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by

Hatice Boylan
Nils-Peter Skoruppa
Haigang Zhou

Max-Planck-Institut für Mathematik
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Hatice Boylan
Nils-Peter Skoruppa
Haigang Zhou

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Istanbul Üniversitesi
Fen Fakültesi
Matematik Bölümü
34134 Vezneciler, Istanbul
Turkey

Universität Siegen
Fachbereich Mathematik
Walter-Flex-Str. 3
57072 Siegen
Germany

Tongji University
School of Mathematical Sciences
1239 Siping Road
Shanghai
P.R. China

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COUNTING ZEROS IN QUATERNION ALGEBRAS USING JACOBI FORMS

HATİCE BOYLAN, NILS-PETER SKORUPPA, AND HAIGANG ZHOU

Abstract. We use the theory of Jacobi forms to study the number of elements in a maximal order of a definite quaternion algebra over the field of rational numbers whose characteristic polynomial equals a given polynomial. A certain weighted average of such numbers equals (up to some trivial factors) the Hurwitz class number $H(4n-r^2)$. As a consequence we obtain new proofs for Eichler's trace formula and for formulas for the class and type number of definite quaternion algebras. As a secondary result we derive explicit formulas for Jacobi Eisenstein series of weight 2 on $\Gamma_0(N)$ and for the action of Hecke operators on Jacobi theta series associated to maximal orders of definite quaternion algebras.

1. Introduction and statement of result

A Jacobi form of scalar index 1 has a Fourier expansion in terms of powers of the elementary functions $q(\tau) = e^{2\pi i\tau}$ and $\zeta(z) = e^{2\pi iz}$. Its Fourier coefficients $c(n, r)$ are indexed by pairs of integers n and r, and it is a basic fact that $c(n, r)$ depends only on $D = 4n-r^2$ and is zero unless $D \geq 0$. Natural sequences $c(n, r)$ of this kind are obtained by counting the zeros of quadratic polynomials $x^2 - rx + n$ with non-positive discriminant in, say, some given ring $\mathcal{O}$, provided, of course, the number of zeros is finite. If we denote the number of zeros of x^2-rx+n by $\rho_{\mathcal{O}}(n, r)$, one can ask when

$$\theta_{\mathcal{O}} := \sum_{\substack{n,r\in\mathbb{Z}\\ 4n-r^2\geq 0}} \rho_{\mathcal{O}}(n, r)\, q^n\zeta^r \tag{1}$$

is a Jacobi form.

We might take as a first test for $\mathcal{O}$ the ring $\mathbb{Z}$ of integers itself. Then, for $D = 4n - r^2 \geq 0$, we have $\rho_{\mathbb{Z}}(n, r) = 0$ unless $D = 0$, and we obtain

$$\theta_{\mathbb{Z}} = \sum_{s\in\mathbb{Z}} q^{s^2}\zeta^{2s},$$

which is indeed a Jacobi form. In fact, $\theta_{\mathbb{Z}}$ is an element of the space $J_{1/2,1}(4)$ of Jacobi forms of weight 1/2 and index 1 on $\Gamma_0(4)$.

As another example consider the maximal order $\mathbb{Z}_\Delta$ of the imaginary quadratic number field of discriminant Δ. Here $\rho_{\mathbb{Z}_\Delta}(n, r) = 0$ unless $D = 4n - r^2 = -\Delta f^2$ for some integer f, and we obtain

$$\theta_{\mathbb{Z}_\Delta} = \sum_{\substack{r,f\in\mathbb{Z}\\ r^2\equiv\Delta f^2 \bmod 4}} q^{\frac{r^2-\Delta f^2}{4}}\zeta^r,$$

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which again is a Jacobi form, this time in the space $J_{1,1}\big(|\Delta|, \big(\frac{\Delta}{\cdot}\big)\big)$ of Jacobi forms on $\Gamma_0(|\Delta|)$ with character $\big(\frac{\Delta}{\cdot}\big)$.

In this note we consider more closely the case where $\mathcal{O}$ is a maximal order in a definite quaternion algebra Q over the field of rational numbers. Here $\rho_{\mathcal{O}}(n,r)$, for $4n-r^2 \geq 0$, equals the number of x in $\mathcal{O}$ whose reduced characteristic polynomial equals $x^2 - rx + n$, i.e. the number of x in $\mathcal{O}$ with reduced trace r and reduced norm n. From the latter it is clear that $\theta_{\mathcal{O}}$ is a Jacobi form (in fact, a Jacobi theta series; see Proposition 4.1 below). The purpose of this note is to use, based on this observation, the theory of Jacobi forms to derive an explicit formula for the weighted average of the numbers $\rho_{\mathcal{O}}(n,r)$ when $\mathcal{O}$ runs through a system of representatives for the $Q^\times$-conjugacy classes of the maximal orders of Q. Recall that the number of $Q^\times$-conjugacy classes of the maximal orders, which is usually called the type number of Q_p, is finite (a formula is recalled below). Note also that the numbers $\rho_{\mathcal{O}}(n,r)$, for fixed r and n depend only on the $Q^\times$-conjugacy class of $\mathcal{O}$. To state the formula for the weighted average we need some notations.

Let $-D$ be a negative discriminant (i.e. a negative integer which is a square modulo 4) and N a squarefree positive integer. We let f be the largest positive integer containing only primes dividing N and whose square divides D such that $-D/f^2$ is still a discriminant, and we set

$$H^{(N)}(D) = H(D/f^2) \prod_{p|N} \left(1 - \left(\frac{-D/f^2}{p}\right)\right). \tag{2}$$

Here $H(D)$, for any $D \geq 0$ is the Hurwitz class number, that is, for $D > 0$, the number of $\mathrm{SL}(2,\mathbb{Z})$-equivalence classes of binary integral positive definite quadratic forms of discriminant $-D$, where forms which are equivalent to a multiple of x^2+y^2 or x^2+xy+y^2 are counted with multiplicity $1/2$ and $1/3$, respectively, and $H(0) = -1/12$. We also set

$$H^{(N)}(0) = \frac{1}{12} \prod_{p|N} (p-1),$$

and $H^{(p)}(D) = 0$ for positive $D \equiv 1,2 \bmod 4$.

For a maximal order $\mathcal{O}$ of a quaternion algebra Q we use $\mathrm{Aut}(\mathcal{O})$ for the group of all classes $x\mathbb{Q}^\times$ in $Q^\times/\mathbb{Q}^\times$ such that $x\mathcal{O}x^{-1} = \mathcal{O}$. If Q is definite the group $\mathrm{Aut}(\mathcal{O})$ is finite (see Prop. 6.1). [1].

Main Theorem. *Let Q be a definite quaternion algebra over the field of rational numbers, let e and N be the number and the product of primes at which Q ramifies. If $\mathcal{O}_\mu$ ($\mu = 1,2,\ldots,t$) denotes a complete set of representatives for the $Q^\times$-conjugacy classes of maximal orders of Q, then one has*

$$\sum_{\mu=1}^{t} \frac{1}{\mathrm{card}\,(\mathrm{Aut}(\mathcal{O}_\mu))}\, \theta_{\mathcal{O}_\mu} = \frac{1}{2^e} \sum_{\substack{n,r\in\mathbb{Z}\\ 4n-r^2\geq 0}} H^{(N)}(4n-r^2)\, q^n \zeta^r, \tag{3}$$

where $\theta_{\mathcal{O}_\mu}$ are the function defined in (1)*,*

If we write out (3) in terms of Fourier coefficients and express $\mathrm{Aut}(\mathcal{O}_\mu)$ in terms of ideal classes (see Prop. 6.1) we obtain

$$\sum_{\mu=1}^{t} \frac{2\,m(\mathcal{O}_\mu)}{\mathrm{card}\,(\mathcal{O}_\mu^\times)}\, \rho_{\mathcal{O}_\mu}(n,r) = H^{(N)}(4n-r^2), \tag{4}$$

[1]The basic notions concerning quaternion algebras as used in the following theorem and the subsequent discussion will be shortly recalled in Section 6.

where $m(\mathcal{O})$, for any maximal order $\mathcal{O}$, denotes the number of left-ideal classes of $\mathcal{O}$ which contain a two-sided ideal. For a given maximal order $\mathcal{O}$, let $\mathfrak{a}_\mu$ ($\mu = 1, 2, \ldots, h$) be a complete set of representatives for the $\mathcal{O}$-left-ideal classes and let $\mathcal{R}_\mu$ be the right order of $\mathfrak{a}_\mu$. Every maximal order $\mathcal{O}'$ is $Q^\times$-conjugate to at least one of the $\mathcal{R}_\mu$ (since the $\mathcal{O}$-left ideal $\mathcal{O}\mathcal{O}'$ is contained in one of the $\mathfrak{a}_\mu$). Moreover, $\mathcal{R}_\mu$ is $Q^\times$-conjugate to exactly $m(\mathcal{R}_\mu)$ of the $\mathcal{R}_\nu$ ($\nu = 1, \ldots, h$) (since $\mathcal{R}_\mu$ is conjugate to $\mathcal{R}_\nu$ if and only if $\mathfrak{a}_\mu^{-1}\mathfrak{a}_\nu$ is right equivalent to a two sided ideal of $\mathcal{R}_\mu$, and since $\mathfrak{a}_\mu^{-1}\mathfrak{a}_\nu$ ($\nu = 1, \ldots, h$) represents the left ideal classes of $\mathcal{R}_\mu$). We can therefore rewrite (4) in the form

$$\sum_{\mu=1}^{h} \frac{2}{\operatorname{card}\left(\mathcal{R}_\mu^\times\right)}\, \rho_{\mathcal{R}_\mu}(n, r) = H^{(N)}(4n - r^2). \tag{5}$$

For $n = r = 0$ we obtain in particular

$$\sum_{\mu=1}^{h} \frac{2}{\operatorname{card}\left(\mathcal{R}_\mu^\times\right)} = \frac{1}{12}\prod_{p|N}(p-1),$$

which is Eichler's mass formula [Eic38, Satz 1].

For the case that N is a prime the identities (5) were stated and proved in [Gro87, §1, p. 123], however, without reference to Jacobi forms. It was used in [Gro87, §1] for sketching a proof of Eichler's trace formula. This is a formula for the trace $\operatorname{tr} B(n)$ of the nth Brandt matrix $B(n) = \big(b_{\mu,\nu}(n)\big)_{\mu,\nu}$ associated to Q, which is the $h \times h$ matrix with

$$b_{\mu,\nu}(n) = \operatorname{card}\left(\left\{b \in \mathfrak{a}_\nu^{-1}\mathfrak{a}_\mu : \mathrm{n}(\mathfrak{a}_\mu^{-1}\mathfrak{a}_\nu b) = n\right\}\right) / \operatorname{card}\left(\mathcal{R}_\mu^\times\right).$$

Indeed, if we take, for a fixed $n \geq 1$, the sum of both sides of (5) over all r with $r^2 \leq 4n$, the sum on the left becomes $2 \operatorname{tr} B(n)$, and we obtain the following corollary.

Corollary. *In the notations of the main theorem the trace of the nth Brandt matrix $B(n)$ of the quaternion algebra Q is given by the formula*

$$\operatorname{tr} B(n) = \frac{1}{2}\sum_{r^2 \leq 4n} H^{(N)}(4n - r^2). \tag{6}$$

This formula is, in a slightly differently stated form, Eichler's trace formula for the quaternion algebra Q [Eic55, Satz 10]. (Gross exposition reduces (5) to a formula [Gro87, Eq. (1.12)] for counting embeddings of orders of imaginary quadratic fields into Q_p as it can be read off from Eichler [Eic55, §6, p. 145]. Latter formula can also be found in a more explicit form in [Vig80, Thme. 5.11] as "formule de trace", where it is derived by adelic considerations.).

