sim(x)$; i.e. the canonical morphism $\mathrm{colim}(H_u \circ H_u^\sim(x) \longrightarrow H_u(\mathrm{colim} H_u^\sim)$ is an isomorphism. But, obviously, the colimit of $H_u \circ H_u^\sim(x)$ is naturally isomorphic to $H_u^\infty(x)$. This implies that the canonical morphism $\tau_u(H_u^\infty(x)) : H_u^\infty(x) \longrightarrow H_u(H_u^\infty(x))$ is an isomorphism. By A1.7.1, this means that $H_u^\infty(x)$ is an $\mathbb{A}$-sheaf.

Let $x \xrightarrow{f} y$ be an arbitrary morphism of A such that y is an $\mathbb{A}$-sheaf. By A1.7.1, the canonical morphism $\tau_u(y) : y \longrightarrow H_u(y)$ is an isomorphism. This implies that $H_u^\infty(y)$ exists and the canonical morphism $\tau_u^\infty(y) : y \longrightarrow H_u^\infty(y)$ is an isomorphism. Thus, the morphism $x \xrightarrow{f} y$ is uniquely represented as the composition of $\tau_u^\infty(x) : x \longrightarrow H_u^\infty(x)$ and a morphism $H_u^\infty(x) \longrightarrow y$. Therefore $H_u^\infty(x)$ is an $\mathbb{A}$-sheaf associated to x.

(b) Suppose $H_u^\infty(x)$ exists. Since x is an $\mathbb{A}$-monopresheaf, the morphism $\tau_u(x)$ is a monomorphism (cf. A1.7.2). It follows from A1.8.2.2 that all morphisms $H_u^n \tau_u^{m-n} : H_u^n(x) \longrightarrow H_u^m(x)$ are monomorphisms. By hypothesis, the functor $u^!$ preserves the colimit of monofunctors $\mathbb{N} \longrightarrow A$, in particular $u^!$ preserves the colimit of the functor $H_u^\sim(x)$. Since $u_!$ preserves all colimits, the functor $H_u = u_! u^!$ preserves colimit of $H_u^\sim(x)$; i.e. the canonical morphism $\mathrm{colim}(H_u \circ H_u^\sim(x) \longrightarrow H_u(colim H_u^\sim)$ is an isomorphism. A colimit of $H_u \circ H_u^\sim(x)$ is naturally isomorphic to $H_u^\infty(x)$. By the argument (a) above, the object $H_u^\infty(x)$ is an $\mathbb{A}$-sheaf associated to x. ■

A1.8.2.5. Proposition. *Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a cosite. And let C be the category* **Sets***, or the category $k - mod$ of k-modules. Then*

(a) For any functor $A \xrightarrow{F} C$, the functor $H_{\widehat{u}}(F)$ is an $\mathbb{A}$-monopresheaf.

(b) If F is a monopresheaf, then $H_{\widehat{u}}(F)$ is a sheaf associated with F.

In particular, for any functor $A \xrightarrow{F} C$, the functor $H^2_{\widehat{u}}(F)$ is a sheaf associated with F.

Proof. In the case $\mathbb{A}$ is a cosite, the functor $H_{\widehat{u}}$ is isomorphic to the *Heller-Row functor*, otherwise called "+-construction" for which the assertion is a known fact. ■

A1.9. Subcanonical Q^o- and Q-categories. We call a Q^o-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ *subcanonical* if every representable presheaf on A is a sheaf on the associated quasi-site.

A Q-category is called *subcanonical* if its dual Q^o-category is subcanonical.

A1.9.1. Classical examples of subcanonical pretopologies. One of the most important examples is the category of (commutative) affine schemes with the Zariski, or étale, or **fpqc** pretopology. Recall that **fpqc** covers are families of affine scheme morphisms $\{\phi_i : Spec(R_i) \longrightarrow Spec(R) |\ i \in J\}$ such that all inverse image functors $\phi_i^* : R - mod \longrightarrow R_i - mod$ are exact, and the family $\{\phi_i^* |\ i \in J\}$ contains a finite conservative (i.e. reflecting isomorphisms) subfamily. In the case of Zariski topology, rings R_i are localizations at finitely generated multiplicative sets and ϕ_i universal morphisms. In the case of étale pretopology, all morphisms ϕ_i are étale.

It is well known that the **fpqc** topology is subcanonical. In particular, Zariski and étale topologies are subcanonical.

A1.9.2. A standard noncommutative example. Fix an associative ring k and take as A the category of *rings over* k, i.e. (unital) ring morphisms $k \longrightarrow R$. Objects of the category $\bar{A}$ are faithfully flat morphisms, i.e. morphisms $\phi : R \longrightarrow T$ of rings over k such that the inverse image functor $\phi^* : R - mod \longrightarrow T - mod$ is exact and faithful. The functor $u_* : \bar{A} \longrightarrow A$ is defined by $(R \xrightarrow{\phi} T) \longmapsto R$.

