A non-Archimedean analogue of the Hodge-$\mathcal{D}$-conjecture for products of elliptic curves

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Abstract

In this paper we show that the map

$$\partial : CH^2(E_1 \times E_2, 1) \otimes \mathbb{Q} \longrightarrow PCH^1(Y)$$

is surjective, where E_1 and E_2 are two non-isogenous semistable elliptic curves over a local field, $CH^2(E_1 \times E_2, 1)$ is one of Bloch's higher Chow groups and $PCH^1(Y)$ is a certain subquotient of a Chow group of the special fibre of a semi-stable model of $E_1 \times E_2$. On one hand, this can be viewed as a non-Archimedean analogue of the Hodge-$\mathcal{D}$-conjecture of Beilinson - which is known to be true in this case by the work of Chen and Lewis [CL05], and on the other, an analogue of the works of Speiß [Spi99], Mildenhall [Mil92] and Flach [Fla92] in the case when the elliptic curves have split multiplicative reduction.

1 Introduction

The aim of this note is to prove a special case of the following conjecture: Let K be a local field of with residue characteristic v and ring of integers $\mathcal{O}$. Let X be a variety over K and let $\mathcal{X}$ be a semi-stable model over $\mathcal{O}$. Then the map

$$CH^a(X, b) \otimes \mathbb{Q} \xrightarrow{\partial} PCH^{a-1}(\mathcal{X}_v, b-1)$$

is surjective. Here, assuming the Parshin-Soulé conjecture, if $b > 1$, $PCH^{a-1}(\mathcal{X}_v, b-1)$ is the higher Chow group $CH^{a-1}(\mathcal{X}_v, b-1) \otimes \mathbb{Q}$. In particular, it is 0 if $\mathcal{X}_v$ is non-singular. If $b = 1$ it is a certain subquotient of the Chow group $CH^{a-1}(\mathcal{X}_v) \otimes \mathbb{Q}$.

A conjecture of Bloch's and the work of Consani [Con98] suggests that the dimension of the group $PCH^{a-1}(\mathcal{X}_v, b-1)$ is the order of the pole of $L_v(H^{2a-b-1}(X), s)$ at $s = (a - b)$. In general this can be non-zero so the surjectivity is non-trivial.

We prove this in the case when $X = E_1 \times E_2$ where E_1 and E_2 are non-isogenous elliptic curves over K, $a = 2$ and $b = 1$. When v is a prime of good reduction, the expected dimension of $PCH^1(\mathcal{X}_v, 0)$ is 4 or 2, depending on whether the special fibres $\mathcal{E}_{1,v}$ and $\mathcal{E}_{2,v}$ are isogenous or not. The surjectivity of the map in this case was shown by Spieß[Spi99]. When v is a prime of good reduction for one of the elliptic curves and semi-stable reduction for the other the expected dimension is 2. In this case it is easy to see what the elements of $CH^2(E_1 \times E_2, 1)$ are. So the only case that remains is the situation when v is a prime of semi-stable reduction for both E_1 and E_2. In this case the expected dimension is 3. It is easy to find elements of $CH^2(X, 1)$ which map on to two of those dimensions but the third seems to require a little more work, which is the purpose of this paper.

Beilinson's Hodge-$\mathcal{D}$-conjecture [Jan88], specialized to our case, states that the map

$$r_{\mathcal{D}} \otimes \mathbb{R} : CH^2(X,1) \otimes \mathbb{R} \longrightarrow H^3_{\mathcal{D}}(X, \mathbb{R}(2))$$

is surjective. This is now a theorem of Chen and Lewis [CL05]. As explained below, the group $PCH^1(Y)$ shares many properties with the Deligne cohomology group, so our statement can be viewed as a non-Archimedean analogue of this. In general the Hodge-$\mathcal{D}$ conjecture is false [MS97].

An S-integral version of the Beilinson conjectures, or a special case of the Tamagawa number conjecture of Bloch and Kato, would assert that, for a variety over a number field, there are elements in the higher Chow group over the number field itself which bound the elements of the Chow groups of the special fibres. The only case for which there is some evidence is the work of Bloch and Grayson [BG86] on $CH^2(E,2)$ of elliptic curves, but even here, as far as I am aware, there is not a single case of an elliptic curve over $\mathbb{Q}$ where it is known that there are as many elements of the group as would be predicted by the full S-integral Beilinson conjecture.

On the other hand, Beilinson [Beĭ84] proved his conjecture for the product of two non-isogenous elliptic curves over $\mathbb{Q}$, and this can be viewed as a statement for the Archimedean prime. The common wisdom [Man91] is that the Archimedean prime behaves like a prime of semi-stable reduction. Further, the work of Mildenhall [Mil92] provides evidence for this conjecture when v is a prime of good reduction and E_1 and E_2 are isogenous elliptic curves over $\mathbb{Q}$. So one might hope that one can extend these results to the case when E_1 and E_2 are not isogenous and v is a prime of semi-stable (that is, split multiplicative) reduction for both of them, but the naive generalization does not seem to work. Hence we were forced to consider the local situation.