Note that the class and type number of Q (i.e. the number h of left ideal classes of a maximal order modulo right-multiplication by $Q^\times$ and the number of maximal orders of Q up to $Q^\times$-conjugacy) equal $h = \operatorname{tr}(B(1))$ and $t = 2^{-e}\sum_{d|N} \operatorname{tr}(B(d))$, respectively. The first formula is obvious. The second follows on writing $\operatorname{tr}(B(d)) = \sum_{\mu=1}^{t} b_{\mu,\mu}(d)\, m(\mathcal{R}_\mu)$, where we order the $\mathfrak{a}_\mu$ in such a way that the $\mathcal{R}_\mu$ ($\mu = 1, 2, \ldots, t$) are pairwise non-conjugate modulo $Q^\times$. The numbers $b_{\mu,\mu}(d)$ count the two-sided principal $\mathcal{R}_\mu$-ideals of norm d. Since $\operatorname{card}(\mathfrak{P}(\mathcal{R}_\mu)/\mathbb{Q}^\times) = 2^e/m(\mathcal{R}_\mu)$ (see Proposition 6.1) we find $\sum_{d|N} b_{\mu,\mu}(d)\, m(\mathcal{R}_\mu) = 2^e$, which implies the claimed formula. Inserting (6) into the formulas for h and t we obtain after some obvious modifications the following corollary.

Corollary. *In the notations of the main theorem the number h of left ideal classes of a given maximal order equals*

$$h = \sum_{t|N} \mu(t) \left(\frac{N/t}{12} + \frac{1}{3}\left(\frac{t}{3}\right) + \frac{1}{4}\left(\frac{-4}{t}\right) \right),$$

and the number t of maximal orders of Q up to $Q^\times$-conjugacy equals

$$t = 2^{-e-1} \sum_{n|N} \sum_{r^2 \leq 4n} H^{(N)}(4n - r^2) = 2^{-e}\Big(h + \frac{1}{2} \sum_{n|N,\, n>1} \sum_{r^2 \leq 4n} H^{(N)}(4n - r^2)\Big).$$

(Note that, for $n \mid N$ and $n > 1$, we have $H^{(N)}(4n - r^2) = 0$ unless $n \mid 4n - r^2$, i.e. unless $r = 0$ or $n, r = 2, \pm 1$ or $n, r = 3, \pm 1$.) For N equal to a prime, a formula for h was given in [Eic38, Satz 2] and a formula for t in [Deu50, Eq. (10)]; for arbitrary N formulas for h and t were stated and proved in [Eic55, Satz 10, 11]. (Formulas for h and t have in fact an extended history in the literature since the formulas loc.cit. and also variants in later publications contained little errors or were somewhat clumsily stated.) It is an elementary though somewhat unpleasant exercise to transform the class and type number formulas given loc.cit. into the form given here.

Our proof of (4), and hence also of the corollaries, uses (apart from some basic facts from the arithmetic of quaternion algebras including Eichler's mass formula) merely the theory of Jacobi forms. In § 2 we recall shortly those tools from the theory of Jacobi forms on $\Gamma_0(N)$ which we need. In § 4 we shall see that the functions $\theta_{\mathcal{O}_\mu}$ of the main theorem are Jacobi forms of a certain subspace $\mathfrak{S}(N)$ of the space $J_{2,1}(N)$ of Jacobi forms of weight 2 and index 1 on $\Gamma_0(N)$. We shall prove, using the arithmetic of Q, that the left hand side of (3) is a Hecke eigenform with the same eigenvalues as the unique Jacobi Eisenstein series in $\mathfrak{S}(N)$. The Eisenstein series in $J_{2,1}(N)$ (for squarefree N) are rapidly constructed in Section 3 from the mock Eisenstein series of weight 3/2 [HZ76, Ch. 2, Thm. 2]. Using the method of Rankin convolutions we shall prove in § 2 that no cusp form in $J_{2,1}(N)$ can have the same eigenvalues as a Jacobi Eisenstein series in this space. The main theorem is then an immediate consequence on comparing the Fourier coefficients $c(0, 0)$ on both sides of (3) and revoking Eichler's mass formula. In § 7 the reader finds various tables and numerical examples concerning the main theorem. In § 6 we recall basic facts concerning quaternion algebras which are used in this article, and we prove two lemmas about the arithmetic in quaternion algebras which we need in the proof that the left hand side (3) is indeed a Hecke eigenform.

We note that the proof of (3) is trivial if $J_{2,1}(N)$ is one-dimensional, which is the case exactly for $N = 2, 3, 5, 7, 13$, when the type number and the class number of Q is 1. In this case it suffices to remark that the left and right hand side of (3) are Jacobi forms in $J_{2,1}(N)$, and the coincidence of the Fourier coefficient $c(0, 0)$ on both sides of (3) can be checked by direct computation (e.g. using [S$^+$13]).

2. Review of Jacobi forms in $J_{2,1}(N)$

If not otherwise stated N denotes in this section an arbitrary positive integer. We use $J_{2,m}(N)$ for the space of Jacobi forms of weight 2 and index m on $\Gamma_0(N)$ as defined in [EZ85, §1]. A formula for the dimension of this space can be found in [Sko06, Thm. 3].

In particular, it turns out that, for a prime N, the dimension of $J_{2,1}(N)$ equals the class number h of the definite quaternion algebra which ramifies exactly at N. For general squarefree N, the dimension of $J_{2,1}(N)$ equals the dimension of the space $M_2(N)$ of modular forms of weight 2 on $\Gamma_0(N)$. However, for the following we do not have any use of this.

We recall that every Jacobi form ϕ in $J_{2,1}(N)$ has a Fourier expansion of the form

$$\phi = \sum_{\substack{D\geq 0, r\in\mathbb{Z}\\ -D\equiv r^2 \bmod 4}} C_\phi(D) q^{\frac{r^2+D}{4}} \zeta^r$$

(cf. [EZ85, Thm. 2.2]). Indeed, $C_\phi(D) = c(\frac{r^2+D}{4}, r)$ if D is a non-negative integer $D \equiv 0, 3 \bmod 4$, where r denotes any integer such that $r \equiv D \bmod 2$, and where $c(n,r)$ are the Fourier coefficients of ψ. We set $C_\phi(D) = 0$ if $-D$ is not a discriminant.

Proposition 2.1. *For squarefree N with, say, e prime divisors, the subspace of cusp forms has co-dimension $2^e - 1$ in $J_{2,1}(N)$.*

Proof. Note that 2^e equals the number of cusps of $\Gamma_0(N)$. Let A_j $(1 \leq j \leq 2^e)$ be matrices in $\mathrm{SL}(2,\mathbb{Z})$ such that the $A_j\infty$ represent the cusps of $\Gamma_0(N)$. If ϕ is a Jacobi form in $J_{2,1}(N)$ the *singular part*[2] of the Fourier development of ϕ at $A_j\infty$ equals $c_j \sum_{s\in\mathbb{Z}} q^{s^2}\zeta^{2s}$ for some constant c_j. Clearly ϕ is a cusp form if $c_j = 0$ for all j. Therefore the co-dimension of the subspace of cusp forms is less or equal to 2^e. However, $\sum_A \phi|_{2,1}A$, where A runs through a set of representatives for $\Gamma_0(p)\backslash \mathrm{SL}(2,\mathbb{Z})$, is an element of $J_{2,1}\big(\mathrm{SL}(2,\mathbb{Z})\big)$, and hence equals 0. If we take as representatives the matrices A_jT^k, where $1 \leq j \leq 2^e$ and $0 \leq k < w_j$ for each j with w_j the cusp width of $A_j\infty$, we obtain $\sum_j w_jc_j = 0$. Hence the co-dimension in question is less or equal to $2^e - 1$. In the next section we shall construct $2^e - 1$ linearly independent non-cusp forms (see the remark after Theorem 3.7). This proves the proposition. $\square$

We need the Hecke operators $T(l)$ on $J_{2,1}\big(N\big)$, where l runs through the primes which do not divide N. These are defined as in [EZ85, §4, eq. (3)], however, with $\Gamma_1\backslash \mathrm{M}_2(\mathbb{Z})$ replaced by the right cosets modulo $\Gamma_0(N)$ of the subset of all matrices $\left[\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right]$ in $\mathrm{M}_2(\mathbb{Z})$ where N divides c. Since the cosets whose determinant is l^2 (and relatively prime to N) are in both cases represented by the same collection of matrices (namely, the matrices $\left[\begin{smallmatrix} l^2/d & b\\ 0 & d\end{smallmatrix}\right]$ with d running through the positive divisors of l^2 and $0 \leq b < d$), [EZ85, Theorem 4.5] and its proof remain literally valid for positive integers l relatively prime to N. In particular, we have for any prime l not dividing N and any ϕ in $J_{2,1}(N)$ the formula

Proposition 2.2. *For any form ϕ in $J_{2,1}(N)$, we have*

$$C_{T(l)\phi}(D) = C_\phi(l^2D) + \left(\frac{-D}{l}\right) C_\phi(D) + lC_\phi(D/l^2). \tag{7}$$

We shall also need the following property of Hecke eigenforms.

Proposition 2.3. *Let ϕ in $J_{2,1}(N)$ be a common eigenform of $T(l)$ for all primes l, which do not divide a given multiple M of N, and let $T(l)\phi = \lambda(l)\phi$. Then, for any negative discriminant $-D$ such that for no square of a prime l outside M the number D/l^2 is a discriminant, we have*

$$\sum_{\substack{n\geq 1\\ \gcd(n,M)=1}} \frac{C_\phi(Dn^2)}{n^s} = C_\phi(D) \prod_{\substack{l \text{ prime}\\ l\nmid M}} \frac{1-\left(\frac{-D}{l}\right)l^{-s}}{1-\lambda(l)l^{-s}+l^{1-2s}}.$$

Proof. The proof is a standard computation which can be found in a similar form as needed here e.g. in [Shi73, Thm.1.9]. For the convenience of the reader we recall

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[2]This is the part $\sum_{r^2\equiv 0 \bmod 4} C_\phi(0)\, q^{\frac{r^2}{4}}\, \zeta^r$ of the Fourier expansion of $\phi|_{k,m}A_j$.

it for our case. From (7) we obtain, for any prime l outside M and any n not divisible by l,

$$\lambda(l)C(Dn^2) = C(Dn^2l^2) + \left(\frac{-D}{l}\right)C(Dn^2),$$
$$\lambda(l)C(Dn^2l^{2m}) = C(Dn^2l^{2m+2}) + lC(Dn^2l^{2m-2}) \quad (m \geq 1),$$

where we suppressed the index ϕ. Setting $F_n(T) = \sum_{m\geq 0} C(Dn^2l^{2m})x^m$ we deduce from the preceding identities

$$\lambda(l)F_n(T) = \big(F_n(T) - C(Dn^2)\big)/x + \left(\frac{-D}{l}\right)C(Dn^2) + lTF_n(T),$$

in other words

$$F_n(x) = C(Dn^2)\frac{1-\left(\frac{-D}{l}\right)T}{1-\lambda(l)T+lT^2}.$$

But

$$\sum_{\substack{n\geq 1\\ \gcd(n,M)=1}} C(Dn^2)n^{-s} = \sum_{\substack{n\geq 1\\ \gcd(n,Ml)=1}} F_n(l^{-s})n^{-s},$$

and therefore

$$\sum_{\substack{n\geq 1\\ \gcd(n,M)=1}} C(Dn^2)n^{-s} = \Big(\sum_{\substack{n\geq 1\\ \gcd(n,Ml)=1}} C(Dn^2)n^{-s}\Big)\frac{1-\left(\frac{-D}{l}\right)T}{1-\lambda(l)T+lT^2}.$$

Applying the same reasoning to the inner sum with an $l' \nmid Ml$ instead of l and repeating this again to the inner sum of the resulting identity and so forth we obtain

$$\sum_{\substack{n\geq 1\\ \gcd(n,M)=1}} C(Dn^2)n^{-s} = \Big(\sum_{\substack{n\geq 1\\ \gcd(n,Ml_1\cdots l_m)=1}} C(Dn^2)n^{-s}\Big)\prod_{n=1}^{m}\frac{1-\left(\frac{-D}{l_n}\right)T}{1-\lambda(l_n)T+l_nT^2}$$

for any m, where $l_1, l_2, \ldots$ denote the primes outside N. Letting m tend to infinity we obtain the claimed formula. □

As a consequence of the preceding proposition we obtain

Proposition 2.4. *There is no non-zero cusp form in $J_{2,1}(N)$ which is a common eigenfunction of all $T(l)$ (l a prime, $\gcd(l,N)=1$) with eigenvalues $l+1$.*

Proof. Assume ϕ would be a nonzero cusp form in $J_{2,1}(N)$ which satisfies $T(l)\phi = (l+1)\phi$ for all primes l not dividing N. From Proposition 2.3 we deduce for every negative discriminant $-D$ such that $-D/l^2$ is not a discriminant for any prime l outside N

$$\sum_{\substack{n\geq 1\\ \gcd(n,N)=1}} \frac{C_\phi(Dn^2)}{n^s} = C_\phi(D)\prod_{\substack{l \text{ prime}\\ l\nmid N}}\frac{(1-\left(\frac{-D}{l}\right)l^{-s}}{(1-l^{-s})(1-l^{1-s})}.$$

The Dirichlet series $D(s)$ on the left is the Rankin convolution of the elliptic cusp forms $h = \sum_N C_\phi(N)q^N$ and $\theta = \sum_{n\in\mathbb{Z}} q^{Dn^2}$ of weights $3/2$ and $1/2$, respectively. It is well-known that such a Rankin convolution possesses a continuation to a meromorphic function on the complex plane which is holomorphic at $s=1$. More precisely, $D(2s)$ multiplied by $(4\pi D)^{-s}\Gamma(s)$ equals (up to a constant) the integral $\int h(-\overline{\tau})\theta(\tau)E_s(\tau)\frac{dxdy}{y^2}$, taken over a fundamental domain of $\Gamma(4M)$ for a sufficiently big M, where

$$E_s = y^{3/2}\sum y^{s-1/2}|_1 A,$$

the sum being over all A in a set of representatives for $\langle\left[\begin{smallmatrix}1 & M\\ 0 & 1\end{smallmatrix}\right]\rangle\backslash\Gamma(M)$, and y denoting the imaginary part of τ. The Eisenstein series $E_s(\tau)$ possesses for each τ a continuation to a meromorphic function in s on the complex plane, and is holomorphic at $s = 1/2$. For details we refer the reader e.g. to [Shi73, pp. 467–469] and the list of references therein.

