Some New Symplectic 4-Manifolds

R. E. Gompf

Dept. of Mathematics
University of Texas
Austin, TX 78712

USA

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI / 92-47

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Robert E. Gompf[1]

In mathematical physics and geometry, there has been much recent interest in the concept of a **symplectic manifold**. This is a smooth manifold M (necessarily of even dimension) endowed with a **symplectic form** ω - that is, a closed 2-form which is nondegenerate as a bilinear form on each tangent space. Equivalently, nondegeneracy means that the top exterior power of ω is a volume form on M. Perhaps the most basic question regarding symplectic manifolds is the following: which manifolds admit symplectic forms? A necessary condition is that M admit an almost-complex structure. Since Gromov [Gr] showed that all open almost-complex manifolds admit symplectic structures, we will henceforth restrict attention to closed manifolds. In this context, relatively little is known. Any Kähler manifold is automatically symplectic (simply use the Kähler form), but non-Kähler examples are somewhat scarce. The objective of this article is to introduce a new technique for constructing symplectic manifolds, and to obtain several families of non-Kähler symplectic manifolds with novel properties. These families solve several major existence problems.

The first non-Kähler examples were described by Thurston [T] in 1976. The simplest of these were 4-manifolds with first Betti number $b_1 = 3$. These were clearly non-Kähler since any Kähler manifold has b_1 even. This raised the question of whether there are simply connected non-Kähler examples. McDuff [Mc1] produced such examples in dimension 10. The corresponding question in dimension 4, however, has remained open, in spite of serious study. For example, this problem was posed by Donaldson at Oberwolfach in 1988 ([Ki], Problem 15). The first application of the technique introduced in the present paper is to answer the question by constructing such symplectic 4-manifolds. We construct an infinite family of homotopy $K3$ surfaces which are symplectic but non-Kähler. In fact, these examples are diffeomorphic to a subfamily of the family described by the author and Mrowka in [GM], which were shown by Mrowka's masterful computation of Donaldson's invariants to be an infinite family of diffeomorphism types of non-complex homotopy $K3$'s.

A related question was asked by Kotschick: What groups can be realized as the fundamental group of a (closed) symplectic manifold? For example, can the integers be realized? Again, little was known about this question. Thurston's example had $b_1 = 3$, and Fernández, Gotay and Gray [FGG] constructed (non-complex) symplectic 4-manifolds with $b_1 = 2$, but examples with $b_1 = 1$ appear to have been previously unknown. Our second application is a purely elementary construction which provides a complete answer to this question: In fact, **any** finitely presented group is the fundamental group of some closed, symplectic 4-manifold. (Of course, the corresponding statement in any higher even dimension follows immediately by taking products with S^2). These examples can be assumed to be spin, in which case most will not have the homotopy type of any complex surface. The examples are also quite different from previously known non-Kähler symplectic 4-manifolds (which are typically bundles over surfaces).

The construction developed in this paper can be generalized in several directions, and each of these generalizations yields new examples of symplectic manifolds. Details and further discussion will appear in [G].

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1. The construction

Our technique rests on a symplectic version of the Tubular Neighborhood Theorem which is due to Weinstein ([W] Theorem 4.1) and follows from Moser's remarkably simple method of proof of the Darboux theorem [Mo]. In particular, suppose we are given (closed) symplectic manifolds (M, ω_M) and (N, ω_N) with a codimension 2 symplectic embedding $j : N \to M$. (That is, j is a smooth embedding with $j^*\omega_M = \omega_N$). Assume that the normal bundle is trivial, and let α be a given trivialization. We require α to respect the orientation determined by the natural volume forms on M and N. Let D_ε denote the open ε-disk in $\mathbf{R}^2$ with the standard symplectic structure. The following lemma is an immediate corollary of Weinstein's theorem.