The outline of the paper is as follows. In the first section we define the group PCH that appears as the target of the boundary map. We then specialize to the product of two semi-stable elliptic curves and describe the fibre of the semi-stable model of the product of the two curves and the group PCH in this case. Then we use the work of Frey and Kani on the existence of curves of genus 2 on products of elliptic curves along with Speiß's work to construct some elements in the higher Chow group. Finally we compute their boundary and show that they suffice to prove surjectivity.

The method of proof is almost identical to that of Spieß, the only difference being that we have to modify his arguments appropriately to work in the case of bad reduction. He obtains some consequences for codimension 2 cycles on $\mathcal{X}$ which follow from the surjectivity of ∂, so they apply in our case as well.

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2 Preliminaries

Let X be a smooth proper variety over a local field K and $\mathcal{O}$ the ring of integers of K with closed point v and generic point η.

By a model $\mathcal{X}$ of X we mean a flat proper scheme $\mathcal{X} \to Spec(\mathcal{O})$ together with an isomorphism of the generic fibre X_η with X. Let Y be the special fibre $\mathcal{X}_v = \mathcal{X} \times Spec(k(v))$. We will always also make the assumption that the model is strictly semi-stable, which means that it is a regular model and the fibre Y is a divisor with normal crossings, the components have multiplicity one and they intersect transversally. Let $i : Y \hookrightarrow \mathcal{X}$ denote the inclusion map.

2.1 Consani's Double Complex

In [Con98], Consani defined a double complex of Chow groups of the components of the special fibre with a monodromy operator N following the work of Steenbrink [Ste76] and Bloch-Gillet-Soulé [BGS95]. Using this complex she was able to relate the higher Chow group of the special fibre at a semi-stable prime to the regular Chow groups of the components. This relation is what is used in defining the group PCH.

Let $Y = \bigcup_{i=1}^{t} Y_i$ be the special fibre of dim n with Y_i its irreducible components. For $I \subset \{1, \ldots, t\}$, define

$$Y_I = \cap_{i \in I} Y_i$$

Let $r = |I|$ denote the cardinality of I. Define

$$Y^{(r)} := \begin{cases} \mathcal{X} & \text{if } r = 0 \\ \coprod_{|I|=r} Y_I & \text{if } 1 \le r \le n \\ \emptyset & \text{if } r > n \end{cases}$$

For u and t with $1 \le u \le t < r$ define the map

$$\delta(u) : Y^{(t+1)} \to Y^{(t)}$$

as follows. Let $I = (i_1, \ldots, i_{t+1})$ with $i_1 < i_2 < \ldots < i_{t+1}$. Let $J = I - \{i_u\}$. This gives an embedding $Y_I \to Y_J$. Putting these together induces the map $\delta(u)$. Let $\delta(u)_*$ and $\delta(u)^*$ denote the corresponding maps on Chow homology and cohomology respectively. They further induce the Gysin and restriction maps on the Chow groups.

Define

$$\gamma := \sum_{u=1}^{r+1} (-1)^{u-1} \delta(u)_*$$

and

$$\rho := \sum_{u=1}^{r+1} (-1)^{u-1} \delta(u)^*$$

These maps have the properties that

- $\gamma^2 = 0$
- $\rho^2 = 0$
- $\gamma \cdot \rho + \rho \cdot \gamma = 0$

2.2 The group PCH

Let a, q be two integers with $q - 2a > 0$.

$$PCH^{q-a-1}(Y, q-2a-1) := \begin{cases} \frac{Ker(i^* i_* : CH_{n-a}(Y^{(1)}) \to CH^{a+1}(Y^{(1)}))}{Im(\gamma : CH_{n-a}(Y^{(2)}) \to CH_{n-a}(Y^{(1)}))} \otimes \mathbb{Q} & \text{if } q - 2a = 1 \\ \\ \frac{Ker(\gamma : CH_{n-(q-a-1)}(Y^{(q-2a)}) \to CH_{n-(q-a-1)}(Y^{(q-2a-1)}))}{Im(\gamma : CH_{n-(q-a-1)}(Y^{(q-2a+1)}) \to CH_{n-(q-a-1)}(Y^{(q-2a)}))} \otimes \mathbb{Q} & \text{if } q - 2a > 1 \end{cases}$$

Here n is the dimension of Y. Note that if $q-2a > 1$ and Y is non-singular, this group is 0, while if Y is singular and semi-stable, the Parshin-Soulé conjecture implies that this group is $CH^{q-a-1}(Y, q-$

$2a-1)\otimes\mathbb{Q}$. If $q-2a=1$ and Y is non-singular, the group is $CH^a(Y)\otimes\mathbb{Q}$. Our interest is in the remaining case, namely when $q-2a=1$ and Y is singular.

