October 16, 2008

PROPERTIES OF WEIGHT POSETS FOR WEIGHT MULTIPLICITY FREE REPRESENTATIONS

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INTRODUCTION

Let $\mathfrak g$ be a semisimple Lie algebra over an algebraically closed field $\Bbbk$ of characteristic zero, $\mathfrak t$ a Cartan subalgebra of $\mathfrak g$, and $\mathcal R$ a finite-dimensional $\mathfrak g$-module. The set of $\mathfrak t$-weights in $\mathcal R$ is denoted by $\mathcal P(\mathcal R)$. Having chosen a set of simple roots for $(\mathfrak g, \mathfrak t)$, we can regard $\mathcal P(\mathcal R)$ as poset with respect to the *root order*. For $\gamma, \mu \in \mathcal P(\mathcal R)$, this means that μ *covers* γ if and only if $\mu - \gamma$ is a simple root. These posets are called *weight posets*. The *Hasse diagram* of $\mathcal P(\mathcal R)$ is a directed graph whose set of vertices is $\mathcal P(\mathcal R)$ and there is the edge directed from γ to μ if and only if μ covers γ. The set of edges in the Hasse diagram is denoted by $\mathcal E(\mathcal R)$. We say that $\mathcal R$ is *weight multiplicity free* (wmf for short) if all $\mathfrak t$-weight spaces in $\mathcal R$ are one-dimensional. Then $\dim \mathcal R = \#\mathcal P(\mathcal R)$. If $\mathcal R$ is wmf, then $\mathcal P(\mathcal R)$ is said to be a wmf-*poset*. Clearly, one can define weight posets and wmf-representations for arbitrary reductive Lie algebras. However, if $\mathcal R$ is a simple $\mathfrak g$-module, then the center of $\mathfrak g$ does not affect the property of being wmf.

In this article, we study wmf-posets; specifically, we are interested in relations between $\dim \mathcal R$ and the number of edges, $\#\mathcal E(\mathcal R)$. The irreducible wmf-representations of *simple* Lie algebras are classified by R. Howe [2, 4.6]. We begin with computing the number of edges for all representations in Howe's list. It is then easy to get formulae for the irreducible wmf-representations of *semisimple* algebras. We also observe that there are non-trivial isomorphisms between weight posets of different irreducible wmf-representations. Therefore, the number of different wmf-posets is considerably smaller than that of wmf-representations.

Our main results concern wmf-posets associated with gradings of simple Lie algebras. We consider two types of gradings:

$$\mathbb Z\text{-grading:}\quad \mathfrak g = \bigoplus_{i\in\mathbb Z} \mathfrak g(i), \qquad \text{periodic or } \mathbb Z_m\text{-grading:}\quad \mathfrak g = \bigoplus_{i\in\mathbb Z_m} \mathfrak g_i .$$

Here $\mathfrak g(0)$ and $\mathfrak g_0$ are reductive Lie algebras. For $\mathbb Z$-gradings, $\mathsf{rk}\,\mathfrak g = \mathsf{rk}\,\mathfrak g(0)$ and each $\mathfrak g(i)$ is a wmf $\mathfrak g(0)$-module. For periodic gradings, it is not always the case that $\mathsf{rk}\,\mathfrak g = \mathsf{rk}\,\mathfrak g_0$. However, if $\mathsf{rk}\,\mathfrak g = \mathsf{rk}\,\mathfrak g_0$, which means that the corresponding periodic automorphism $\vartheta \in \mathrm{Aut}(\mathfrak g)$ is inner, then each $\mathfrak g_i$ is a wmf $\mathfrak g_0$-module. After work of Vinberg [5], it is known that the representations $(\mathfrak g(0) : \mathfrak g(i))$ and $(\mathfrak g_0 : \mathfrak g_i)$ have nice invariant-theoretic properties. (Actually, it suffices to consider the representations with $i = 1$.) Let $\mathcal E(i)$ (resp. $\mathcal E_i$) denote the set of edges in the Hasse diagram of $\mathcal{P}(\mathfrak{g}(i))$ (resp. $\mathcal{P}(\mathfrak{g}_i)$). Our main result can be regarded as combinatorial manifestation of 'niceness' of representations considered by Vinberg.

Theorem 0.1. *Let $\mathfrak{g}$ be a simple Lie algebra.*

1) In case of $\mathbb{Z}$-gradings, we always have $0 < 2\dim\mathfrak{g}(1) - \#(\mathcal{E}(1)) \leqslant h$, where h is the Coxeter number of $\mathfrak{g}$. Furthermore, if $m = \max\{j \mid \mathfrak{g}(j) \ne 0\} > 1$, then $2\dim\mathfrak{g}(1) - \#\mathcal{E}(1) < h$.

2) For periodic gradings of inner type, we have $0 \leqslant 2\dim\mathfrak{g}_1 - \#(\mathcal{E}_1)$. The equality $2\dim\mathfrak{g}_1 = \#(\mathcal{E}_1)$ holds if and only if $\mathfrak{g}$ is simply-laced and $\mathfrak{g}_0$ is semisimple.

It is worth noting that $\mathfrak{g}_0$ is semisimple if and only if $\mathfrak{g}_1$ is a simple $\mathfrak{g}_0$-module (Vinberg [5]).

The most interesting constraint in the theorem is that $\#\mathcal{E}(1)/\dim\mathfrak{g}(1) < 2$ (or $\leqslant 2$ for periodic gradings). This is a real condition, since the ratio $\#\mathcal{E}(\mathcal{R})/\dim\mathcal{R}$ can be arbitrarily large even for irreducible wmf-representations $\mathcal{R}$ of simple Lie algebras.

Most of the proofs are based on case-by-case considerations. Some *a priori* proofs occur in the simply-laced case. Let us say that a $\mathbb{Z}$-grading is *standard* if $\mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ generate $\mathfrak{g}$ as Lie algebra. (Another, but equivalent definition is given in Section 3.)

Theorem 0.2. *Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}}\mathfrak{g}(i)$ be a standard $\mathbb{Z}$-grading. If $\mathfrak{g}$ is simply-laced and $\mathsf{rk}\,[\mathfrak{g}(0),\mathfrak{g}(0)] = \mathsf{rk}\,\mathfrak{g} - k$, then $\sum_{i\geqslant 1}(2\dim\mathfrak{g}(i) - \#\mathcal{E}(i)) = k{\cdot}h$.*

There is also a partial converse to Theorem 0.1 which is valid in the simply-laced case, see Theorem 4.6.

Yet another property of wmf-posets associated with $\mathbb{Z}$-gradings is expressed in terms of (upper) covering polynomials $\mathcal{K}(t)$ [4], see the definition in §1.1. We show that, for the posets $\mathcal{P}(\mathfrak{g}(i))$, $\deg\mathcal{K}(t)$ is at most 3. Because we compute the upper covering polynomials for all wmf-posets, this provides another necessary combinatorial condition for a wmf-representation to occur in connection with a $\mathbb{Z}$-grading. The degree bound also yields a simple interpretation of inequality $0 < 2\dim\mathfrak{g}(1) - \#(\mathcal{E}(1))$ in terms of coefficients of $\mathcal{K}(t)$.

Here is a brief description of the article. In § 1, we fix main notation and recall some results on the poset of positive roots from [3, 4]. In § 2, we compute the number of edges for the wmf-posets and point out wmf-representations with isomorphic weight posets. In § 3 and 4, we consider wmf-posets associated with $\mathbb{Z}$- and periodic gradings, respectively. The upper covering polynomials of wmf-posets are discussed in § 5.

Acknowledgements. This work was done during my stay at the Max-Planck-Institut für Mathematik (Bonn). I am grateful to this institution for the warm hospitality and support.

1. Generalities

1.1. **Posets, edges, Hasse diagrams.** Let $(\mathcal{P},\succcurlyeq)$ be a finite poset (partially ordered set). We say that μ *covers* ν if $\mu \ne \nu$, $\mu \succcurlyeq \nu$, and if γ satisfies $\mu \succcurlyeq \gamma \succcurlyeq \nu$, then either $\gamma = \mu$ or

$\gamma = \nu$. The *Hasse diagram* of $\mathcal{P}$ is the directed graph whose set of vertices is $\mathcal{P}$, and the edges are the pairs $(\mu, \nu) \in \mathcal{P} \times \mathcal{P}$ such that μ covers ν. For brevity, such a pair (μ, ν) will also be referred to as an edge of $\mathcal{P}$. We say that $\mathcal{P}$ is *connected* if the Hasse diagram of $\mathcal{P}$ is.

In [4], we considered two statistics on a finite poset $\mathcal{P}$ and thereby two generating functions. Namely, for $a \in \mathcal{P}$, one can count the number of elements that either are covered by a or covers a. In particular, set $\mathcal{P}^{(j)} = \{a \in \mathcal{P} \mid a \text{ covers } j \text{ elements of } \mathcal{P}\}$. Then the *upper covering polynomial* of $\mathcal{P}$ is

$$\mathcal{K}_{\mathcal{P}}(t) = \sum_{j \geqslant 0} \#(\mathcal{P}^{(j)}) t^j .$$

Notice that $\mathcal{K}_{\mathcal{P}}(1) = \#\mathcal{P}$ and $\mathcal{K}'_{\mathcal{P}}(1) = \#\mathcal{E}(\mathcal{P})$, the number of edges of $\mathcal{P}$. The related notion of the *lower covering polynomial* is not needed in this article, since these two polynomials coincide for the weight posets.

1.2. **Root systems and weight posets.** Let $\mathfrak{g}$ be a reductive algebraic Lie algebra of semisimple rank n. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{t} \oplus \mathfrak{n}^+$, where $\mathfrak{t}$ is a Cartan subalgebra. Associated with this choice, one obtains

- the set of positive roots Δ^+;
- the set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \Delta^+$;
- the set of dominant weights $\mathcal{X}_+$ and of fundamental weights $\{\varpi_1, \dots . \varpi_n\}$.