But the product on the right possesses an analytic continuation with a pole at $s = 1$. It follows that $C_\phi(Dn^2) = 0$ for all n which are relatively prime to N. But every negative discriminant $-\Delta$ can be written in the form $-\Delta = -D_0 f^2 n^2$ with a fundamental discriminant $-D_0$, a positive integer f containing only prime factors of N and a positive integer n relatively prime to N. Applying the preceding argument to $D = D_0 f^2$ shows that $C_\phi(\Delta) = 0$. This shows that $\phi = 0$, hence yields a contradiction. □

3. The Eisenstein series in $J_{2,1}(N)$

We say that a function ϕ defined on $\mathbb{H}\times\mathbb{C}$ (where $\mathbb{H}$ denotes the upper half plane of complex numbers) *transforms like a Jacobi form of weight k and index* 1 *on* Γ if it satisfies $\Phi|_{k,m}g = \phi$ for all g in Γ and all g in $\mathbb{Z}^2$. Here the right action $(\phi, g) \mapsto \phi|_{k,m}g$ is as defined in [EZ85, §1, eqs. (2), (3)].

Theorem 3.1. *Let*

$$E^*_{2,1}(\tau, z) = \sum_{n,r\in\mathbb{Z}} H(4n - r^2)\, q^n \zeta^r + v^{-1/2} \sum_{\substack{r,f\in\mathbb{Z}\\ r\equiv f \bmod 2}} \beta(\pi v f^2)\, q^{\frac{r^2-f^2}{4}} \zeta^r,$$

where

$$\beta(x) := \frac{1}{16\pi}\int_1^\infty u^{-3/2} e^{-xu}\, du \quad (x \geq 0).$$

Then $E^*_{2,1}$ transforms like a Jacobi form of weight 2 *and index* 1 *on* $\mathrm{SL}(2,\mathbb{Z})$.

Proof. Set $\mathcal{H} := \sum_{D\geq 0} H(D)\, q^D$. In [HZ76, Ch. 2, Thm. 2] it was shown that

$$\mathcal{F}(\tau) := \mathcal{H}(\tau) + v^{-1/2}\sum_{f\in\mathbb{Z}} \beta(4\pi f^2 v)\, q^{-f^2}$$

transforms under $\Gamma_0(4)$ like a modular form of weight $3/2$. Setting

$$\mathcal{F}_0(\tau) := \frac{1}{4}\sum_{x \bmod 4} \mathcal{F}(\frac{\tau + x}{4}),$$
$$\mathcal{F}_1(\tau) := \frac{1}{4}\sum_{x \bmod 4} i^x \mathcal{F}(\frac{\tau + x}{4}),$$

it follows

$$\mathcal{F}_0(\tau + 1) = \mathcal{F}_0(\tau), \qquad \mathcal{F}_0(\tfrac{-1}{\tau}) = \sqrt{\tau}^3\, \tfrac{1+i}{2}\big(\mathcal{F}_0(\tau) + \mathcal{F}_1(\tau)\big),$$
$$\mathcal{F}_1(\tau + 1) = -i\mathcal{F}_1(\tau), \qquad \mathcal{F}_1(\tfrac{-1}{\tau}) = \sqrt{\tau}^3\, \tfrac{1+i}{2}\big(\mathcal{F}_0(\tau) - \mathcal{F}_1(\tau)\big).$$

The identities on the left are obvious. For the two others note, first of all, that $\mathcal{F}(\tau) = \mathcal{F}_0(4\tau) + \mathcal{F}_1(4\tau)$. From $\mathcal{F}(\frac{w}{4w+1}) = (4w + 1)^{3/2}\mathcal{F}(w)$ we therefore deduce, on writing $4w + 1 = \tau$ (so that $\frac{4w}{4w+1} = 1 - \frac{1}{\tau}$), the identity

$$\mathcal{F}_0(\tfrac{-1}{\tau}) - i\mathcal{F}_1(\tfrac{-1}{\tau}) = \sqrt{\tau}^3\big(\mathcal{F}_0(\tau) + i\mathcal{F}_1(\tau)\big).$$

Replacing τ by $-1/\tau$ shows that this identity also holds true if we replace on both sides i by $-i$ and multiply the right hand side by i. Summing up we have

$$\big(\mathcal{F}_0(\tfrac{-1}{\tau}), \mathcal{F}_1(\tfrac{-1}{\tau})\big)\begin{bmatrix}1 & 1\\ -i & i\end{bmatrix} = \sqrt{\tau}^3\big(\mathcal{F}_0, \mathcal{F}_1\big)\begin{bmatrix}1 & i\\ i & 1\end{bmatrix},$$

from which the claimed transformation laws become obvious.

As in [EZ85, Thm. 5.4, p. 64] it follows from the transformation laws for $\mathcal{F}_0$ and $\mathcal{F}_1$ that

$$\mathcal{F}_0(\tau)\Big(\sum_{\substack{r\in\mathbb{Z}\\ r\equiv 0 \bmod 2}} q^{r^2/4}\zeta^r\Big) + \mathcal{F}_1(\tau)\Big(\sum_{\substack{r\in\mathbb{Z}\\ r\equiv 1 \bmod 2}} q^{r^2/4}\zeta^r\Big)$$

transforms like a Jacobi form of weight 2 and index 1 on $\mathrm{SL}(2,\mathbb{Z})$. But this function is $E^*_{2,1}$, which proves the proposition. □

The holomorphic part of $E^*_{2,1}$ is an eigenform for all Hecke operators $T(l)$ with eigenvalues $l+1$. This statement is essentially equivalent to the following formula.

Proposition 3.2. *For any negative discriminant $-D$, say $D = D_0F^2$ with a fundamental discriminant $-D_0$ and a perfect square F^2, and for any exact divisor f of F, one has*

$$H(D) = H(D/f^2)\,\gamma_{D_0}(f) \quad \textit{where} \quad \gamma_{D_0}(f) = \sum_{d|f}\mu(d)\left(\frac{-D_0}{d}\right)\sigma_1(f/d).$$

Proof. This is an easy consequence of the well-known formula

$$h(D_0f^2) = \frac{h(D_0)}{[\mathfrak{o}_1{}^\times : \mathfrak{o}_f{}^\times]}\, f\prod_{p|f}\left(1-\left(\frac{-D_0}{p}\right)\frac{1}{p}\right), \tag{8}$$

where $h(D_0f^2)$ denotes the number of modulo $\mathrm{SL}(2,\mathbb{Z})$ inequivalent integral primitive binary quadratic forms of discriminant $-D_0f^2$, and $\mathfrak{o}_f$ denotes the order of conductor f of the field $\mathbb{Q}(\sqrt{-D_0})$. Indeed, obviously

$$H(D) = \sum_{f|F}\frac{h(D_0f^2)}{\operatorname{card}(\mathfrak{o}_f{}^\times)/2},$$

and inserting (8) we obtain

$$H(D) = \frac{h(D_0)}{\operatorname{card}(\mathfrak{o}_1{}^\times)/2}\sum_{f|F} f\prod_{p|f}\left(1-\left(\frac{-D_0}{p}\right)\frac{1}{p}\right) = H(D_0)\,\gamma_{D_0}(F).$$

Since $\gamma_{D_0}(F)$ is multiplicative in F this implies then also $H(D_0F^2/f^2)\gamma_{D_0}(f)$ for any exact divisor of F.

A complete proof of (8), relating the class groups of $\mathfrak{o}_f$ and $\mathfrak{o}_1$, can be found in [Lan73, Ch. 8, §1, Thm. 7]. However, the formula is much older. Loc.cit. hints to [Web08, §98] (in fact, it is (10) in §98 and (15) in §100). Some refer for (8) to [Ded77, end of §9], who himself contributes the case of an imaginary quadratic number field (as needed here) to [GWC86, Art. 256, V, VI]), where the historically interested reader might try to find its origin. □

Corollary 3.3. *For all negative discriminants and all primes l one has*

$$H(l^2D) + \left(\frac{-D}{l}\right)H(l) + lH(D/l^2) = (l+1)H(D)$$

(with the convention $H(D/l^2)$ if D/l^2 is not a discriminant).

Proof. Write $D = D_0f^2$ where f is a the maximal power of l whose square divides D such that $-D/f^2$ is still a discriminant. The claimed identity is then equivalent to

$$\gamma(fl) + \left(\frac{-D}{l}\right)\gamma(l) + l\gamma(f/l) = (l+1)\gamma(f)$$

(where $\gamma = \gamma_{D_0}$ and $\gamma(l/f) = 0$ if $l \nmid f$), which can be quickly checked. □

The proof of the following two lemmas is straightforward and we leave them to the reader. (For the invariance of $U(p)\phi$ under $\Gamma_0(p)$ in the first lemma one needs that, for any $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in $\Gamma_o(p)$, one has $\left[\begin{smallmatrix} 1/p & x/p \\ & p \end{smallmatrix}\right] A = \left[\begin{smallmatrix} cx+a & B/p^2 \\ p^2c & -cy+d \end{smallmatrix}\right]\left[\begin{smallmatrix} 1/p & y/p \\ & p \end{smallmatrix}\right]$, where $B = -(cx+a)y + (dx+b)$ and y is the integer modulo p^2 such that $y \equiv \frac{dx+b}{cx+a} \bmod p^2$, and that the map $\xi \mapsto \frac{d\xi+b}{c\xi+a}$ defines a bijection of $\mathbb{Z}/p^2\mathbb{Z}$ onto itself.)

Lemma 3.4. *Suppose that ϕ is a function defined on $\mathbb{H} \times \mathbb{C}$ which transforms like a Jacobi form of weight 2 and index 1 on $\Gamma_0(N)$. Then, for any prime p,*

$$U(p)\phi := \frac{1}{p} \sum_{\mu \bmod p} \sum_{x \bmod p^2} \phi|_{2,1}\big(\tfrac{1}{p}\left[\begin{smallmatrix} 1 & x \\ & p^2 \end{smallmatrix}\right]\big)[0,\mu]$$

transforms like a Jacobi form of weight 2 and index 1 on $\Gamma_0(Np)$. For every negative discriminant $-D \le 0$, one has

$$C_{U(p)\phi}(D) = C_\phi(p^2 D). \tag{9}$$

Lemma 3.5. *For any prime p, one has*

$$U(p)E^*_{2,1} = \sum_{n,r\in\mathbb{Z}} H\big(p^2(4n-r^2)\big)\, q^n\zeta^r + pR,$$

where R is the second term on the right of the defining equation of $E^*_{2,1}$ (in Theorem 3.1).

Using these two lemmas it is now obvious how to construct from $E^*_{2,1}$ a holomorphic Jacobi form in $\Gamma_0(p)$.

Lemma 3.6. *For any prime p, the function*

$$E_{2,1,p} := U(p)E^*_{2,1} - pE^*_{2,1}$$

defines a Jacobi form in $J_{2,1}(p)$. Its n,rth Fourier coefficient equals the quantity $H^{(p)}(4n-r^2)$ defined in (2).