The 'Real' Deligne cohomology has the property that its dimension is the order of the pole of the Archimedean factor of the L-function at a certain point on the left of the critical point. The group $PCH^1(Y)$ has a similar property. Let F^* be the geometric Frobenius and $N(v)$ the number of elements of $k(v)$. The local L-factor of the $(q-1)^{st}$-cohomology group is then

$$L_v(H^{q-1}(X),s)=(det(I-F^*N(v)^{-s}|H^{q-1}(\bar{X},\mathbb{Q}_\ell)^I))^{-1}$$

Theorem 2.1 (Consani). *Let v be a place of semistable reduction. Assuming the weight-monodromy conjecture, the Tate conjecture for the components and the injectivity of the cycle class map on the components Y_I, the Parshin-Soulé conjecture and that F^* acts semisimply on $H^*(\bar{X},\mathbb{Q}_\ell)^I$. we have*

$$\dim_{\mathbb{Q}} PCH^{q-a-1}(Y,q-2a-1)=-\operatorname{ord}_{s=a} L_v(H^{q-1}(X),s):=d_v$$

Proof. [Con98], Cor 3.6. □

From this point of view the group $PCH^{q-a-1}(Y,q-2a-1)$ can be viewed as a non-Archimedean analogue of the 'Real' Deligne cohomology. Since the L-factor at a prime of good reduction does not have a pole at $s=a$ when $q-2a>1$, the Parshin-Soulé conjecture can be interpreted as the statement that this non-Archimedean Deligne cohomology has the correct dimension, namely 0, even at a prime of good reduction.

Remark 2.2. *As is clear from the definition, the group PCH depends on the choice of the semi-stable model of X. However, Consani's theorem says that the dimension does not. So to a large extent one can work with any semi-stable model. Perhaps the correct definition is one obtained by taking a limit of semi-stable models as in the work of Bloch,Gillete and Soulé [BGS95] on non-Archimedean Arakelov theory.*

From this point on we specialize to the case when X is a surface and further $n=2$,$q=3$ and $a=1$. We will be interested in group $CH^2(X,1)$ and the map to $PCH^1(Y):=PCH^1(Y,0)$. This is related to the order of the pole of the L-function of $H^2(X)$ at $s=1$. Soon we will further specialize to the case when $X=E_1\times E_2$.

2.3 Elements of the higher chow group

The group $CH^2(X,1)$ has the following presentation [Ram89]. It is generated by formal sums of the type

$$\sum_i (C_i,f_i)$$

where C_i are curves on X and f_i are $\bar{K}$-valued functions on the C_i satisfying the cocycle condition

$$\sum_i \operatorname{div} f_i=0.$$

Relations in this group are give by the tame symbol of pairs of functions on X.

There are some decomposable elements of this group coming from the product structure

$$CH^1(X)\otimes CH^1(X,1)\longrightarrow CH^2(X,1)$$

A theorem of Bloch's [Blo86] says that $CH^1(X,1)$ is simply K^* where K is the field of definition of X so such an element looks like a sum of elements of the type (C,a) where C is a curve on X and

a is in K^*. More generally, an element is said to be decomposable if it can be written as a sum of products as above over possibly an extension of the base field. Elements which are not decomposable are said to be indecomposable.

The group $CH^2(X,1) \otimes \mathbb{Q}$ is the same as the $\mathcal{K}$-cohomology group $H^1_{Zar}(X,\mathcal{K}_2) \otimes \mathbb{Q}$ and the motivic cohomology group $H^3_{\mathcal{M}}(X,\mathbb{Q}(2))$.

2.4 The boundary map

The usual Beilinson regulator maps the higher chow group to the Real Deligne cohomology. In the non-Archimedean context, it appears that the boundary map

$$\partial : CH^2(X,1) \longrightarrow PCH^1(Y)$$

plays a similar role. It is defined as follows

$$\partial(\sum_i (C_i, f_i)) = \sum_i \operatorname{div}_{\bar{C}_i}(f_i)$$

where $\bar{C}_i$ denotes the closure of C_i in the semi-stable model $\mathcal{X}$ of X. By the cocycle condition, the 'horizontal divisors' namely, the closure of $\sum_i \operatorname{div}_{C_i}(f_i)$ cancel out and the result is supported on the special fibre. Further, since the boundary ∂ of an element is the sum of divisors of functions, it lies in $Ker(i^* i_*)$.

For a decomposable element of the form (C,a) the regulator map is particularly simple to compute,

$$\partial((C,a)) = \operatorname{ord}_v(a) C_v$$

3 Products of Elliptic Curves

From now on we specialize to the case when $X = E_1 \times E_2$ where E_1 and E_2 are elliptic curves over a local field K of residue characteristic v with semi-stable reduction at v. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ denote the Néron minimal models of E_1 and E_2 over $S = Spec(\mathcal{O})$ respectively. The special fibre at v of the E_i are Néron polygons -

$$\mathcal{E}_{i,v} = \cup_{j=0}^{k_i-1} \mathcal{E}^j_{i,v}$$

where k_i denotes the number of components of the special fibre of $\mathcal{E}_i$. Each $\mathcal{E}^j_{i,v} \simeq \mathbb{P}^1$. Let $\mathcal{E}^0_{i,v}$ denote the identity component - that is the component which intersects the 0-section.

