The explicit factorization of the Cremona Transformation which is an extension of the Nagata Automorphism into elementary links

By

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Abstract

We consider a Cremona transformation $\Phi_\sigma : \mathbf{P}^3 \cdots \to \mathbf{P}^3$ which is an extension of the famous Nagata automorphism σ on the affine 3-space $\mathbf{A}^3$. In this paper we shall factorize this Cremona transformation Φ_σ explicitly into a finite number of elementary links according to the Sarkisov Program. As a result, in our factorization of Φ_σ, it can be decomposed into eight elementary links.

1. Introduction

Throughout the present article we work over the field of complex numbers $\mathbf{C}$. In affine algebraic geometry it is an important problem to understand the structure of the groups $\mathrm{Aut}_{\mathbf{C}}(\mathbf{C}^{[n]})$ of automorphims on the polynomial rings $\mathbf{C}^{[n]}$. Let A_n (resp. J_n) be the subgroup of $\mathrm{Aut}_{\mathbf{C}}(\mathbf{C}^{[n]})$ consisting of all affine transformations (resp. de Jonquiére transformations), where an automorphism τ is said to be *affine* (resp. *de Jonquiére*) if $\tau(x_i) = \sum_{j=1}^{n} a_{ij}x_j + b_i$ with $a_{ij}, b_i \in \mathbf{C}$ $(1 \leq i \leq n)$ such that $\det(a_{ij}) \neq 0$ (resp. $\tau(x_i) = a_i x_i + f_i(x_{i+1}, \cdots, x_n)$ with $a_i \neq 0$ and $f_i \in \mathbf{C}[x_{i+1}, \cdots, x_n]$ for $1 \leq i \leq n$). An automorphism τ is called *tame* if it is contained in the subgroup generated by A_n and J_n. Classically it is well known that all the automorphisms $\tau \in \mathrm{Aut}_{\mathbf{C}}\mathbf{C}[x, y]$ are tame (cf. Nagata [Na72], Abhyankar-Moh [A-M75], Miyanishi [Miy78]). In order to grope the structure of $\mathrm{Aut}_{\mathbf{C}}\mathbf{C}[x, y]$ by geometric method, it played a vital role to factorize Cremona transformations on $\mathbf{P}^2$ which are obtained as extensions of the automorphisms on the inside affine plane $\mathbf{A}^2$ (cf. [Miy78, Chap. 2, Section 3], [Ma01]).

On the other hand, for the case $n \geq 3$, the structures of $\mathrm{Aut}_{\mathbf{C}}(\mathbf{C}^{[n]})$ are not understood completely at the present time. We shall pay attention to the case

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$n = 3$. For instance, we consider the following famous *Nagata automorphism* $\sigma \in \mathrm{Aut}_{\mathbf{C}}(\mathbf{C}[x, y, z])$ on the affine 3-space $\mathbf{A}^3$ (x, y, z) (cf. Nagata [Na78, p. 16], Alev [Al95]):

$$\begin{cases} \sigma(x) &= x - 2y(xz + y^2) - z(xz + y^2)^2, \\ \sigma(y) &= y + z(xz + y^2), \\ \sigma(z) &= z. \end{cases}$$

The Nagata automorphism σ seems to represent the difficulty of the structure of $\mathrm{Aut}_{\mathbf{C}}(\mathbf{C}[x, y, z])$. For instance, at the present time, we do not know whether or not σ is tame.[*1] The Nagata automorphism σ is extended naturally to a Cremona transformation Φ_σ on $\mathbf{P}^3$ of the following form:

$$\begin{cases} x &\mapsto xw^4 - 2yw^2(xz + y^2) - z(xz + y^2)^2, \\ y &\mapsto yw^4 + zw^2(xz + y^2), \\ z &\mapsto zw^4, \\ w &\mapsto w^5, \end{cases}$$

where $H_\infty := \{w = 0\}$ is a hyperplane at infinity with respect to the canonical open immersion $\iota : \mathbf{A}^3$ $(x, y, z) \hookrightarrow \mathbf{P}^3$. Our main interest in this article is to factorize Φ_σ explicitly into a finite number of elementary links according to the Sarkisov Program. *Elementary links* are some kinds of birational maps between 3-dimensional Mori fiber spaces (see Section 2 for the details). In my opinion we belive that this explicit factorization of Φ_σ (cf. Theorem 1.1) is important in order to analyze whether or not the Nagata automorphism σ is tame from a point of view of algebro-geometric methods as well as in two-dimensional case (cf. [Miy78]).

We shall state the scheme of this article. In Section 2 we recall the definitions of Sarkisov Program and elementary links in dimension three. Though the Sarkisov Program guarantees that any birational map between 3-dimensional Mori fiber spaces can be factorized into elementary links, it is, in general, difficult and complicated to perform its explicit factorization. This obstacle seems to result from the fact that there are no complete classification of the 3-dimensional terminal divisorial contractions. However, fortunately, in our factorization of Φ_σ, the varieties keep smoothness, and the terminal divisorial contractions to smooth threefolds are completely understood due to Mori and Kawakita (cf. Theorems 2.1, 2.2).

In Section 3 we shall factorize the Cremona transformation Φ_σ explicitly according to Sarkisov Program. We consider the proper transform $\mathcal{H}$ of the complete linear system $|\mathcal{O}_{\mathbf{P}^3}(1)|$ by $\Phi_\sigma : \mathbf{P}^3 \cdots \to \mathbf{P}^3$ and investigate the singularities of a general member of $\mathcal{H}$ along base locus $\mathrm{Bs}\,\mathcal{H}$. After that we take a maximal divisorial blow-up and observe the configurations of general members of the proper transform of $\mathcal{H}$ concretely and so on. In this way,

[*1] Recently, Shestakov and Umirbaev [S-U02] seem to obtain an algorithm in order to decide when an automorphism in $\mathrm{Aut}_{\mathbf{C}}(\mathbf{C}[x, y, z])$ is tame by algebraic argument. By making use of this algorithm they seem to prove that the Nagata automorphism σ is not tame.

we continue to investigate the singularities of a general member of the proper transform of $\mathcal{H}$ in each step of the factorization. If a log pair associated with the proper transform of $\mathcal{H}$ is not canonical, then we consider where is the maximal center and take the maximal divisorial blow-up. If a log pair is canonical but not nef, then we take an extremal ray which is negative with respect to this log pair and contract the 2-dimensional extremal face spanned by this ray and other ray yielding a Mori fiber space (see Section 2 for the detailed mechanism of the Sarkisov Program). In any case, we shall construct an elementary link to decrease the Sarkisov degree (see Definition 2.1) and describe the concrete configurations of general members of the proper transform of $\mathcal{H}$, and prepare the next step of the factorization until we reach the situation where a log pair is canonical and nef to obtain the factorization. As a result in our factorization of Φ_σ, it can be decomposed explicitly as in the following Theorem 1.1 (= Theorem 3.1) which is our main result in this article.

Theorem 1.1. *Let σ be the Nagata automorphism on the affine 3-space $\mathbf{A}^3$ and let $\Phi_\sigma : \mathbf{P}^3 \cdots \to \mathbf{P}^3$ be a Cremona transformation which is an extension of σ. Then Φ_σ can be factorized into eight elementary links explicitly, say $\Phi_\sigma = \Phi_8 \circ \cdots \circ \Phi_1$. More precisely, each elementary link Φ_j $(1 \leq j \leq 8)$ is described as follows (here we shall follow the notations used in Section 3 and each number of type is the type of elementary link; see Section 2):*

(i) $\Phi_1 : \mathbf{P}^3 \overset{p_0}{\leftarrow} Z_0 \overset{q_0}{\rightarrow} X_1 \cong \mathbf{Q}^3$, *where p_0 is the blow-up along the conic C_0 with an exceptional divisor $E_0 \cong \mathbf{F}_2$ and q_0 is the contraction of $H_{\infty Z_0} \cong \mathbf{P}^2$ (cf. Figure 1) to a smooth point (Type II),*

(ii) $\Phi_2 : X_1 \overset{p_1}{\leftarrow} X_2$ *is the blow-up along the curve C_1 (cf. Figure 2) with an exceptional divisor $E_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$ (Type I),*

(iii) $\Phi_3 : X_2 \overset{p_2}{\leftarrow} Z_2 \overset{q_2}{\rightarrow} X_3$, *where p_2 is the blow-up along the curve L_1 (cf. Figure 3) with an exceptional divisor $E_2 \cong \mathbf{F}_2$ and q_2 is the contraction of $E_{0\,Z_2} \cong \mathbf{F}_2$ (cf. Figure 4) onto a curve (Type II),*

(iv) $\Phi_4 : X_3 \overset{p_3}{\leftarrow} Z_3 \overset{q_3}{\rightarrow} X_4$, *where p_3 is the blow-up along the curve L_3 (cf. Figure 5) with an exceptional divisor $E_3 \cong \mathbf{F}_2$ and q_3 is the contraction of $E_{2\,Z_3} \cong \mathbf{F}_2$ (cf. Figure 6) onto a curve (Type II),*