Proof. From Lemma 3.4 we see that $E_{2,1,p}$ transforms like a Jacobi form in $J_{2,1}(p)$, and from Lemma 3.5 we see that $E_{2,1,p}(\tau,z)$ is holomorphic in τ and z, and also holomorphic at infinity.

We also need to show that $E_{2,1,p}$ is also holomorphic at the cusp 0. For this note that the matrices $\left[\begin{smallmatrix} 1 & x \\ & p^2 \end{smallmatrix}\right]\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]$ $(0 \le x < p^2)$ represent the same cosets in $\mathrm{SL}(2,\mathbb{Z})\backslash \mathrm{M}_2(\mathbb{Z})$ as the set $\mathcal{C}$ of matrices $\left[\begin{smallmatrix} 1 & x_1 \\ & p^2 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} p & x_2 \\ & p \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} p^2 & \\ & 1 \end{smallmatrix}\right]$, where x_1 and x_2 run through representatives for the primitive residue classes modulo p^2 and p, respectively. Therefore $\big(U(p)E^*_{2,1}\big)|_{2,1}\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] = V(p)Q + V(p)R$, where Q and R denote the first and the second sum on the right of the defining equation of $E^*_{2,1}$ (Theorem 3.1), and where $V(p)$ denotes the operator

$$V(p)\phi = \frac{1}{p} \sum_{\mu \bmod p} \sum_{M\in\mathcal{C}} \phi|_{2,1}(\tfrac{1}{p}M)[\mu, 0].$$

One verifies that $V(p)Q$ has a Fourier development in terms of $q^n\zeta^r$ with $4n \ge r^2$, and that $V(p)R = pR$, which implies the claimed holomorphicity at 0.

A simple calculation shows that its n,rth Fourier coefficient equals $H(p^2D) - pH(D)$, where $D = 4n - r^2$. But this equals $H^{(p)}(D)$, as follows immediately from the formula in Proposition 3.2. □

Theorem 3.7. *For any squarefree positive integer $N > 1$, the series*

$$E_{2,1,N} := \sum_{n,r\in\mathbb{Z}} H^{(N)}(4n-r^2)\, q^n\zeta^r$$

defines a Jacobi form in $J_{2,1}(N)$*, which is a Hecke eigenform for* $T(l)$ *with eigenvalue* $l+1$ *for any prime* $l \nmid N$*.*

Remark. Let e be the number of primes in N. The $2^e - 1$ series $E_{2,1,t}$ $(t \mid N, t > 1)$ are linearly independent as follows from the fact that the Dth Fourier coefficient of $E_{2,1,t}$ is zero if and only if $\left(\frac{-D_0}{p}\right) = 1$ for all primes $p \mid t$, where $-D_0$ is the fundamental discriminant which equals $-D$ up to a perfect square. We define the *subspace* $J_{2,1}^{\mathrm{Eis}}(N)$ *of Eisenstein series in* $J_{2,1}(N)$ as the subspace of $J_{2,1}(N)$ spanned by these series. Since every Eisenstein series is a Hecke eigenform with eigenvalue $l+1$ under $T(l)$ we deduce from Proposition 2.4 that $J_{2,1}^{\mathrm{Eis}}(N)$ does not contain any cusp form. Since the co-dimension of the subspace $J_{2,1}^{(\mathrm{cusp}}(N)$ of cusp forms in $J_{2,1}(N)$ equals $2^e - 1$ (Proposition 2.1) we conclude

$$J_{2,1}(N) = J_{2,1}^{\mathrm{Eis}}(N) \oplus J_{2,1}^{\mathrm{cusp}}(N). \tag{10}$$

Proof of Theorem 3.7. Indeed, $E_{2,1,N} = \prod_{p \mid N} \left(U(p) - p\right) E_{2,1}^*$ as follows from the formula $\gamma_{D_0}(fp) - p\gamma_{D_0}(f) = \left(1 - \left(\frac{-D_0}{p}\right)\right)$ (valid for any p-power f and any discriminant $-D_0$). The first statement is therefore a consequence of Lemmas 3.4 and 3.5. Since $T(l)$ and $U(p)$ obviously commute for $l \neq p$, we read off $T(l)E_{2,1,N} = (l+1)E_{2,1,N}$ from Corollary 3.3. □

4. The theta functions $\theta_{\mathcal{O}}$

In this section N is a squarefree positive integer with an odd number of prime factors, and Q_N the quaternion algebra over $\mathbb{Q}$ which is ramified exactly at the primes dividing N and at infinity. For a given quaternion algebra Q over $\mathbb{Q}$, we use $\mathrm{t}(x)$, $\mathrm{n}(x)$ and $\overline{x} = \mathrm{t}(x) - x$ for the reduced trace, reduced norm and the conjugate of an element x of Q. Thus $\mathrm{n}(x) = x\overline{x}$ and $\mathrm{t}(x\overline{x}) = 2\,\mathrm{n}(x)$.

Proposition 4.1. *For any maximal order* $\mathcal{O}$ *of* Q_N*, the series* $\theta_{\mathcal{O}}$ *in* (1) *defines a Jacobi form in* $J_{2,1}(N)$*.*

Proof. The $\mathbb{Z}$-module $\mathcal{O}$ becomes an even integral positive definite lattice of level N and determinant N^2, when equipped with the bilinear form $(x, y) \mapsto \mathrm{t}(x\overline{y})$.

Though this is an old and well-known result, we do not have any reference. It can be easily proven though. For this observe that the *discriminant module* $\mathrm{Disc}_{\mathcal{O}}$ (which is the quotient of the dual of $\mathcal{O}$ with respect to the bilinear form $(x, y) \mapsto \mathrm{t}(x\overline{y})$ modulo $\mathcal{O}$ equipped with the induced quadratic form $\underline{\mathrm{n}}$ which maps $x + \mathcal{O}$ to $\mathrm{n}(x) + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z}$) has order equal to the determinant of the lattice $\mathcal{O}$, and its level (i.e. the smallest positive integer ℓ such that $\ell\underline{n} = 0$) equals the level of the lattice $\mathcal{O}$. But the discriminant module decomposes as direct sum of the discriminant modules $\mathrm{Disc}_{\mathbb{Z}_p \otimes \mathcal{O}}$ (where we view $\mathbb{Q}_p/\mathbb{Z}_p$ as subgroup of $\mathbb{Q}/\mathbb{Z}$ via the natural map. Finally, $\mathrm{Disc}_{\mathbb{Z}_p \otimes \mathcal{O}} = 0$ if $p \nmid N$, i.e. if $\mathbb{Z}_p \otimes \mathcal{O} \approx \mathrm{M}_2(\mathbb{Z}_p)$, and $\mathrm{Disc}_{\mathbb{Z}_p \otimes \mathcal{O}}$ has order p^2 and level p if $p \mid N$. Indeed, for $p \mid N$ we have $\mathbb{Q}_p \otimes Q_N = \left(\frac{a,p}{\mathbb{Q}_p}\right)$ with $\left(\frac{a}{p}\right) = -1$ (see Section 6 for the notations) and the unique maximal order of $\mathbb{Q}_p \otimes Q_N$ is $\mathfrak{o} + j\mathfrak{o}$ with $\mathfrak{o} = \mathbb{Z}_p + i\mathbb{Z}_p$ for odd p and $\mathfrak{o} = \mathbb{Z}_2 + \mathbb{Z}_2\frac{1+i}{2}$ for $p = 2$. Therefore $\mathrm{Disc}_{\mathbb{Z}_p \otimes \mathcal{O}}$ equals the discriminant module of the ring of integers in $\mathbb{Q}_p(\sqrt{a})$ equipped with the bilinear form $(x, y) \mapsto p\,\mathrm{t}_{K/\mathbb{Q}_p}(x\overline{y})$, which implies the claim.

The series

$$\theta_{\mathcal{O}} = \sum_{x \in \mathcal{O}} q^{\mathrm{n}(x)} \zeta^{\mathrm{t}(x)}$$

is the Jacobi theta series associated to the lattice $\mathcal{O}$, and defines therefore a Jacobi form in $J_{2,1}(N)$ [Klo46, Thm. 1]. (It is somewhat cumbersome to deduce the fact that $\theta_{\mathcal{O}}$ is invariant under $\Gamma_0(N)$ from the given reference, and the reader might find it more convenient to use instead [Eic66, App. to Ch. 1, §3, 3., pp. 48].) □

Lemma 4.2. *Let x be an integral element of the (not necessarily definite) quaternion algebra Q over $\mathbb{Q}$, let $-D = \mathrm{t}(x)^2 - 4\,\mathrm{n}(x)$, and p a prime at which Q ramifies. Then $\left(\frac{-D}{p}\right) \neq +1$.*

Proof. Indeed, $\left(\frac{-D}{p}\right) = +1$ would imply that $-D = a^2$ for an a in $\mathbb{Z}_p$. (Note that $-D$ is a discriminant, so that, for $p = 2$, the assumption $\left(\frac{-D}{p}\right) = +1$ implies $-D \equiv +1 \bmod 8$). With such an a we would have $a^2 = \mathrm{t}(x)^2 - 4\,\mathrm{n}(x)$, in other words $\mathrm{n}(a - \mathrm{t}(x) + 2x) = 0$. But $a - \mathrm{t}(x) + 2x \neq 0$ (since $\mathrm{t}(x)^2 \neq 4\,\mathrm{n}(x)$) in contradiction to the assumption that $\mathbb{Q}_p \otimes Q$ is a skewfield. □

As an immediate consequence we obtain the following.

Proposition 4.3. *For any maximal order $\mathcal{O}$ of Q_N the series $\theta_{\mathcal{O}}$ is an element of the subspace $\mathfrak{S}(N)$ of $J_{2,1}(N)$ of forms ϕ such that, for every negative discriminant $-D$, one has $C_{\theta_O r}(D) = 0$ unless $\left(\frac{-D}{p}\right) \neq 1$ for all primes p dividing N.*

The subspace $\mathfrak{T}(N)$ of $\mathfrak{S}(N)$ generated by the theta functions $\theta_{\mathcal{O}}$, where $\mathcal{O}$ runs through the maximal orders in Q is invariant under all Hecke operators as we shall see in a moment. Note that $\theta_{\mathcal{O}}$ depends only on the $Q^{\times}$-conjugacy class of $\theta_{\mathcal{O}}$. In particular, the dimension of $\mathfrak{T}(N)$ is bounded to above by the type number t_N of Q. For N equal to a prime p its precise dimension can be deduced from [Gro87, Cor. 13.6], the first instance where the dimension is smaller than t_p being $p = 389$ [Gro87, p. 181].

Theorem 4.4. *Let $\mathcal{O}$ be a maximal order of the definite quaternion algebra Q over $\mathbb{Q}$, and let l be a prime at which Q does not ramify. Then*

$$T(l)\theta_{\mathcal{O}} = \sum_{\substack{\mathcal{O}' \neq \mathcal{O} \\ l\mathcal{O}' \subseteq \mathcal{O}}} \theta_{\mathcal{O}'},$$

where the sum is over all maximal orders $\mathcal{O}' \neq \mathcal{O}$ of Q such that $l\mathcal{O}' \subseteq \mathcal{O}$.

The number of maximal orders $\mathcal{O}' \neq \mathcal{O}$ of Q such that $l\mathcal{O}' \subseteq \mathcal{O}$ is finite. In fact, it equals $l + 1$ (see Prop. 6.3 below). For the special case of a definite quaternion algebra ramified at only one prime number q a formula for the action of $T(l)$ ($l \neq q$) on the $\theta_{\mathcal{O}}$ is given (in terms of Brandt matrices) in [Kra86, Part I, §2, Satz 2]. It is not difficult to show that both formulas coincide in this case.