Lemma 1.1. *Under the above hypotheses, there is a symplectic embedding $\varphi : N \times D_\varepsilon \to M$ ($\varepsilon > 0$ sufficiently small) such that $\varphi|N \times 0 = j$ and φ maps the product normal framing of $N \times 0$ onto α (up to isotopy).*

Proof. Since ω_N is nondegenerate, we may specify the normal bundle of N in M using ω_M-orthogonality. The Tubular Neighborhood Theorem for smooth manifolds produces a smooth embedding $\varphi_0 : N \times D_\varepsilon \to M$ with all of the required properties except for preservation of the symplectic form. By our choice of the normal bundle, $\varphi_0^*\omega_M$ splits as a product form at each point x of $N \times 0$ in $N \times D_\varepsilon$. In the first factor we obtain the form $j^*\omega_M = \omega_N$. Since the second factor is the 2-dimensional space $T_x D_\varepsilon$, $\varphi_0^*\omega_M$ on this subspace differs from the standard form at 0 on D_ε only by a scale factor. This scale factor is positive because of the orientation condition on α, so by an isotopy of φ_0 rescaling each D_ε fiber, we may assume that $\varphi_0^*\omega_M$ agrees with the product symplectic form on $N \times D_\varepsilon$ everywhere on $N \times 0$. This is the hypothesis of Weinstein's theorem, which produces a diffeomorphic embedding of a neighborhood of $N \times 0$ into $N \times D_\varepsilon$ (rel $N \times 0$) whose composite with φ_0 is the symplectic embedding φ. It is immediate from Weinstein's proof that the embedding in $N \times D_\varepsilon$ is isotopic to the identity rel $N \times 0$, and our lemma follows. q.e.d.

We now give the main construction. Suppose we are given symplectic manifolds (N, ω_N) and (M_i, ω_i), $i = 1, 2$, with codimension 2 symplectic embeddings $j_i : N \to M_i$ and normal framings α_i as above. By Lemma 1.1 we obtain symplectic embeddings $\varphi_i : N \times D_\varepsilon \to M_i$. Let $\psi : D_\varepsilon - 0 \to D_\varepsilon - 0$ be a symplectomorphism sending neighborhoods of 0 to neighborhoods of the boundary of D_ε. (For example, take $\psi(r, \theta) = \left(\sqrt{\varepsilon^2 - r^2}, -\theta\right)$).

Definition 1.2. *The* **symplectic sum** *of M_1 and M_2 along N (with respect to $j_1, j_2, \alpha_1, \alpha_2$) is obtained by gluing together the manifolds $M_i - j_i(N)$ $(i = 1, 2)$ along the regions $\varphi_i(N \times (D_\varepsilon - 0))$ via the symplectomorphism $id_N \times \psi$.*

The resulting diffeomorphism type is clearly determined by j_i and α_i $(i = 1, 2)$, but the symplectic form is not. (In particular, the volume of the manifold depends on ε).

With more effort, this construction can be done in greater generality. In particular, one can deal with twisted normal bundles. It is also possible to symplectically sum manifold **pairs**. For details and additional applications, see [G].

2. Simply connected examples

In order to apply our symplectic sum construction, we must begin with some basic building blocks. For this purpose, **elliptic surfaces** are particularly useful. The literature contains numerous expositions of the topology of these 4-manifolds; one reference is [GM]. We now sketch what is needed for our present purposes.

An elliptic surface is a complex surface (hence, smooth 4 -manifold) which admits an **elliptic fibration** - a holomorphic map onto a complex curve (i.e. Riemann surface) such that generic fibers (point pre-images) are tori. The simplest simply connected examples are rational - in fact, diffeomorphic to $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$. Such an example may be constructed from a generic pencil of cubic curves in $\mathbb{C}P^2$ by blowing up the nine points where the curves intersect. The blow-ups separate the cubic curves, which become the fibers of the elliptic fibration. The blow-ups also force the complement of a generic fiber to be simply connected. Other examples of elliptic surfaces include the **Dolgachev surfaces** which are obtained from rational elliptic surfaces by applying a cut-and-paste procedure called **logarithmic transform** (determined up to diffeomorphism by a positive integer **multiplicity**) to several generic fibers. These examples may be assumed to be Kähler. Many Dolgachev surfaces are homeomorphic but not diffeomorphic to $\mathbb{C}P^2 \# 9\overline{\mathbb{C}P}^2$.

We are now ready to apply the symplectic sum construction.