3.1 Semi-stable models of elliptic curves

In this section we describe the semi-stable model of the product of elliptic curves. The product of semi-stable models of E_1 and E_2 is unfortunately not semi-stable - one has to blow up certain points lying on the intersection of the products of the components . Locally, one has the following description [Con99]:

Lemma 3.1. *Let $z_1 z_2 = w_1 w_2$ be a local description of $\mathcal{E}_1 \times_S \mathcal{E}_2$ around the point (P,Q) where P and Q are double points lying on the intersection of two components of the special fibre of $\mathcal{E}_{i,v}$, say $P \in \mathcal{E}^0_{1,v} \cap \mathcal{E}^1_{1,v}$ and $Q \in \mathcal{E}^0_{2,v} \cap \mathcal{E}^1_{2,v}$. After a blow up of $\mathcal{E}_1 \times \mathcal{E}_2$ with center at the origin (z_1,z_2,w_1,w_2) the resulting degeneration $\psi : \mathcal{Z} \rightarrow S$ is normal crossings. The special fibre Y is the union of five irreducible components $Y = \cup_{i=1}^5 Y_i$. We label them as follows $Y_1 = (\mathcal{E}^0_{1,v} \times \mathcal{E}^0_{2,v})$, $Y_2 = (\mathcal{E}^0_{1,v} \times \mathcal{E}^1_{2,v})$, $Y_3 = (\mathcal{E}^1_{1,v} \times \mathcal{E}^0_{2,v})$, $Y_4 = (\mathcal{E}^1_{1,v} \times \mathcal{E}^1_{2,v})$ and Y_5 is the exceptional divisor. The $Y_i' s, 1 = \{1,\dots,4\}$ are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ while Y_5 is isomorphic to $\mathbb{P}^2$.*

Proof. [Con99]- Lemma 4.1. □

Needless to say, we have to repeat this construction at every double point. We use this description to compute the group $PCH^1(Y)$ in our case. So the group in question is

$$PCH^1(Y) = \frac{Ker(i^*i_* : CH_1(Y^{(1)}) \to CH^2(Y^{(1)}))}{Im(\gamma : CH_1(Y^{(2)}) \to CH_1(Y^{(1)}))} \otimes \mathbb{Q}$$

$Y^{(1)}$ consists of the disjoint union of the components Y_i and $Y^{(2)}$ consists of their pairwise intersections. From the description above we see that it is made up of 'horizontal components' $\mathcal{E}^j_{1,v} \times Q$ or 'vertical components' $P \times \mathcal{E}^j_{2,v}$ or curves $\mathfrak{F} \simeq \mathbb{P}^1$ lying on the exceptional divisor, one for each blow up.

Lemma 3.2. . *When $X = E_1 \times E_2$ the group $PCH^1(Y)$ is three dimensional, generated by $\mathfrak{E}_1 = \psi^*(\mathcal{E}^0_{1,v} \times Q)$,$\mathfrak{E}_2 = \psi^*(P \times \mathcal{E}^0_{2,v})$ and a curve on an exceptional divisor $\mathfrak{F}$.*

Proof. Notice that it suffices to restrict ourselves to the local situation above restricted further to just one of the Y_i from $i = 1 \dots 4$ and the exceptional divisor. This is because the cycles of the form $\mathcal{E}^i_{1,v} \times Q - \mathcal{E}^j_{1,v} \times Q'$, namely differences between two 'horizontal' components, similarly for the 'vertical' components and the difference of two curves in the exceptional fibres over two different points, all lie in the image of the Gysin map γ. As we are dividing out by this image, they are equivalent and it suffices to consider only the Chow group of the product of the identity components and the exceptional divisor over the origin of the product. This Chow group is generated by the three cycles $\mathfrak{E}_1, \mathfrak{E}_2$ and $\mathfrak{F}$.

□

Remark 3.3. *When $E_1 \times E_2$ are elliptic curves over $\mathbb{Q}$ the dimension of the Real Deligne cohomology, which is the target of the Beilinson regulator map, is also three dimensional. In that case it is easy to find cycles which bound two of the three generators. The third requires more work - one has to use the modular parametrization [Beĭ84].*

To prove surjectivity, therefore, we have to find three elements of the Chow group $CH^2(E_1 \times E_2, 1) \otimes \mathbb{Q}$ which bound these three generators.

3.2 Genus two curves on products of elliptic curves

Speiß [Spi99] constructed an element of the higher chow group using a genus two curve on the generic fibre. We show that his construction can be used in our case of semistable reduction as well. We have to use some work of Frey and Kani [FK91] on the existence of irreducible genus two curves whose Jacobian is isogenous to a product of elliptic curves.

Theorem 3.4 (Frey and Kani). *Let K be a local field with residual characteristic v and E_1 and E_2 two elliptic curves over K. Let n be an odd integer and $\phi : E_1[n] \to E_2[n]$ which is a K-rational anti-isometry with respect to the Weil pairings, that is $e_n(\phi(x), \phi(y)) = e_n(x, y)^{-1}$. Let $J = E_1 \times E_2/graph(\phi)$ and $p : E_1 \times E_2 \to J$ the projection. Then there exists a unique curve $C \subset J$ defined over K such that the following holds.*

- *C is a stable curve of genus two in the sense of Deligne and Mumford.*
- *$-id^*(C) = C$.*

- *Let λ_C denote the map from $J \to \check{J}$ induced by the line bundle corresponding to C. Then the composite maps,*
$$\pi_i : C \xrightarrow{j} J \xrightarrow{\lambda_C} \check{J} \xrightarrow{\check{p}} E_1 \times E_2 \longrightarrow E_i \ \textit{i=1,2}$$
are finite morphisms of degree n.