For $\gamma \in \Delta^+$, $[\gamma : \alpha_i]$ is the coefficient of α_i in the expression of γ via the simple roots. The *height* of γ is $\mathsf{ht}(\gamma) = \sum_{i=1}^n [\gamma : \alpha_i]$. If $\mathfrak{g}$ is simple, then $h = h(\mathfrak{g})$ is the *Coxeter number* and θ is the highest root of Δ^+. Recall that $\mathsf{ht}(\theta) = h - 1$. We use the numbering of simple roots as in [6, Tables].

We endow Δ^+ with the usual *root order*. This means that γ covers β if and only if $\gamma - \beta \in \Pi$. If (γ, β) is an edge of Δ^+ and $\gamma - \beta = \alpha_i$, then this edge is said to be *of type* α_i. The set of all edges is denoted by $\mathcal{E}(\Delta^+)$ and the subset of edges of type α_i is denoted by $\mathcal{E}_{\alpha_i}(\Delta^+)$. For $\mathfrak{g}$ non-simple, the Hasse diagram of Δ^+ is the disjoint union of the Hasse diagrams of simple ideals of $\mathfrak{g}$. Therefore, it suffices to understand the structure of Hasse diagram if $\mathfrak{g}$ is simple. We say that $\mathfrak{g}$ (or Δ) is *simply-laced* if so is the Dynkin diagram of Δ.

Theorem 1.1 ([3, Sect. 1])**.** *Suppose $\mathfrak{g}$ is simple. The number of edges of type α_i depends only on length of α_i. If Δ is simply laced, then $\#(\mathcal{E}_{\alpha_i}(\Delta^+)) = h - 2$. In particular, $\#\mathcal{E}(\Delta^+) = n(h - 2)$.*

Remark 1.2. If Δ is multiply-laced and α_i is long, then $\#(\mathcal{E}_{\alpha_i}(\Delta^+)) = h^* - 2$, where h^* is the *dual Coxeter number*. For α_i short, the formula (and the proof!) is not so nice, see [3, Theorem 1.2].

For a finite-dimensional $\mathfrak{g}$-module $\mathcal{R}$, let $\mathcal{P}(\mathcal{R})$ be the set of $\mathfrak{t}$-weights of $\mathcal{R}$. Again, we regard $\mathcal{P}(\mathcal{R})$ as poset with respect to the root order and call it the *weight poset* of $\mathcal{R}$. The set of all edges of $\mathcal{P}(\mathcal{R})$ (resp. edges of type α_i) is denoted by $\mathcal{E}(\mathcal{R})$ (resp. $\mathcal{E}_{\alpha_i}(\mathcal{R})$). If $\mathcal{R} = \mathcal{R}(\lambda)$ is the simple $\mathfrak{g}$-module with highest weight $\lambda \in \mathcal{X}_+$, then we write $\mathcal{P}(\lambda)$ and $\mathcal{E}(\lambda)$ for the sets of weights and edges, respectively. Note that λ is the unique maximal element of $\mathcal{P}(\lambda)$, and the lowest weight of $\mathcal{R}(\lambda)$ is the unique minimal element. If we wish to stress the dependance of either of the previous objects on $\mathfrak{g}$, then we write $\Pi(\mathfrak{g})$ or $\mathcal{R}(\mathfrak{g}, \lambda)$ or $\mathcal{E}(\mathfrak{g}, \lambda)$, etc. Recall that $\mathcal{P}(\lambda)$ is called a wmf-*poset* if $\mathcal{R}(\lambda)$ is wmf $\mathfrak{g}$-module.

Theorem 1.3 ([3, Theorem 2.2])**.** *Suppose $\mathfrak{g}$ is simple and $\mathcal{P}(\lambda)$ is a wmf-poset. Then $\#(\mathcal{E}_{\alpha_i}(\lambda))$ depends only on length of α_i. In particular, if Δ is simply-laced, then $\mathsf{rk}\,\mathfrak{g}$ divides $\#(\mathcal{E}(\lambda))$.*

The following property of upper covering polynomials is proved in [4, Sect. 2].

Theorem 1.4. *For any root system Δ^+, we have $\deg \mathcal{K}_{\Delta^+} \leqslant 3$, and $\deg \mathcal{K}_{\Delta^+} = 3$ if and only if Δ is of type $\mathbf{D}_n$ or $\mathbf{E}_n$ or $\mathbf{F}_4$. Furthermore, suppose that γ covers three other roots, i.e., $\gamma - \alpha_{i_j} \in \Delta^+$ for some $\alpha_{i_1}, \alpha_{i_2}, \alpha_{i_3} \in \Pi$. Then these simpe roots are pairwise orthogonal.*

2. On edges of weight posets of wmf-representations

In this section, we compute the number of edges in (the Hasse diagrams of) connected wmf-posets and describe some isomorphisms between weight posets.

For the simple Lie algebras, the list of all irreducible wmf-representations is obtained by R. Howe [2, 4.6]. This information is contained in the first two columns of Table 1 (using our convention on the numbering of ϖ_i).

The computation of the number of edges is simplified by the fact that different weight posets can naturally be isomorphic. Clearly, the weight poset does not change, if we replace $\mathcal{R}(\lambda)$ with its dual (use the longest element of the Weyl group). Two important non-trivial isomorphisms are described below.

Theorem 2.1.

1) $\mathcal{P}(\mathbf{A}_n, m\varpi_1) \simeq \mathcal{P}(\mathbf{A}_{n+m-1}, \varpi_n) \simeq \mathcal{P}(\mathbf{A}_{n+m-1}, \varpi_m) \simeq \mathcal{P}(\mathbf{A}_m, n\varpi_1)$. *In particular, for $m = 2$ we have* $\mathcal{P}(\mathbf{A}_2, n\varpi_1) \simeq \mathcal{P}(\mathbf{A}_n, 2\varpi_1) \simeq \mathcal{P}(\mathbf{A}_{n+1}, \varpi_2)$

2) $\mathcal{P}(\mathbf{B}_n, \varpi_n) \simeq \mathcal{P}(\mathbf{D}_{n+1}, \varpi_{n+1})$.

Proof. 1) Since $\mathcal{R}(\mathbf{A}_{n+m-1}, \varpi_n)^* = \mathcal{R}(\mathbf{A}_{n+m-1}, \varpi_m)$, it suffices to establish the first isomorphism.

Let $(e_1, \dots, e_{n+m})$ be the standard weight basis of the tautological representation, $\mathcal{R}(\varpi_1)$, of $\mathbf{A}_{n+m-1} = \mathfrak{sl}_{n+m}$. That is, the weight of e_i is ε_i, and $\sum_i \varepsilon_i = 0$. The simple roots are $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. The polyvectors $e_{i_1} \wedge \dots \wedge e_{i_n}$, $1 \leqslant i_1 < \ldots < i_n \leqslant n+m$, form a weight basis for $\wedge^n(\mathcal{R}(\varpi_1)) = \mathcal{R}(\varpi_n)$, and the weight of $e_{i_1} \wedge \dots \wedge e_{i_n}$ is $\varepsilon_{i_1} + \ldots + \varepsilon_{i_n}$. The above polyvector is identified with the sequence $\boldsymbol{i} = (i_1, \dots, i_n)$. It will be convenient to assume that $i_0 = 0$ and $i_{n+1} = n + m + 1$.

On the other hand, let $x_1, \dots, x_{n+1}$ be the standard weight basis of the tautological representation, $\mathcal{R}(\varpi_1)$, of $\mathbf{A}_n = \mathfrak{sl}_{n+1}$. Then the monomials $x_1^{j_1} \dots x_{n+1}^{j_{n+1}}$, $j_1 + \dots + j_{n+1} = m$, form a weight basis for $\mathcal{S}^m(\mathcal{R}(\varpi_1)) = \mathcal{R}(m\varpi_1)$. The above monomial has weight $\sum_k j_k\varepsilon_k$ and is identified with the sequence $\boldsymbol{j} = (j_1, \dots, j_{n+1})$.

Define a correspondence between the two weight bases as follows:

$$(i_1, \dots, i_n) \mapsto (n+m-i_n, i_n - i_{n-1} - 1, \dots, i_2 - i_1 - 1, i_1 - 1). \tag{2.1}$$

In other words, $j_k = i_{n+2-k} - i_{n+1-k} - 1,\ k = 1, 2, \dots, n+1$. Clearly, this is a bijection. Let us verify that this also provides an isomorphism of weight posets.

For $\nu = \varepsilon_{i_1} + \dots + \varepsilon_{i_n} \in \mathcal{P}(\varpi_n)$ and $\alpha_j \in \Pi(\mathbf{A}_{n+m-1})$, one has $\nu - \alpha_j \in \mathcal{P}(\varpi_n)$ if and only if $j = i_k$ and $i_{k+1} - i_k \geqslant 2$ for some k. On the level of $\boldsymbol{i}$-sequences, this means that i_k is replaced with $i_k + 1$, while all other components remain intact. In terms of the corresponding $\boldsymbol{j}$-sequences, we make transformation $j_{n-k+1} \mapsto j_{n-k+1} - 1$ and $j_{n-k+2} \mapsto j_{n-k+2} + 1$, which corresponds to subtraction the simple root $\alpha_{n-k+1} \in \Pi(\mathbf{A}_n)$. Therefore, mapping (2.1) yields a bijection between two sets of edges.

2) The weights of $\mathcal{R}(\mathbf{B}_n, \varpi_n)$ are $(\pm\varepsilon_1 \pm \varepsilon_2 \dots \pm \varepsilon_n)/2$, where all combinations of signs are allowed. The weights of $\mathcal{R}(\mathbf{D}_{n+1}, \varpi_{n+1})$ are $(\pm\varepsilon_1 \pm \varepsilon_2 \dots \pm \varepsilon_{n+1})/2$, where the total number of minuses is even. The bijection between the weights is quite obvious. For $\nu \in \mathcal{P}(\mathbf{B}_n, \varpi_n)$, the corresponding weight for $\mathbf{D}_{n+1}$ is $\nu' = \nu \pm (\varepsilon_{n+1}/2)$, where the sign of ε_{n+1} is determined by the condition that the total number of "$-$" in ν' to be even.