(v) $\Phi_5 : X_4 \overset{p_4}{\leftarrow} Z_4 \overset{q_4}{\rightarrow} X_5$, *where p_4 is the blow-up along the curve L_4 (cf. Figure 7) with an exceptional divisor $E_4 \cong \mathbf{F}_2$ and q_4 is the contraction of $E_{3\,Z_4} \cong \mathbf{F}_2$ (cf. Figure 8) onto a curve (Type II),*

(vi) $\Phi_6 : X_5 \overset{p_5}{\leftarrow} Z_5 \overset{q_5}{\rightarrow} X_6$, *where p_5 is the blow-up along the curve L_5' (cf. Figure 9) with an exceptional divisor $E_5 \cong \mathbf{F}_2$ and q_5 is the contraction of $E_{4\,Z_5} \cong \mathbf{F}_2$ (cf. Figure 10) onto a curve (Type II),*

(vii) $\Phi_7 : X_6 \overset{q_6}{\rightarrow} X_7 \cong \mathbf{Q}^3$ *is the contraction of $E_{1\,X_6} \cong \mathbf{P}^1 \times \mathbf{P}^1$ (cf. Figure 11) onto a curve (Type III),*

(viii) $\Phi_8 : X_7 \overset{p_7}{\leftarrow} Z_7 \overset{q_7}{\rightarrow} X_8 \cong \mathbf{P}^3$, *where p_7 is the blow-up at the point P_7 (cf. Figure 12) with an exceptional divisor $E_7 \cong \mathbf{P}^2$ and q_7 is the contraction of $E_{5\,Z_7} \cong \mathbf{F}_2$ onto a conic (Type II).*

We shall employ the following notations in this paper:

Notation.

- K_V : canonical divisor of a variety V;
- $D|_W$: scheme-theoretic restriction of a Cartier divisor D on V to a subvariety $W \subset V$;
- D^2 : the self-intersection of an irreducible Cartier divisor D on V;
- $\mathrm{mult}_W D$: multiplicity of a Cartier divisor D on V along a subvariety $W \subset V$;
- $D_1 \sim D_2$: D_1 is linearly equivalent to D_2;
- $\mathbf{Q}^3$: a smooth quadric hypersurface in $\mathbf{P}^4$;
- $\mathbf{Q}_0^2$: a quadric cone in $\mathbf{P}^3$;
- $\mathbf{F}_k$: Hirzebruch surface of degree k;
- M_k : the minimal section on $\mathbf{F}_k$;
- l_k : the class of a fiber on $\mathbf{F}_k$;
- $a(E; V, \alpha\mathcal{H}_V)$: the discrepancy of a discrete valuation E of the function field $\mathbf{C}(V)$ of V with respect to $K_V + \alpha\mathcal{H}_V$;

When a Cartier divisor D on a variety V and a birational map $V \cdots \to W$ are given, we denote by D_W the proper transform on W of D consistently in this article if there are no confusions.

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2. Overview of the Sarkisov Program in dimension Three

A program called the *Sarkisov Program* was developed by Sarkisov, Reid and Corti in order to factorize birational maps between 3-dimensional Mori fiber spaces as a composite of so called *elementary links* (cf. [Cor95]). There are four types **I**$\sim$**IV** for these elementary links appearing in the Sarkisov Program as follows:

$$
\begin{tikzcd}
 & Z \arrow[dl, "p"'] \arrow[r, dashed] & X_1 \arrow[dd, "\phi_1"] \\
X \arrow[d, "\phi"'] & & \\
S & & S_1 \arrow[ll]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & Z \arrow[dl, "p"'] \arrow[r, dashed] & Z_1 \arrow[dr, "p_1"] & \\
X \arrow[d, "\phi"'] & & & X_1 \arrow[d, "\phi_1"] \\
S \arrow[rrr, equal] & & & S_1
\end{tikzcd}
$$

Type I **Type II**

$$\begin{tikzcd}
X \arrow[r, dashed] \arrow[ddd, "\phi"'] & Z_1 \arrow[dr, "p_1"] & \\
& & X_1 \arrow[d, "\phi_1"] \\
& & {} \\
S \arrow[rr] & & S_1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[rr, dotted] \arrow[d, "\phi"'] & & X_1 \arrow[d, "\phi_1"] \\
S \arrow[dr] & & S_1 \arrow[dl] \\
& T &
\end{tikzcd}$$

Type III **Type IV**

where $p : Z \to X$ and $p_1 : Z_1 \to X_1$ are the *divisorial contractions* (or the *maximal divisorial blow-up*; see Definition 2.2 below) and each of the birational maps denoted by the horizontal dotted arrows, which may be an identity, is a composite of log flips (i.e., flips, flops and inverse flips), $\phi : X \to S$, $\phi_1 : X_1 \to S_1$ are the 3-dimensional Mori fiber spaces and $\rho(S/T) = \rho(S_1/T) = 1$.

In fact, to perform the Sarkisov Program, we need to define the *Sarkisov degree* with respect to a suitably chosen very ample complete linear system and verify that it decreases after untwisting each elementary link (see [Cor95], [Cor00], [Ma01] for the detail).

We shall recall the definition of the Sarkisov degree. Let $\varphi : X/S \cdots \to X'/S'$ be a given birational map between 3-dimensional Mori fiber spaces $\phi : X \to S$ and $\phi' : X' \to S'$.

Definition 2.1. Let $\mathcal{H}' = |-\mu' K_{X'} + \phi'^*(A')|$ be a very ample complete linear system on X', where μ' is a positive integer and A' is a sufficiently ample Cartier divisor on S'. This choice of $\mathcal{H}'$ is made once and for all at the beginning of the factorization process and $\mathcal{H}'$ remains unchanged throughoutly. Let $\mathcal{H} := \varphi_*^{-1}(\mathcal{H}')$ be the proper transform of $\mathcal{H}'$ by φ. Since $\phi : X \to S$ is a Mori fiber space, $\mathcal{H}$ is written uniquely as $\mathcal{H} \equiv -\mu K_X + \phi^*(A)$, where μ is a positive rational number and A is a $\mathbf{Q}$-Cartier divisor on S. This μ is called the *quasi-effective threshold* of the pair $(X, \mathcal{H})$. Let $c := \sup\{\lambda \in \mathbf{Q}; K_X + \lambda\mathcal{H}$ is canonical$\}$, which we call *canonical threshold* of $(X, \mathcal{H})$. (In the case $\mathrm{Bs}\,\mathcal{H} = \emptyset$ we put $c = \infty$.) Finally, let e be the number of discrete valuations $\{E\}$ of $\mathbf{C}(X)$ which are exceptional over X and have discrepancy $a(E; X, c\mathcal{H}) = 0$. (Note that $e \geq 0$ is finite.) The triplet (μ, c, e) is called as *Sarkisov degree* of $(X, \mathcal{H})$.

We shall define the lexicographical order on the set of Sarkisov degree as follows. When two birational maps $X/S \overset{\varphi}{\cdots\to} X'/S' \overset{\varphi_1}{\leftarrow\cdots} X_1/S_1$ are given, we have two Sarkisov degree (μ, c, e) and (μ_1, c_1, e_1) of the pairs $(X, \mathcal{H})$ and $(X_1, \mathcal{H}_1)$ as defined just above, where $\mathcal{H}$ (resp. $\mathcal{H}_1$) is the proper transform of $\mathcal{H}$ by φ (resp. φ_1). We write $(\mu, c, e) > (\mu_1, c_1, e_1)$ if one of the following (i)$\sim$(iii) are satisfied:

(i) $\mu > \mu_1$;

(ii) $\mu = \mu_1$ and $c < c_1$;
(iii) $\mu = \mu_1$, $c = c_1$ and $e > e_1$.

The essential in the Sarkisov Program lies in untwisting the original birational map φ by suitable elementary links to decrease Sarkisov degree and to use Noether-Fano-Iskovskih inequality (cf. [Cor95, Cor00, Ma01]). We shall sketch the mechanism of the Sarkisov Program to factorize $\varphi : X/S \cdots \to X'/S'$. The reader refer to [Cor95, Cor00, Ma01] for the detail mechanism.

With the notations as above, at first, we ask whether or not $K_X + (1/\mu)\mathcal{H}$ is canonical and nef. If it is so, then φ is an isomorphsim as Mori fiber spaces by virtue of Noether-Fano-Iskovskih inequality (cf. [Cor95, Cor00]) and there are nothing to do. Otherwise, we need to consider according as $K_X + (1/\mu)\mathcal{H}$ is canonical or not.

In the case where $K_X + (1/\mu)\mathcal{H}$ is canonical but not nef, we find a $\big(K_X + (1/\mu)\mathcal{H}\big)$-nagative extremal ray $\mathbf{P}$ by the Cone Theorem (cf. [K-M-M87]), and consider the extremal face $\mathbf{F}$ spanned by $\mathbf{P}$ and $\mathbf{R}$, where $\mathbf{R}$ is an extremal ray yielding a Mori fiber space $\phi : X \to S$. Let $\mathrm{cont}_{\mathbf{F}} : X \to T$ be the contraction of $\mathbf{F}$. Note that since $\mathbf{R} \subset \mathbf{F}$, the contraction $\mathrm{cont}_{\mathbf{F}}$ factors through $\phi = \mathrm{cont}_{\mathbf{R}} : X \to S$. We run $\big(K_X + (1/\mu)\mathcal{H}\big)$-MMP over T. Then we reach at the end either a log minimal model or a log Mori fiber space (with respect to $K_X + (1/\mu)\mathcal{H}$ over T). In any case we obtain the link of type **III** or **IV** according as the last birational contraction in this $\big(K_X + (1/\mu)\mathcal{H}\big)$-MMP over T is divisorial or not.