Proof. [FK91], Proposition [1.3]. □

We now apply this criterion in a special case, choosing n judiciously so as to ensure that we bound the right cycle. This is a variation of the method used in [Spi99], Lemma [3.3].

Let a be an integer and $n = a^2 + 1$. Choose a such that $(a, v) = (n, v) = 1$ and n is odd. Extend K to a field where all the n-torsion of E_1 and E_2 are defined. From the theory of Néron models, we have that n then divides the number of components k_i of the special fibres $\mathcal{E}_{i,v}$ of the Néron models $\mathcal{E}_i$ of E_i.

As a group, the special fibre $\mathcal{E}_{i,v}$ is isomorphic to $\mathbb{G}_m \times \mathbb{Z}/k_i\mathbb{Z}$. We will denote an element of $\mathcal{E}_{i,v}$ by $(x; m)$ with $x \in \mathbb{G}_m$ and $m \in \mathbb{Z}/k_i\mathbb{Z}$. Let h_a denote the isogeny

$$h_a : \mathcal{E}_{1,v} \longrightarrow \mathcal{E}_{2,v}$$

$$h_a((x; m)) = (x^a; am)$$

where multiplication by a is to be understood as the action of the class of a in $\mathbb{Z}/(k_1, k_2)\mathbb{Z}$ which is identified with $Hom(\mathbb{Z}/k_1\mathbb{Z}, \mathbb{Z}/k_2\mathbb{Z})$. Since $n|(k_1, k_2)$ this group is non-trivial.

Let $h_a[n] : \mathcal{E}_{1,v}[n] \longrightarrow \mathcal{E}_{2,v}[n]$ denote the restriction of h_a to the n-torsion points. Since $(n, v) = 1$ the groups $E_i[n]$ and $\mathcal{E}_{i,v}[n]$ are isomorphic [ST68]. So the map $h_a[n]$ lifts to a map $\phi_a : E_1[n] \to E_2[n]$.

Lemma 3.5. *The map ϕ_a is an anti-isometry with respect to the Weil pairing e_n*

Proof. If X and Y are two points in $E_1[n]$ mapping to $(x; m_x)$ and $(y; m_y)$ in $\mathcal{E}_{1,v}[n]$ respectively we have
$$e_n(\phi_a(X), \phi_a(Y)) = e_n(h_a((x; m_x)), h_a((y; m_y))) = e_n((x; m_x), \check{h_a} \circ h_a((y; m_y)))$$
$$= e_n((x; m_x), (y^{a^2}; a^2 m_y)) = e_n((x; m_x), (y^{n-1}; (n-1)m_y)) = e_n((x; m_x), (y^{-1}; -m_y)) = e_n(x, y)^{-1}$$
as Y is in the n torsion, so $(n-1)m_y = -m_y$ and $y^{n-1} = y^{-1}$. The third equality is because the dual isogeny $\check{h_a}$ is the adjoint of h_a with respect to the Weil pairing. Further, $\check{h_a} \circ h_a$ is simply multiplication by $a^2 = n - 1$.

□

From the theorem of Frey and Kani with $\phi = \phi_a$ we get a corresponding stable genus 2 curve C and finite morphisms $\pi_i : C \to E_i$ of degree n. C is a principal polarization on $J = E_1 \times E_2/(graph\phi)$. It satisfies the additional property that $p^*(C) \sim n\Theta$, where $\Theta = E_1 \times 0 \cup 0 \times E_2$. Further, it is the unique curve satisfying that as well as $-id^*(C) = C$.

We would like to understand the special fibre of the closure of this curve in a semistable model of $E_1 \times E_2$. We first describe what happens in the product of the two Néron models $\mathcal{E}_1 \times \mathcal{E}_2$ and then describe its image in the semi-stable model of $E_1 \times E_2$ constructed in section 3.1. Let $\mathcal{C}$ denote the closure of C in the product of $\mathcal{E}_1 \times \mathcal{E}_2$ of the Néron models of E_1 and E_2.

Proposition 3.6. *The special fibre $\mathcal{C}_v$ of $\mathcal{C}$ is reducible and is isomorphic to $\mathcal{E}_{1,v} \sqcup_1 \mathcal{E}_{2,v}$, namely a union of two curves isomorphic to $\mathcal{E}_{1,v}$ and $\mathcal{E}_{2,v}$ which meet transversally at $((1; 0), (1; 0))$, the identity, in the product of $\mathcal{E}^0_{1,v}$ and $\mathcal{E}^0_{2,v}$, the product of the identity components, and nowhere else. The finite maps $\tilde{\pi}_i : \mathcal{C}_v \longrightarrow \mathcal{E}_{i,v}$ are given by $\tilde{\pi}_1 = id \sqcup_1 -\check{h}_a$ and $\tilde{\pi}_2 = h_a \sqcup_1 id$.*

Proof. We follow the argument in Spieß[Spi99] mutatis mutandis. The idea is to show that the special fibre has the property that it is a genus two curve such that it is invariant under $-id$ and then appeal to the uniqueness property of Frey-Kani [FK91], Proposition 1.1.