We also use 'prime' to mark simple roots of $\mathbf{D}_{n+1}$. Recall that

$$\Pi(\mathbf{B}_n) = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}, \quad \Pi(\mathbf{D}_{n+1}) = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n - \varepsilon_{n+1}, \varepsilon_n + \varepsilon_{n+1}\}.$$

It follows that, for $1 \leqslant i \leqslant n-1$, the edges of type α_i are the "same" in both posets; i.e., $\nu - \alpha_i \in \mathcal{P}(\mathbf{B}_n, \varpi_n)$ if and only if $\nu' - \alpha_i' \in \mathcal{P}(\mathbf{D}_n, \varpi_{n+1})$. Something not entirely trivial only happens for α_n. If $\nu - \alpha_n \in \mathcal{P}(\mathbf{B}_n, \varpi_n)$, then the last sign in ν must be 'plus'. Now, we have a fork. If $\nu' = \nu + (\varepsilon_{n+1}/2)$, then $\nu' - \alpha_{n+1}' \in \mathcal{P}(\mathbf{D}_n, \varpi_{n+1})$. If $\nu' = \nu - (\varepsilon_{n+1}/2)$, then $\nu' - \alpha_n' \in \mathcal{P}(\mathbf{D}_n, \varpi_{n+1})$. $\square$

Theorem 2.2.

1. $\#\mathcal{E}(\mathbf{A}_n, \varpi_m) = m\binom{n}{m} = n\binom{n-1}{m-1}$*;*

2. $\#\mathcal{E}(\mathbf{D}_n, \varpi_n) = n2^{n-3}$*.*

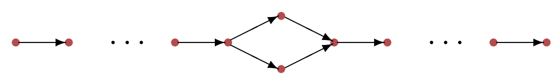
Proof. 1. Let $\varepsilon_1, \dots, \varepsilon_{n+1}$ be the weights of the tautological representation of $\mathbf{A}_n$. Then the weights of $\mathcal{R}(\varpi_m)$ are $\varepsilon_{i_1} + \dots + \varepsilon_{i_m}$, where $1 \leqslant i_1 < i_2 < \dots < i_m \leqslant n+1$ (cf. the proof of Theorem 2.1). Let $\nu \in \mathcal{P}(\varpi_m)$ be a weight such that $\nu - \alpha_1 \in \mathcal{P}(\varpi_m)$. Then ν must be of the form $\varepsilon_1 + \varepsilon_{i_2} + \dots + \varepsilon_{i_m}$ with $i_2 \geqslant 3$. Obviously, the number of such weights ν is equal to $\binom{n-1}{m-1}$, and this is the number of edges of type α_1. Making use of Theorem 1.3, we conclude that the total number of edges is $n\binom{n-1}{m-1}$.

2. Let $\nu \in \mathcal{P}(\varpi_n)$ be a weight such that $\nu - \alpha_1 \in \mathcal{P}(\varpi_n)$. Then ν must be of the form $(\varepsilon_1 - \varepsilon_2 \pm \varepsilon_3 \ldots \pm \varepsilon_n)/2$. Since the number of minuses is supposed to be even, there are 2^{n-3} possibilities for such ν, and hence 2^{n-3} edges of type α_1. Making use of Theorem 1.3, we conclude that the total number of edges is $n2^{n-3}$. $\square$

For the simplest representations of $\mathbf{B}_n$, $\mathbf{C}_n$, and $\mathbf{G}_2$ (i.e., for $\lambda = \varpi_1$), $\mathcal{P}(\lambda)$ is a chain. Hence $\#\mathcal{E}(\lambda) = \dim \mathcal{R}(\lambda) - 1$. This also provides the poset isomorphisms:

$$\mathcal{P}(\mathbf{B}_n, \varpi_1) \simeq \mathcal{P}(\mathbf{A}_{2n}, \varpi_1), \quad \mathcal{P}(\mathbf{C}_n, \varpi_1) \simeq \mathcal{P}(\mathbf{A}_{2n-1}, \varpi_1), \quad \mathcal{P}(\mathbf{G}_2, \varpi_1) \simeq \mathcal{P}(\mathbf{A}_6, \varpi_1).$$

The Hasse diagram of $\mathcal{P}(\mathbf{D}_n, \varpi_1)$ is



The remaining cases can be handled directly or using some theory from the next section. Our computations are gathered in the table. The last column will be needed in Section 4.

TABLE 1. The number of edges for the wmf representations

$\mathfrak{g}$	λ	$\dim \mathcal{R}(\lambda)$	$\#\mathcal{E}(\lambda)$	Reference	$\#\mathcal{E}(\lambda)/\dim \mathcal{R}(\lambda)$
$\mathbf{A}_n$	ϖ_m	$\binom{n+1}{m}$	$m\binom{n}{m}$	Theorem 2.2	$\frac{m(n+1-m)}{n+1}$
	$m\varpi_1, m\varpi_n$	$\binom{n+m}{m}$	$m\binom{n+m-1}{m}$	Theorems 2.1 & 2.2	$\frac{nm}{n+m}$
$\mathbf{B}_n$	ϖ_n	2^n	$(n+1)2^{n-2}$	Theorems 2.1 & 2.2	$\frac{n+1}{4}$
	ϖ_1	$2n+1$	$2n$	$\mathcal{P}(\lambda)$ is chain	$1 - \frac{1}{2n+1}$
$\mathbf{C}_n$	ϖ_1	$2n$	$2n-1$	$\mathcal{P}(\lambda)$ is chain	$1 - \frac{1}{2n}$
	ϖ_3 $(n=3)$	14	17	directly	17/14
$\mathbf{D}_n$	ϖ_1	$2n$	$2n$	see figure	1
	ϖ_{n-1}, ϖ_n	2^{n-1}	$n2^{n-3}$	Theorem 2.2	$\frac{n}{4}$
$\mathbf{E}_6$	ϖ_1	27	36	Example 3.6	4/3
$\mathbf{E}_7$	ϖ_1	56	84	Example 3.7	3/2
$\mathbf{G}_2$	ϖ_1	7	6	$\mathcal{P}(\lambda)$ is chain	7/6

Below, we list all weight posets occurring in Table 1, up to isomorphism:

(2.2) Serial cases: $\mathcal{P}(\mathbf{A}_n, \varpi_m)$, $1 \leqslant m \leqslant (n+1)/2$; $\mathcal{P}(\mathbf{D}_n, \varpi_n)$, $n \geqslant 5$; $\mathcal{P}(\mathbf{D}_n, \varpi_1)$, $n \geqslant 4$;
Sporadic cases: $\mathcal{P}(\mathbf{E}_6, \varpi_1)$, $\mathcal{P}(\mathbf{E}_7, \varpi_1)$, $\mathcal{P}(\mathbf{C}_3, \varpi_3)$;

The irreducible wmf representations of a semisimple algebra $\mathfrak{s}$ are tensor products of irreducible wmf representations of different simple ideals of $\mathfrak{s}$. It is therefore easy to compute the number of edges for them using information on the simple ideals.

Lemma 2.3. *Let $(\mathfrak{g}', \mathcal{R}')$, $(\mathfrak{g}'', \mathcal{R}'')$ be two wmf representations. Then $(\mathfrak{g}' \times \mathfrak{g}'', \mathcal{R}' \otimes \mathcal{R}'')$ is also wmf and*

$$\#\mathcal{E}(\mathcal{R}' \otimes \mathcal{R}'') = \dim \mathcal{R}' \cdot \#\mathcal{E}(\mathcal{R}'') + \dim \mathcal{R}'' \cdot \#\mathcal{E}(\mathcal{R}').$$

Proof. The Hasse diagram of $\mathcal{P}(\mathcal{R}' \otimes \mathcal{R}'')$ is determined by the following conditions:

- $\mathcal{P}(\mathcal{R}' \otimes \mathcal{R}'') = \mathcal{P}(\mathcal{R}') \times \mathcal{P}(\mathcal{R}'')$;
- Suppose $a_1, a_2 \in \mathcal{P}(\mathcal{R}')$ and $b_1, b_2 \in \mathcal{P}(\mathcal{R}'')$. Then $((a_1, b_1), (a_2, b_2)) \in \mathcal{E}(\mathcal{R}' \otimes \mathcal{R}'')$ if and only if either $a_1 = a_2$ and $(b_1, b_2) \in \mathcal{E}(\mathcal{R}'')$ or $b_1 = b_2$ and $(a_1, a_2) \in \mathcal{E}(\mathcal{R}')$.

Therefore $\mathcal{E}(\mathcal{R}' \otimes \mathcal{R}'') \simeq (\mathcal{E}(\mathcal{R}') \times \mathcal{P}(\mathcal{R}'')) \sqcup (\mathcal{E}(\mathcal{R}'') \times \mathcal{P}(\mathcal{R}'))$. □

In terminology of Graph Theory, the Hasse diagram of $\mathcal{P}(\mathcal{R}' \otimes \mathcal{R}'')$ is called the *cartesian product* of Hasse diagrams of $\mathcal{P}(\mathcal{R}')$ and $\mathcal{P}(\mathcal{R}'')$. We also apply this term to the posets themselves.

Corollary 2.4. $\dfrac{\#\mathcal{E}(\mathcal{R}' \otimes \mathcal{R}'')}{\dim(\mathcal{R}' \otimes \mathcal{R}'')} = \dfrac{\#\mathcal{E}(\mathcal{R}')}{\dim(\mathcal{R}')} + \dfrac{\#\mathcal{E}(\mathcal{R}')}{\dim(\mathcal{R}')}.$

We say that two wmf-representations are *equivalent* if the corresponding weight posets are isomorphic. It follows from the preceding discussion that if one of the factors in $(\mathfrak{g}' \times \mathfrak{g}'', \mathcal{R}' \otimes \mathcal{R}'')$ is being replaced with an equivalent one, then the resulting tensor products are equivalent.