In the case where $K_X + (1/\mu)\mathcal{H}$ is not canonical, i.e., $c < 1/\mu$, we have to take a suitable maximal divisorial blow-up as defined below:

Definition 2.2. We call $p : Z \to X$ a *maximal divisorial blow-up* with respect to $K_X + c\mathcal{H}$ if $p : Z \to X$ is a projective birational morphism from Z with only $\mathbf{Q}$-factorial and terminal singularities such that:
(i) $\rho(Z/X) = 1$;
(ii) the exceptional locus of p is a prime divisor;
(iii) p is $\big(K_X + c\mathcal{H}\big)$-crepant, i.e., $K_Z + c\mathcal{H}_Z = p^*\big(K_X + c\mathcal{H}\big)$, where $\mathcal{H}_Z$ is the proper transform of $\mathcal{H}$ on Z.

A discrete valuation E of $\mathbf{C}(X)$ is a *maximal singularity* of $\mathcal{H}$ if E is the exceptional divisor of a maximal divisorial blow-up. A *maximal center* is the center $\mathrm{Center}_X(E)$ on X of a maximal singularity E.

It is known that a maximal divisorial blow-up with respect to $K_X + c\mathcal{H}$ exists (cf. [Ma01, Prop. 13-1-8]), but in general it is not necessarily unique. After taking a maximal divisorial blow-up $p : Z \to X$ with respect to $K_X + c\mathcal{H}$, we run $\big(K_Z + c\mathcal{H}_Z\big)$-MMP over S. Then we necessarily reach a log Mori fiber space $\phi_1 : X_1 \to S_1$. (Note that the case where we reach a log minimal model over S does not occur.) According as the last birational contraction in the $\big(K_Z + c\mathcal{H}_Z\big)$-MMP over S is divisorial or not we obtain an elementary link of type **II** or type **I**.

Thus we can construct an elementary link $\varphi_1 : X/S \cdots \to X_1/S_1$ of type **I$\sim$IV**. It can be verified that the Sarkisov degree of $(X_1, \mathcal{H}_1)$, say (μ_1, c_1, e_1), decreases properly compared with that of $(X, \mathcal{H})$. Therefore we have anew birational map $\varphi \circ \varphi_1^{-1} : X_1/S_1 \cdots \to X'/S'$ between Mori fiber spaces with strictly smaller Sarkisov degree (μ_1, c_1, e_1). We then ask again whether or not $K_{X_1} + (1/\mu_1)\mathcal{H}_1$ is canonical and nef. If the answer is yes, then $\varphi \circ \varphi_1^{-1} : X_1/S_1 \cdots \to X'/S'$ is an isomorphism by Noether-Fano-Iskovskih inequality and to see $\varphi = \varphi_1$. On the other hand, if the answer is no, we repeat the above mentioned process with $\varphi \circ \varphi_1^{-1}$. Each time we untwist, the Sarkisov degree should strictly drop. Finally, it is known that this process has to come to an end after finitely many repetitions to obtain the expression of φ as a composite of elementary links.

This is why, in order to understand the mechanism of the Sarkisov Program well, it is important to understand maximal divisorial blow-ups and flips. Note that flips in dimension three were understood by Kollár and Mori (cf. [Ko-Mo92]). Let $f : (E \subset Y) \to (\Gamma \subset X)$ be a 3-dimensional maximal divisorial blow-up. Note that there are two ways to deal with f, namely, one starting from Y, and the other from X. In the former case f is said to be a *divisorial contraction*, on the other hand, in the latter case f is said to be a *maximal divisorial blow-up* as defined in Definition 2.2. The three-dimensional divisorial contractions (or maximal divisorial blow-up) are not completely understood at the present time. However, fortunately, the divisorial contractions to smooth threefolds are completely understood due to the works of Mori and Kawakita. In our factorization of Φ_σ, which is an extension of the Nagata automorphism σ, performed in the next section we need only such contractions.

Let $f : (E \subset Y) \to (\Gamma \subset X)$ be a 3-dimensional divisorial contraction. The following result is well known by virtue of Mori's classification.

Theorem 2.1. ([Mori82]) *In the case where* $\dim \Gamma = 1$ *and* X *is smooth along the curve* Γ*,* $f : (E \subset Y) \to (\Gamma \subset X)$ *is the blow-up of* X *along* Γ*, namely,* $Y \cong \mathit{Proj}_X\left(\bigoplus_{d \geq 0} \mathcal{I}^d_{\Gamma,X}\right)$*, where* $\mathcal{I}_{\Gamma,X} \subset \mathcal{O}_X$ *is the ideal sheaf of the curve* $\Gamma \subset X$*.*

On the other hand, in the case where Γ is a smooth point on X, the following result due to Kawakita is known.

Theorem 2.2. ([Kaw01]) *In the case where* $\dim \Gamma = 0$*, i.e.,* $\Gamma = \{P\}$ *is a point, and* X *is smooth at* P*,* $f : (E \subset Y) \to (P \in X)$ *is a weighted blow-up. More precisely, we can take local coordinates* x, y, z *at* P *and coprime positive integers* a *and* b*, such that* f *is the weighted blow-up of* X *with its weights* $\mathrm{wt}(x, y, z) = (1, a, b)$*.*

3. The explicit factorization of the Cremona transformation which is an extension of the Nagata Automorphism

Let $\sigma \in \mathrm{Aut}_{\mathbf{C}}(\mathbf{C}[x,y,z])$ be the Nagata automorphism and let $\Phi_\sigma : \mathbf{P}^3 \cdots \to \mathbf{P}^3$ be the Cremona transformation which is a natural extension of σ (see Section 1, Introduction). Namely, Φ_σ is defined as follows:

$$\begin{cases} x & \mapsto & xw^4 - 2yw^2(xz+y^2) - z(xz+y^2)^2, \\ y & \mapsto & yw^4 + zw^2(xz+y^2), \\ z & \mapsto & zw^4, \\ w & \mapsto & w^5, \end{cases}$$

where $w = 0$ defines a hyperplane at infinity H_∞ with respect to the canonical open immersion $\iota : \mathbf{A}^3\ (x,y,z) \hookrightarrow \mathbf{P}^3$. In this section we shall factorize Φ_σ into a finite number of elementary links explicitly according to the Sarkisov Program (cf. Theorem 1.1).

Let $\mathcal{H} := {\Phi_\sigma}_*^{-1}(|\mathcal{O}_{\mathbf{P}^3}(1)|)$ be the proper transform by Φ_σ of the complete linear system $|\mathcal{O}_{\mathbf{P}^3}(1)|$. (In the subsequent of this section the proper transform of $\mathcal{H}$ on a variety V is denoted by $\mathcal{H}_V$ otherwise mentioned.) Then $\mathcal{H}$ is spanned by S, $T + 2H_\infty$, $H + 4H_\infty$ and $5H_\infty$ independently, where $S := (xw^4 - 2yw^2(xz+y^2) - z(xz+y^2)^2 = 0)$, $T := (yw^2 + z(xz+y^2) = 0)$ and $H := (z = 0)$. It is obvious that $\mathrm{Bs}\,\mathcal{H} = C_0 \cup L_0$, where $C_0 := (xz+y^2 = w = 0)$ is a conic and $L_0 := (z = w = 0)$ is a line.

We consider the open neighborhood of the point $P_0 := (1:0:0:0)$, say $U_0 := \mathbf{P}^3 \backslash (x = 0)$ with affine coordinates (y,z,w) where a general member S of $\mathcal{H}$ and T, H have the local equations $f_0 := w^4 - 2yw^2(z+y^2) - z(z+y^2)^2$, $g_0 := yw^2 + z(z+y^2)$ and $h_0 := z$, respectively. Moreover the curves L_0 and C_0 are defined by the ideals (z,w) and $\mathfrak{a}_0 := (z+y^2, w) \subset \Gamma(\mathcal{O}_{U_0}) = \mathbf{C}[y,z,w]$, respectively on U_0. We shall calculate the Sarkisov degree (see Definition 2.1) of the pair $(\mathbf{P}^3, \mathcal{H})$.