Proof of Theorem 4.4. We have, for any negative discriminant $-D$, that

$$C_{T(l)\theta_{\mathcal{O}}}(D) = c_{\mathcal{O}}(l^2 n, lr) + \left(\frac{-D}{l}\right) c_{\mathcal{O}}(n, r) + l c_{\mathcal{O}}(n/l^2, r/l),$$

where r and n is any solution of $D := 4n - r^2 \geq 0$. If $-D/l^2$ is a discriminant we assume that n and r are chosen so that $l^2 \mid n$ and $l \mid r$, and we set $c_{\mathcal{O}}(n/l^2, r/l) = 0$ otherwise. We have obviously $c_{\mathcal{O}}(n/l^2, r/l) = c_{l\mathcal{O}}(n, r)$. Moreover, $c_{\mathcal{O}}(l^2 n, lr) = c_{\mathcal{O}(l)}(n, r)$, where

$$\mathcal{O}(l) := \{x \in Q : x \text{ integral and } lx \in \mathcal{O}\}.$$

Therefore we have to prove

$$c_{\mathcal{O}(l)}(n, r) = \sum_{\substack{\mathcal{O}' \neq \mathcal{O} \\ l\mathcal{O}' \subseteq \mathcal{O}}} c_{\mathcal{O}'}(n, r) - \left(\frac{-D}{l}\right) c_{\mathcal{O}}(n, r) - l c_{l\mathcal{O}}(n, r).$$

For this we need to study $\mathcal{O}(l)$. By Proposition 6.4 (1) below we know that $\mathcal{O}(l)$ is the union of all maximal orders $\mathcal{O}'$ such that $l\mathcal{O}'$ is contained in $\mathcal{O}$. From

Prop. 6.3 (1) and (4) (in Section 6) we deduce

$$\mathcal{O}(l) = \mathcal{O} \sqcup \bigsqcup_{l\mathcal{O}' \subseteq \mathcal{O}} [\mathcal{O}' \setminus (\mathcal{O}' \cap \mathcal{O})],$$

where $\sqcup$ stands for disjoint union, and where $\mathcal{O}'$ runs over all maximal orders such that $l\mathcal{O}'$ is contained in $\mathcal{O}$. As a consequence we have

$$c_{\mathcal{O}(l)}(n,r) = \sum_{l\mathcal{O}' \subseteq \mathcal{O}} c_{\mathcal{O}'}(n,r) - \sum_{\substack{\mathcal{O}' \neq \mathcal{O} \\ l\mathcal{O}' \subseteq \mathcal{O}}} c_{\mathcal{O}' \cap \mathcal{O}}(n,r).$$

Let S denote the second sum on the right.

For analyzing S, let for an x in $\mathcal{O}$ satisfying $x^2 + xr + n = 0$, denote $\nu(x)$ the number of $\mathcal{O}' \neq \mathcal{O}$ in the sum defining S such that $\mathcal{O}'$ contains x, so that $S = \sum_x \nu(x)$. The formula for $\nu(x)$ from Proposition 6.4 (2) implies $S = c_{\mathcal{O}}(n,r) + \left(\frac{-D}{l}\right) c_{\mathcal{O}}(n,r) + l c_{l\mathcal{O}}(n,r)$ (note that $\left(\frac{-D}{l}\right) \neq 0$, i.e. $l \nmid r^2 - 4n$, implies $c_{l\mathcal{O}}(n,r) = 0$). This proves the theorem. □

The group $Q^\times/\mathbb{Q}^\times$ acts on Q via conjugation. By Skolem-Noether's theorem (e.g. [Vig80, Ch. 1, §2, Thm. 2.1]) this action yields an isomorphism of $Q^\times/\mathbb{Q}^\times$ with the group of automorphisms of Q. We see in particular that the group $\mathrm{Aut}(\mathcal{O})$ of ring automorphisms of an order $\mathcal{O}$ of Q equals $\{x\mathbb{Q}^\times \in Q^\times/\mathbb{Q}^\times : x\mathcal{O}x^{-1} = \mathcal{O}\}$. Since conjugation leaves the reduced norm invariant we also recognize that $\mathrm{Aut}(\mathcal{O})$, as group of isometries of the lattice $\mathcal{O}$ with respect to the norm form, is finite if Q is definite.

Proposition 4.5. *Let Q be a definite quaternion algebra over $\mathbb{Q}$, let $\mathcal{O}_\mu$ ($\mu = 1, 2, \ldots, t$) be a complete set of representatives for the maximal orders of Q modulo $Q^\times$-conjugation. Then*

$$\Theta_Q := \sum_{\mu=1}^{t} \frac{1}{\mathrm{card}\,(\mathrm{Aut}(\mathcal{O}_\mu))} \theta_{\mathcal{O}_\mu},$$

is a eigenform in $J_{2,1}(N)$ for each $T(l)$ with eigenvalue $l+1$, where l is a prime at which Q does not ramify.

Proof. Fix a prime l at which Q does not ramify. For each $\mathcal{O}_\mu$ let $I_{\mu,\nu}$ the set of integral $\mathcal{O}_\mu$ left ideals of norm l whose right order is $Q^\times$-conjugate to $\mathcal{O}_\nu$, and let $i_{\mu,\nu} = \mathrm{card}\,(I_{\mu,\nu})$. Then $T(l)\theta_{\mathcal{O}_\mu} = \sum_\nu i_{\mu,\nu}\theta_{\mathcal{O}_\nu}$ by Prop. 6.3 (2). By (1) of the same lemma, we know $\sum_\nu i_{\mu,\nu} = l+1$ for every μ. In other words, $Me = (l+1)e$, where $M = (i_{\mu,\nu})_{\mu,\nu}$ and e denotes the column vector of length h whose entries are all 1.

We want to determine the left-eigenvector of M for the eigenvalue $l+1$. For this let

$$a_\mu = \mathrm{card}\left(\{x \in Q^\times : x\mathcal{O}_\mu = \mathcal{O}_\mu x\}/\mathbb{Q}^\times\right) = \mathrm{card}\,(\mathrm{Aut}(\mathcal{O}_\mu)),$$

and let A be the diagonal matrix with $1/a_1,\ 1/a_2,\ \ldots$ as diagonal elements. We shall show in a moment that $AMA^{-1} = M'$ (where $'$ denotes transposition).

From this and $Me = (l+1)e$ and we deduce $(l+1)e' = e'M' = e'AMA^{-1}$, i.e. that $e'A$ is left eigenvector of M with eigenvalue $l+1$. This implies then that $\sum_\mu \frac{1}{a_\mu}\theta_{\mathcal{O}_\mu}$ is indeed eigenvector of $T(l)$ with eigenvalue $l+1$.

For proving $MA^{-1} = A^{-1}M'$, i.e. $i_{\mu,\nu}a_\nu = i_{\nu,\mu}a_\mu$, let $\widetilde{I}_{\mu,\nu}$, for every μ, ν, denote the set of pairs $(\mathfrak{a}, x)$, where $\mathfrak{a}$ is in $I_{\mu,\nu}$ and x in Q such that $x\mathcal{O}_\nu x^{-1}$ is the right order of $\mathfrak{a}$. Note that $\mathrm{card}\left(\widetilde{I}_{\mu,\nu}\right) = i_{\mu,\nu}a_\nu$. We need therefore bijections between $\widetilde{I}_{\mu,\nu}$ and $\widetilde{I}_{\nu,\mu}$.

For each μ,ν, a bijection $\alpha_{\mu,\nu} : \widetilde{I}_{\mu,\nu} \to \widetilde{I}_{\nu,\mu}$ is given by the application $(a,x) \mapsto (x^{-1}l\mathfrak{a}^{-1}x, x^{-1})$, respectively. Note that $x^{-1}l\mathfrak{a}^{-1}x$ is indeed integral, i.e. $\mathfrak{b} := l\mathfrak{a}^{-1} \subseteq \mathcal{O}' := x\mathcal{O}_\mu x^{-1}$, since $l\mathcal{O} \subset \mathfrak{a}$ (see the beginning of the proof of Prop. 6.3), hence $\mathfrak{b} \subseteq \mathcal{O}$, so that $\mathfrak{b}\mathfrak{b} \subset \mathfrak{b}$ which implies that $\mathfrak{b}$ is part of the left order $\mathcal{O}'$ of $\mathfrak{b}$. These maps are bijective since $\alpha_{\mu,\nu}$ and $\alpha_{\nu,\mu}$ are inverse to each other. This concludes the proof of the proposition. □

5. Proof of the main theorem

If the space $J_{2,1}(N)$ is one dimensional, we obtain directly from the fact that $\Theta_Q = \Theta_{Q_N}$ and $E_{2,1,N}$ are elements of $J_{2,1}(N)$ (Proposition 4.1 and Theorem 3.7), i.e. without any reference to Hecke theory, that Θ_Q and $E_{2,1,N}$ are equal up to multiplication by a constant. Note that $J_{2,1}(N)$ is one dimensional if and only if N is in $\{2,3,5,7,13\}$ as follows from the dimension formulas in [Sko06, Thm. 3]. As we saw in Section 1 these are the primes where the class number and the type number of Q_N coincide. In other words, $J_{2,1}(N)$ is one dimensional if and only if Q_N contains only one maximal order, say, $\mathcal{O}$, up to $Q_N^\times$-conjugacy. If the reader likes he can check directly without invocation of Eichler's mass formula that the number $2/\operatorname{card}(\mathcal{O}^\times)$ equals $H^{(N)}(0) = (N-1)/12$. (One can use e.g. [S+13] to obtain a bases for $\mathcal{O}$ and to find the units).

In the general case we know that Θ_{Q_N} is a eigenform for $T(l)$ with eigenvalue $l+1$ for all $l \nmid N$ (Prop. 4.5) and hence, by the decomposition (10) the sum of an Eisenstein series plus a cusp form which is eigenform for $T(l)$ with eigenvalue $l+1$ for all $l \nmid N$. But the latter must be 0 by Prop. 2.4. It follows that Θ_{Q_N} is an Eisenstein series. By Prop. 4.3 it is a member of the subspace $\mathfrak{S}(N)$. We show in a moment that $E_{2,1,N}$ is the only Eisenstein series in $\mathfrak{S}(N)$. Therefore Θ_{Q_N} is a multiple of $E_{2,1,N}$, and comparing the constant terms of their Fourier development proves the theorem. So let $E := \sum_{t|N,\ 1<t<N} c(t)\, E_{2,1,t}$ be a member of $\mathfrak{S}(N)$. Note that $C_E(D) = 0$ for any D such that $\left(\frac{-D}{p}\right) = +1$ for some $p \mid N$. For a given prime $q \mid N$ choose a discriminant $-D < 0$ such that $\left(\frac{-D}{q}\right) = -1$ and $\left(\frac{-D}{p}\right) = +1$ for all $p \mid N$, $p \neq q$. Then $0 = C_E(D) = c(q)H^{(q)}(D)$, whence $c(q) = 0$. Next, for a given pair of primes $q \neq r$ dividing N, choose a discriminant $-D < 0$ such that $\left(\frac{-D}{q}\right) = \left(\frac{-D}{r}\right) = -1$ and $\left(\frac{-D}{p}\right) = +1$ for all $p \mid N$, $p \neq q, r$. Again $0 = C_E(D) = c(qr)H^{(qr)}(D)$, whence $c(qr) = 0$. Continuing in this way we eventually find $c(t) = 0$ for all t. This completes the proof of the theorem.

6. Appendix: Some lemmas on quaternion algebras

In this section we recall briefly the basics of the theory of quaternion algebras over $\mathbb{Q}$, and we prove some facts concerning the arithmetic in quaternion algebras which we needed for the proof of Theorem 4.4.

For a given field F of characteristic 0 and elements a,b in $F^\times$, we use $\left(\frac{a,b}{F}\right)$ for the F-algebra which possesses a basis $1, i, j, k = ij = -ji$ with $i^2 = a$ and $j^2 = b$. These are the quaternion algebras over F. For the field of p-adic numbers $\mathbb{Q}_p$ or of real numbers $\mathbb{Q}_\infty$, there are up to isomorphism only two such equivalence classes, namely the class of the algebra $\mathrm{M}_2(\mathbb{Q}_p)$ of 2×2-matrices over $\mathbb{Q}_p$, and the class containing the skew-fields [Vig80, Ch.I, Cor. 2.4, Ch.II, Thm. 1.1]. If one identifies these classes with $+1$ and -1, respectively, then the class of $\left(\frac{a,b}{\mathbb{Q}_p}\right)$ is the usual p-adic Hilbert symbol $(a,b)_p$ [Vig80, Ch.II, Cor. 1.2]. In the case of a quaternion algebra over $\mathbb{Q}$ the algebras $\left(\frac{a,b}{\mathbb{Q}}\right)$ and $\left(\frac{a',b'}{\mathbb{Q}}\right)$ are isomorphic if and only if $(a,b)_p = (a',b')_p$ for all p (including ∞) [Vig80, Ch.III, Thm. 3.1]. A quaternion algebra Q over $\mathbb{Q}$ *ramifies at* p if $\mathbb{Q}_p \otimes Q$ is a skew-field. If it ramifies at ∞ it is called *definite*. By what we have said, Q ramifies only at finitely many primes; in fact, this number is always even (by the product formula for the p-adic Hilbert symbols). Vice versa, given a set R of an even number of primes, we can find a, b in $\mathbb{Q}^\times$ such that $(a,b)_p = -1$ exactly for p in S, and then $\left(\frac{a,b}{\mathbb{Q}}\right)$ is a (up to isomorphism, *the*) quaternion algebra which ramifies exactly at the primes in R.