3. Weight posets for wmf-representations associated with $\mathbb{Z}$-gradings

In this section, $\mathfrak{g}$ is a simple Lie algebra. Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ be a $\mathbb{Z}$-grading. Then there is a semisimple $s \in \mathfrak{g}$ such that $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [s, x] = ix\}$. Consequently, $\mathfrak{g}(0)$ is reductive, $\mathsf{rk}\,\mathfrak{g}(0) = \mathsf{rk}\,\mathfrak{g}$, and each $\mathfrak{g}(i)$ is a wmf $\mathfrak{g}(0)$-module. Our objective is to prove that such wmf-representations possess a special property. Without loss of generality, we may assume that $s \in \mathfrak{t}$ and $\alpha_i(s) \geqslant 0$ for all $\alpha_i \in \Pi$. Then $\mathfrak{t} \subset \mathfrak{g}(0)$, and $\mathfrak{g}(0)$ inherits the triangular decomposition from $\mathfrak{g}$.

By [5, § 1.2, § 2.1], if one is interested in possible $\mathfrak{g}(0)$-modules $\mathfrak{g}(i)$, then it is enough to consider the $\mathfrak{g}(0)$-modules $\mathfrak{g}(1)$ for all simple $\mathfrak{g}$. (For $i > 1$, the question is reduced to considering the induced grading of a certain simple subalgebra of $\mathfrak{g}$.) For this reason, it suffices to consider $s \in \mathfrak{t}$ such that $\alpha_i(s) \in \{0, 1\}$. Then $\Pi = \Pi(0) \sqcup \Pi(1)$, where $\Pi(i) = \{\alpha \in \Pi \mid \alpha(s) = i\}$. The corresponding $\mathbb{Z}$-gradings are said to be *standard*. More precisely, if $\#\Pi(1) = k$, then we call it a *k-standard* grading. A standard $\mathbb{Z}$-grading will be represented by the Dynkin diagram of $\mathfrak{g}$, where the vertices in $\Pi(1)$ are coloured. If $\Pi(1) = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$, then the α_{i_j}'s are precisely the lowest weights of the simple $\mathfrak{g}(0)$-submodules in $\mathfrak{g}(1)$. Therefore, $\mathfrak{g}(1)$ is the sum of k simple $\mathfrak{g}(0)$-modules.

Let $\Delta(i)$ denote the set of roots ($\mathfrak{t}$-weights) in $\mathfrak{g}(i)$. Then $\Delta(0)$ is the root system of $\mathfrak{g}(0)$ and $\Pi(0)$ is the set of simple roots in $\Delta(0)^+$. We regard $\Delta(i)$ as weight poset of the

$\mathfrak g(0)$-module $\mathfrak g(i)$ and write $\mathcal E(i)$ for the set of edges in (the Hasse diagram of) $\Delta(i)$. If $m=\max\{j \mid \mathfrak g(j)\ne 0\}$, then $\theta\in\Delta(m)$.

Remark 3.1. The Hasse diagrams of posets $\Delta(i)$ are obtained as follows. Take the Hasse diagram of Δ^+ and remove all the edges of types $\alpha_{i_1},\dots,\alpha_{i_k}$. The remaining (disconnected) graph is the union of Hasse diagrams of $\Delta(i)$, $i\geqslant 1$, and $\Delta(0)^+$.

Theorem 3.2. *Let $\mathfrak g=\bigoplus_{i\in\mathbb Z}\mathfrak g(i)$ be a k-standard grading. Suppose $\mathfrak g$ is simply laced. Then*

$$\sum_{i\geqslant 1}(2\dim\mathfrak g(i)-\#\mathcal E(i))=k{\cdot}h. \tag{3.1}$$

Proof. Recall that $\dim\mathfrak n^+=nh/2$. Let $\{n_j,h_j\}_{j\in J}$ be the ranks and Coxeter numbers, respectively, for all simple ideals of $\mathfrak g(0)$. Then $\dim(\mathfrak g(0)\cap\mathfrak n^+)=\sum_{j\in J}n_jh_j/2$. Note that $\sum_{j\in J}n_j=n-k$. It follows that

$$2\sum_{i\geqslant 1}\dim\mathfrak g(i)=nh-\sum_{j\in J}n_jh_j=kh+\sum_{j\in J}n_j(h-h_j)\ . \tag{3.2}$$

If $\Pi(1)=\{\alpha_{i_1},\dots,\alpha_{i_k}\}$, then the posets $\Delta(i)$, $i\geqslant 1$, do not contain edges of types $\alpha_{i_1},\dots,\alpha_{i_k}$ and each simple ideal of $\mathfrak g(0)$ is also simply-laced. Therefore

$$\begin{aligned}\sum_{i\geqslant 1}\#\mathcal E(i)&=\#\mathcal E(\Delta^+)-\#\mathcal E(\Delta(0)^+)-\sum_{s=1}^{k}\#\{\text{the edges of type }\alpha_{i_s}\}=\\ &=n(h-2)-\sum_{j\in J}n_j(h_j-2)-kh=\sum_{j\in J}n_j(h-h_j).\end{aligned}\tag{3.3}$$

Taking the difference of Eq. (3.2) and (3.3) yields the assertion. □

Remark 3.3. In the multiply-laced case, there is no such a nice formula. The reason is that the total number of edges in both Δ^+ and $\Delta(0)^+$ does not admit a simple closed expression, see Remark 1.2. Furthermore, $\sum_{i\geqslant 1}(2\dim\mathfrak g(i)-\#\mathcal E(i))$ can be more than $k{\cdot}h$ and it does not depend only on $\#\Pi(1)$, see example below.

Example 3.4. For $\mathfrak g=\mathbf F_4$, consider all 1-standard $\mathbb Z$-gradings (i.e., with $k=1$). In these four cases, the left hand side of Eq. (3.1) equals:

α_1	α_2	α_3	α_4
16	18	15	13

. In particular, each sum is larger than $h=12$.

The subspace $\mathfrak p=\bigoplus_{i\geqslant 0}\mathfrak g(i)$ is a parabolic subalgebra, and it is maximal parabolic if and only the grading is 1-standard. Then each $\mathfrak g(i)$ is a simple $\mathfrak g(0)$-module [6, 3.5]. Conversely, if $\#\Pi(1)>1$, then $\mathfrak g(1)$ is not a simple module. Without loss of generality, we may restrict ourselves with 1-standard $\mathbb Z$-gradings. In other words, every *simple* $\mathfrak g(0)$-submodule of $\mathfrak g(1)$ can be obtained as the component of degree 1 of a simple subalgebra of $\mathfrak g$ endowed with induced grading. Indeed, suppose that $\#\Pi(1)>1$ and $\alpha_j\in\Pi(1)$. Let $\mathcal R$ be the simple

$\mathfrak{g}(0)$-submodule of $\mathfrak{g}(1)$ with lowest weight α_j. Let us remove the vertices $\Pi(1) \setminus \{\alpha_j\}$ from the Dynkin diagram and take the connected subgraph containing α_j. This subgraph is the Dynkin diagram of a simple subalgebra $\mathfrak{s} \subset \mathfrak{g}$. The vertex α_j determines a $\mathbb{Z}$-grading $\mathfrak{s} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{s}(i)$, and $\mathcal{R}$ is isomorphic to the $\mathfrak{s}(0)$-module $\mathfrak{s}(1)$. If $\Pi(1) = \{\alpha_i\}$, then $\max\{j \mid \mathfrak{g}(j) \neq 0\} = [\theta : \alpha_i]$. For $[\theta : \alpha_i] = 1$, we have $\mathfrak{g} = \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1)$, and the nilpotent radical of $\mathfrak{p}$ is abelian. Such $\mathbb{Z}$-gradings are said to be *short*.

Theorem 3.5. *For any short grading, one has* $2\dim\mathfrak{g}(1) - \#\mathcal{E}(1) = h$.

Proof. If Δ is simply-laced, then the assertion follows from Theorem 3.2. If Δ is multiply-laced, then we have only two possibilities: α_n for $\mathbf{C}_n$ and α_1 for $\mathbf{B}_n$, and everything can be counted directly.

For $(\mathbf{C}_n, \alpha_n)$, he semisimple part of $\mathfrak{g}(0)$, denoted $\mathfrak{g}(0)'$, is $\mathbf{A}_{n-1}$, and the $\mathbf{A}_{n-1}$-module $\mathfrak{g}(1)$ is $\mathcal{R}(2\varpi_1)$. Here $\dim\mathfrak{g}(1) = n(n+1)/2$ and $\#\mathcal{E}(2\varpi_1) = \#\mathcal{E}(1) = n(n-1)$, see Table 1. The case of $\mathbf{B}_n$ is left to the reader. □

Example 3.6. If $\mathfrak{g} = \mathbf{E}_7$, then $[\theta : \alpha_1] = 1$. For the respective short grading, we have $\mathfrak{g}(0)' = \mathbf{E}_6$ and $\mathfrak{g}(1)$ is the $\mathbf{E}_6$-module $\mathcal{R}(\varpi_1)$, of dimension 27. Therefore $\#\mathcal{E}(\varpi_1) = 2\dim\mathcal{R}(\varpi_1) - h(\mathfrak{g}) = 36$.

Example 3.7. Every simple Lie algebra has a unique (up to conjugation) $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{-2\leqslant i\leqslant 2} \mathfrak{g}(i)$ such that $\dim\mathfrak{g}(2) = 1$. (The grading corresponding to the minimal nilpotent orbit.) Then $\Delta(2) = \{\theta\}$ and $\Delta(i) = \{\gamma \mid (\gamma, \theta^\vee) = i\}$. Obviously, $\mathcal{E}(2) = \varnothing$. Suppose Δ is simply-laced. By Theorem 3.2,

$$2\dim\mathfrak{g}(1) - \#\mathcal{E}(1) + 2 = kh,$$

where $k = \#\Pi(1)$ is the number of simple roots that are not orthogonal to θ. Furthermore, $\dim\mathfrak{g}(1) + \dim\mathfrak{g}(2) = \#\{\gamma \in \Delta^+ \mid (\gamma, \theta) > 0\}$, and the latter is equal to $2h - 3$ [1, Ch. VI, § 1, n. 11, prop. 32]. Hence $\dim\mathfrak{g}(1) = 2h - 4$ and $\#\mathcal{E}(1) = (4-k)h - 6$. For $\mathbf{D}_n$ and $\mathbf{E}_n$, θ is fundamental, i.e., $k = 1$. Hence $\#\mathcal{E}(1) = 3h - 6 = \frac{3}{2}\dim\mathfrak{g}(1)$. In particular, if $\mathfrak{g} = \mathbf{E}_8$, then $\mathfrak{g}(0)' = \mathbf{E}_7$ and $\mathfrak{g}(1) = \mathcal{R}(\varpi_1)$. Whence $\#\mathcal{E}(\mathbf{E}_7, \varpi_1) = (3/2)\cdot 56 = 84$.