Lemma 3.1.
(1) *The quasi-effective threshold* μ_0 *of the pair* $(\mathbf{P}^3, \mathcal{H})$ *equals* $5/4$*, and* $K_{\mathbf{P}^3} + (1/\mu_0)\mathcal{H}$ *is not canonical,*
(2) *The canonical threshold* c_0 *of the pair* $(\mathbf{P}^3, \mathcal{H})$ *equals* $1/2$ *and* C_0 *is a maximal center of* $(\mathbf{P}^3, \mathcal{H})$ *(cf. Definition* 2.2*).*

Proof. It is clear that $\mu_0 = 5/4$ and $K_{\mathbf{P}^3} + (1/\mu_0)\mathcal{H}$ is nef. As Φ_σ is not an isomorphism, $K_{\mathbf{P}^3} + (1/\mu_0)\mathcal{H}$ is not canonical by virtue of Noether-Fano-Iskovskih inequality (cf. [Cor95, Cor00]). In consideration of the local equation f_0 of a general member $S \in \mathcal{H}$, we know that the possibility for a maximal center of $(\mathbf{P}^3, \mathcal{H})$ is either the conic C_0 itself or the point P_0. In the case where C_0 (resp. P_0) is the maximal center, the maximal divisorial blow-up with respect to $K_{\mathbf{P}^3} + c_0\mathcal{H}$ is the blow-up along C_0 (resp. a weighted blow-up at P_0 with suitable weights) by Theorem 2.1 (resp. by Theorem 2.2). Let E_0 (resp. $E(a,b,c)$) be an exceptional divisor obtained by the blow-up along C_0

(resp. by the weighted blow-up at P_0 of weight $\mathrm{wt}(y,z,w)=(a,b,c)$, where one of a, b, c is 1 and the other two are coprime positive integers; see Theorem 2.2). It is easy to see that $a(E_0;\mathbf{P}^3,(1/2)\mathcal{H})=0$, so $c_0\le 1/2$. On the other hand, we have $a(E(a,b,c);\mathbf{P}^3,(1/2)\mathcal{H})=a+b-c-1$ (resp. $(1/2)(a+b)-1$, $-(a/2)+b-1$, $a-(b/2)+c-1$, $c-1$ or $-a+(b/2)+c-1$) if $4c$ (resp. $a+b+2c$, $3a+2c$, $3b$, $2a+2b$, $4a+b$) is minimum among $4c$, $a+b+2c$, $3a+2c$, $3b$, $2a+2b$ and $4a+b$. Then, by the straightforward computations, we can see $a(E(a,b,c);\mathbf{P}^3,(1/2)\mathcal{H})>0$ in any case. Thus we complete the proof. Q.E.D.

By Theorem 2.1 and Lemma 3.1, we have to blow-up $p_0: E_0\subset Z_0\to C_0\subset \mathbf{P}^3$ along the conic C_0 with an exceptional divisor $E_0\cong\mathbf{F}_2$. It is not difficult to see that the proper transform $\mathcal{H}_{Z_0}$ of $\mathcal{H}$ by p_0 is spanned by S_{Z_0}, $T_{Z_0}+2H_{\infty Z_0}$, $H_{Z_0}+4H_{\infty Z_0}+2E_0$ and $5H_{\infty Z_0}+3E_0$ (here we follow **Notation** in Section 1, Introduction). We use the same notations L_0 and C_0 to represent, respectively, the proper transform of L_0 on Z_0 and the intersection curve $H_{\infty Z_0}\cap E_0$. Note that C_0 is a conic on $H_{\infty Z_0}\cong\mathbf{P}^2$ and the minimal section on $E_0\cong\mathbf{F}_2$. We have the following (cf. Figure 1):

Lemma 3.2.
(1) $K_{Z_0}\sim -4H_{\infty Z_0}-3E_0$;
(2) $H_{\infty Z_0}^2\sim\mathcal{O}_{\mathbf{P}^2}(-1)$ *and* $E_0^2\sim -M_2+2l_2$;
(3) $S_{Z_0}|_{H_{\infty Z_0}}=T_{Z_0}|_{H_{\infty Z_0}}=H_{Z_0}|_{H_{\infty Z_0}}=L_0$;
(4) $S_{Z_0}|_{E_0}\sim 2M_2+6l_2$, $T_{Z_0}|_{E_0}\sim M_2+4l_2$ *and* $H_{Z_0}|_{E_0}\sim 2l_2$.

Proof. The assertions (1) and (2) are easy to see. Since $S_{Z_0}\sim 5H_{\infty Z_0}+3E_0$, $T_{Z_0}\sim 3H_{\infty Z_0}+2E_0$ and $H_{Z_0}\sim H_{\infty Z_0}+E_0$, we can obtain the linearly equivalent relations as stated in (4) by virtue of (2). By the similar argument it follows that $S_{Z_0}|_{H_{\infty Z_0}}$, $T_{Z_0}|_{H_{\infty Z_0}}$ and $H_{Z_0}|_{H_{\infty Z_0}}$ are linearly equivalent to L_0. However, since S_{Z_0}, T_{Z_0} and H_{Z_0} contain L_0, we see the assertion (3). Q.E.D.

Though we have the linearly equivalent relations of $S_{Z_0}|_{E_0}$, $T_{Z_0}|_{E_0}$ and $H_{Z_0}|_{E_0}$ (cf. Lemma 3.2, (4)), we further need to investigate the concrete configurations of these intersections. Indeed, we have the following:

Lemma 3.3. *$S_{Z_0}|_{E_0}=2C_1+2L_1$, $T_{Z_0}|_{E_0}=C_1+2L_1$ and $H_{Z_0}|_{E_0}=2L_1$, where C_1 is a cross-section on $E_0\cong\mathbf{F}_2$ linearly equivalent to M_2+2l_2 and L_1 is a fiber on E_0 intersecting L_0 (cf. Figure 1).*

Proof. We have to observe the blow-up $p_0: Z_0\to\mathbf{P}^3$ concretely. In the open affine subset U_0 (y,z,w), the center C_0 of p_0 is defined by an ideal $\mathfrak{a}_0:=(z+y^2,w)\subset\Gamma(\mathcal{O}_{U_0})=\mathbf{C}[y,z,w]$. The inverse image $p_0^{-1}(U_0)$ is isomorphic to $\mathrm{Proj}\bigl(\bigoplus_{d\ge 0}\mathfrak{a}_0^d\bigr)$ which is covered by two affine 3-spaces, say $U_0^{(1)}$ (y,z,w) and $U_0^{(2)}$ (y,z,w). More precisely, one $U_0^{(1)}\to U_0$ is associated to the algebra homomorphism $y\mapsto y$, $z\mapsto zw-y^2$, $w\mapsto w$. On the other hand, the other

$U_0^{(2)} \to U_0$ is associated to $y \mapsto y$, $z \mapsto z$, $w \mapsto w(z+y^2)$. On $U_0^{(1)}$, the surfaces S_{Z_0}, T_{Z_0}, H_{Z_0} and E_0 are defined by $f_0^{(1)} = w^2 - 2yzw - z^2(zw-y^2)$, $g_0^{(1)} = yw + z(zw-y^2)$, $h_0^{(1)} = zw - y^2$ and w, respectively. On the other hand, on $U_0^{(2)}$, these are defined by $f_0^{(2)} = w^4(z+y^2)^2 - 2yw^2(z+y^2) - z$, $g_0^{(2)} = yw^2(z+y^2) + z$, $h_0^{(2)} = z$ and $z+y^2$, respectively. In combination of these facts with Lemma 3.2, we have the assertion. Q.E.D.

To perform the Sarkisov Program, we have to run $\left(K_{Z_0} + c_0\mathcal{H}_{Z_0}\right)$-MMP (cf. Section 2 or [Cor95, Cor00], [Ma01] for the datail). The cone $\overline{\mathrm{NE}}\,(Z_0) \subset \mathbf{R}^2$ is spanned by two rays one of which comes from the rulings on an exceptional divisor $E_0 \cong \mathbf{F}_2$ and the other comes from lines on $H_{\infty Z_0} \cong \mathbf{P}^2$. The latter ray is $\left(K_{Z_0} + c_0\mathcal{H}_{Z_0}\right)$-nagative. Since $H_{\infty}{}^2_{Z_0} \sim \mathcal{O}_{\mathbf{P}^2}(-1)$, it can be contracted to a smooth point. Let $q_0 : Z_0 \to X_1$ be the contraction of $H_{\infty Z_0}$. The composite $\Phi_1 := q_0 \circ p_0^{-1}$ is an elementary link of type **II**. Then $X_1 \cong \mathbf{Q}^3$ and $K_{X_1} \sim -3E_{0X_1}$. The proper transform $\mathcal{H}_{X_1}$ of $\mathcal{H}_{Z_0}$ on X_1 is spanned by S_{X_1}, $T_{X_1} + E_{0X_1}$, $H_{X_1} + 2E_{0X_1}$ and $3E_{0X_1}$. By virtue of Lemma 3.3, we can readily see the following:

Lemma 3.4.
(1) $S_{X_1}|_{E_{0X_1}} = 2C_1 + 2L_1$, $T_{X_1}|_{E_{0X_1}} = C_1 + 2L_1$, $H_{X_1}|_{E_{0X_1}} = 2L_1$, *where we use the same notations* C_1 *and* L_1 *to represent their proper transforms (cf. Figure 2);*
(2) $\mathrm{Bs}\,\mathcal{H}_{X_1} = C_1 \cup L_1$*;*
(3) *On the open affine subset* $U_1 := q_0(U_0^{(1)})\ (y,z,w)$*, the surfaces* S_{X_1}*,* T_{X_1}*,* H_{X_1} *and* E_{0X_1} *are defined by* $f_1 := w^2 - 2yzw - z^2(zw-y^2)$*,* $g_1 := yw + z(zw-y^2)$*,* $h_1 := zw - y^2$ *and* w*, respectively. On* U_1*, the curves* C_1 *and* L_1 *are defined by the ideals* $\mathfrak{a}_1 := (z,w)$ *and* $(y,w) \subset \Gamma(\mathcal{O}_{U_1}) = \mathbf{C}[y,z,w]$*, respectively.*

We shall calculate the Sarkisov degree of the pair $(X_1, \mathcal{H}_{X_1})$.