Example 2.1. (Warm-up). Let M_1 and M_2 be Kähler elliptic surfaces, and let $N = \mathbf{R}^2/\mathbf{Z}^2$ be the standard symplectic torus. Pick a generic fiber in each M_i. This will be a Kähler (hence, symplectic) submanifold of M_i. Moser [Mo] showed that any two symplectic surfaces with the same genus and area are symplectomorphic (i.e. there is a diffeomorphism between them preserving the symplectic forms). Thus, we have (after rescaling M_i) a symplectic embedding $j_i : N \to M_i$ onto the given fiber. The fibration on M_i determines a canonical normal framing α_i of j_i, so we may form the symplectic sum of M_1 and M_2 along N. At the level of smooth manifolds, this operation is easily recognized as **fiber sum**, a technique for producing (in the smooth category) new elliptic surfaces from old ones. Any simply connected elliptic surface is diffeomorphic to a fiber sum of Dolgachev and rational surfaces - for example, the $K3$ surface is the fiber sum of two rational elliptic surfaces. Thus, we have produced symplectic structures on many elliptic surfaces. Of course, this is no surprise, since the manifolds obtained from Kähler elliptic surfaces by fiber sum are typically known to admit Kähler structures. However, a slight modification of our construction yields a much different answer:

Example 2.2. (Simply connected, non-Kähler examples). For $i = 1, 2$, let M_i denote a simply connected (relatively minimal) Dolgachev surface, given by relatively prime multiplicities p_i, $q_i \geq 1$. We repeat the previous construction, to obtain a symplectic manifold M by symplectically summing M_1 and M_2 along a fiber in each. This time, however, we replace the canonical framing α_2 by a new framing α_2' which differs from α_2 by a twist. The result is to change the isotopy class of the gluing map by $id_{S^1} \times$ (Dehn twist) on $N \times D_\varepsilon = S^1 \times \left(S^1 \times D_\varepsilon\right)$.

To analyze the diffeomorphism type of M, we think of M as being obtained from the compact manifolds $M_i^0 = M_i - \varphi_i\left(N \times D_{\varepsilon/\sqrt{2}}\right)$ by gluing along the boundaries. We identify the boundaries as $S^1 \times S^1 \times \pm\partial D_{\varepsilon/\sqrt{2}}$ using the canonical framings α_i, so that the gluing of Example 2.1 is given by the identity matrix with respect to the corresponding basis for H_1. Our twisted gluing map is now given by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

But Moishezon ([M], remark after Lemma 7) showed that if an orientation-preserving self-diffeomorphism of the boundary of M_i^0 preserves the elliptic fibration, then it can be extended over the entire manifold M_i^0. This allows us to change A without altering the diffeomorphism type of M, by composing on either side with any matrix in $SL(3, \mathbf{Z})$ which has bottom row $[0\ 0\ 1]$. In particular, given the matrices

$$B = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

the gluing map given by BAC still produces M. But this map is simply a cyclic permutation of the three S^1 factors. The manifold M is now easily recognized as the manifold $K(p_1, q_1; 1, 1; p_2, q_2)$ described in [GM]. In that paper, it was shown that this manifold is homeomorphic to the $K3$ surface if all multiplicities are odd, and to $3\mathbf{C}P^2 \# 19\overline{\mathbf{C}P}^2$ otherwise. Using Donaldson's invariants, it was shown that M is not diffeomorphic to any complex manifold (with either orientation), unless $p_i = q_i = 1$ for some i. Thus, M is our required simply connected, non-Kähler symplectic 4-manifold. Furthermore, it was shown in [GM] that M is irreducible (i.e. not a connected sum of two manifolds with $b_2 \neq 0$), so it is not even a connected sum of complex manifolds. By varying the multiplicities, we obtain infinitely many diffeomorphism types of such manifolds M within each homeomorphism type: According to [GM] the products $p_i q_i\,(i = 1, 2)$ are both diffeomorphism invariants (up to order) unless both are even (in which case the product $p_1\, q_1\, p_2\, q_2$ is still an invariant).

Remark. The exotic nature of the diffeomorphism type of M is lost under product with S^2 : Since M is h-cobordant to a Kähler elliptic surface, $M \times S^2$ is h-corbordant, and hence diffeomorphic, to a Kähler manifold. (It is an open question whether the product form on $M \times S^2$ is realized as a Kähler form). This observation raises the question of whether simply connected, non-Kähler symplectic manifolds can be found in dimensions 6 and 8. (Such examples were constructed in dimensions 10 and higher by McDuff [Mc1]). In fact, 8-dimensional examples can be constructed using pairwise symplectic sum [G]. As of this writing, the question is still open in dimension 6.