For a Q-category $\mathbb{A}$, let $\mathfrak{T}\mathbb{A}$ denote the quasi-cosite associated with $\mathbb{A}$ (cf. 2.4 and 3.9).

A1.9.2.1. Proposition. *(a) The Q-category $\mathbb{A}$ defined above is subcanonical.*

(b) For any $\bar{y} \in Ob\bar{A}$, the canonical morphism $\mathfrak{r}_u : u_*(\bar{y}) \longrightarrow u_!(\bar{y})$ induces a strict epimorphism of the corresponding $\mathbb{A}$-spaces.

Proof. The assertions follow from [R4], Proposition 4.3.2. ■

A1.9.3. Subcanonical Q^o-categories in terms of covers. Consider a Q^o-category $\mathbb{A}_\tau$. We say that τ is subcaninical if $\mathbb{A}_\tau$ is subcanonical.

Let τ be a subcanonical quasi-pretopology. Then (4) is an exact diagram of sheaves of sets. Taking in (5) $F = A(-, x)$, we obtain the diagram

$$A(y,x) \longrightarrow \prod_{i \in I} A(y_i, x) \rightrightarrows \prod_{i,j \in I} A(y_i \times_y y_j, x). \tag{6}$$

Since τ is subcanonical, the presheaf $A(-, x)$ is a sheaf for any $x \in ObA$, i.e. the diagram (6) is exact for any $x \in ObA$. But this means that the diagram (4) is an exact diagram of sheaves for any $\bar{y} \in ObA_\tau$.

A1.9.3.2. Note. Suppose the object $\bar{y} = (y_i \to y|\ i \in I)$ of A_τ is such that there exist coproducts $\coprod_{i \in I} y_i$ and $\coprod_{i,j \in I} y_i \times_y y_j$. Then the diagram (6) is isomorphic to the diagram

$$A(y,x) \longrightarrow A(\coprod_{i \in I} y_i, x) \rightrightarrows A(\coprod_{i,j \in I} y_i \times_y y_j, x). \tag{7}$$

In this case, the condition "τ is subcanonical" implies the that the diagram

$$\coprod_{i,j \in I} y_i \times_y y_j \rightrightarrows \coprod_{i \in I} y_i \longrightarrow y \tag{8}$$

is exact. Conversely, if for any $\bar{y} = (y_i \to y|\ i \in I)$ in $\mathfrak{T}$, the coproducts $\coprod_{i \in I} y_i$ and $\coprod_{i,j \in I} y_i \times_y y_j$ exist and the diagram (8) is exact, then τ is subcanonical.

A1.10. Q-categories and sites.

A1.10.1. Topologies on a given category. Let A be a category. Consider the set $A^\wedge_{ex}$ of strictly full subcategories B of $A^\wedge$ (*strictly* means that any object of $A^\wedge$ isomorphic to an object of B belongs to B) such that the inclusion functor $B \hookrightarrow A^\wedge$ has a left adjoint, i^*_B, which is *left exact*, i.e. it preserves finite limits. Since i^*_B preserves all colimits, it is exact. Recall the latter means that i^*_B is a localization functor at a class of morphisms which admit left and right fractions [GZ, I.3.4].

On the other hand, denote by $\mathfrak{Top}/A$ the set of topologies on A. There is a canonical map

$$\Psi : A^\wedge_{ex} \longrightarrow \mathfrak{Top}/A \tag{1}$$

defined as follows. To each subcategory B of $A^\wedge$ such that the inclusion functor $i_* : B \hookrightarrow A^\wedge$ has an exact left adjoint, $i^* : A^\wedge \longrightarrow B$, Ψ assigns a topology $\mathfrak{T}_B$ such that for any object X of A, $\mathfrak{T}_B(X)$ consists of all subobjects R of $A(-X)$ such that $i^*(R \hookrightarrow A(-, X))$ is an isomorphism.

A1.10.2. Theorem (Giraud) *The map (1) is a bijection. The inverse map assigns to each topology $\mathfrak{T}$ on A the corresponding subcategory of sheaves of sets on the site $(A, \mathfrak{T})$.*

Proof. See [SGA4, Exp.II], Theorem 5.5. ∎

A1.10.3. The case of arbitrary Q-categories. Denote by $A^\wedge_{rex}$ the set of all strictly full subcategories of $A^\wedge$ such that the inclusion functor $B \hookrightarrow A^\wedge$ has a (not necessarily exact) left adjoint, i^*_B. Let B be a subcategory from $A^\wedge_{rex}$. To any $X \in ObA$, we assign the set $\bar{A}_B(X)$ of all subfunctors R of $A(-,X)$ such that $i^*_B(R \hookrightarrow A(-X))$ is an isomorphism. Denote by $\bar{A}_B$ the category whose objects are pairs (X,R), where $R \in \bar{A}_B(X)$. Morphisms from (X,R) to (X',R') are given by morphisms $f : X \to X'$ such that there is a commutative diagram

$$\begin{array}{ccc} R & \hookrightarrow & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ R' & \hookrightarrow & X' \end{array}$$