Looking at Eq. (3.1), one might suspect that each summand in the left hand side in non-negative. Actually, each summand appears to be positive, and this property is, in a sense, characteristic for wmf-representations associated with $\mathbb{Z}$-gradings.

Theorem 3.8. *Let* $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ *be an arbitrary* $\mathbb{Z}$*-grading. Then* $0 < 2\dim\mathfrak{g}(1) - \#\mathcal{E}(1) \leqslant h(\mathfrak{g})$. *Moreover, if* $m = \max\{j \mid \mathfrak{g}(j) \neq 0\} > 1$, *then* $2\dim\mathfrak{g}(1) - \#\mathcal{E}(1) < h$.

Proof. As is explained above, it suffices to consider 1-standard $\mathbb{Z}$-gradings. If $\Pi(1) = \{\alpha_i\}$, then $m = [\theta : \alpha_i]$.

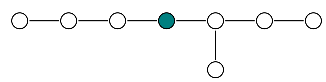
- Suppose Δ is simply-laced.

a) For $[\theta : \alpha_i] = 1$, the assertion follows from Theorem 3.2 with $k = 1$.

b) For $[\theta : \alpha_i] = 2$, it is sufficient to prove that $0 < 2 \dim \mathfrak{g}(2) - \#\mathcal{E}(2) < h$.
We know that $\mathfrak{g}(2)$ is a simple $\mathfrak{g}(0)$-module. Let $\mathfrak{s}(0)$ be the reductive subalgebra of $\mathfrak{g}(0)$ that contains the (one-dimensional) centre of $\mathfrak{g}(0)$ and all simple ideals acting non-triviallly on $\mathfrak{g}(2)$. Then $\mathfrak{s} := \mathfrak{g}(-2) \oplus \mathfrak{s}(0) \oplus \mathfrak{g}(2)$ is a simple subalgebra of $\mathfrak{g}$ and this decomposition is a short grading of $\mathfrak{s}$. Hence $2 \dim \mathfrak{g}(2) - \#\mathcal{E}(2) = h(\mathfrak{s}) > 0$ (Theorem 3.2). It remains to notice that the highest root $\theta \in \Delta(2) \subset \Delta^+$ is the highest root for $\mathfrak{s}$ as well, but the height of θ in $\Delta(\mathfrak{s})$ is strictly less than that in $\Delta(\mathfrak{g})$, i.e., $h(\mathfrak{s}) < h = h(\mathfrak{g})$.

c) So far the argument was satisfactory. In fact, it completely covers the series $\mathbf{A}_n$ and $\mathbf{D}_n$. But for $[\theta : \alpha_i] \geqslant 3$, we have to resort to case-by-case considerations. There is one such case for $\mathbf{E}_6$, three for $\mathbf{E}_7$, and six for $\mathbf{E}_8$.

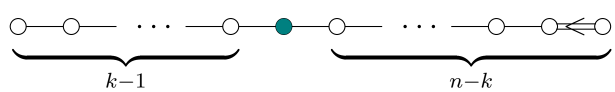
The output of our computations for all 1-standard gradings of all simple algebras is presented after the proof. Here is a sample of required computations. For $\mathfrak{g} = \mathbf{E}_8$ and $\alpha_4 \in \Pi$, we have $[\theta : \alpha_4] = 5$. The corresponding coloured Dynkin diagram is:



This diagram shows that, for the $\mathbb{Z}$-grading determined by α_4, $\mathfrak{g}(0)'$ is $\mathbf{A}_3 \times \mathbf{A}_4$ and the highest weight of $\mathfrak{g}(1)$ is $\varpi_1 + \varpi_2'$. In other words, $\mathfrak{g}(1) = \mathcal{R}(\varpi_1) \otimes \mathcal{R}(\varpi_2')$ is the tensor product of the tautological representation of $\mathbf{A}_3$ and the second fundamental representation of $\mathbf{A}_4$. Hence $\dim \mathfrak{g}(1) = 40$. Using Lemma 2.3 and data in Table 1, we compute that $\#\mathcal{E}(1) = 12{\cdot}4 + 10{\cdot}3 = 78$.

- Suppose Δ is multiply-laced. Here our argument is fully computational.

– The case of $\mathbf{C}_n$. For $k \leqslant n-1$, $[\theta : \alpha_k] = 2$ and the coloured Dynkin diagram is

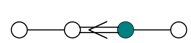


Therefore, $\mathfrak{g}(0)'$ is $\mathbf{A}_{k-1} \times \mathbf{C}_{n-k}$ and $\mathfrak{g}(1) = \mathcal{R}(\varpi_1 + \varpi_1') = \mathcal{R}(\varpi_1) \otimes \mathcal{R}(\varpi_1')$. Hence $\dim \mathfrak{g}(1) = k(2n-2k)$ and $\#\mathcal{E}(1) = (k-1)(2n-2k) + k(2n-2k-1) = 2 \dim \mathfrak{g}(1) - (2n-k)$.

Finally, $[\theta : \alpha_n] = 1$ and this case is considered in Theorem 3.5.

– Computations for $\mathbf{B}_n$ are quite similar. For $k \geqslant 2$, $[\theta : \alpha_k] = 2$. Then $\mathfrak{g}(0)'$ is $\mathbf{A}_{k-1} \times \mathbf{B}_{n-k}$ and $\mathfrak{g}(1) = \mathcal{R}(\varpi_1) \otimes \mathcal{R}(\varpi_1')$. Therefore, $\dim \mathfrak{g}(1) = k(2n-2k+1)$ and $\#\mathcal{E}(1) = (k-1)(2n-2k+1) + k(2n-2k) = 2 \dim \mathfrak{g}(1) - (2n-k+1)$.

– Consider one possibility for $\mathbf{F}_4$. Here $[\theta : \alpha_3] = 3$ and the coloured Dynkin diagram is



Therefore, $\mathfrak{g}(0)' = \mathbf{A}_2 \times \mathbf{A}_1$ with $\mathfrak{g}(1) = \mathcal{R}(2\varpi_1) \otimes \mathcal{R}(\varpi_1')$. Hence $\dim \mathfrak{g}(1) = 12$ and $\#\mathcal{E}(1) = 18$. $\square$

Hopefully, there is a better proof of Theorem 3.8. The following table presents the numbers $2\dim\mathfrak g(1) - \#\mathcal{E}(1)$ for all 1-standard gradings of the exceptional Lie algebras.

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8		α_1	α_2	α_3	α_4
$\mathbf{E}_6$	12	6	3	6	12	10	–	–	$\mathbf{F}_4$	8	5	6	11
$\mathbf{E}_7$	18	8	4	2	5	16	10	–	$\mathbf{G}_2$	3	5	–	–
$\mathbf{E}_8$	28	9	4	2	1	3	16	7					

Below, for the 1-standard $\mathbb{Z}$-grading determined by the simple root α_k, the difference $2\dim\mathfrak g(1) - \#\mathcal{E}(1)$ is denoted by $\mathcal{Z}(\alpha_k)$. Here are data for the classical series:

$\mathbf{A}_n$: $\mathcal{Z}(\alpha_k) = n+1 = h$ for any k;
$\mathbf{B}_n$: $\mathcal{Z}(\alpha_k) = 2n-k+1$ for any k;
$\mathbf{C}_n$: $\mathcal{Z}(\alpha_k) = 2n-k$ if $k \leqslant n-1$; $\mathcal{Z}(\alpha_n) = 2n$.
$\mathbf{D}_n$: $\mathcal{Z}(\alpha_k) = 2n-2k$ if $k \leqslant n-2$; $\mathcal{Z}(\alpha_{n-1}) = \mathcal{Z}(\alpha_n) = 2n-2$;

It is curious that if α_k is any vertex for $\mathbf{A}_n$ or the branch vertex for $\mathbf{D}_n$ or $\mathbf{E}_n$, then $\mathcal{Z}(\alpha_k)$ equals the determinant of the corresponding Cartan matrix.

4. $\mathbb{Z}_m$-GRADINGS VERSUS $\mathbb{Z}$-GRADINGS

We continue to assume that $\mathfrak g$ is simple. For a $\mathbb{Z}$-grading, we prove in Theorem 3.8 that $2\dim\mathfrak g(1) > \#\mathcal{E}(1)$. That is to say, the number of edges cannot be too large. In the simply-laced case it turns out, rather surprisingly, that if we "replace" a $\mathbb{Z}$-grading with a periodic grading, this inequality turns into equality!

More precisely, consider the following situation. Let $\vartheta \in \mathrm{Aut}(\mathfrak g)$ be an *inner* automorphism of order $m > 1$. Choose a primitive root of unity $\zeta = \sqrt[m]{1}$ and set $\mathfrak g_i = \{x \in \mathfrak g \mid \vartheta(x) = \zeta^i x\}$. This yields a $\mathbb{Z}_m$-grading of $\mathfrak g$:

$$\mathfrak g = \bigoplus_{i\in\mathbb{Z}_m} \mathfrak g_i.$$

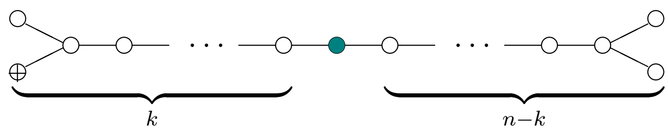
Here $\mathfrak g_0$ is reductive. As ϑ is inner, $\mathsf{rk}\,\mathfrak g = \mathsf{rk}\,\mathfrak g_0$ and each $\mathfrak g_i$ is a wmf $\mathfrak g_0$-module. Furthermore, we may assume that $\mathfrak t \subset \mathfrak g_0$ and then obtain the decomposition $\Delta = \sqcup_{i\in\mathbb{Z}_m}\Delta_i$, where Δ_i is the set of weights of the $\mathfrak g_0$-module $\mathfrak g_i$. Let $\mathcal{E}_i$ denote the set of edges in the Hasse diagram of the weight poset of $\mathfrak g_i$.