3. General fundamental groups

We now answer Kotschick's question about realizing fundamental groups.

Theorem 3.1. *Let G be any finitely presented group. Then there is a closed, symplectic 4-manifold M with $\pi_1(M) \cong G$. Furthermore, M may be chosen to be spin or nonspin.*

Remark. The corresponding assertion in any even dimension > 4 follows immediately by taking products with S^2.

Proof. Fix a finite presentation $\langle g_1, \ldots, g_k | r_1, \ldots, r_l \rangle$ for G. Let F be a (closed, oriented) surface of genus k, with a standard collection of oriented circles $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k$ in F representing a symplectic basis for $H_1(F)$ (so $\alpha_i \cdot \beta_j = \delta_{ij}$). Then the quotient $\pi_1(F)/\langle \beta_1, \ldots, \beta_k \rangle$ is a free group generated by $\alpha_1, \ldots, \alpha_k$ (suitably attached to a base point). For $i = 1, \ldots, \ell$, let γ_i be a smoothly immersed, oriented circle in F, representing the word r_i in this group (substituting $\alpha_1, \ldots, \alpha_k$ for the generators $g_1, \ldots, g_k$ in r_i). For $i = 1, \ldots, k$, let $\gamma_{\ell+i} = \beta_i$. Now $\pi_1(F)/\langle \gamma_1, \ldots, \gamma_{k+\ell} \rangle \cong G$.

We would like to construct a closed 1-form ρ on F which restricts to a volume form on each oriented circle γ_i. Clearly, this is not possible in full generality (for homological reasons), so we must modify our picture somewhat. Let $T = S^1 \times S^1$ and let x, y denote distinct points in S^1. Let $\alpha = S^1 \times x$, $\beta = x \times S^1$ and $\gamma = y \times S^1$ (oriented parallel to β). Let D be a disk in T disjoint from $\alpha \cup \beta$ and intersecting γ in an arc. Now assume the collection $\{\gamma_1, \ldots, \gamma_{k+\ell}\}$ in F is in general position, so that the union of these curves forms an oriented graph embedded in F, together with (possibly) some isolated embedded circles. For each edge (or isolated circle) e, form the connected sum of F with a copy of T by matching D with a similar disk in F centered on an interior point of e. Thus, the edge e will be summed with γ. Perform the gluing so that the orientations on e and γ match. Continue to call the surface F, and let $\{\gamma_1, \ldots, \gamma_m\}$ denote the original curves γ_i (suitably summed with copies of γ) together with the new circles α and β in each copy of T. Clearly, we still have $\pi_1(F)/\langle \gamma_1, \ldots, \gamma_m \rangle \cong G$. However, the new oriented graph $\Gamma = \cup_{i=1}^m \gamma_i$ has an additional property: each edge e of Γ has a segment which lies in α, β or γ in some copy of T.

It is now easy to construct the required 1-form. First, note that T admits a closed 1-form ρ_0 which vanishes near D and has positive integral on each edge of the oriented graph $\alpha \cup \beta \cup (\gamma - \text{int } D)$. (For example, collapse a neighborhood of D to a point, project T diagonally onto S^1 (mapping α, β and γ with degree $+1$) and pull back the volume form of S^1). Let ρ^* be the closed 1-form on F obtained by putting ρ_0 on each copy of $T - D$ and extending by zero. Then $\int_e \rho^* > 0$ for each oriented edge e in Γ. For $i = 1, \ldots, m$, this assertion implies that we can find a volume form θ_i on γ_i (i.e. on the domain circle) with $\int_e \theta_i = \int_e \rho^*$ for each edge e of Γ lying in γ_i. It follows that $\theta_i - \rho^* | \gamma_i = df_i$ for some smooth function $f_i : \gamma_i \to \mathbf{R}$ which vanishes on each vertex of Γ in γ_i. Together, the functions $f_1, \ldots, f_m$ define a function on Γ which extends to a smooth $f : F \to \mathbf{R}$. The form $\rho = \rho^* + df$ on F is the required closed 1-form with $\rho | \gamma_i = \theta_i$ a volume form on γ_i for each i.