(with uniquely defined arrow g). The composition is defined in an obvious way. There is a natural fully faithful functor $A \longrightarrow \bar{A}_B,\ X \longmapsto (X,X)$, which is right adjoint to the functor $\bar{A}_B \longrightarrow A,\ (X,R) \longmapsto X$. This defines a Q-category $\mathbb{A}_B = (\bar{A}^{op}_B \rightleftarrows A^{op})$.

The map $B \longmapsto \mathbb{A}_B$ defines a functor

$$\Psi_A : A^\wedge_{rex} \longrightarrow QCat/A^{op} \tag{1}$$

Here $QCat/A^{op}$ is the category whose objects are Q-categories of the form $(\bar{A} \rightleftarrows A^{op})$ (with fixed A) and morphisms are morphisms of Q-categories identical on A^{op}.

Let $\mathbb{A} = (\bar{A} \rightleftarrows A^{op})$ be a Q-category. The map which assigns to the Q-category $\mathbb{A}$ the subcategory $Sp_{\mathbb{A}}$ of $A^\wedge$ formed by $\mathbb{A}$-spaces defines a functor

$$\Phi_A : QCat/A^{op} \longrightarrow A^\wedge_{rex} \tag{2}$$

A1.10.4. Proposition. *The functor Φ_A is left adjoint to Ψ_A, and $\Phi_A \circ \Psi_A = Id_{A^\wedge_{rex}}$.*

Proof. The functor $\Psi_A \circ \Phi_A$ assigns to any Q-category $\mathbb{A} = (\bar{A} \rightleftarrows A^{op})$ the Q-category associated with the category of $\mathbb{A}$-spaces. The adjunction morphism,

$$Id_{QCat/A^{op}} \longrightarrow \Psi_A \circ \Phi_A$$

assigns to each Q-category $\mathbb{A} = (\bar{A} \rightleftarrows A^{op})$ the canonical morphism $\mathbb{A} \longrightarrow \Psi_A(Sp_{\mathbb{A}})$ which is identical on A and sends each $\bar{y} \in \bar{A}$ to the image of the canonical morphism $\bar{A}(\bar{y}, u^*(-)) \longrightarrow A(u_*(\bar{y}), -)$. ∎

Appendix 2: Finiteness conditions.

In (commutative) algebraic geometry, there are essentially two notions of "global finiteness": the notion of a quasi-compact morphism and the notion of a quasi-separated morphism. These are derivatives of the notion of a *quasi-compact object*. In noncommutative geometry, there is more than one meaningful interpretation of quasi-compactness (like there is more than one meaningful interpretation of smoothness). By this reason, we discuss here a *relative* version of a quasi-compact and a quasi-separated morphisms. "Relative" means that they depend on a subcategory, $\mathcal{P}$, thought as a category of "quasi-compact objects". Notice that this relative version is mentioned in [SGA4, Expose VI, Remarque 1.9.1].

A2.1. Setting. Fix a category $\mathfrak{B}$ and its subcategory $\mathcal{P}$ such that any object of $\mathfrak{B}$ isomorphic to an object of $\mathcal{P}$ belongs to $\mathcal{P}$. We shall sometimes assume that the following condition holds:

(#) If $Y \xrightarrow{f} X$ is a split monomorphism in $\mathfrak{B}$ (i.e. $g \circ f = id_Y$ for some $X \xrightarrow{g} Y$) and $X \in \mathcal{P}$, then $f \in \mathcal{P}$.

A2.2. Examples.

A2.2.1. Objects of finite type and finitely presentable objects. Suppose $\mathfrak{B}$ has colimits of filtered inductive systems. An object X of a category $\mathfrak{B}$ is said to be *of finite type* (resp. *finitely presentable*) if for any filtered inductive systeme $D \xrightarrow{\mathcal{D}} \mathfrak{B}$, the canonical map

$$\mathrm{colim}\mathfrak{B}(X,\mathcal{D}) \longrightarrow \mathfrak{B}(X,\mathrm{colim}\mathcal{D}) \qquad (1)$$

is injective (resp. bijective).

We denote by $\mathfrak{B}_{ft}$ the full subcategory of $\mathfrak{B}$ formed by objects of finite type and by $\mathfrak{B}_{fp}$ its full subcategory formed by finitely presented objects. The following assertion shows that the subcategories $\mathfrak{B}_{ft}$ and $\mathfrak{B}_{fp}$ satisfy the condition A2.1(#).