Theorem 4.1. *Let $\mathfrak g = \bigoplus_{i\in\mathbb{Z}_m}\mathfrak g_i$ be a periodic grading such that $\mathfrak g_1$ is a simple $\mathfrak g_0$-module. Suppose that Δ is simply-laced. Then $2\dim\mathfrak g_1 = \#(\mathcal{E}_1)$.*

Proof. We proceed in a case-by-case fashion. The periodic gradings such that $\mathfrak g_1$ is a simple $\mathfrak g_0$-module have a nice explicit description, see [5, § 8], [6, 3.7]. Any such grading is determined by one vertex of the *extended Dynkin diagram* of $\mathfrak g$, denoted $\tilde{\mathcal{D}}(\mathfrak g)$ or $\widetilde{\mathbf{X}}_n$ if $\mathbf{X}_n$ is the Cartan type of $\mathfrak g$. The extra vertex of $\tilde{\mathcal{D}}(\mathfrak g)$ is denoted by α_0 and we formally set $[\theta : \alpha_0] = 1$. In figures below, the extra vertex is marked with 'cross'. A choice of vertex α_i,

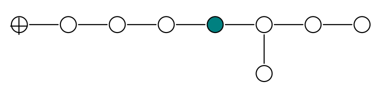
$0 \leqslant i \leqslant n$, yields a $\mathbb{Z}_m$-grading with $m = [\theta : \alpha_i]$. Having removed this vertex, one obtains a union of Dynkin diagrams that represents $\mathfrak{g}_0$. In this case $\mathfrak{g}_0$ is semisimple. In fact, $\mathfrak{g}_0$ is semisimple if and only if $\mathfrak{g}_1$ is a simple $\mathfrak{g}_0$-module, see [5, Prop. 18]. (For $[\theta : \alpha_i] = 1$, one obtains the initial Dynkin diagram and the trivial automorphism. So, this case is excluded from further considerations.) The bonds between α_i and the adjacent vertices of $\widetilde{\mathcal{D}}(\mathfrak{g})$ determine the structure of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$, see [5, Prop. 17] for the details.

D) Consider the periodic grading of $\mathbf{D}_n$ corresponding to α_k, $2 \leqslant k \leqslant n-2$. Here $[\theta : \alpha_k] = 2$ and the extended diagram is the following:

$\widetilde{\mathbf{D}}_n$: 

Therefore $\mathfrak{g}_0 = \mathbf{D}_k \times \mathbf{D}_{n-k}$ and $\mathfrak{g}_1 = \mathcal{R}(\varpi_1) \otimes \mathcal{R}(\varpi'_1)$. Hence $\dim \mathfrak{g}_1 = 4k(n-k)$ and, by Lemma 2.3, $\#(\mathcal{E}_1) = 8k(n-k)$.

E) For the exceptional Lie algebras, E.B. Vinberg gives the table of all periodic gradings such that $\mathfrak{g}_0$ is semisimple [5, § 9]. It is not hard to check the required equality for the inner automorphisms of $\mathbf{E}_n$, $n = 6, 7, 8$ (the first 14 items in Vinberg's table). As a sample, we consider the automorphism of order 5 for $\mathbf{E}_8$ (cf. the case considered in the proof of Theorem 3.8). Now the extended coloured Dynkin diagram is:

$\widetilde{\mathbf{E}}_8$: 

Therefore $\mathfrak{g}_0 = \mathbf{A}_4 \times \mathbf{A}_4$ and $\mathfrak{g}_1 = \mathcal{R}(\varpi_1) \otimes \mathcal{R}(\varpi'_2)$. Hence $\dim \mathfrak{g}_1 = 5{\cdot}10 = 50$ and $\#(\mathcal{E}_1) = 12{\cdot}5 + 10{\cdot}4 = 100$. $\square$

However, in the multiply-laced case, we still obtain the strict inequality $2 \dim \mathfrak{g}_1 > \#(\mathcal{E}_1)$, see below. Probably, the relation $2 \dim \mathfrak{g}_1 = \#(\mathcal{E}_1)$ could be adjusted somehow, but I have no idea. Nevertheless, we have the following general assertion:

Theorem 4.2. *Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i$ be an arbitrary periodic grading. Then $2 \dim \mathfrak{g}_1 \geqslant \#(\mathcal{E}_1)$ and the equality occurs if and only if the assumptions of Theorem 4.1 holds.*

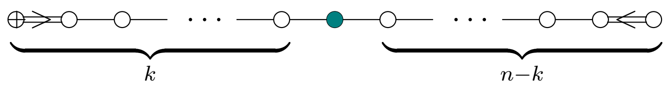
Proof. By Vinberg's theory [5, § 8], if we wish to check all possible representations $(\mathfrak{g}_0 : \mathfrak{g}_1)$ associated with periodic gradings, it suffices to deal with periodic gradings determined by arbitrary subsets Π_1 of the extended Dynkin diagram of $\mathfrak{g}$. If $\Pi_1 = \{\alpha_{i_1}, \dots, \alpha_{i_k}\}$, $0 \leqslant i_j \leqslant n$, then the order of ϑ equals $\sum_j [\theta : \alpha_{i_j}]$. (Recall that $\mathfrak{g}_0$ is semisimple if and only if $\#(\Pi_1) = 1$.)

a) $\#(\Pi_1) = 1$ and Δ is simply-laced. This case is considered in Theorem 4.1.

b) $\#(\Pi_1) = 1$ and Δ is multiply-laced.

• Consider the case of $\mathbf{C}_n$ and $\Pi_1 = \{\alpha_k\}$, $1 \leqslant k \leqslant n-1$. Since $[\theta : \alpha_k] = 2$, the automorphism ϑ is an involution.

$\widetilde{\mathbf{C}}_n$:

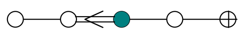


Here $\mathfrak{g}_0 = \mathbf{C}_k \times \mathbf{C}_{n-k}$ and $\mathfrak{g}_1 = \mathcal{R}(\varpi_1) \otimes \mathcal{R}(\varpi_1')$. Therefore $\dim \mathfrak{g}_1 = 2k(2n-2k)$ and $\#(\mathcal{E}_1) = (2k-1)(2n-2k) + 2k(2n-2k-1)$. Hence $2\dim \mathfrak{g}_1 - \#(\mathcal{E}_1) = 2n$.

• In the similar situation for $\mathbf{B}_n$, with $\Pi_1 = \{\alpha_k\}$, $2 \leqslant k \leqslant n$, we obtain $\mathfrak{g}_0 = \mathbf{D}_k \times \mathbf{B}_{n-k}$ and $\mathfrak{g}_1 = \mathcal{R}(\varpi_1) \otimes \mathcal{R}(\varpi_1')$. Then $\dim \mathfrak{g}_1 = 2k(2n-2k+1)$ and $\#(\mathcal{E}_1) = 2k(2n-2k+1) + 2k(2n-2k)$. Hence $2\dim \mathfrak{g}_1 - \#(\mathcal{E}_1) = 2k$.

• For $\mathbf{F}_4$, we consider the example with $\Pi_1 = \{\alpha_3\}$. The coloured extended diagram is

$\widetilde{\mathbf{F}}_4$:



and ϑ is of order 3. Here $\mathfrak{g}_0 = \mathbf{A}_2 \times \mathbf{A}_2$ and $\mathfrak{g}_1 = \mathcal{R}(2\varpi_1) \otimes \mathcal{R}(\varpi_1')$. Therefore $\dim \mathfrak{g}_1 = 18$ and $\#(\mathcal{E}_1) = 30$. We omit consideration of other vertices for $\mathbf{F}_4$ and the $\mathbf{G}_2$-case.

c) $\#(\Pi_1) = k \geqslant 2$.

Recall that $\tilde{\mathcal{D}}(\mathfrak{g})$ is the extended Dynkin diagram of $\mathfrak{g}$. Suppose that k vertices of $\tilde{\mathcal{D}}(\mathfrak{g})$ are coloured. Removing these vertices yields a union of Dynkin diagrams, and this is the semisimple part of $\mathfrak{g}_0$. The centre of $\mathfrak{g}_0$ is $(k-1)$-dimensional. Vinberg's theory says that each vertex in Π_1 gives rise to an irreducible constituent of $\mathfrak{g}_1$. Namely, for $\alpha_i \in \Pi_1$, let us take the largest connected subdiagram of $\tilde{\mathcal{D}}(\mathfrak{g})$ that contains α_i and no other vertices from Π_1. (Practically, this means that we consider only those simple components of $\mathfrak{g}_0$ that act nontrivially on the selected simple submodule of $\mathfrak{g}_1$.) Since $\#(\Pi_1) \geqslant 2$, we obtain a *proper* subdiagram of $\tilde{\mathcal{D}}(\mathfrak{g})$, which is thereby a Dynkin diagram. That is, we get a *usual* Dynkin (sub)diagram with one coloured vertex α_i. As is explained in Section 3, this gives rise to a standard $\mathbb{Z}$-grading of a certain simple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$. If $\mathfrak{s} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{s}(i)$ is the grading determined by α_i, then our selected simple submodule of $\mathfrak{g}_1$ is isomorphic to the $\mathfrak{s}(0)$-module $\mathfrak{s}(1)$. However, we have already verified inequality $2\dim \mathfrak{s}(1) > \#\mathcal{E}(1)$ for all standard $\mathbb{Z}$-gradings.