We can now construct our example. For $i = 1, \ldots, m$, let T_i denote the immersed torus $\gamma_i \times \alpha$ in the 4-manifold $F \times T$. Put a product symplectic structure ω on $F \times T$; then T_i will be an immersed Lagrangian submanifold (i.e. $\omega | T_i \equiv 0$). However, if θ denotes a volume form on α (pulled back over T by projection), the closed 2-form $\eta = \rho \wedge \theta$ on $F \times T$ restricts to a symplectic form on each T_i. Since the nondegeneracy of ω is an open condition and $F \times T$ is compact, for any sufficiently small t the form $\omega' = \omega + t\eta$ will be symplectic on $F \times T$ and on $z \times T$ (z any preassigned point in $F - \Gamma$). Clearly, it is also symplectic on each T_i (for $t \neq 0$). Now we perturb the tori T_i so that they become disjointly embedded: Write $F \times T$ as $(F \times \beta) \times \alpha$. Perturb the curves γ_i in the 3-manifold $F \times \beta$ to make them disjointly embedded, then cross the picture with α. We can keep the perturbation sufficiently C^1-small so that the perturbed tori T_i' are still symplectic and disjoint from $z \times T$. Furthermore, the normal bundles of the T_i' will be trivial (by the corresponding assertion for γ_i in $F \times \beta$). Let V denote a rational elliptic surface with a generic fiber N, and let M be the symplectic manifold obtained from $(F \times T, \omega')$ by symplectic sum with suitably scaled copies of V along each T_i' and $z \times T$. Since $V - N$ is simply connected, the sums have the effect of killing each γ_i and $\pi_1(T)$ in $\pi_1(F \times T)$, yielding $\pi_1(M) \cong G$.

To arrange for M to be non-spin, simply note that signature adds under symplectic sum. (The symplectic sum of two manifolds is clearly cobordant to the disjoint union). Since $\sigma(V) = -8$, we can sum M with an extra copy of V if necessary so that $\sigma(M)$ is not divisible by 16, then invoke Rohlin's Theorem. (Alternatively, we could simply do a symplectic blow-up, but the previous construction seems likely to yield minimal examples). Achieving spinness is not much harder: First note that $F \times T$ is spin. Also the fiber sum of V with itself is the $K3$ surface, which is spin. Replace V in the above construction by the $K3$ surface. Now $T^2 \times D^2$ has exactly 4 spin structures, and any two of these are related by a self-diffeomorphism which projects to the identity on T^2. Thus, we may choose the normal framing on each of our embedded tori so that the spin structure inherited from the ambient manifold is induced by the product framing on $S^1 \times S^1 \times D_\varepsilon$. If we symplectically sum using these normal framings, the spin structures on $F \times T$ and $K3$ will match up and M will be spin. q.e.d.

Proposition 3.2. *Suppose G is not the fundamental group of any minimal elliptic surface with positive Euler characteristic. Then the manifold M of Theorem 3.1 cannot be homotopy equivalent to any minimal complex surface. In particular, if M is spin it cannot be homotopy equivalent to* **any** *complex surface.*

Remark. The class of groups which we have ruled out is quite restricted. In fact, it is precisely the class of fundamental groups of closed, orientable 2-orbifolds [U]. For example, Proposition 3.2 can be applied whenever $b_1(G)$ is odd. If M is spin, we can add a further restriction, requiring all multiplicities in the elliptic surface to be odd.

Proof. For symplectic sums along tori, both the Euler characteristic and signature add. Thus, $\chi(M) = 12r$ and $\sigma(M) = -8r$, where $r > 0$ is the number of V summands (counting each $K3$ surface as 2). If S is a minimal complex surface with the homotopy type of M, it follows that $c_1^2(S) = 2c_2(S) + p_1(S) = 2\chi(S) + 3\sigma(S) = 0$ and $c_2(S) = 12r > 0$. Also $b_+(S) = b_+(M) > 0$ because the cohomology class of the symplectic form of M has positive square. By the Kodaira classification (see [BPV] Table 10, p. 188), any minimal complex surface with $c_1^2 = 0$ and $c_2 > 0$ is either diffeomorphic to a minimal elliptic surface or of Class VII. In our situation the first case is ruled out by hypothesis. In the other case, the Kodaira dimension of S is $-\infty$, so $p_g(S) = 0$. Also, $b_1(S) = 1$, so $b_+(S) = 2p_g(S) = 0$. (See [BPV] Section IV, Theorem 2.6 (iii)). This gives the required contradiction. q.e.d.