A2.2.1.1. Lemma. *Any retract of an object of finite type (resp. of a finitely presentable object) is of finite type (resp. finitely presentable).*

Proof. Let X be of finite type, and let Y be a retract of X; i.e. there exist morphisms $Y \xrightarrow{\phi} X$ and $X \xrightarrow{\psi} Y$ such that $\psi \circ \phi = id_X$. Let $D \xrightarrow{\mathcal{D}} \mathfrak{B}$ be a filtered inductive system. Consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{colim}\mathfrak{B}(Y,\mathcal{D}) & \longrightarrow & \mathfrak{B}(Y,\mathrm{colim}\mathcal{D}) \\
\psi_\bullet \downarrow & & \downarrow \psi_\bullet \\
\mathrm{colim}\mathfrak{B}(X,\mathcal{D}) & \longrightarrow & \mathfrak{B}(X,\mathrm{colim}\mathcal{D})
\end{array} \qquad (2)$$

in which vertical arrows are given by functor morphism

$$\mathfrak{B}(Y,-) \xrightarrow{\psi_\bullet} \mathfrak{B}(X,-), \quad g \longmapsto g \circ \psi.$$

Since $\psi \circ \phi = id_Y$, $\phi_\bullet \circ \psi_\bullet = id_{\mathfrak{B}(Y,-)}$. In particular, vertical arrows in (2) are injective.

(a) If X is of finite type, the lower horizontal arrow is injective. Therefore, the canonical morphism $\mathrm{colim}\mathfrak{B}(Y, \mathcal{D}) \longrightarrow \mathfrak{B}(Y, \mathrm{colim}\mathcal{D})$ is injective, i.e. Y is of finite type.

(b) Suppose X is finitely presentable. Note that Y is a cokernel of the pair of morphisms $id_X, \phi \circ \psi : X \rightrightarrows X$. Thus we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{colim}\mathfrak{B}(Y, \mathcal{D}) & \longrightarrow & \mathfrak{B}(Y, \mathrm{colim}\mathcal{D}) \\
\psi_\bullet \downarrow & & \downarrow \psi_\bullet \\
\mathrm{colim}\mathfrak{B}(X, \mathcal{D}) & \xrightarrow{\sim} & \mathfrak{B}(X, \mathrm{colim}\mathcal{D}) \\
\downarrow\downarrow & & \downarrow\downarrow \\
\mathrm{colim}\mathfrak{B}(X, \mathcal{D}) & \xrightarrow{\sim} & \mathfrak{B}(X, \mathrm{colim}\mathcal{D})
\end{array}$$

such that the vertical diagrams $\cdot \to \cdot \rightrightarrows \cdot$ are exact and two (lower) horizontal arrows are isomorphisms. Therefore, the canonical morphism $\mathrm{colim}\mathfrak{B}(Y, \mathcal{D}) \longrightarrow \mathfrak{B}(Y, \mathrm{colim}\mathcal{D})$ is bijective, i.e. Y is finitely presentable. ∎

A2.2.2. Example: quasi-compact objects of a site and quasi-compact objects of a topos. Let $\mathfrak{B}$ be a site. An object X of $\mathfrak{B}$ is called *quasi-compact* if for any cover $\{X_i \to X \mid i \in J\}$, there exists a finite subset I of J such that $\{X_i \to X \mid i \in I\}$ is still a cover.

Let $\mathfrak{B}^\sim$ denote the topos of sheaves of sets on $\mathfrak{B}$, and let $h^+ : \mathfrak{B} \longrightarrow \mathfrak{B}^\sim$ be the composition of the Ioneda embedding $\mathfrak{B} \longrightarrow \mathfrak{B}^\wedge$, $X \longmapsto \mathfrak{B}(-, X)$, and the sheafification functor $\mathfrak{B}^\wedge \longrightarrow \mathfrak{B}^\sim$.

A2.2.2.1. Proposition. *An object X of $\mathfrak{B}$ is quasi-compact iff its image, $h^+(X)$, in the topos $\mathfrak{B}^\sim$ is of finite type (in the sense of A2.2.1).*

Proof. The fact follows from [SGA4], Exp.VI, Proposition 1.2 and Theorem 1.23. ∎

Notice that if $\mathfrak{B}$ itself is a topos with the canonical topology, then the functor h^+ is a category equivalence. Therefore quasi-compact objects of a topos are exactly objects of finite type.

It follows from A2.2.2.1 and A2.2.1.1 that for any site $\mathfrak{B}$, the class $\mathfrak{B}_{qc}$ of quasi-compact objects of $\mathfrak{B}$ satisfies the condition A2.1(#).