Lemma 3.5.
(1) *The quasi-effective threshold* μ_1 *of* $(X_1, \mathcal{H}_{X_1})$ *equals* 1 *and* $K_{X_1} + (1/\mu_1)\mathcal{H}_{X_1}$ *is not canonical;*
(2) *The canonical threshold* c_1 *of* $(X_1, \mathcal{H}_{X_1})$ *equals* $1/2$ *and* C_1 *is the maximal center of* $(X_1, \mathcal{H}_{X_1})$*;*

Proof. The assertion (1) is easy to see. Of course, the maximal center of $(X_1, \mathcal{H}_{X_1})$ is contained in $\mathrm{Bs}\,\mathcal{H}_{X_1} = C_1 \cup L_1$. In consideration of the local equation f_1 of a general member $S_{X_1} \in \mathcal{H}_{X_1}$ on U_1 (cf. Lemma 3.4, (3)), we can see (2) by the similar argument as in Lemma 3.1. Q.E.D.

By virtue of Lemma 3.5, we have to blow-up X_1 along the curve C_1, say $p_1 : E_1 \subset Z_1 \to C_1 \subset X_1$, where $E_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$ is the exceptional divisor of p_1. The proper transform $\mathcal{H}_{Z_1}$ of $\mathcal{H}_{X_1}$ is spanned by S_{Z_1}, $T_{Z_1} + E_{0Z_1}$, $H_{Z_1} + 2E_{0Z_1}$ and $3E_{0Z_1} + E_1$. Note that the proper transforms of the hyperplane sections on $X_1 \cong \mathbf{Q}^3 \subset \mathbf{P}^4$ containing C_1 yield a Del Pezzo fibration $Z_1 \to \mathbf{P}^1$ with a general fiber isomorphic to a smooth quadric $\mathbf{P}^1 \times \mathbf{P}^1$ and $E_{0Z_1} \cong \mathbf{Q}_0^2$ as a singular fiber. We have the following (cf. Figure 3):

Lemma 3.6.
(1) $K_{Z_1} \sim -3E_{0Z_1} - 2E_1$;
(2) $E_{0Z_1}^2 \sim 0$ *and* $E_1^2 \sim -M_0 + 2l_0$, *where* M_0 *(resp.* l_0*) is a fiber on* $E_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$ *which maps onto a curve* C_1 *(resp. a point on* C_1*) via* $p_1 : Z_1 \to X_1$*;*
(3) $S_{Z_1}|_{E_{0Z_1}} = T_{Z_1}|_{E_{0Z_1}} = H_{Z_1}|_{E_{0Z_1}} = 2L_1$, *where we use the notation* L_1 *to represent its proper transform on* Z_1*;*
(4) $S_{Z_1}|_{E_1} \sim 2M_0 + 2l_0$, $T_{Z_1}|_{E_1} \sim M_0 + 2l_0$ *and* $H_{Z_1}|_{E_1} \sim 2l_0$.

Proof. All the assertions are readily seen by the similar argument as in Lemma 3.2. Q.E.D.

Now we need to observe not only the linearly equivalent relations as stated in Lemma 3.6 (4) but also the concrete configurations of these intersection curves. Indeed, we have the following:

Lemma 3.7. $S_{Z_1}|_{E_1} = 2C_2$, $T_{Z_1}|_{E_1} = C_2 + L_2$ *and* $H_{Z_1}|_{E_1} = 2L_2$, *where* C_2 *(resp.* L_2*) is a curve on* $E_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$ *linearly equivalent to* $M_0 + l_0$ *(resp. to* l_0*) intersecting* L_1 *(cf. Figure 3).*

Proof. For the proof we need to describe the blow-up $p_1 : E_1 \subset Z_1 \to C_1 \subset X_1$ around the open affine subset U_1 (y, z, w) explicitly. The center C_1 of p_1 is defined by an ideal $\mathfrak{a}_1 = (z, w) \subset \Gamma(\mathcal{O}_{U_1}) = \mathbf{C}[y, z, w]$ on U_1. Hence the inverse image $p_1^{-1}(U_1)$ is isomorphic to $\mathrm{Proj}(\bigoplus_{d \geq 0} \mathfrak{a}_1^d)$ which is covered by two affine 3-spaces, say $U_1^{(1)}$ (y, z, w) and $U_1^{(2)}$ (y, z, w). More precisely, $U_1^{(1)} \to U_1$ is associated to the algebra homomorphism $y \mapsto y$, $z \mapsto z$, $w \mapsto zw$, on the other hand, $U_1^{(2)} \to U_1$ is associated to $y \mapsto y$, $z \mapsto zw$, $w \mapsto w$. On $U_1^{(1)}$, the surfaces S_{Z_1}, T_{Z_1}, H_{Z_1}, E_{0Z_1} and E_1 are defined by $f_1^{(1)} := (y - w)^2 - z^2 w$, $g_1^{(1)} := y(y-w) - z^2 w$, $h_1^{(1)} := z^2 w - y^2$, w and z, respectively. By virtue of these local equations and Lemma 3.6 (4), we can obtain the concrete configurations of $S_{Z_1}|_{E_1}$, $T_{Z_1}|_{E_1}$ and $H_{Z_1}|_{E_1}$ as desired. Q.E.D.

As in the previous step, in order to perform the Sarkisov Program, we need to run $(K_{Z_1} + c_1\mathcal{H}_{Z_1})$-MMP. The cone $\overline{\mathrm{NE}}(Z_1) \subset \mathbf{R}^2$ is spanned by two rays one of which comes from rulings on an exceptional divisor $E_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$, and the other comes from the rulings on a general fiber $\cong \mathbf{P}^1 \times \mathbf{P}^1$ of a Del Pezzo fibration $Z_1 \to \mathbf{P}^1$ (see the argument before Lemma 3.6). The latter ray is $(K_{Z_1} + c_1\mathcal{H}_{Z_1})$-negative. Thus we put $X_2 := Z_1$ and the contraction of the latter ray $\phi_2 : X_2 \to B_2 = \mathbf{P}^1$ is a Mori fiber space. Put $\Phi_2 := p_1^{-1} : X_1 \cdots \to X_2$, which is an elementary link of type **I**.

In the subsequent, we put $U_2 := U_1^{(1)}$, $f_2 := f_1^{(1)}$, $g_2 := g_1^{(1)}$ and $h_2 := h_1^{(1)}$ (see the proof of Lemma 3.7 for the notations). It is readily seen that $\mathrm{Bs}\,\mathcal{H}_{X_2} = L_1$. By the similar arguments as in Lemmas 3.1, 3.5, we have the following result concerning the Sarkisov degree of $(X_2, \mathcal{H}_{X_2})$ in consideration of the local equation f_2 of a general member $S_{X_2} \in \mathcal{H}_{X_2}$.

Lemma 3.8.

(1) *The quasi-effective threshold* μ_2 *of* $(X_2, \mathcal{H}_{X_2})$ *equals* $1/2$ *and* $K_{X_2} + (1/\mu_2)\mathcal{H}_{X_2}$ *is not canonical;*

(2) *The canonical threshold* c_2 *of* $(X_2, \mathcal{H}_{X_2})$ *equals* $1/2$ *and* L_1 *is the maximal center.*

Hence we have to blow-up X_2 along the curve L_1, say $p_2 : E_2 \subset Z_2 \to L_1 \subset X_2$, where $E_2 \cong \mathbf{F}_2$ is the exceptional divisor of p_2. The configurations of the boundary surfaces $E_{0\,Z_2}$, $E_{1\,Z_2}$ and E_2 on Z_2 are described as in Figure 4. The restriction $E_{1\,Z_2} \to E_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$ of p_2 onto $E_{1\,Z_2}$ is just the blow-up at the point $P_2 := L_1 \cap E_1$ with an exceptional curve $e_0 := E_2 \cap E_{1\,Z_2}$. Let M_0 and l_0 be the rulings on $E_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$ (see Lemma 3.6, (2)) such that they meet each other at P_2. We use the same notations M_0 and l_0 to represent their proper transforms on $E_{1\,Z_2}$. By noting $\mathrm{mult}_{L_1} S_{X_2} = \mathrm{mult}_{L_1} T_{X_2} = \mathrm{mult}_{L_1} H_{X_2} = 1$, we know that the proper transform $\mathcal{H}_{Z_2}$ is spanned by S_{Z_2}, $T_{Z_2} + E_{0\,Z_2} + E_2$, $H_{Z_2} + 2E_{0\,Z_2} + 2E_2$ and $3E_{0\,Z_2} + E_{1\,Z_2} + 2E_2$. With these notations, by the similar argument as in Lemma 3.2, we can readily see the following result:

Lemma 3.9.
(1) $K_{Z_2} \sim -3E_{0\,Z_2} - 2E_{1\,Z_2} - 2E_2$;
(2) $E_{0\,Z_2}^2 \sim -M_2 - l_2$, $E_{1\,Z_2}^2 \sim -M_0 + 2l_0 + e_0$ *and* $E_2^2 \sim -M_2 + l_2$;
(3) $S_{Z_2}|_{E_{0\,Z_2}} = T_{Z_2}|_{E_{0\,Z_2}} = H_{Z_2}|_{E_{0\,Z_2}} = L_1$, *where we use the same notation* L_1 *to represent the curve contained in* $E_{0\,Z_2} \cap E_2$ *which is mapped onto* L_1 *via* p_2;
(4) $S_{Z_2}|_{E_{1\,Z_2}} \sim 2M_0 + 2l_0 + 3e_0$, $T_{Z_2}|_{E_{1\,Z_2}} \sim M_0 + 2l_0 + 2e_0$ *and* $H_{Z_2}|_{E_{1\,Z_2}} \sim 2l_0 + e_0$;
(5) $S_{Z_2}|_{E_2} \sim M_2 + 2l_2$, $T_{Z_2}|_{E_2} \sim M_2 + 2l_2$ *and* $H_{Z_2}|_{E_2} \sim M_2 + 2l_2$.

Though we have the linearly equivalent relations concerning $S_{Z_2}|_{E_{1\,Z_2}}$, $T_{Z_2}|_{E_{1\,Z_2}}$, $H_{Z_2}|_{E_{1\,Z_2}}$ on $E_{1\,Z_2}$ and $S_{Z_2}|_{E_2}$, $T_{Z_2}|_{E_2}$, $H_{Z_2}|_{E_2}$ on $E_2 \cong \mathbf{F}_2$ as stated in Lemma 3.9 (4) and (5), we now need to observe the concrete configurations of these intersection curves. In fact, we have the following:

Lemma 3.10. $S_{Z_2}|_{E_{1\,Z_2}} = 2C_2 + L_3$, $T_{Z_2}|_{E_{1\,Z_2}} = C_2 + L_2 + L_3$, $H_{Z_2}|_{E_{1\,Z_2}} = 2L_2 + L_3$ *and* $S_{Z_2}|_{E_2} = T_{Z_2}|_{E_2} = H_{Z_2}|_{E_2} = L_1 + 2L_3$ *(cf. Figure 4), where* C_2 *is an irreducible curve linearly equivalent to* $M_0 + l_0 + e_0$ *on* $E_{1\,Z_2}$.

Proof. We need to describe the blow-up $p_2 : E_2 \subset Z_2 \to L_1 \subset X_2$ explicitly on the open affine subset U_2 (y, z, w). On U_2, the center L_1 of p_2 is defined by an ideal $\mathfrak{a}_2 = (y, w) \subset \Gamma(\mathcal{O}_{U_2}) = \mathbf{C}[y, z, w]$. Hence $p_2^{-1}(U_2) \cong \mathrm{Proj}\big(\bigoplus_{d \geq 0} \mathfrak{a}_2^d\big)$ which is covered by two affine 3-spaces, say $U_2^{(1)}$ (y, z, w) and $U_2^{(2)}$ (y, z, w). We may assume that a morphism $U_2^{(1)} \to U_2$ (resp. $U_2^{(2)} \to U_2$) is associated to the algebra homomorphism $y \mapsto y$, $z \mapsto z$, $w \mapsto yw$ (resp. $y \mapsto yw$, $z \mapsto z$, $w \mapsto w$). By the straightforward computations, on $U_2^{(1)}$ (resp. $U_2^{(2)}$), the surfaces S_{Z_2}, T_{Z_2}, H_{Z_2} and E_2 are defined by $f_2^{(1)} = y(w-1)^2 - z^2 w$, $g_2^{(1)} = y(w-1) + z^2 w$, $h_2^{(1)} = y - z^2 w$ and y (resp. $f_2^{(2)} = (y-1)^2 w - z^2$, $g_2^{(2)} = y(y-1)w - z^2$,

$h_2^{(2)} = y^2w - z^2$ and w). By these equations and Lemma 3.9 we have the assertion. Q.E.D.

We perform $(K_{Z_2} + c_2\mathcal{H}_{Z_2})$-MMP over B_2. The cone $\overline{\mathrm{NE}}(Z_2/B_2) \subset \mathbf{R}^2$ is spanned by two rays one of which comes from the rulings on the exceptional divisor $E_2 \cong \mathbf{F}_2$ and the other comes from those on $E_{0\,Z_2} \cong \mathbf{F}_2$ (cf. Lemma 3.9 (2)). The latter ray is $(K_{Z_2} + c_2\mathcal{H}_{Z_2})$-nagative. The contraction of $E_{0\,Z_2}$, say $q_2 : Z_2 \to X_3$, induces a Del Pezzo fibration $\phi_3 : X_3 \to B_3 = B_2 \cong \mathbf{P}^1$ on X_3 with $E_{2\,X_3} \cong \mathbf{Q}_0^2$ as a singular fiber. The composite $\Phi_3 := q_2 \circ p_2^{-1} : X_2/B_2 \cdots \to X_3/B_3$ is an elementary link of type **II**. We have $K_{X_3} \sim -2E_{1\,X_3} - 2E_{2\,X_3}$. The proper transform $\mathcal{H}_{X_3}$ is spanned by S_{X_3}, $T_{X_3} + E_{2\,X_3}$, $H_{X_3} + 2E_{2\,X_3}$ and $E_{1\,X_3} + 2E_{2\,X_3}$. By virtue of Lemma 3.10, we can readily see the following:

Lemma 3.11.
(1) $S_{X_3}|_{E_{1\,X_3}} = 2C_2 + L_3$, $T_{X_3}|_{E_{1\,X_3}} = C_2 + L_2 + L_3$ *and* $H_{X_3}|_{E_{1\,X_3}} = 2L_2 + L_3$, *where we use the same notations* C_2, L_2, L_3 *to represent their proper transforms (cf. Figure 5);*
(2) $\mathrm{Bs}\,\mathcal{H}_{X_3} = L_3$*;*
(3) *On the open affine subset* $U_3 := q_2(U_2^{(2)})$ (y,z,w), *the surfaces* S_{X_3}, T_{X_3}, H_{X_3}, $E_{1\,X_3}$ *and* $E_{2\,X_3}$ *are defined respectively by* $f_3 := (y-1)^2w - z^2$, $g_3 := y(y-1)w - z^2$, $h_3 := y^2w - z^2$, z *and* w, *and the curves* C_2, L_2, L_3 *are defined by the ideals* $(y-1, z)$, (y,z), $\mathfrak{a}_3 := (z,w) \subset \Gamma(\mathcal{O}_{U_3}) = \mathbf{C}[y,z,w]$, *respectively.*

We shall check the Sarkisov degree of $(X_3, \mathcal{H}_{X_3})$.

Lemma 3.12.
(1) *The quasi-effective threshold* μ_3 *of* $(X_3, \mathcal{H}_{X_3})$ *equals* $1/2$ *and* $K_{X_3} + (1/\mu_3)\mathcal{H}_{X_3}$ *is not canonical;*
(2) *The canonical threshold* c_3 *of* $(X_3, \mathcal{H}_{X_3})$ *equals* 1 *and the maximal centers are* L_3 *and the point* $Q_3 := (1,0,0) \in U_3$ (y,z,w). *More precisely, such discrete valuations of* $\mathbf{C}(X_3)$ *which are the maximal singularity of* $\mathcal{H}_{X_3}$ *(cf. Definition* 2.2*) are obtained by the blow-up along* L_3 *and the (ordinary) blow-up at* Q_3.

Proof. We can readily see $\mu_3 = 1/2$. In order to see that $K_{X_3} + (1/\mu_3)\mathcal{H}_{X_3}$ is not canonical, let $p_3 : E_3 \subset Z_3 \to L_3 \subset X_3$ be the blow-up of X_3 along $\mathrm{Bs}\,\mathcal{H}_{X_3} = L_3$ with an exceptional divisor E_3. Since $\mathrm{mult}_{L_3} S_{X_3} = 1$ by virtue of Lemma 3.11 (3), we have $a(E_3; X_3, (1/\mu_3)\mathcal{H}_{X_3}) = -1$, so $K_{X_3} + (1/\mu_3)\mathcal{H}_{X_3}$ is not canonical. It is easily verified that $a(E_3; X_3, \mathcal{H}_{X_3}) = 0$, hence $c_3 \le 1$. By Lemma 3.11, S_{X_3} is smooth along $L_3 \backslash \{Q_3\}$ and $\mathrm{mult}_{Q_3} S_{X_3} = 2$, and the possibility of the maximal centers are Q_3 and L_3. Let $E(a,b,c)$ be the exceptional divisor of the weighted blow-up at Q_3 with weight $\mathrm{wt}(y-1, z, w) = (a,b,c)$, where one of a, b, c is 1 and the other two are coprime positive integers (cf. Theorem 2.2). Then $a(E(a,b,c); X_3, \mathcal{H}_{X_3})$ equals $-a+b-1$ (resp. $a-b+c-1$) if $2a + c \le 2b$ (resp. if $2b \le 2a + c$). Suppose that $a(E(a,b,c); X_3, \mathcal{H}_{X_3}) \le 0$.