Thus, for all periodic gradings with $\#(\Pi_1) \geqslant 2$, we have $2\dim \mathfrak{g}_1 > \#(\mathcal{E}_1)$. □

Remark 4.3. If $\vartheta \in \mathrm{Aut}(\mathfrak{g})$ is outer, then it may occasionally happen that $\mathfrak{g}_1$ is a wmf $\mathfrak{g}_0$-module. For instance, $\mathfrak{g} = \mathbf{D}_4$ has an automorphism of order 3 such that $\mathfrak{g}_0 = \mathbf{G}_2$ and $\mathfrak{g}_1 = \mathcal{R}(\varpi_1)$; $\mathfrak{g} = \mathbf{A}_{2n+2m-1}$ has an automorphism of order 4 such that $\mathfrak{g}_0 = \mathbf{D}_n \times \mathbf{C}_m$ and $\mathfrak{g}_1 = \mathcal{R}(\varpi_1) \otimes \mathcal{R}(\varpi_1')$.

The relations of Theorems 3.8, 4.1, and 4.2 suggest us to determine all irreducible wmf representations such that $\#\mathcal{E}(\lambda)/\dim \mathcal{R}(\lambda) \leqslant 2$. By Corollary 2.4, the ratio

$$\mathsf{R} := \#\mathcal{E}(\lambda)/\dim \mathcal{R}(\lambda)$$

is additive with respect to tensor products. Therefore we can start with the irreducible representations of simple Lie algebras. Using the last column of Table 1, we can find all suitable serial cases.

Example 4.4. – For $\mathcal{R}(\mathbf{A}_n, \varpi_m)$, we can assume that $m \leqslant (n+1)/2$. Then

- $\mathsf{R} < 2$ if and only if $m = 1, 2$ and n is arbitrary or $m = 3$ and $n = 5, 6, 7$;
- $\mathsf{R} = 2$ if and only if $(n, m) = (7, 4)$ or $(8, 3)$.

– For $\mathcal{R}(\mathbf{D}_n, \varpi_n)$, we see that $\mathsf{R} < 2$ if $n \leqslant 7$ and $\mathsf{R} = 2$ if $n = 8$.

– For $\mathcal{R}(\mathbf{D}_n, \varpi_1)$, we always have $\mathsf{R} = 1$.

For these three series of representations, $\mathcal{R}(\lambda)$ occurs as $\mathfrak{g}(1)$ for some 1-standard $\mathbb{Z}$-grading (resp. as $\mathfrak{g}_1$ for some periodic grading) if and only if $\mathsf{R} < 2$ (resp. $\mathsf{R} = 2$).

Repr.	R	grading	Repr.	R	grading
$\mathcal{R}(\mathbf{A}_n, \varpi_1)$	< 2	$(\mathbf{A}_{n+1}, \alpha_1)$	$\mathcal{R}(\mathbf{A}_8, \varpi_3)$	2	$(\widetilde{\mathbf{E}}_8, \alpha_8)$
$\mathcal{R}(\mathbf{A}_n, \varpi_2)$	< 2	$(\mathbf{D}_{n+1}, \alpha_{n+1})$	$\mathcal{R}(\mathbf{D}_n, \varpi_n)$, $n = 5, 6, 7$	< 2	$(\mathbf{E}_{n+1}, \alpha_n)$
$\mathcal{R}(\mathbf{A}_n, \varpi_3)$, $n = 5, 6, 7$	< 2	$(\mathbf{E}_{n+1}, \alpha_{n+1})$	$\mathcal{R}(\mathbf{D}_8, \varpi_8)$	2	$(\widetilde{\mathbf{E}}_8, \alpha_7)$
$\mathcal{R}(\mathbf{A}_7, \varpi_4)$	2	$(\widetilde{\mathbf{E}}_7, \alpha_7)$	$\mathcal{R}(\mathbf{D}_n, \varpi_1)$	1	$(\mathbf{D}_{n+1}, \alpha_1)$

In column "grading", we point out the type of Dynkin diagram (usual or extended) and the coloured vertex.

Example 4.5. Consider tensor products of simplest representations of algebras $\mathbf{A}_n$.

– If $\mathfrak{s} = \mathbf{A}_{n_1-1} \times \mathbf{A}_{n_2-1}$ and $\mathcal{R}(\varpi_1) \otimes \mathcal{R}(\varpi_1')$, then $\mathsf{R} = \frac{n_1-1}{n_1} + \frac{n_2-1}{n_2} < 2$. It is also seen that this representation is associated with a short $\mathbb{Z}$-grading of $\mathfrak{g} = \mathbf{A}_{n_1+n_2-1}$.

– If $\mathfrak{s} = \mathbf{A}_{n_1-1} \times \mathbf{A}_{n_2-1} \times \mathbf{A}_{n_3-1}$ and $\mathcal{R}(\varpi_1) \otimes \mathcal{R}(\varpi_1') \otimes \mathcal{R}(\varpi_1'')$, then $\mathsf{R} = \frac{n_1-1}{n_1} + \frac{n_2-1}{n_2} + \frac{n_3-1}{n_3}$. In this case, condition $\mathsf{R} < 2$ can be rewritten as

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} > 1. \tag{4.1}$$

This inequality often appears in classification problems; e.g., in classifying finite subgroups of $SL_2(\mathbb{C})$ or quivers of finite type. The well-known solutions are: $(2,2,n)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$. The corresponding representations are associated with a $\mathbb{Z}$-grading of $\mathbf{D}_{n+2}$ or $\mathbf{E}_n$, $n = 6, 7, 8$ (the branch vertex of the Dynkin diagram should be coloured).

The solutions of equation $\mathsf{R} = 2$ are $(3,3,3)$, $(2,4,4)$, $(2,3,6)$. The corresponding representations are associated with a periodic grading of $\mathbf{E}_n$ (the branch vertex of the extended Dynkin diagram $\widetilde{\mathbf{E}}_n$ should be coloured).

– For the product of four representations, the only possibility is $n_1 = n_2 = n_3 = n_4 = 2$ with $\mathsf{R} = 2$. The corresponding representation is associated with the branch vertex of $\widetilde{\mathbf{D}}_4$, i.e., with a $\mathbb{Z}_2$-grading of $\mathbf{D}_4$. Thus, we again observe the following phenomenon:

If $\mathsf{R} < 2$ (resp. $\mathsf{R} = 2$), then the wmf-representations in question are associated with $\mathbb{Z}$-gradings (resp. periodic gradings).

It is tempting to suggest that this is always true. However, the poset isomorphisms in Theorem 2.1 limit one's optimism. Indeed, $\mathcal{P}(\mathbf{A}_n, \varpi_m) \simeq \mathcal{P}(\mathbf{A}_{n-m+1}, m\varpi_1)$. For $m = 2$, the representations $\mathcal{R}(\mathbf{A}_n, \varpi_2)$ and $\mathcal{R}(\mathbf{A}_{n-1}, 2\varpi_1)$ have isomorphic weight posets and both are associated with short $\mathbb{Z}$-gradings (for $\mathfrak{g} = \mathbf{D}_{n+1}$ and $\mathbf{C}_n$, respectively). But for $m = 3$ and $n = 5, 6, 7$, the representation $\mathcal{R}(\mathbf{A}_{n-2}, 3\varpi_1)$ is not associated with a grading, whereas $\mathcal{R}(\mathbf{A}_n, \varpi_3)$ is associated with a $\mathbb{Z}$-grading of $\mathbf{E}_{n+1}$. Similar phenomenon occurs for the isomorphism $\mathcal{P}(\mathbf{D}_n, \varpi_n) \simeq \mathcal{P}(\mathbf{B}_{n-1}, \varpi_{n-1})$ with $n = 6, 7, 8$.

Therefore, a correct statement should be given in terms of weight posets.

Theorem 4.6. *Let $\mathcal{P}$ be the weight poset of an irreducible wmf representation of a semisimple Lie algebra $\mathfrak{s}$. Suppose every simple ideal of $\mathfrak{s}$ is simply-laced. Then $\mathcal{P}$ occurs as either $\Delta(1)$ for some $\mathbb{Z}$-grading or Δ_1 for some periodic grading of a simple Lie algebra $\mathfrak{g}$ if and only if $\mathsf{R} = \frac{\#\mathcal{E}(\mathcal{P})}{\#\mathcal{P}} \leqslant 2$. Furthermore, $\frac{\#\mathcal{E}(\mathcal{P})}{\#\mathcal{P}} = 2$ if and only if $\mathcal{P} = \Delta_1$ for a periodic grading of a simply-laced algebra $\mathfrak{g}$.*

Sketch of the proof. In view of Theorems 3.8 and 4.2, only "if" part should be verified, i.e., if $\mathcal{P}$ is the weight poset of an irreducible wmf-representation such that $\frac{\#\mathcal{E}(\mathcal{P})}{\#\mathcal{P}} \leqslant 2$, then $\mathcal{P} \simeq \Delta(1)$ or $\mathcal{P} \simeq \Delta_1$ for some grading of $\mathfrak{g}$.

All connected posets arising as $\Delta(1)$ or Δ_1 are readily determined via Vinberg's theory: one should remove one vertex from either the usual or extended Dynkin diagram of $\mathfrak{g}$ and consider the representations obtained, modulo to equivalence described at the end of Section 2. This yields the *first list* of posets.

On the other hand, we begin with posets pointed out in Eq. (2.2), except $\mathcal{P}(\mathbf{C}_3, \varpi_3)$. The serial cases satisfying condition $\mathsf{R} \leqslant 2$ are determined in Example 4.4. All these "initial" posets are contained in the first list. Then, using Corollary 2.4, we determine the cartesian products of them that still satisfy condition $\mathsf{R} \leqslant 2$. Example 4.5 can be regarded as part of relevant argument. We omit other routine considerations. Finally, we will see that all admissible cartesian products of the initial posets belong to the first list. □

Remark 4.7. If we omit the condition that all simple factors are simply-laced, then the assertion of Theorem 4.6 becomes wrong. Consider the wmf-representation $\mathcal{R}(\varpi_3)\otimes\mathcal{R}(\varpi_1')$ of $\mathfrak{s} = \mathbf{C}_3 \times \mathbf{A}_2$. Here $\mathsf{R} = 17/14 + 2/3 < 2$, but $\mathcal{P}(\varpi_3 + \varpi_1')$ is not associated with a grading.