A2.3. Weakly $\mathcal{P}$-representable, $\mathcal{P}$-quasi-separated, and $\mathcal{P}$-coherent morphisms. We call a morphism $X \xrightarrow{f} Y$ of $\mathfrak{B}$ *weakly $\mathcal{P}$-representable* if for any commutative diagram

$$\begin{array}{ccc}
Z & \xrightarrow{s} & X \\
t \downarrow & & \downarrow f \\
T & \xrightarrow{g} & Y
\end{array}$$

with $T \in \mathcal{P}$, there exist a commutative diagram

$$\begin{array}{ccc}
Z' & \xrightarrow{s'} & X \\
t' \downarrow & & \downarrow f \\
T & \xrightarrow{g} & Y
\end{array}$$

and a morphism $\gamma : Z \longrightarrow Z'$ such that $t' \in \mathcal{P}$, $s = s' \circ \gamma$ and $t = t' \circ \gamma$.

We denote the class of all weakly $\mathcal{P}$-representable morphisms of $\mathfrak{B}$ by $\Sigma^{\mathcal{P}}$.

A2.3.1. Proposition. *Suppose fiber products exist in $\mathfrak{B}$ and the subcategory $\mathcal{P}$ satisfies A2.1(#). Then a morphism $X \xrightarrow{f} Y$ belongs to $\Sigma^{\mathcal{P}}$ iff for any morphism $T \longrightarrow Y$ such that $T \in \mathcal{P}$, the projection $X \times_Y T \longrightarrow T$ belongs to $\mathcal{P}$. In other words, $\Sigma^{\mathcal{P}}$ is the class of $\mathcal{P}$-representable morphism.*

Proof. Suppose $(X \xrightarrow{f} Y) \in \mathcal{P}$, and let $g : T \longrightarrow Y$ be a morphism with $T \in \mathcal{P}$. By definition, there exists a commutative diagram

$$\begin{array}{ccc} Z' & \xrightarrow{s'} & X \\ {\scriptstyle t'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & Y \end{array}$$

and a morphism $\gamma : X \times_Y T \longrightarrow Z'$ such that $t' \in \mathcal{P}$, $s = s' \circ \gamma$ and $t = t' \circ \gamma$. Here s, t denote the canonical projections $T \longleftarrow X \times_Y T \longrightarrow X$. By the universal property of fiber products, there exists a unique morphism $\phi : Z' \longrightarrow X \times_Y T$ such that $s' = s \circ \phi$ and $t' = t \circ \phi$. Thus $s = s \circ \phi \circ \gamma$ and $t = t \circ \phi \circ \gamma$ which implies that $\phi \circ \gamma = id$. By (#), $X \times_Y T \longrightarrow T$ is a composition of two morphisms of $\mathcal{P}$, hence it belongs to $\mathcal{P}$. ∎

A2.3.2. $\mathcal{P}$-quasi-separated morphisms. We call a morphism $X \xrightarrow{f} Y$ *$\mathcal{P}$-quasi-separated* if for any pair of morphisms $t_1, t_2 : T \longrightarrow X$ such that $T \in \mathcal{P}$ and $f \circ t_1 = f \circ t_2$, any morphism $g : Z \longrightarrow T$ which equalizes t_1, t_2 factorizes through a morphism $(g' : Z' \longrightarrow T) \in \mathcal{P}$ such that $t_1 \circ g' = t_2 \circ g'$.

Denote by $\Sigma^{\mathcal{P}}_{qs}$ the class of all $\mathcal{P}$-quasi-separated morphisms.

A2.3.3. Propostion. *(a) Any monomorphism of $\mathfrak{B}$ is $\mathcal{P}$-quasi-separated.*

(b) Suppose the $\mathcal{P}$ has the property A2.1(#) and $X \times_Y X$ exists. Then $f \in \Sigma^{\mathcal{P}}_{qs}$ iff the diagonal morphism, $\Delta_{X/Y} : X \longrightarrow X \times_Y X$ belongs to $\Sigma^{\mathcal{P}}$.

Proof. (a) If $X \longrightarrow Y$ is a monomorphism, then the morphism $\Delta_{X/Y} : X \longrightarrow X \times_Y X$ is an isomorphism, hence the assertion.

(b) The argument is similar to that of A2.3.1. ∎

A2.3.4. $\mathcal{P}$-coherent morphisms. We call a morphism *$\mathcal{P}$-coherent* if it belongs to the class $\Sigma^{\mathcal{P}}_{coh} = \Sigma^{\mathcal{P}} \cap \Sigma^{\mathcal{P}}_{qs}$.

A2.3.5. Proposition. *(a) Any isomorphism is $\mathcal{P}$-coherent.*

(b) The classes $\Sigma^{\mathcal{P}}$ and $\Sigma^{\mathcal{P}}_{qs}$ are closed under composition. In particular $\Sigma^{\mathcal{P}}_{coh}$ is closed under composition.