Then by the straightforward computations we know that the only possibility is $(a, b, c) = (1, 1, 1)$ and $a(E(1, 1, 1); X_3, \mathcal{H}_{X_3}) = 0$. Thus we have the assertion.
Q.E.D.

As in the proof of Lemma 3.12, we consider the blow-up $p_3 : E_3 \subset Z_3 \to L_3 \subset X_3$ along L_3 with an exceptional divisor $E_3 \cong \mathbf{F}_2$. Then the configurations of the boundary surfaces ${E_1}_{Z_3}$, ${E_2}_{Z_3}$, E_3 are described as in Figure 6, where we use the same notations for the curves C_2 and L_2 to represent their proper transforms. Note that C_4 is a fiber (resp. the minimal section) on ${E_2}_{Z_3} \cong \mathbf{F}_2$ (resp. on $E_3 \cong \mathbf{F}_2$). The proper transform $\mathcal{H}_{Z_3}$ of $\mathcal{H}_{X_3}$ is spanned by S_{Z_3}, $T_{Z_3} + {E_2}_{Z_3} + E_3$, $H_{Z_3} + 2{E_2}_{Z_3} + 2E_3$ and ${E_1}_{Z_3} + 2{E_2}_{Z_3} + 2E_3$. We have the following result. Since the proof for this lemma is similar to that for Lemma 3.2, we shall omit it.

Lemma 3.13.
(1) $K_{Z_3} \sim -2{E_1}_{Z_3} - 2{E_2}_{Z_3} - 3E_3$;
(2) ${E_1}_{Z_3}^2 \sim 2l_0$ *and* ${E_2}_{Z_3}^2 \sim -M_2 - l_2$ *and* ${E_3}^2 \sim -M_2 - l_2$;
(3) $S_{Z_3}|_{{E_1}_{Z_3}} \sim 2M_0 + 2l_0 + 2e_0$, $T_{Z_3}|_{{E_1}_{Z_3}} \sim M_0 + 2l_0 + e_0$ *and* $H_{Z_3}|_{{E_1}_{Z_3}} \sim 2l_0$, *where we use the same notations* M_0, l_0 *and* e_0 *to represent their proper transforms (see the argument before Lemma* 3.9 *and Figure 6);*
(4) $S_{Z_3}|_{{E_2}_{Z_3}} \sim T_{Z_3}|_{{E_2}_{Z_3}} \sim H_{Z_3}|_{{E_2}_{Z_3}} \sim l_2$;
(5) $S_{Z_3}|_{E_3} \sim T_{Z_3}|_{E_3} \sim H_{Z_3}|_{E_3} \sim M_2 + 2l_2$.

We further need to observe the concrete configurations of the scheme-theoretic intersections. Indeed, we have the following:

Lemma 3.14.
(1) $S_{Z_3}|_{{E_1}_{Z_3}} = 2C_2$, $T_{Z_3}|_{{E_1}_{Z_3}} = C_2 + L_2$ *and* $H_{Z_3}|_{{E_1}_{Z_3}} = 2L_2$;
(2) $S_{Z_3}|_{{E_2}_{Z_3}} = T_{Z_3}|_{{E_2}_{Z_3}} = H_{Z_3}|_{{E_2}_{Z_3}} = C_4$;
(3) $S_{Z_3}|_{E_3} = C_4 + 2L_4$, $T_{Z_3}|_{E_3} = C_4 + L_4 + L'_4$ *and* $H_{Z_3}|_{E_3} = C_4 + 2L'_4$ *(cf. Figure 6).*

Proof. The proof is similar to that for Lemma 3.10. We shall describe the blow-up $p_3 : E_3 \subset Z_3 \to L_3 \subset X_3$ explicitly around U_3 (y, z, w). On this open affine subset U_3, the center L_3 of the blow-up p_3 is defined by an ideal $\mathfrak{a}_3 = (z, w) \subset \Gamma(\mathcal{O}_{U_3}) = \mathbf{C}[y, z, w]$. Hence $p_3^{-1}(U_3) \cong \mathrm{Proj}\big(\bigoplus_{d \geq 0} \mathfrak{a}_3^d\big)$ which is covered by two affine 3-spaces, say $U_3^{(1)}$ (y, z, w) and $U_3^{(2)}$ (y, z, w). We may assume that $U_3^{(1)} \to U_3$ (resp. $U_3^{(2)} \to U_3$) is a morphism associated to the algebra homomorphism $y \mapsto y$, $z \mapsto zw$, $w \mapsto w$ (resp. $y \mapsto y$, $z \mapsto z$, $w \mapsto zw$). Then the surfaces S_{Z_3}, T_{Z_3}, H_{Z_3} and E_3 are defined by $f_3^{(1)} := (y-1)^2 - z^2 w$, $g_3^{(1)} := y(y-1) - z^2 w$, $h_3^{(1)} := y^2 - z^2 w$ and w on $U_3^{(1)}$ (resp. $f_3^{(2)} := (y-1)^2 w - z$, $g_3^{(2)} := y(y-1)w - z$, $h_3^{(2)} := y^2 w - z$ and z on $U_3^{(2)}$). Thus we have the assertion as desired in consideration of Lemma 3.13.
Q.E.D.

We perform $(K_{Z_3} + c_3\mathcal{H}_{Z_3})$-MMP over B_3. The cone $\overline{\mathrm{NE}}(Z_3/B_3)$ is spanned by two rays one of which comes from the rulings on the exceptional divisor $E_3 \cong \mathbf{F}_2$ and the other comes from those on $E_{2Z_3} \cong \mathbf{F}_2$. The latter ray is $(K_{Z_3} + c_3\mathcal{H}_{Z_3})$-negative. Let $q_3 : Z_3 \to X_4$ be the contraction of E_{2Z_3}. Note that there exists a Del Pezzo fibration, say $\phi_4 : X_4 \to B_4 = B_3 \cong \mathbf{P}^1$, with a general fiber isomorphic to a smooth quadric and with $E_{3X_4} \cong \mathbf{Q}_0^2$ as a singular fiber. The composite $\Phi_4 := q_3 \circ p_3^{-1} : X_3/B_3 \cdots \to X_4/B_4$ is an elementary link of type **II**. The canonical divisor K_{X_4} is linearly equivalent to $-2E_{1X_4} - 3E_{3X_4}$ and the proper transform $\mathcal{H}_{X_4}$ is spanned by S_{X_4}, $T_{X_4} + E_{3X_4}$, $H_{X_4} + 2E_{3X_4}$ and $E_{1X_4} + 2E_{3X_4}$. We can readily see the following result in consideration of Lemma 3.14.

Lemma 3.15.
(1) $S_{X_4}|_{E_{1X_4}} = 2C_2$, $T_{X_4}|_{E_{1X_4}} = C_2 + L_2$, $H_{X_4}|_{E_{1X_4}} = 2L_2$, $S_{X_4}|_{E_{3X_4}} = 2L_4$, $T_{X_4}|_{E_{3X_4}} = L_4 + L'_4$ *and* $H_{X_4}|_{E_{3X_4}} = 2L'_4$, *where we use the same notations* C_2, L_2, L_4, L'_4 *etc. to represent their proper transforms (cf. Figure 7).* $E_{1X_4} \cong \mathbf{F}_1$ *and the curves* C_2 *(resp.* L_2*) is a cross-section on* E_{1X_4} *with self-intersection number* 1 *(resp.* -1*);*
(2) $\mathrm{Bs}\,\mathcal{H}_{X_3} = L_4$*;*
(3) *On the open affine subset* $U_4 := q_3(U_3^{(1)})$ (y, z, w), *the surfaces* S_{X_4}, T_{X_4}, H_{X_4}, E_{1X_4} *and* E_{3X_4} *are defined respectively by* $f_4 := (y-1)^2 - z^2w$, $g_4 := y(y-1) - z^2w$, $h_4 := y^2 - z^2w$, z *and* w. *Furthermore, the base locus* $\mathrm{Bs}\,\mathcal{H}_{X_4} = L_4$ *is defined by an ideal* $\mathfrak{a}_4 := (y-1, w) \subset \Gamma(\mathcal{O}_{U_4}) = \mathbf{C}[y, z, w]$ *on* U_4.

The Sarkisov degree of $(X_4, \mathcal{H}_{X_4})$ is written as follows.

Lemma 3.16.
(1) *The quasi-effective threshold* μ_4 *of* $(X_4, \mathcal{H}_{X_4})$ *equals* $1/2$ *and* $K_{X_4} + (1/\mu_4)\mathcal{H}_{X_4}$ *is not canonical;*
(2) *The canonical threshold* c_4 *of* $(X_4, \mathcal{H}_{X_4})$ *equals* 1 *and the maximal center is* L_4.