For any element a of quaternion algebra Q, we can consider its characteristic polynomial $x^2 - tx + n = 0$ when we view a as endomorphism of the $\mathbb{Q}$-vector space Q via left multiplication. The numbers t and n are called the reduced trace $\mathrm{t}(a)$ and reduced norm $\mathrm{n}(a)$ of a. The map $a \mapsto \overline{a} := a - \mathrm{t}(a)$ defines an anti-involution of Q. Obviously, $\mathrm{t}(a) = a + \overline{a}$ and $\mathrm{n}(a) = a \cdot \overline{a}$.

Assume that F is $\mathbb{Q}$ or $\mathbb{Q}_p$ with finite p and Q a quaternion algebra over F. An element a is called integral if $\mathrm{t}(a)$ and $\mathrm{n}(a)$ are integral. A maximal order is a maximal subring of Q consisting of integral elements only. In the algebra $\mathrm{M}_2(\mathbb{Q}_p)$ the maximal orders are the rings $x\,\mathrm{M}_2(\mathbb{Z}_p)x^{-1}$ ($\mathrm{n}(x) = \det(x) \neq 0$), and if Q is a skew-field over $\mathbb{Q}_p$ the integral elements form an order, and so Q has only one maximal order in this case. If $F = \mathbb{Q}$ the number of maximal orders of Q modulo conjugation with elements from $Q^\times$ is finite [Vig80, Ch.III, Cor. 5.4]. This is the *type number of* Q.

A *fractional ideal* $\mathfrak{a}$ of Q is a submodule of rank 4 over the ring of integers of F such that its left order $\{a \in Q : a\mathfrak{a} \subseteq \mathfrak{a}\}$ and its right order $\{a \in Q : \mathfrak{a}a \subseteq \mathfrak{a}\}$ are maximal orders. (If one of both is maximal the other is too [Vig80, p. 86]). An ideal is called integral if it is contained in its left order (it is then also contained in its right order and vice versa [Vig80, Ch. I, Lemme 4.3 (2)]). The fractional ideals form a groupoid: If the right order of $\mathfrak{a}$ equals the left order of $\mathfrak{b}$ then $\mathfrak{ab}$ (the module of finite sums $\sum a_j b_j$ with a_j in $\mathfrak{a}$ and b_j in $\mathfrak{b}$) is again a fractional ideal, and $\mathfrak{a}^{-1} := \{x \in Q : \mathfrak{a}x\mathfrak{a} \subseteq \mathfrak{a}\}$ is a fractional ideal and $\mathfrak{a}\mathfrak{a}^{-1}$ and $\mathfrak{a}^{-1}\mathfrak{a}$ equals the left and right order of $\mathfrak{a}$, respectively. Let $\mathcal{O}$ be a maximal order of Q. For $F = \mathbb{Q}_p$ all $\mathcal{O}$-left ideals are principal, i.e. of the form $\mathcal{O}x$ ($x \in Q^\times$). For $F = \mathbb{Q}$ the number of $\mathcal{O}$-left ideals modulo multiplication with elements of $Q^\times$ from the right is finite [Vig80, Ch.III, Thm. 5.4]. This number does not depend on the choice of $\mathcal{O}$ and is the *class number of* Q.

The norm $\mathrm{n}(\mathfrak{a})$ of a fractional ideal is defined as the fractional ideal in F generated by the reduced norms of the elements of $\mathfrak{a}$. The norm is multiplicative [Vig80, p.24]). Moreover, the norm function on integral ideals is strictly decreasing. This means that for any two integral $\mathcal{O}$-left-ideals $\mathfrak{a}$, $\mathfrak{b}$ the inclusion $\mathfrak{a} \subsetneq \mathfrak{b}$ implies $\mathrm{n}(\mathfrak{b}) \mid \mathrm{n}(\mathfrak{a})$, $\mathrm{n}(\mathfrak{b}) \neq \mathrm{n}(\mathfrak{a})$ (as follows e.g. from [Vig80, Ch. I, §4, Lemme 4.4]).

Let Q be a quaternion algebra and X a fixed lattice of Q (i.e. a $\mathbb{Z}$-module of rank 4 in Q). The $\mathbb{Z}$-lattices L in Q are in one to one correspondence with the families $\{L_p\}_p$ (p a prime number) of $\mathbb{Z}_p$-lattices L_p in $\mathbb{Q}_p \otimes_{\mathbb{Q}} Q$ such that $L_p = \mathbb{Z}_p \otimes X$ for almost all p. The correspondence is given by $L \mapsto \{\mathbb{Z}_p \otimes_{\mathbb{Z}} L\}_p$ in one and $\{L_p\} \mapsto \bigcap_p (Q \cap L_p)$ in the other direction [Vig80, Ch. III, §5, Thm. 5.1]. Under this correspondence maximal orders correspond to maximal orders and fractional ideals to fractional ideals. In particular, one has for any fractional ideal $\mathfrak{a}$ of $\mathbb{Q}$, $\mathfrak{a} = \bigcap_p \mathbb{Z}_{(p)} \otimes \mathfrak{a}$, the indices taken over all prime numbers, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p (the ring of rational numbers whose denominator in shortest terms is not divisible by p).

Proposition 6.1. *Let $\mathcal{O}$ be a maximal order of the definite quaternion algebra Q. Then*

$$\mathrm{card}\,(\mathrm{Aut}(\mathcal{O})) = \frac{2^e\,\mathrm{card}\,(\mathcal{O}^\times)}{2\,m(\mathcal{O})},$$

where $m(\mathcal{O})$ denotes the number of $\mathcal{O}$-left ideal classes containing a two-sided ideal, and where e denotes the number of primes at which Q ramifies.

Lemma 6.2. *Let the notations be as in the preceding proposition.*

1. *The number of two-sided $\mathcal{O}$-ideals modulo multiplication by $\mathbb{Q}^\times$ equals 2^e.*
2. *The number of two-sided principal $\mathcal{O}$-ideals modulo multiplication by $\mathbb{Q}^\times$ equals $2^e/m(\mathcal{O})$.*

Proof. For (1) we use that the group $\mathfrak{I}(\mathcal{O})$ of two-sided ideals of $\mathcal{O}$ is generated by the ideals with norm p, were p runs through the set R of primes at which Q is ramified, and the subgroup of ideals $\mathcal{O}q$ where q is in $Q^\times$ [Vig80, p. 86]. The norm map induces therefore an injection $\mathfrak{I}(\mathcal{O})/\mathbb{Q}^\times \to \mathbb{Q}^\times/{\mathbb{Q}^\times}^2$ whose image is the subgroup generated by the classes $p{\mathbb{Q}^\times}^2$ $(p \in R)$.

For (2) let $\mathfrak{P}(\mathcal{O})$ denote the group of two-sided ideals of $\mathcal{O}$. It is clear that

$$m(\mathcal{O}) = \operatorname{card}\left(\mathfrak{I}(\mathcal{O})/\mathfrak{P}(\mathcal{O})\right) = \frac{\operatorname{card}\left(\mathfrak{I}(\mathcal{O})/\mathbb{Q}^\times\right)}{\operatorname{card}\left(\mathfrak{P}(\mathcal{O})/\mathbb{Q}^\times\right)}$$

(For the first equality note that, for any two-sided ideals $\mathfrak{a}$ and $\mathfrak{b}$, one has $\mathfrak{a} = \mathfrak{b}x$ for some x in $Q^\times$ if and only if $\mathcal{O}x$ is two-sided.) The claimed formula follows now from (1). □

Proof of Proposition 6.1. Note, first of all, that $x \mapsto \mathcal{O}x$ induces a surjective homomorphism $\operatorname{Aut}(\mathcal{O}) \to \mathfrak{P}(\mathcal{O})/\mathbb{Q}^\times$, where $\mathfrak{P}(\mathcal{O})$ is the group of nonzero two-sided principal ideals of $\mathcal{O}$. The kernel of this map equals $\mathcal{O}^\times\mathbb{Q}^\times/\mathbb{Q}^\times \cong \mathcal{O}^\times/\{\pm 1\}$. Therefore

$$\operatorname{card}\left(\operatorname{Aut}(\mathcal{O})\right) = \frac{1}{2}\operatorname{card}\left(\mathcal{O}^\times\right) \cdot \operatorname{card}\left(\mathfrak{P}(\mathcal{O})/\mathbb{Q}^\times\right).$$

The claimed formula follows now with Lemma 6.2 (2). □

Proposition 6.3. *Let Q be a (not necessarily definite) quaternion algebra over $\mathbb{Q}$, let $\mathcal{O}$ be a any maximal order of Q' and l a prime at which Q does not ramify.*

1. *$\mathcal{O}$ possesses exactly $l+1$ integral ideals of norm l.*
2. *The application $\mathfrak{a} \mapsto$ right order of $\mathfrak{a}$ defines a bijection between the integral $\mathcal{O}$-left ideals of norm l and the maximal orders $\mathcal{O}' \neq \mathcal{O}$ such that $l\mathcal{O}' \subseteq \mathcal{O}$.*
3. *For a maximal order $\mathcal{O}'$ the inclusions $l\mathcal{O}' \subseteq \mathcal{O}$ and $l\mathcal{O} \subseteq \mathcal{O}'$ are equivalent.*
4. *For any two different right orders $\mathcal{O}'$ and $\mathcal{O}''$ with $l\mathcal{O}', l\mathcal{O}'' \subseteq \mathcal{O}$ one has $\mathcal{O}' \cap \mathcal{O}'' \subseteq \mathcal{O}$.*

Proof. We show, first of all, that an integral $\mathcal{O}$ left-ideal $\mathfrak{a}$ has norm l if and only if $l\mathcal{O} \subsetneq \mathfrak{a} \subsetneq \mathcal{O}$. Indeed, let $\mathfrak{a}$ be integral with norm l. As ideal $\mathfrak{a}$ contains with an element x also $\mathrm{n}(x) = x(2\,\mathrm{t}(x) - x)$, and therefore also the gcd of the norms of its elements, i.e. l. The inequalities hold true since $l\mathcal{O}$ and $\mathcal{O}$ have norm l^2 and 1, respectively. The inverse implication follows from the fact that the norm function on integral $\mathcal{O}$-left ideals is strictly decreasing.

For (1) we note that $\mathcal{O}/l\mathcal{O}$ is isomorphic to $\mathrm{M}_2(\mathbb{F}_l)$ (since it is isomorphic to $(\mathbb{Z}_l \otimes \mathcal{O})/l(\mathbb{Z}_l \otimes \mathcal{O})$ and $\mathbb{Z}_l \otimes \mathcal{O}$ is isomorphic to $\mathrm{M}_2(\mathbb{Z}_l)$), and so the proper left-ideals of $\mathcal{O}$ containing but not equal to $l\mathcal{O}$ are in one to one correspondence with the proper left ideals of $\mathrm{M}_2(\mathbb{Z}_l)$. These in turn are the ideals $\mathrm{M}_2(\mathbb{F}_l)R$, where R runs through a complete set of representatives for the $l+1$ cosets in $\mathrm{GL}(2,\mathbb{F}_l)\backslash\,\mathrm{M}_2(\mathbb{F}_l)_0$ different from $\{0\}$, where $\mathrm{M}_2(\mathbb{F}_l)_0$ denotes the set of matrices in $\mathrm{M}_2(\mathbb{F}_l)$ with determinant 0. Note that $\mathrm{M}_2(\mathbb{F}_l)_0$ is the only proper two-sided ideal of $\mathrm{M}_2(\mathbb{F}_l)$, so that, in particular, none of the left-ideals $\mathrm{M}_2(\mathbb{F}_l)R$ is two-sided.