(c) Let $X \xrightarrow{f} Y$ and $g : T \longrightarrow Y$ be morphisms such that $X \times_Y T$ exists. If $f \in \Sigma^{\mathcal{P}}$ (resp. $f \in \Sigma^{\mathcal{P}}_{qs}$, resp. $f \in \Sigma^{\mathcal{P}}_{coh}$), then the canonical projection $X \times_Y T \longrightarrow T$ belongs to the same class.

(d) Let $X \xrightarrow{f} Y$, $Y \xrightarrow{g} Z$ be morphisms.

(i) If $g \circ f \in \Sigma^{\mathcal{P}}_{qs}$, then $f \in \Sigma^{\mathcal{P}}_{qs}$.

(ii) Suppose $g \in \Sigma^{\mathcal{P}}_{qs}$. If $g \circ f \in \Sigma^{\mathcal{P}}$ (resp. $g \circ f \in \Sigma^{\mathcal{P}}_{coh}$), then $f \in \Sigma^{\mathcal{P}}$ (resp. $f \in \Sigma^{\mathcal{P}}_{coh}$).

Proof. (a) Obvious.

(b) (i) Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xleftarrow{s} S$ be a diagram such that $f, g \in \Sigma^{\mathcal{P}}$ and $S \in Ob\mathcal{P}$. Let

$$\begin{array}{ccc} W & \xrightarrow{\phi} & X \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle gf} \\ S & \xrightarrow{s} & Z \end{array}$$

be a commutative diagram. Since $g \in \Sigma^{\mathcal{P}}$, there exists a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\phi} & X \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{t} & Y \\ {\scriptstyle \psi'}\downarrow & & \downarrow{\scriptstyle g} \\ S & \xrightarrow{s} & Z \end{array}$$

such that $\psi' \circ \xi = \psi$ and $T \in \mathcal{P}$. Since $f \in \Sigma^{\mathcal{P}}$, there exists a commutative diagram

$$\begin{array}{ccc} W' & \xrightarrow{\phi'} & X \\ {\scriptstyle \xi'}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{t} & Y \end{array}$$

and a morphism $\gamma : W \longrightarrow W'$ such that $\xi = \xi' \circ \gamma$, $\phi = \phi' \circ \gamma$ and $W' \in \mathcal{P}$, hence the assertion.

(ii) Suppose morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are $\mathcal{P}$-quasi-separated. Let

$$\begin{array}{ccccc} W & \xrightarrow{h} & T & \xrightarrow{t_1} & X \\ {\scriptstyle h}\downarrow & & {\scriptstyle t_2}\downarrow & & \downarrow{\scriptstyle gf} \\ T & \xrightarrow{t_1} & X & \xrightarrow{gf} & Z \end{array} \qquad (1)$$

be a commutative diagram such that $T \in \mathcal{P}$. Consider instead of (1) the commutative diagram

$$\begin{array}{ccccc} W & \xrightarrow{h} & T & \xrightarrow{ft_1} & Y \\ {\scriptstyle h}\downarrow & & {\scriptstyle ft_2}\downarrow & & \downarrow{\scriptstyle g} \\ T & \xrightarrow{ft_1} & Y & \xrightarrow{g} & Z \end{array} \qquad (1)$$

Since $g \in \Sigma^{\mathcal{P}}_{qs}$, the morphism $h : W \longrightarrow T$ is a composition of a morphism $(h' : S \longrightarrow T) \in \mathcal{P}$ such that $f \circ t_1 \circ h' = f \circ t_2 \circ h'$ and a morphism $\xi : W \longrightarrow S$. Since h equalizes t_1, t_2, the morphism ξ equalizes $t'_1 = t_1 \circ h'$ and $t'_2 = t_2 \circ h'$. Since f is $\mathcal{P}$-quasi-separated,

ξ factors through a morphism $(\xi' : U \longrightarrow S) \in \mathcal{P}$ such that $t_1' \circ \xi' = t_2' \circ \xi'$. Thus $t_1 \circ (h' \circ \xi') = t_2 \circ (h' \circ \xi')$, hence the assertion.

(c) Let $X \xrightarrow{f} Y$, $g : T \longrightarrow Y$ be morphisms such that $X \times_Y T$ exists. Suppose $f \in \Sigma^{\mathcal{P}}$. Consider a commutative diagram

$$\begin{array}{ccccc} W & \xrightarrow{\phi} & X \times_Y T & \xrightarrow{t_1} & X \\ \psi \downarrow & & f_1 \downarrow & & \downarrow f \\ S & \xrightarrow{s} & T & \xrightarrow{t} & Y \end{array} \qquad (3)$$

such that the right square is cartesian and $S \in \mathcal{P}$. Since $f \in \Sigma^{\mathcal{P}}$, there exists a commutative diagram