Let

$$p = \begin{pmatrix} id & \check{h_a} \\ -h_a & id \end{pmatrix} : \mathcal{E}_{1,v} \times \mathcal{E}_{2,v} \to \mathcal{E}_{1,v} \times \mathcal{E}_{2,v}$$

Let $X = (x; m_x)$ be an element of $\mathcal{E}_{1,v}[n]$. Then

$$p(X, h_a[n](X)) = (X.\check{h_a} \circ h_a(X), -h_a(X).h_a(X))$$

$$= ((x^{1+a^2}; (1+a^2)m_x), (x^{-a+a}, (-a+a)m_x)) = ((1;0),(1;0))$$

as $a^2 + 1 = n$ and X is in the n-torsion. So $graph(h_a[n]) \subset ker(p)$. Similarly, we can see that the kernel of

$$\check{p} = \begin{pmatrix} id & -\check{h_a} \\ h_a & id \end{pmatrix} : \mathcal{E}_{1,v} \times \mathcal{E}_{2,v} \to \mathcal{E}_{1,v} \times \mathcal{E}_{2,v}$$

contains $graph(\check{h_a}[n])$. Since $\check{p} \circ p = [n]$ we have

$$|ker(\check{p} \circ p)| = |ker(p)|^2 = |ker([n])| = n^4 = |graph(h_a[n])|^2$$

so $ker(p) = graph(h_a[n])$ and we can identify the image of p with $\mathcal{E}_{1,v} \times \mathcal{E}_{2,v}/(graph(h_a[n])$.

Let $\Theta_v = \mathcal{E}_{1,v} \times (1;0) \cup (1;0) \times \mathcal{E}_{2,v}$. This is a stable genus 2 curve on $\mathcal{E}_{1,v}\mathcal{E}_{2,v}$. Clearly $-id^*(\Theta_v) = \Theta_v$

Lemma 3.7. $p^*(\Theta_v) \sim n\Theta_v$.

Proof. Let Γ_{h_a}, respectively $\Gamma^t_{-\check{h_a}}$ denote the graphs of the maps (id, h_a), respectively $(-\check{h_a}, id)$ from $\mathcal{E}_{1,v}$, respectively $\mathcal{E}_{2,v}$ to $\mathcal{E}_{1,v} \times \mathcal{E}_{2,v}$. Since the diagrams

$$\begin{array}{ccc} \mathcal{E}_{1,v} & \xrightarrow{(id,h_a)} & \mathcal{E}_{1,v} \times \mathcal{E}_{2,v} \\ {\scriptstyle [n]}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{E}_{1,v} & \xrightarrow{(id,0)} & \mathcal{E}_{1,v} \times \mathcal{E}_{2,v} \end{array} \qquad \begin{array}{ccc} \mathcal{E}_{2,v}, & \xrightarrow{(-\check{h_a},id)} & \mathcal{E}_{1,v} \times \mathcal{E}_{2,v} \\ {\scriptstyle [n]}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{E}_{2,v} & \xrightarrow{(id,0)} & \mathcal{E}_{1,v} \times \mathcal{E}_{2,v} \end{array}$$

are cartesian, we have

$$p^*(\Theta_v) = \Gamma_{h_a} \cup \Gamma^t_{-\check{h_a}}$$

Since the closure of $\mathcal{E}_{1,v} \times \mathcal{E}_{2,v}$ is a union of $\mathbb{P}^1 \times \mathbb{P}^1$'s the divisor $p^*(\Theta)$ can be written as a sum $p_1^*(D_1) + p_2^*(D_2)$ where p_1 and p_2 are the projection maps to $\mathcal{E}_{1,v}$ and $\mathcal{E}_{2,v}$ respectively and D_1 and D_2 are divisors on $\mathbb{P}^1$.

We have

$$[\Gamma_{h_a}].[\mathcal{E}_{1,v} \times (1;0)] = [Ker(h_a) \times (1;0)]$$

$$[\Gamma_{h_a}].[(1;0) \times \mathcal{E}_{2,v}] = [(1;0) \times (1;0)] = [\Gamma^t_{-\check{h_a}}].[\mathcal{E}_{1,v} \times (1;0)]$$

$$[\Gamma^t_{-h_a}].[(1;0) \times Ker(-\check{h_a})]$$

From this we get,

$$D_1 \sim (p_1)_*(p_1^*(D_1) + p_2^*(D_2)) = [Ker(h_a)] + [(1;0)]$$

and similarly $D_2 \sim [(1;0)] + [Ker(-\check{h_a})]$. Since any two points on $\mathbb{P}^1$ are equivalent, we have D_1 and D_2 are equivalent to $(a^2+1)[(1;0)] = n[(1;0)]$. So we have

$$p^*(\Theta_v) = p_1^*(n[(1;0)]) + p_2^*(n[(1;0)]) = n(\mathcal{E}_{2,v} + \mathcal{E}_{1,v}) = n\Theta_v$$

□

The map λ_C and λ_Θ, which are isomorphisms from $J \to \check{J}$ and $E_1 \times E_2 \to (E_1 \check{\times} E_2)$ induced by the principal polarizations C and Θ extend to isomorphisms of the Néron models and in particular, induce isomorphisms of the special fibres. The map p induces a homomorphism $\check{p} : \check{J} \to (E_1 \check{\times} E_2)$ which is the same as p^* on the divisors of degree 0. Let p' be the homomorphism $p' = \Theta^{-1} \circ \check{p} \circ \lambda_C$ extended to induce a homomorphism of the special fibres. From the definition, it is easy to see $Ker(p')$ is contained in the n-torsion.