Not only are these manifolds far from being Kähler, but they have rather different topology from known non-Kähler symplectic 4-manifolds. The latter examples are typically surface bundles over surfaces. For such a manifold, the long exact sequence in homotopy shows that π_2 is generated as an abelian group by at most two elements. However, for any of our manifolds M from Theorem 3.1, $\pi_2(M)$ contains a free $\mathbf{Z}[G]$ module on $8r \geq 8$ generators. To see this, simply note that each copy of $V - N$ contains an E_8 plumbing, which contributes a rank 8 $\mathbf{Z}[G]$ submodule. (There are no relations because the E_8 plumbing is bounded by a homology sphere).

The higher dimensional manifolds obtained from Theorem 3.1 by product with S^2's are also usually non-Kähler. Whenever $b_1(G) \neq 0$, the given symplectic form cannot be cohomologous to a Kähler form. Furthermore, the manifolds will frequently not be homotopy equivalent to any Kähler manifold. This is clear when $b_1(G)$ is odd, but it frequently also holds when $b_1(G)$ is even. It suffices for the given presentation $\langle g_1, \ldots, g_k | r_1, \ldots, r_l \rangle$ of G to satisfy $k - \ell = b_1(G)$, with $b_1(G) \neq 0$. (For example, consider free groups, with $\ell = 0$). These assertions follow from an examination of the Lefschetz map (which vanishes on H^1). See [G] for details.

It is useful to arrange our examples to have integral symplectic forms. This is quite easy, by the following observation (which has appeared in various guises in the literature).

Observation 3.3. Any (closed) symplectic manifold (M, ω) admits another symplectic form ω' whose cohomology class lies in $H^2(M; \mathbf{Z})$.

Proof. Fix any metric on M, and let B_ε denote the ε-ball about 0 in the space of harmonic 2-forms on M. For ε sufficiently small, every element of $\omega + B_\varepsilon$ will be a symplectic form. Since $\omega + B_\varepsilon$ covers an open set of $H^2_{DR}(M)$, it contains an element ω'' with $[\omega''] \in H^2(M; \mathbf{Q})$. Multiplying by a suitable integer, we obtain ω'. q.e.d.

This observation has several applications. First, we note that any of our 4-dimensional examples, endowed with an integral symplectic form, can be embedded symplectically in $\mathbf{C}P^5$. This results in new symplectic manifolds in dimensions ≥ 10, by the method of McDuff [Mc1]. (See also [G]). Second, we obtain (as was pointed out to the author by Kotschick - see [K]) an alternate proof of a theorem of A'Campo and Kotschick:

Theorem 3.4. *(A'Campo, Kotschick [AK]): Any finitely presented group G is realized as the fundamental group of a contact 5-manifold.*

Remark. As observed in [K], the higher dimensional version of this theorem is trivially true, since cotangent sphere bundles of manifolds always admit contact structures.

Proof. We follow the argument sketched by Kotschick in [K]. Let (M, ω) be a symplectic 4-manifold with $\pi_1(M) \cong G$ and $[\omega] \in H^2(M; \mathbf{Z})$. Symplectically blow up M to obtain (M', ω') with $\pi_1(M') \cong G$, $[\omega'] \in H^2(M'; \mathbf{Z})$ and an embedded 2-sphere $S \subset M'$ with $\langle [\omega'], S \rangle = 1$. (Since $\int_S \omega'$ depends on the size of the ball removed from M in the blow-up [Mc2], we may have to enlarge ω by an integer scale factor first). Let P denote the principal S^1-bundle over M' with Chern class $[\omega']$. Since $P|S$ is the Hopf bundle, the fibers of P are π_1-trivial and $\pi_1(P) \cong G$. However, a result of Boothby and Wang ([BW], Theorem 3) asserts that P admits a contact structure. (In fact, this is essentially the connection form on P whose curvature is $2\pi i \omega'$). q.e.d.

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