$$\begin{array}{ccccc} W' & \xrightarrow{\phi'} & X \times_Y T & \xrightarrow{t_1} & X \\ \psi' \downarrow & & & & \downarrow f \\ S & \xrightarrow{s} & T & \xrightarrow{t} & Y \end{array} \qquad (4)$$

with $\psi' \in \mathcal{P}$ and a morphism $\gamma : W \longrightarrow W'$ such that $\phi = \phi' \circ \gamma$, $\psi = \psi' \circ \gamma$. If follows from the universal property of fiber products that the diagram

$$\begin{array}{ccccc} W' & \xrightarrow{\phi'} & X \times_Y T & \xrightarrow{t_1} & X \\ \psi' \downarrow & & f_1 \downarrow & & \downarrow f \\ S & \xrightarrow{s} & T & \xrightarrow{t} & Y \end{array}$$

commutes.

The remaining assertions are proved in a similar way. Details are left to the reader. ■

A2.4. $\mathcal{P}$-quasi-separated and $\mathcal{P}$-coherent objects.

A2.4.1. Definition. We call an object X of $\mathfrak{B}$ *$\mathcal{P}$-quasi-separated* if any morphism $T \longrightarrow X$ with $T \in \mathcal{P}$, belongs to $\Sigma^{\mathcal{P}}$.

A2.4.2. Definition. We call an object X *$\mathcal{P}$-coherent* if it is $\mathcal{P}$-quasi-separated and belongs to $\mathcal{P}$. We denote by $\mathcal{P}_{coh}$ the full subcategory of $\mathcal{P}$ formed by $\mathcal{P}$-coherent objects.

A2.4.3. Proposition. *Let $X \xrightarrow{f} Y$ be a morphism of $\mathfrak{B}$.*

(a) If $Y \in \mathcal{P}$ and $f \in \Sigma^{\mathcal{P}}$, then $X \in \mathcal{P}$.

(b) If Y is $\mathcal{P}$-quasi-separated and $X \in \mathcal{P}$, then $f \in \Sigma^{\mathcal{P}}$.

(c) If Y is $\mathcal{P}$-coherent, then $f \in \Sigma^{\mathcal{P}}$ iff $X \in \mathcal{P}$.

(d) If $Y \in \mathcal{P}$ (resp. Y is $\mathcal{P}$-quasi-separated, resp. Y is $\mathcal{P}$-coherent) and if $f \in \Sigma^{\mathcal{P}}$ (resp. $f \in \Sigma^{\mathcal{P}}_{qs}$, resp. $f \in \Sigma^{\mathcal{P}}_{coh}$), then $X \in \mathcal{P}$ (resp. X is $\mathcal{P}$-quasi-separated, resp. X is $\mathcal{P}$-coherent).

Proof. (a) Since $Y \in \mathcal{P}$, $X \simeq X \times_Y Y \in \mathcal{P}$ (see A2.3.1).

(b) Follows from definition.

(c) Follows from (a) and (b).

(d) (i) Let Y and $X \xrightarrow{f} Y$ be $\mathcal{P}$-quasi-separated. Let $S \xrightarrow{g} X$ be a morphism with $S \in \mathcal{P}$. Since Y is $\mathcal{P}$-quasi-separated and $S \in \mathcal{P}$, the composition $f \circ g$ belongs to $\Sigma^{\mathcal{P}}$. By A2.3.5(d)(ii), $g \in \Sigma^{\mathcal{P}}$, hence X is $\mathcal{P}$-quasi-separated.

Suppose Y and $X \xrightarrow{f} Y$ are $\mathcal{P}$-coherent. Then by (i), X is $\mathcal{P}$-quasi-separated and by (c), $X \in \mathcal{P}$, hence X is $\mathcal{P}$-coherent. ■

A2.4.4. Proposition. *Subobjects of $\mathcal{P}$-quasi-separated objects are $\mathcal{P}$-quasi-separated.*

Proof. Let $Y \xrightarrow{f} X$ be a monomorphism, and let X be a $\mathcal{P}$-quasi-separated object. Let

$$\begin{array}{ccc} Z & \xrightarrow{\phi} & S \\ {\scriptstyle \psi}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ T & \xrightarrow{t} & Y \end{array}$$

be a commutative diagram such that $S, T \in \mathcal{P}$. Since X is $\mathcal{P}$-quasi-separated, there is a commutative diagram

$$\begin{array}{ccc} Z' & \xrightarrow{\phi'} & S \\ {\scriptstyle \psi'}\big\downarrow & & \big\downarrow{\scriptstyle f \circ s} \\ T & \xrightarrow{f \circ t} & X \end{array}$$

with $\psi' \in \Sigma^{\mathcal{P}}$ and a morphism $Z \xrightarrow{h} Z'$ such that $\phi = \phi' \circ h$, $\psi = \psi' \circ h$. Since f is a monomorphism, the diagram

$$\begin{array}{ccc} Z' & \xrightarrow{\phi'} & S \\ {\scriptstyle \psi'}\big\downarrow & & \big\downarrow{\scriptstyle s} \\ T & \xrightarrow{t} & Y \end{array}$$

is commutative. ■

A2.5. $\mathcal{P}$-constructible objects. Suppose the category $\mathfrak{B}$ has a final object. An object X of $\mathfrak{B}$ is called *constructible* if it is coherent over a final object. We denote by $\mathfrak{B}_{\mathcal{P}-cons}$ the full subcategory of $\mathfrak{B}$ formed by $\mathcal{P}$-constructible objects.