For (2) let $\mathfrak{a}$ be an integral $\mathcal{O}$-left ideal of norm l. Then the right-order of $\mathfrak{a}$ is maximal (since its left-order is maximal [Vig80, p. 86]). Since $l \in \mathfrak{a}$ we have $l\mathcal{O}' \subseteq \mathfrak{a}\mathcal{O}' \subseteq \mathfrak{a} \subseteq \mathcal{O}$. We have $\mathcal{O}' \neq \mathcal{O}$ since $\mathfrak{a}/l\mathcal{O}$ is not a two-sided ideal of $\mathcal{O}/l\mathcal{O}$ as we saw in the preceding paragraph.

The application is surjective. Namely, if $l\mathcal{O}' \subseteq \mathcal{O}$ for a maximal $\mathcal{O}' \neq \mathcal{O}$ then $\mathfrak{a} := l\mathcal{O}\mathcal{O}'$ is an $\mathcal{O}$ left-ideal which has $\mathcal{O}'$ as its right-order and contains l. Clearly,

$\mathfrak{a} \subseteq \mathcal{O}$, and $\mathfrak{a} \neq \mathcal{O}$ (since $l\mathcal{O}\mathcal{O}' = \mathcal{O}$ would imply that $\mathcal{O}$ and $\mathcal{O}'$ are both right-orders of $\mathfrak{a}$, both in contradiction to the assumption $\mathcal{O}' \neq \mathcal{O}$ and their maximality as orders). Also $l\mathcal{O} \neq \mathfrak{a}$ since $l\mathcal{O} = l\mathcal{O}\mathcal{O}'$ would imply $l\mathcal{O} \supseteq l\mathcal{O}'$ in contradiction with the maximality of $\mathcal{O}' \neq \mathcal{O}$. Hence $\mathfrak{a}$ has norm l.

Finally, the application is injective. If $\mathfrak{a}$ is an integral $\mathcal{O}$ left-ideal of norm l having right order $\mathcal{O}'$, then $l\mathcal{O} \subsetneq l\mathcal{O}\mathcal{O}' \subseteq \mathfrak{a}$, the latter inclusion being true since $l\mathcal{O} \subseteq \mathfrak{a}$, and the former inequality being true since $l\mathcal{O} = l\mathcal{O}\mathcal{O}'$ would imply that $\mathcal{O}$ and $\mathcal{O}'$ were both right orders of $l\mathcal{O}\mathcal{O}'$, and hence by maximality were equal. It follows $\mathfrak{a} = l\mathcal{O}\mathcal{O}'$.

For (3) let $\mathcal{O}'$ be a maximal order with $l\mathcal{O}' \subseteq \mathcal{O}$. Let $\mathfrak{a}$ be the $\mathcal{O}$-left ideal with right order $\mathcal{O}'$. But then $l\mathfrak{a}^{-1}$ is an integral $\mathcal{O}'$-left ideal of norm l with right order $\mathcal{O}$, and hence $l\mathcal{O} \subseteq \mathcal{O}'$. Exchanging the roles of $\mathcal{O}$ and $\mathcal{O}'$ in the preceding argument, we see that inversely $l\mathcal{O} \subseteq \mathcal{O}'$ implies $l\mathcal{O}' \subseteq \mathcal{O}$.

For (4) we can suppose that $\mathcal{O}'$ and $\mathcal{O}''$ are both different from $\mathcal{O}$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be the $\mathcal{O}$-left-ideals with right-orders $\mathcal{O}'$ and $\mathcal{O}''$, respectively. Both ideals are different, so that $l\mathcal{O} \subseteq \mathfrak{a} \cap \mathfrak{b} \subsetneq \mathfrak{a}$, and hence $\mathfrak{a} \cap \mathfrak{b} = l\mathcal{O}$. Since $\mathfrak{a} \cap \mathfrak{b}$ is invariant under right-multiplication by $\mathcal{O}' \cap \mathcal{O}''$, so is then $\mathcal{O}$. In particular, $1 \cdot \mathcal{O}' \cap \mathcal{O}'' \subseteq \mathcal{O}$, which is the claim. □

Proposition 6.4. *Let $\mathcal{O}$ be an order of the (not necessarily definite) quaternion algebra Q and l a prime at which Q does not ramify.*

(1) *One has $\{x \in Q : x$ integral and $lx \in \mathcal{O}\} = \bigcup_{l\mathcal{O}' \subseteq \mathcal{O}} \mathcal{O}'$.*
(2) *For a given x in $\mathcal{O}$ with trace r and norm n the number $\nu(x)$ of maximal orders $\mathcal{O}' \neq \mathcal{O}$ with $l\mathcal{O}' \subseteq \mathcal{O}$ containing x equals*

$$\nu(x) = \begin{cases} 1 + \left(\frac{-D}{l}\right) & \text{if } x \notin l\mathcal{O}, \\ 1 + l & \text{if } x \in l\mathcal{O}, \end{cases}$$

where $-D = r^2 = 4n$.

Proof. We use $\mathfrak{O}$ for the set of maximal orders $\mathcal{O}'$ such that $l\mathcal{O}' \subseteq \mathcal{O}$. For a prime m, let $\mathbb{Z}_{(m)}$ denote the localization of $\mathbb{Z}$ at m (i.e. the ring of rational numbers whose denominator in lowest terms does not contain m). If $m \neq l$ then $\mathbb{Z}_{(m)} \otimes \mathcal{O}' = \mathbb{Z}_{(m)} \otimes \mathcal{O}$ for all $\mathcal{O}'$ in $\mathfrak{O}$ since l is invertible in $\mathbb{Z}_{(m)}$.

If $m = l$ then $\mathbb{Q}_l \otimes Q$ is isomorphic to the algebra $\mathrm{M}_2(\mathbb{Q}_l)$ of 2×2-matrices with entries from $\mathbb{Q}_l$. The reduced trace and norm are the usual determinant and trace of matrices. The subring $\mathfrak{o} := \mathrm{M}_2(\mathbb{Z}_l)$ of 2×2-matrices with entries from $\mathbb{Z}_l$ is a maximal order, and every maximal order is conjugate to $\mathfrak{o}$. The left-ideals of norm l are $\mathfrak{o}\rho$, where ρ runs through a set of representatives $\mathfrak{R}$ for $\mathrm{SL}(2, \mathbb{Z}_l)\backslash\mathfrak{o}_l$ and $\mathfrak{o}_l$ denotes the set of elements in $\mathfrak{o}$ of determinant l. For $\mathfrak{R}$ one can take the set of $l + 1$ matrices $\left[\begin{smallmatrix} 1 & u \\ 0 & l \end{smallmatrix}\right]$ with u running through a complete set of residues modulo l, and $\left[\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right]$. The right order of $\mathfrak{o}\rho$ is $\rho^{-1}\mathfrak{o}\rho$. Since $l\rho^{-1}$ has entries in $\mathbb{Z}_l$ we see that $l\rho^{-1}\mathfrak{o}\rho \subseteq \mathfrak{o}$. (For a proof of these facts see [Vig80, Chap. II, §2, Thm. 2.3].)

For proving (1) let x be an integral element in Q such that lx is in $\mathcal{O}$. Clearly, x is in $\mathbb{Z}_{(m)} \otimes \mathcal{O} = \mathbb{Z}_{(m)} \otimes \mathcal{O}'$ for all $\mathcal{O}'$ in $\mathfrak{O}$ and all primes $m \neq l$. Fix an isomorphism of $\mathbb{Q}_l$-algebras $\mathbb{Q}_l \otimes Q \approx \mathrm{M}_2(\mathbb{Q}_l)$ which takes $\mathbb{Z}_l \otimes \mathcal{O}$ onto $\mathfrak{o}$, and let ξ be the image of x under this isomorphism. Then ξ has trace and determinant in $\mathbb{Z}_l$ and $l\xi$ is in $\mathfrak{o}$. If ξ is not in $\mathfrak{o}$ we can find a ρ in $\mathfrak{R}$ such that $\rho\xi\rho^{-1}$ is in $\mathfrak{o}$ (i.e. ξ is in $\rho^{-1}\mathfrak{o}\rho$). Namely, let $l\xi = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, so that $r := a + d \equiv 0 \bmod l$ and $n := ad - bc \equiv 0 \bmod l^2$. If c is divisible by l^2 we can choose ρ in $\mathrm{SL}(2, \mathbb{Z})\left[\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix}\right]$. If l is exact divisor of c then l divides also a, d, b (as follows from $l \mid r$ and $l^2 \mid n$), and hence ξ is in $\mathfrak{o}$. If c is not divisible by l we choose ρ in $\mathrm{SL}(2, \mathbb{Z})\left[\begin{smallmatrix} 1 & u \\ 0 & l \end{smallmatrix}\right]$ with a u such that

$$\left[\begin{smallmatrix} 1 & u \\ 0 & l \end{smallmatrix}\right]\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & -u/l \\ 0 & 1/l \end{smallmatrix}\right] = \left[\begin{smallmatrix} a+uc & X/l \\ lc & d-uc \end{smallmatrix}\right] \tag{11}$$

has entries in $l\mathbb{Z}_l$. Since

$$X = -cu^2 + (d-a)u + b = -c\left(u - \frac{d-a}{2c}\right)^2 + \frac{r^2-4n}{4c},$$

so that we can take $u \equiv \frac{d-a}{2c} \bmod l$ for odd l and $u \equiv d/c \bmod 4$ if $l = 2$.

We conclude that x is in one of the orders $\mathbb{Z}_l \otimes \mathcal{O}'$ for a suitable $\mathcal{O}'$ in $\mathfrak{O}$, and since it is also in Q we conclude that x is in fact contained in $\mathbb{Z}_{(l)} \otimes \mathcal{O}'$. As we saw x is also contained in $\mathbb{Z}_{(m)} \otimes \mathcal{O}'$ for all $m \neq l$. But the intersection of all $\mathbb{Z}_{(m)} \otimes \mathcal{O}'$ equals $\mathcal{O}'$, which proves that x is in $\mathcal{O}'$, i.e. in the right hand side of the claimed identity in (1). That the right hand side of this identity is contained in $\mathcal{O}$ is obvious.

For (2) let again x in $\mathcal{O}$ and $\xi = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in $\mathfrak{o}$ the image of x as before. We want to count the number $\nu(x)$ of $\mathcal{O}'$ in $\mathfrak{O}$, $\mathcal{O}' \neq \mathcal{O}$, such that x is in $\mathcal{O}'$, or, equivalently, such that x is in $\mathbb{Z}_{(m)} \otimes \mathcal{O}'$ for all primes m. Since $\mathbb{Z}_{(m)} \otimes \mathcal{O}' = \mathbb{Z}_{(m)} \otimes \mathcal{O}$ we see that $\nu(x)$ equals the number of ρ in $\mathfrak{R}$ such that ξ is contained in $\rho^{-1}\mathfrak{o}\rho$, i.e. such that $\rho x \rho^{-1}$ is in $\mathfrak{o}$. In view of (11) this number equals $l+1$ if ξ is in $l\mathfrak{o}$, and otherwise $1 + \left(\frac{r^2-4n}{l}\right)$ (this is immediate if c is not divisible by l, but holds also for $l \mid c$, the verification of which we leave to the reader). This completes the proof of the lemma. □

7. Appendix: Tables and examples

For a squarefree integer N with an odd number of primes let Q_N denote the definite quaternion algebra which ramifies exactly at the prime divisors of N. Its type number equals 1 if and only if N is one of the nine values $2, 3, 5, 7, 13, 30, 42, 70, 78$ as can be seen from Table 1. For a rigorous proof note that the type number t of

Table 1. Type and class numbers t and h of definite quaternion algebras ramified at the prime divisors of a given integer N.