Remark 4.8. The appearance of inequality (4.1) as an equivalent of the condition $\#\mathcal{E}(1)/\dim\mathfrak{g}(1) < 2$ for some $\mathbb{Z}$-gradings suggests that at least in the simply-laced case there could be a direct relationship between the determinant of the Cartan matrix and the number $2\dim\mathfrak{g}(1) - \#\mathcal{E}(1)$. This is also confirmed by the following coincidence. The periodic gradings, where $\mathfrak{g}_1$ is a simple $\mathfrak{g}_0$-module, satisfy the condition $2\dim\mathfrak{g}_1 - \#(\mathcal{E}_1) = 0$; on the other hand, periodic grading are described via extended Dynkin diagrams, and the extended Cartan matrix has zero determinant.

5. $\mathbb{Z}$-GRADINGS AND UPPER COVERING POLYNOMIALS

For $\mathbb{Z}$-gradings, yet another property of the $\mathfrak{g}(0)$-modules $\mathfrak{g}(i)$ can be expressed in terms of upper covering polynomials (see Section 1) of posets $\Delta(i)$.

Proposition 5.1. *For any $\mathbb{Z}$-grading of $\mathfrak{g}$, $\deg \mathcal{K}_{\Delta(i)}(t) \leqslant 3$.*

Proof. The posets $\Delta(i)$ are obtained from Δ^+ by removing certain edges. Clearly, this procedure does not increase the degree of upper covering polynomial, and we use Theorem 1.4. $\square$

Example 5.2. Consider the $\mathbf{A}_n$-module $\mathcal{R}(\varpi_4)$ for $n \geqslant 7$. Here the weight $\varepsilon_1+\varepsilon_3+\varepsilon_5+\varepsilon_7 =: (1357)$ covers four weights: $(2357), (1457), (1367), (1258)$. It follows that $\deg \mathcal{K}_{\mathcal{P}(\varpi_4)} \geqslant 4$. Therefore these modules cannot occur in connection with $\mathbb{Z}$-gradings. (Another reason is that $\#\mathcal{E}(\varpi_4)/\dim \mathcal{R}(\varpi_4) \geqslant 2$.)

We have computed the upper covering polynomials for all the weight posets associated with Table 1.

Theorem 5.3. *Discarding the repetition of posets, the upper covering polynomials are:*

$\mathbf{A}_n$: $\mathcal{K}_{\mathcal{P}(\varpi_m)}(t) = \sum_{r\geqslant 0} \binom{m}{r}\binom{n-m+1}{r} t^r$;

$\mathbf{D}_n$: $\mathcal{K}_{\mathcal{P}(\varpi_n)}(t) = \sum_{r\geqslant 0} \binom{n}{2r} t^r$; $\mathcal{K}_{\mathcal{P}(\varpi_1)}(t) = 1 + (2n-2)t + t^2$;

$\mathbf{C}_3$: $\mathcal{K}_{\mathcal{P}(\varpi_3)}(t) = 1 + 9t + 4t^2$;

$\mathbf{E}_6$: $\mathcal{K}_{\mathcal{P}(\varpi_1)}(t) = 1 + 16t + 10t^2$;

$\mathbf{E}_7$: $\mathcal{K}_{\mathcal{P}(\varpi_1)}(t) = 1 + 27t + 27t^2 + t^3$.

Proof. The only non-trivial cases are the first two. We have bijective proofs that use the description of weights given in the proof of Theorem 2.1.

1) $\mathcal{P}(\mathbf{A}_n, \varpi_m)$. As in Section 2, let $\boldsymbol{i} = (i_1, \dots, i_m)$ be a weight. Here $1 \leqslant i_1 < \dots < i_m \leqslant n+1$. Our goal is to realise when it is possible to subtract exactly r simple roots from $\boldsymbol{i}$. A string (of length a) is a subsequence of $\boldsymbol{i}$ of the form $(i, i+1, \dots, i+a-1)$, $a \geqslant 1$, where $i-1, i+a$ are not in $\boldsymbol{i}$. Regard $\boldsymbol{i}$ as a the disjoint union of strings, separated by gaps. The string is said to be *proper* if it does not contain $n+1$. Clearly, each proper string provides a possibility to subtract a simple root, and vice versa. Therefore, $\boldsymbol{i}$ covers exactly r weights if and only if it contains r proper strings, and perhaps one non-proper string. Make the transform

$$\boldsymbol{i} = (i_1, \dots, i_m) \mapsto (i_1, i_2 - 1, \dots, i_m - m + 1) =: \tilde{\boldsymbol{i}}.$$

Under this transform each string in $\boldsymbol{i}$ squeezes into one element of $[n-m+2]$, and different strings squeeze into different elements. The non-proper string, if it occurs, is squeezed into $\{n-m+2\}$. Using the usual notation for repetitions, we see that $\boldsymbol{i}$ covers exactly r weights if and only if the resulting multiset is of the form

$$\tilde{\boldsymbol{i}} = (j_1^{a_1}, \dots, j_r^{a_r}, (n-m+2)^{a_{r+1}}),$$

where $1 \leqslant j_1 < j_2 < \ldots < j_r \leqslant n-m+1$, $\sum_i a_i = m$, $a_i \geqslant 1$ if $i \leqslant r$, and $a_{r+1} \geqslant 0$. Here the a_i's are the lengths of the strings in $\boldsymbol{i}$. Such a multiset is fully determined by two subsets $\{j_1, \ldots, j_r\} \subset [n-m+1]$ and $\{a_1, a_1+a_2, \ldots, a_1+\ldots+a_r\} \subset [m]$. Thus, the number of possibilities for such multisets $\tilde{\boldsymbol{i}}$ equals $\binom{m}{r}\binom{n-m+1}{r}$.

2) $\mathcal{P}(\mathbf{D}_n, \varpi_n)$. Using the isomorphism of Theorem 2.1(2), we have to prove that, for $\mathbf{B}_n$, $\mathcal{K}_{\mathcal{P}(\varpi_n)}(t) = \sum_{r\geqslant 0}\binom{n+1}{2r}t^r$. Recall that $\Pi(\mathbf{B}_n) = \{\varepsilon_1-\varepsilon_2, \ldots, \varepsilon_{n-1}-\varepsilon_n, \varepsilon_n\}$. A weight $\mu = \frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \ldots \pm \varepsilon_n) \in \mathcal{P}(\mathbf{B}_n, \varpi_n)$ can be regarded as arbitrary sequence of n signs '$+$' and '$-$'.

For $i < n$, we have $\mu - \alpha_i \in \mathcal{P}(\mathbf{B}_n, \varpi_n)$ if and only if the i-th sign is '$+$' and the next one is '$-$'; and $\mu - \alpha_n \in \mathcal{P}(\mathbf{B}_n, \varpi_n)$ if and only if the last sign is '$+$'. It follows that if μ covers exactly r weights, then this can be achieved in two ways:

(a) μ has exactly r changes of signs of the form '$+-$' and the last sign is '$-$'.

(b) μ has exactly $r-1$ changes of signs of the form '$+-$' and the last sign is '$+$'.

In case (a), $\mu = (\underbrace{-\cdots-}_{c}\underbrace{+\cdots+}_{a_1}\underbrace{-\cdots-}_{b_1}\cdots\underbrace{+\cdots+}_{a_r}\underbrace{-\cdots-}_{b_r})$, where $c \geqslant 0$, $a_i, b_i > 0$ and $c + \sum(a_i+b_i) = n$. The starting positions of a- and b-strings can be arbitrary. Hence there are $\binom{n}{2r}$ possibilities for such μ.

In case (b), $\mu = (\underbrace{-\cdots-}_{c}\underbrace{+\cdots+}_{a_1}\underbrace{-\cdots-}_{b_1}\cdots\underbrace{+\cdots+}_{a_r})$, and here we have $\binom{n}{2r-1}$ possibilities. Altogether, we obtain $\binom{n}{2r} + \binom{n}{2r-1} = \binom{n+1}{2r}$, as required. □

To compute upper covering polynomials for irreducible wmf-representations of semisimple Lie algebras, one can use the following refinement of Lemma 2.3.

Proposition 5.4. *Let $(\mathfrak{g}', \mathcal{R}')$, $(\mathfrak{g}'', \mathcal{R}'')$ be two wmf representations. Then*

$$\mathcal{K}_{\mathcal{P}(\mathcal{R}'\otimes\mathcal{R}'')}(t) = \mathcal{K}_{\mathcal{P}(\mathcal{R}')}(t)\cdot\mathcal{K}_{\mathcal{P}(\mathcal{R}'')}(t).$$

Proof. This readily follows from the definition of upper covering polynomials and the cartesian product of graphs. □

Properties of upper covering polynomials provide another possible approach to the proof of inequality $2\dim\mathfrak{g}(1) - \#\mathcal{E}(1) > 0$ in Theorem 3.8. In view of Proposition 5.1, we can write $\mathcal{K}_{\Delta(1)}(t) = a_0 + a_1t + a_2t^2 + a_3t^3$. Here $a_i = \#\{\mu \in \Delta(1) \mid \mu \text{ covers } i \text{ elements}\}$. Then

$$2\dim\mathfrak{g}(1) - \#\mathcal{E}(1) = 2\mathcal{K}_{\Delta(1)}(1) - \mathcal{K}'_{\Delta(1)}(1) = 2 + a_1 - a_3.$$

This integer is automatically positive if $a_3 = 0$, i.e., $\deg\mathcal{K}_{\Delta(1)} \leqslant 2$. Since the latter is the case for $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$, and $\mathbf{G}_2$, we obtain a proof of the above inequality for these series. In general, it would be desirable to find an *a priori* proof of the inequality $2 + a_1 - a_3 > 0$ for all posets $\Delta(1)$.

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