A2.6. Finiteness conditions in a Q-category. Fix a Q-category $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$.

A2.6.1. Lemma. *Let $\bar{y} \in Ob\bar{A}$ be such an object that $u_!(\bar{y})$ exists. If $\bar{y}$ is of finite type (resp. finitely presentable), then $u_!(\bar{y})$ has the same property.*

Proof. Let $\mathcal{D} : D \longrightarrow A$ be a filtered inductive system. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{colim} A(u_!(\bar{y}), \mathcal{D}) & \longrightarrow & A(u_!(\bar{y}), \mathrm{colim}\mathcal{D})) \\ \big\downarrow & & \big\uparrow \\ \mathrm{colim}\bar{A}(\bar{y}, u^* \circ \mathcal{D}) & \longrightarrow & \bar{A}(\bar{y}, \mathrm{colim}(u^* \circ \mathcal{D})) \end{array} \qquad (1)$$

Here the right vertical arrow is the composition of the canonical morphism

$$\bar{A}(\bar{y}, \mathrm{colim}(u^* \circ \mathcal{D})) \longrightarrow \bar{A}(\bar{y}, u^*(\mathrm{colim}\mathcal{D})) \qquad (2)$$

and the isomorphism $\bar{A}(\bar{y}, u^*(\mathrm{colim}\mathcal{D})) \longrightarrow A(u_!(\bar{y}), \mathrm{colim}\mathcal{D})$. Since u^* has a right adjoint, it preserves colimits. In particular, the morphism (2) is an isomorphism. If $\bar{y}$ is of finite type (resp. finitely presentable), the lower horizontal arrow is injective (resp. bijective), hence the upper one is injective (resp. bijective). ∎

A2.7. Quasi-compact spaces and quasi-separated morphisms. Let $\mathbb{A} = (\bar{A} \overset{u}{\rightleftarrows} A)$ be a Q-category, and let τ be a quasi-topology on the category $\mathcal{E}sp_{\mathbb{A}}$ of $\mathbb{A}$-spaces (cf. 8.1 and 8.2). We call an $\mathbb{A}$-space X *τ-quasi-compact* if any τ-cover of X has a finite subcover. Denote by $\mathfrak{K}_\tau$ the full subcategory of the category $\mathcal{E}sp_{\mathbb{A}}$ whose objects are τ-quasi-compact spaces. Applying to the subcategory $\mathfrak{K}_\tau$ the formalism presented above, we obtain the notions of *weakly $\mathfrak{K}_\tau$-representable*, *τ-quasi-separated*, and *τ-coherent* morphisms and spaces (see A2.3, A.2.3.2, A2.3.4, A2.4).

A2.7.1. Proposition. *(a) Any isomorphism is τ-coherent.*

(b) The class of weakly $\mathfrak{K}_\tau$-representable morphisms and the class of τ-quasi-separated morphisms are closed under composition and base change. In particular, the class of τ-coherent morphisms (which is the intersection of other two) is closed under composition and base change.

(c) Suppose the composition $g \circ f$ of morphisms is a weakly $\mathfrak{K}_\tau$-representable morphism (resp. a τ-quasi-separated morphism). Then f is weakly $\mathfrak{K}_\tau$-representable (resp. τ-quasi-separated). In particular, if $g \circ f$ is τ-coherent, then f is quasi-coherent.

Proof. The assertion is a special case of A2.3.5. ∎

A2.7.2. Proposition. *(i) Let $X \xrightarrow{f} Y$ be an $\mathbb{A}$-space morphism. If Y is τ-quasi-compact (resp. Y is τ-quasi-separated, resp. Y is τ-coherent) and if f is weakly $\mathfrak{K}_\tau$-representable (resp. f is τ-quasi-separated, resp. f is τ-coherent), then X is τ-quasi-compact (resp. X is τ-quasi-separated, resp. X is τ-coherent).*

(ii) Subspaces of τ-quasi-separated $\mathbb{A}$-spaces are τ-quasi-separated.

Proof. The assertion (i) is a special case of the assertion (d) in A2.4.3. The assertion (ii) is a specialization of A2.4.4. ∎

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