If $\mathcal{C}'_v$ is a stable genus 2 curve satisfying $p^*(\mathcal{C}'_v) = n\Theta_v$ then one has $\mathcal{C}'_v = T_x(\mathcal{C}_v)$ for some x in $Ker(p')$. As n is odd, if $\mathcal{C}'_v$ further satisfies the condition that $-id^*(\mathcal{C}'_v) = \mathcal{C}'_v$ then $\mathcal{C}'_v \simeq \mathcal{C}_v$, otherwise it would imply that there is an element of 2-torsion in $Ker(p')$. Since Θ_v satisfies this additional condition, $\Theta_v \simeq \mathcal{C}_v$. The rest of the theorem follows by chasing the definitions of the various maps.

□

3.3 A new element

Using the genus 2 curve constructed above we can get several new elements of $CH^2(E_1 \times E_2, 1)$ - one for every choice of pair of Weierstrass points on the genus two curve. The construction is as follows. Let $\mathcal{C}'$ be a minimal regular model of C. From Parshin [Par72] we have a description of the special fibre as well as a description of the closure of the Weierstraß points on the special fibre. In our case the special fibre has the following description (VI, in Parshin's notation) - there are two genus 0 curves, B_1 and B_2 with self intersection -3. To each of these is attached a chain of genus 0 curves $X_i, i = \{1, \dots, r\}$ and $Z_k, k = \{1 \dots t\}$,with t and r odd, respectively with self intersection -2, such that each curve intersects the neighboring two curves at a single point. In other words, these are the Néron special fibres of semistable elliptic curves. The two semi-stable fibres of elliptic curves are joined by a chain of genus 0 curves $Y_j, j = \{1, \dots, s\}$ with self intersection -2 which meet at the identity components. So in particular, $r = k_1 - 1$, $t = k_2 - 1$ and B_1 and B_2 correspond to the identity components of $\mathcal{E}_{1,v}$ and $\mathcal{E}_{2,v}$ respectively.

The closure of the Weierstaß points is as follows - one point lies on each B_1 and B_2 and two points each intersect the components $X_{\frac{s+1}{2}}$ and $Z_{\frac{t+1}{2}}$.

In particular, we have a function $f_{P,Q}$ on C such that the closure of P lies on B_1 and the closure of Q lies on B_2. The divisor of $f_{P,Q}$ on $\mathcal{C}'$ can be expressed in terms of the components above

$$\mathrm{div}_{\mathcal{C}'}(f_{P,Q}) = \mathcal{H} + a_1 B_1 + \sum_{i=1}^{r} b_i X_i + \sum_{j=1}^{s} c_j Y_j + \sum_{k=1}^{t} d_k Z_k + a_2 B_2$$

where $\mathcal{H}$ is the closure of the divisor $2(P) - 2(Q)$ - that is, the horizonal component. Multiplying $f_{P,Q}$ by a power of the uniformizer π one can assume that $a_2 = 0$ as $\mathrm{div}_{\mathcal{C}'}(\pi) = \mathcal{C}'_v$.

Lemma 3.8. *If $f_{P,Q}$ is as above with $a_2 = 0$ then $a_1 \neq 0$.*

Proof. Since $\mathcal{C}'$ is a minimal regular model we can use the intersection theory of arithmetic surfaces described, for example, in Lang [Lan88], Chapter 3. In particular, we have that the intersection number

$$(\mathrm{div}_{\mathcal{C}'}(f_{P,Q}).D) = 0$$

for *any* divisor D contained in the special fibre. Applying this to different choices of D, namely $D = B_i, X_i, Y_i, Z_i$ and using what we know of their intersections and self-intersections gives us the following set of equations -

$$-3a_1 + b_1 + b_r + c_1 + 2 = 0$$

$$a_1 - 2b_1 + b_2 = 0$$

$$b_{i-1} - 2b_i + b_{i+1} = 0 \quad \{i = 2 \ldots r-1\}$$

$$b_{r-1} - 2b_r + a_1 = 0$$

$$a_1 - 2c_1 + c_2 = 0$$

$$c_{j-1} - 2c_j + cj + 1 = 0 \quad \{j = 2 \ldots s-1\}$$

$$c_{s-1} - 2c_s = 0$$

$$-2d_1 + d_2 = 0$$

$$d_{k-1} - 2d_k + d_{k+1} = 0 \quad \{k = 2 \ldots t-1\}$$

$$d_{t-1} - 2d_t = 0$$

$$c_s + d_1 + d_t - 2 = 0$$

Solving these equations shows $d_k = 0, k = \{1, \ldots, t\}$, $c_j = 2(s+1-j)$, so in particular $c_s = 2$,$c_1 = 2s$ and finally $a_1 = b_i, i = \{1 \ldots r\} = 2(s+1)$. In particular, since $s \geq 0$ we have $a_1 \neq 0$.