N	h	t	N	h	t	N	h	t	N	h	t	N	h	t
2	1	1	**66**	4	2	**130**	4	2	**191**	17	15	**258**	8	3
3	1	1	**67**	6	4	**131**	12	11	**193**	16	9	**263**	23	18
5	1	1	**70**	2	1	**137**	12	8	**195**	8	3	**266**	10	3
7	1	1	**71**	7	7	**138**	6	3	**197**	17	11	**269**	23	17
11	2	2	**73**	6	4	**139**	12	9	**199**	17	13	**271**	23	17
13	1	1	**78**	2	1	**149**	13	10	**211**	18	12	**273**	12	5
17	2	2	**79**	7	6	**151**	13	10	**222**	6	2	**277**	23	13
19	2	2	**83**	8	7	**154**	6	3	**223**	19	13	**281**	24	17
23	3	3	**89**	8	7	**157**	13	8	**227**	20	15	**282**	10	4
29	3	3	**97**	8	5	**163**	14	8	**229**	19	12	**283**	24	15
30	2	1	**101**	9	8	**165**	8	4	**230**	10	3	**285**	12	3
31	3	3	**102**	4	2	**167**	15	13	**231**	12	3	**286**	10	3
37	3	2	**103**	9	7	**170**	8	3	**233**	20	13	**290**	12	4
41	4	4	**105**	4	2	**173**	15	11	**238**	8	2	**293**	25	17
42	2	1	**107**	10	8	**174**	6	2	**239**	21	18	**307**	26	16
43	4	3	**109**	9	6	**179**	16	13	**241**	20	13	**310**	10	4
47	5	5	**110**	6	2	**181**	15	10	**246**	8	3	**311**	27	23
53	5	4	**113**	10	7	**182**	6	2	**251**	22	18	**313**	26	15
59	6	6	**114**	4	2	**186**	6	2	**255**	12	4	**317**	27	16
61	5	4	**127**	11	8	**190**	6	2	**257**	22	15	**318**	10	3

Q_N is greater or equal to $h/2^e$, where h is the class number of Q_N and e the number of prime divisors of N, and that $h/2^e \geq \frac{1}{12}\prod_{p|N}\frac{p-1}{2}$ (cf. the formulas for h and t in the second corollary to the main theorem). Therefore one has to check the table only for those N such that $\nu_N := \prod_{p|N}\frac{p-1}{2} \leq 12$, and an easy estimate shows that this implies $e \leq 3$ (since, for $e \geq 5$, one has $\nu_N \geq \frac{2-1}{2}\frac{3-1}{2}\frac{5-1}{2}\frac{7-1}{2}\frac{11-1}{2} = 15 > 12$) and that the largest prime divisor of N is obviously less or equal to 23, and then $N \leq 2\cdot 7\cdot 17 = 238$, which is the maximum of all integers which are primes or products of 3 primes all of which are ≤ 23.

Table 2 lists the first Fourier coefficients $H^{(N)}(D)$ of the series $\frac{2}{\mathrm{card}(\mathrm{Aut}(\mathcal{O}))}\theta_{\mathcal{O}}$ for those Q_N with type number one (where $\mathcal{O}$ is any maximal order of Q_N).

TABLE 2. The first coefficients $H^{(N)}(D)$ for all definite quaternion algebras with type number one.

$D\backslash N$	2	3	5	7	13	30	42	70	78
0	1/12	1/6	1/3	1/2	1	2/3	1	2	2
3	2/3	1/3	2/3	0	0	4/3	0	0	0
4	1/2	1	0	1	0	0	2	0	0
7	0	2	2	1	2	0	0	0	0
8	1	0	2	2	2	0	0	4	0
11	2	0	0	2	2	0	0	0	0
12	2/3	4/3	8/3	0	0	4/3	0	0	0
15	0	2	2	4	4	0	0	0	0
16	1/2	3	0	3	0	0	2	0	0
19	2	2	0	0	2	0	0	0	8
20	2	0	2	0	4	0	0	0	0
23	0	0	6	6	0	0	0	0	0
24	2	2	0	0	4	0	0	0	4
27	8/3	1/3	8/3	0	0	4/3	0	0	0
28	0	4	4	2	4	0	0	0	0
31	0	6	0	0	6	0	0	0	0
32	1	0	6	6	6	0	0	4	0
35	4	0	2	2	0	0	0	4	0
36	5/2	1	0	5	0	0	2	0	0
39	0	4	0	8	4	0	0	0	0
40	2	4	2	0	0	4	0	0	0
43	2	2	2	2	0	8	8	8	0
44	2	0	0	8	8	0	0	0	0
47	0	0	10	0	10	0	0	0	0
48	2/3	10/3	20/3	0	0	4/3	0	0	0

We finally illustrate the main theorem by choosing an N such that Q_N has type number 2. This ensures that the space spanned by the corresponding $\theta_{\mathcal{O}}$ possesses exactly one cusp form (if the space does not happen to be one-dimensional), which has then rational Fourier coefficients and eigenvalues. This cups form is given by

$$S_N = \frac{1}{2}\left(\theta_{\mathcal{O}_1} - \theta_{\mathcal{O}_2}\right),$$

where $\mathcal{O}_1$ and $\mathcal{O}_2$ are orders representing the two types of Q_N, respectively. Table 1 shows that Q_N has type number 2 exactly for each of the 16 values

$$11, 17, 19, 37, 66, 102, 105, 110, 114, 130, 174, 182, 186, 190, 222, 238.$$

We pick the first composite number, which is $N = 66$. The class number of Q_{66} is 4. The two inequivalent maximal orders have groups of units of order 4 and 6. (Recall that the unit group of a maximal order in a definite quaternion algebra over $\mathbb{Q}$ different from Q_2 and Q_3 is always isomorphic to the group of units of order 2, 4 or 6 [Hey29, p. 43]. Hence Eichler's mass formula reads $a+\frac{b}{2}+\frac{c}{3}=\frac{1}{12}\prod_{p|N}(p-1)$, where a, b and c denote the number the right orders of a complete set of ideal classes of a given order which have unit groups of order 2, 4 or 6, respectively. Thus, for $N=66$, we have $a+b+c=4$ and $a+\frac{b}{2}+\frac{c}{3}=\frac{5}{3}$, which has as only solution $a=0$, $b=c=2$).

We can take $Q_{66}=\left(\frac{-1,-33}{\mathbb{Q}}\right)$, i.e. Q_{66} has a basis $1,i,j,k=ij=-ji$ with $i^2=-1$ and $j^2=-33$. Maximal orders are $\mathcal{O}_4$ which has $1,i,j,(1+i+j+k)/2$ as $\mathbb{Z}$-basis and $\mathcal{O}_4^\times=\langle i\rangle$, and the order $\mathcal{O}_6$ which has $(5+i+3j+13k)/10,(i+3j+13k)/5,j+4k,5k$ as $\mathbb{Z}$-basis and $\mathcal{O}_6^\times=\langle t\rangle$ with $t=(5+3i-j-k)/10$ as 6th root of unity. Tables 3 and 4 list the first non-zero Fourier coefficients of $E_{2,1,66}$ and S_{66} (recall that the Dth coefficients of these forms are 0 if $\left(\frac{-D}{p}\right)=1$ for any $p\in\{2,3,11\}$). The first Hecke eigenvalues $\lambda(l)$ of S_{11} are $\lambda(5)=-4$, $\lambda(7)=-2$, $\lambda(13)=4$, $\lambda(17)=-2$, $\lambda(19)=0$. These are the eigenvalues of the newform of weight 2 on $\Gamma_0(66)$ whose L-series equals the L-series of the elliptic curve $y^2+xy=x^3-45x+81$.

TABLE 3. The first coefficients $C_{E_{2,1,66}}(D)$ of $E_{2,1,66}$.

D	$C(D)$	D	$C(D)$	D	$C(D)$	D	$C(D)$	D	$C(D)$
0	5/12	**3**	1/3	**4**	1/2	**12**	1/3	**16**	1/2
27	1/3	**36**	1/2	**48**	1/3	**64**	1/2	**67**	2
75	7/3	**88**	1	**91**	4	**100**	5/2	**108**	1/3
115	4	**132**	1	**136**	4	**144**	1/2	**147**	7/3
148	2	**163**	2	**168**	2	**187**	2	**192**	1/3
196	9/2	**232**	2	**235**	4	**243**	1/3	**256**	1/2
264	2	**267**	2	**268**	2	**276**	4	**280**	4
291	4	**300**	7/3	**312**	2	**324**	1/2	**328**	4
331	6	**339**	6	**352**	1	**355**	8	**363**	1/3
364	4	**372**	2	**379**	6	**388**	4	**400**	5/2

TABLE 4. The first coefficients $C_{S_{66}}(D)$ of S_{66}.

D	$C(D)$	D	$C(D)$	D	$C(D)$	D	$C(D)$	D	$C(D)$
3	−1	**4**	1	**12**	−1	**16**	1	**27**	−1
36	1	**48**	−1	**64**	1	**67**	4	**75**	3
88	−3	**91**	−2	**100**	−5	**108**	−1	**115**	−2
132	2	**136**	−2	**144**	1	**147**	3	**148**	4
163	−6	**168**	4	**187**	4	**192**	−1	**196**	−1
232	4	**235**	−2	**243**	−1	**256**	1	**264**	−1
267	−6	**268**	4	**276**	−2	**280**	−2	**291**	−2
300	3	**312**	−6	**324**	1	**328**	8	**331**	2
339	2	**352**	−3	**355**	6	**363**	−1	**364**	−2
372	−6	**379**	2	**388**	−2	**400**	−5	**408**	4

REFERENCES

[Ded77] Richard Dedekind. *Über die Anzahl der Ideal-Classen in den verschiedenen Ordnungen eines endlichen Körpers*. Vieweg, 1877.

[Deu50] Max Deuring. Die Anzahl der Typen von Maximalordnungen einer definiten Quaternionenalgebra mit primer Grundzahl. *Jber. Deutsch. Math. Verein.*, 54:24–41, 1950.

[Eic38] M. Eichler. Über die Idealklassenzahl total definiter Quaternionenalgebren. *Math. Z.*, 43(1):102–109, 1938.

[Eic55] Martin Eichler. Zur Zahlentheorie der Quaternionen-Algebren. *J. Reine Angew. Math.*, 195:127–151 (1956), 1955.

[Eic66] Martin Eichler. *Introduction to the theory of algebraic numbers and functions*. Translated from the German by George Striker. Pure and Applied Mathematics, Vol. 23. Academic Press, New York-London, 1966.

[EZ85] Martin Eichler and Don Zagier. *The theory of Jacobi forms*, volume 55 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1985.

[Gro87] Benedict H. Gross. Heights and the special values of L-series. In *Number theory (Montreal, Que., 1985)*, volume 7 of *CMS Conf. Proc.*, pages 115–187. Amer. Math. Soc., Providence, RI, 1987.

[GWC86] C.F. Gauss, W.C. Waterhouse, and A.A. Clarke. *Disquisitiones Arithmeticae: Engl. Ed*. Springer London, Limited, 1986.

[Hey29] Käte Hey. *Analytische Zahlentheorie in Systemen hyperkomplexer Zahlen*. PhD thesis, Uiversität Hamburg, 1929. available at the universities of Köln and Hamburg.

[HZ76] F. Hirzebruch and D. Zagier. Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus. *Invent. Math.*, 36:57–113, 1976.

[Klo46] H. D. Kloosterman. The behaviour of general theta functions under the modular group and the characters of binary modular congruence groups. I. *Ann. of Math. (2)*, 47:317–375, 1946.

[Kra86] Jürg Kramer. Jacobiformen und Thetareihen. *Manuscripta Math.*, 54(3):279–322, 1986.

[Lan73] Serge Lang. *Elliptic functions*. Addison-Wesley Publishing Co., Inc., Reading, Mass.-London-Amsterdam, 1973. With an appendix by J. Tate.

[S$^+$13] W. A. Stein et al. *Sage Mathematics Software (Version x.y.z)*. The Sage Development Team, 2013. `http://www.sagemath.org`.

[Shi73] Goro Shimura. On modular forms of half integral weight. *Ann. of Math. (2)*, 97:440–481, 1973.

[Sko06] Nils-Peter Skoruppa. Memorandum on dimension formulas for spaces of Jacobi forms. In *Proceedings of the conference on automorphic representations, L-functions and periods*, pages 172–183. Research Institute for Mathematical Sciences, Kyoto, 2006. arXiv:0711.0632 [math.NT].

[Vig80] Marie-France Vignéras. *Arithmétique des algèbres de quaternions*, volume 800 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[Web08] Heinrich Weber. *Lehrbuch der Algebra: Elliptische Funktionen und Algebraische Zahlen*, volume 3. Friedrich Vieweg und Sohn, 2nd edition, 1908.

İstanbul Üniversitesi, Fen Fakültesi, Matematik Bölümü, 34134 Vezneciler, İstanbul, Turkey

Current address: Max Planck Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
E-mail address: `hatice.boylan@gmail.com`

Universität Siegen, Siegen, Germany
Current address: Tongji University, School of Mathematical Sciences, Shanghai, China
E-mail address: `nils.skoruppa@gmail.com`

Tongji University, School of Mathematical Sciences, Shanghai, China
E-mail address: `haigangz@tongji.edu.cn`