$\square$

Recall that we have maps $\pi_i : C \to E_i$. Let $P_i = \pi_i(P)$ and $Q_i = \pi_i(Q)$. There are functions f_1 on $E_1 \times P_2$ and f_2 on $Q_1 \times E_2$ with

$$\mathrm{div}(f_1) = 2(P_1, P_2) - 2(Q_1, P_2) \text{ and } \mathrm{div}(f_2) = 2(Q_1, P_2) - 2(Q_1, Q_2)$$

On the closure, by the description of the maps π_i on the special fibre, both P and Q map to the identity components of the special fibres $\mathcal{E}_i^0$ of E_i. Hence the divisors of f_i in the semistable model of $E_1 \times E_2$ do not contain any components of the special fibre.

Define $\Xi = \Xi_{P,Q}$ be the cycle

$$\Xi = (C, f_{P,Q}) + (E_1 \times P_2, f_1^{-1}) + (Q_1 \times E_2, f_2^{-1})$$

From the definition of P_i and Q_i we have

$$\mathrm{div}_C(f_{P,Q})) - \mathrm{div}(f_1) - \mathrm{div}(f_2)$$

$$= 2(P_1, P_2) - 2(Q_1, Q_2) - 2(P_1, P_2) + 2(Q_1, P_2) - 2(Q_1, P_2) + 2(Q_1, Q_2) = 0$$

hence Ξ is an element of $CH^2(E_1 \times E_2, 1)$. In the next section we compute its boundary.

3.4 Surjectivity of the boundary map

In this section we compute the image of the elements under the boundary map

$$\partial : CH^2(E_1 \times E_2, 1) \to PCH^1(Y)$$

We have two decomposable elements, $(E_1 \times 0, \pi)$ and $(0 \times E_2, \pi)$, whose boundary is

$$\partial((E_1 \times 0, \pi)) = \psi^*(\mathcal{E}_{1,v} \times (1;0)) = k_1 \mathfrak{E}_1$$

$$\partial((0 \times E_2, \pi)) = \psi^*((1;0) \times \mathcal{E}_{2,v}) = k_2 \mathfrak{E}_2$$

We also have the third element $\Xi_{P,Q}$. To compute its boundary observe that under the map $\mathcal{C}' \to \mathcal{C}$ the components of the special fibre Y_i collapse to a point. Further, the X_i and B_1 map on to the graph Γ_{h_a} and similarly, the Z_j and B_2 map to $\Gamma^t_{-\check{h}_a}$. Since the points at which the curves X_i meet each other or B_1 are being blown up, in the semi-stable fibre one gets a copy of $\mathfrak{F}$ for each such point. There are k_1 components, and in each component $\mathcal{E}^i_{1,v} \times \mathcal{E}^{ai}_{2,v}$ the component of Γ_{h_a} is $\mathcal{E}^i_{1,v} + a^2 \mathcal{E}^{ai}_{2,v}$. So combining this with Lemma 3.8, we have

$$\partial(\Xi_{P,Q}) = (2s+2)k_1(\mathfrak{E}_1 + a^2 \mathfrak{E}_2 + \mathfrak{F})$$

as the horizonal divisor $\mathcal{H}$ gets canceled out by the divisors of f_i. From this and a little bit of elementary linear algebra we have

Theorem 3.9. *Suppose E_1 and E_2 are two non-isogenous elliptic curves over a local field K with split multiplicative reduction at the closed point v. Let $\mathcal{X}$ be a semi-stable model of the product $E_1 \times E_2$ and $\mathcal{X}_v$ denote the special fibre. Then the map*

$$\partial : CH^2(E_1 \times E_2, 1) \otimes \mathbb{Q} \longrightarrow PCH^1(\mathcal{X}_v) \otimes \mathbb{Q}$$

is surjective.

4 Final Remarks

Let Σ be the group,

$$\Sigma = Ker(CH^2(\mathcal{X}) \to CH^2(X)).$$

Spieß [Spi99], Section 4, describes some consequence of the assumption that it is a torsion group. Surjectivity of the map ∂ implies this, in fact, it implies finiteness, so all the consequences apply in our case.

This paper began as an attempt to prove the S-integral Beilinson conjecture when X is a product of two non-isogenous modular elliptic curves over $\mathbb{Q}$. This remains to be done - unfortunately, while our and Spieß' elements can be lifted to number fields, they cannot be used to produce surjectivity as there may be several primes at which their boundary is non-trivial, so at best one can get relations between codimension 2 cycles of the type described in the previous paragraph. In the case of isogenous elliptic curves, Mildenhall's elements have a boundary at precisely one prime. Unfortunately, a direct generalization of his work does not seem to work as while one can construct elements in the product of modular curves which have a boundary at precisely one prime, the projections of these elements on to the product of non-isogenous elliptic curves seems to be trivial [Sch00].

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