Proof. It is easy to see that $\mu_4 = 1/2$ and $K_{X_4} + (1/\mu_4)\mathcal{H}_{X_4} \sim E_{3X_4}$ is nef. Hence we know that $K_{X_4} + (1/\mu_4)\mathcal{H}_{X_4}$ is not canonical by Noether-Fano-Iskovskih inequality (cf. [Cor95, Cor00]). In fact, a general member $S_{X_4} \in \mathcal{H}_{X_4}$ has multiplicity 1 along the base locus curve L_4. Hence we obtain $a(E_4; X_4, (1/\mu_4)\mathcal{H}_{X_4}) = -1$ for an exceptional divisor E_4 which is obtained by the blow-up of X_4 along L_4. It is obvious that the maximal center of $\mathcal{H}_{X_4}$ is contained in the base locus $\mathrm{Bs}\,\mathcal{H}_{X_4} = L_4$. By virtue of the local equation f_4 of $S_{X_4} \in \mathcal{H}_{X_4}$ (see Lemma 3.15) and by the similar arguments as in Lemmas 3.1 and 3.12, we know that L_3 itself is a maximal center of $\mathcal{H}_{X_4}$. Q.E.D.

By Lemma 3.16, we perform the blow-up $p_4 : E_4 \subset Z_4 \to L_4 \subset X_4$ along the curve L_4, where $E_4 \cong \mathbf{F}_2$ is an exceptional divisor. Then the configuration of the boundary surfaces E_{1Z_4}, E_{3Z_4} and E_4 is described as in Figure 8, where we use the same notations for the curves C_2, L_2, L'_4 to represent their proper tranforms. Furthermore, we use the same notation L_4 to represent the curve which is contained in $E_{3Z_4} \cap E_4$ and mapped onto L_4 via p_4. Note that L_4 and L_5 are, respectively, a fiber and the minimal section (resp. the minimal section and a fiber) on $E_{3Z_4} \cong \mathbf{F}_2$ (resp. on $E_4 \cong \mathbf{F}_2$). The proper transform $\mathcal{H}_{Z_4}$ is spanned by S_{Z_4}, $T_{Z_4} + E_{3Z_4} + E_4$, $H_{Z_4} + 2E_{3Z_4} + E_4$ and $E_{1Z_4} + 2E_{3Z_4} + E_4$. The restriction $E_{1Z_4} \to E_{1X_4} \cong \mathbf{F}_1$ of p_4 onto E_{1Z_4} is just the blow-up at the point $P_4 := L_4 \cap E_{1X_4}$ with an exceptional curve $e_1 := E_4 \cap E_{1Z_4}$. Let l_1 be the fiber on $E_{1X_4} \cong \mathbf{F}_2$ passing through P_4 and we use the same notations M_1, l_1 to represent their proper transforms on E_{1Z_4} in the sequel if there are no confusions (cf. Figures 7 and 8). The following result is not difficult to verify by the similar argument as in Lemma 3.2 and we shall omit it.

Lemma 3.17.
(1) $K_{Z_4} \sim -2E_{1Z_4} - 3E_{3Z_4} - 2E_4$;
(2) $E_{1Z_4}^2 \sim 2M_1$, $E_{3Z_4}^2 \sim -M_2 - l_2$ *and* $E_4{}^2 \sim -M_2 - l_2$;
(3) $S_{Z_4}|_{E_{1Z_4}} \sim 2M_1 + 2l_1 + e_1$, $T_{Z_4}|_{E_{1Z_4}} \sim 2M_1 + l_1$ *and* $H_{Z_4}|_{E_{1Z_4}} \sim 2M_1$;
(4) $S_{Z_4}|_{E_{3Z_4}} \sim l_2$, $T_{Z_4}|_{E_{3Z_4}} \sim l_2$ *and* $H_{Z_4}|_{E_{3Z_4}} \sim M_2 + 2l_2$;
(5) $S_{Z_4}|_{E_4} \sim T_{Z_4}|_{E_4} \sim M_2 + 2l_2$ *and* $H_{Z_4}|_{E_4} \sim l_2$.

As in the previous steps, we have to determine the concrete configurations of the restrictions of S_{Z_4}, T_{Z_4} and H_{Z_4} onto the boundary surfaces E_{1Z_4}, E_{3Z_4} and E_4. In fact, we see the following:

Lemma 3.18.
(1) $S_{Z_4}|_{E_{1Z_4}} = 2C_2 + L'_5$, $T_{Z_4}|_{E_{1Z_4}} = C_2 + L_2$ *and* $H_{Z_4}|_{E_{1Z_4}} = 2L_2$;
(2) $S_{Z_4}|_{E_{3Z_4}} = L_4$, $T_{Z_4}|_{E_{3Z_4}} = L'_4$ *and* $H_{Z_4}|_{E_{3Z_4}} = L_5 + 2L'_4$;
(3) $S_{Z_4}|_{E_4} = L_4 + 2L'_5$, $T_{Z_4}|_{E_4} = C_5$ *and* $H_{Z_4}|_{E_4} = L_5$ *(cf. Figure 8).*

Proof. We investigate the blow-up $p_4 : Z_4 \to X_4$ over the open affine subset U_4 (y,z,w). On U_4, the center L_4 of p_4 is defined by an ideal $\mathfrak{a}_4 = (y-1, w) \subset \mathbf{C}[y,z,w]$. Therefore $p_4^{-1}(U_4) \cong \mathrm{Proj}\big(\bigoplus_{d\geq 0} \mathfrak{a}_4^d\big)$ which is covered by two affine 3-spaces, say $U_4^{(1)}$ (y,z,w) and $U_4^{(2)}$ (y,z,w). We may and shall assume that a morphism $U_4^{(1)} \to U_4$ (resp. $U_4^{(2)} \to U_4$) is associated to the algebra homomorphism $y \mapsto y$, $z \mapsto z$, $w \mapsto (y-1)w$ (resp. $y \mapsto yw+1$, $z \mapsto z$, $w \mapsto w$). Then the surfaces S_{Z_4}, T_{Z_4}, H_{Z_4} and E_4 are defined by $f_4^{(1)} := (y-1) - z^2 w$, $g_4^{(1)} := y - z^2 w$, $h_4^{(1)} := y^2 - (y-1)z^2 w$ and $y-1$ on $U_4^{(1)}$ (resp. $f_4^{(2)} := y^2 w - z^2$, $g_4^{(2)} := y(yw+1) - z^2$, $h_4^{(2)} := (yw+1)^2 - z^2 w$ and w on $U_4^{(2)}$). Then we can readily see the assertion in combination of Lemma 3.17. Q.E.D.

We then run the $\big(K_{Z_4} + c_4\mathcal{H}_{Z_4}\big)$-MMP over B_4. The cone $\overline{\mathrm{NE}}\,(Z_4/B_4) \subset \mathbf{R}^2$ is spanned by two rays one of which comes from the rulings on the exceptional divisor $E_4 \cong \mathbf{F}_2$ and the other comes from those on $E_{3Z_4} \cong \mathbf{F}_2$. The latter ray is $\big(K_{Z_4} + c_4\mathcal{H}_{Z_4}\big)$-negative. We contract E_{3Z_4}, say $q_4 : Z_4 \to X_5$, to get a Del Pezzo fibration $\phi_5 : X_5 \to B_5 = B_4 \cong \mathbf{P}^1$ having $E_{4\,X_5} \cong \mathbf{Q}_0^2$ as a singular fiber. The composite $\Phi_5 := q_4 \circ p_4^{-1} : X_4/B_4 \cdots \to X_5/B_5$ is an elementary link of type **II**. It follows that $K_{X_5} \sim -2E_{1\,X_5} - 2E_{4\,X_5}$ and the proper transform $\mathcal{H}_{X_5}$ of $\mathcal{H}_{Z_4}$ is spanned by S_{X_5}, $T_{X_5} + E_{4\,X_5}$, $H_{X_5} + E_{4\,X_5}$ and $E_{1\,X_5} + E_{4\,X_5}$. By virtue of Lemma 3.18, we can easily see the following:

Lemma 3.19.
(1) $S_{X_5}|_{E_{1\,X_5}} = 2C_2 + L_5'$, $T_{X_5}|_{E_{1\,X_5}} = C_2 + L_2$, $H_{X_5}|_{E_{1\,X_5}} = 2L_2$, $S_{X_5}|_{E_{4\,X_5}} = 2L_5'$, $T_{X_5}|_{E_{4\,X_5}} = C_5$ *and* $H_{X_5}|_{E_{4\,X_5}} = 2L_5$, *where we use the same notations* C_2, L_2, L_5', C_5 *to represent their proper transforms and we put* $L_5 := q_4(L_5)$ *(cf. Figure 9);*
(2) $\mathrm{Bs}\,\mathcal{H}_{X_5} = L_5'$*;*
(3) *On the open affine subset* $U_5 := q_4(U_4^{(2)})\ (y,z,w)$, *the surfaces* S_{X_5}, T_{X_5}, H_{X_5}, $E_{1\,X_5}$ *and* $E_{4\,X_5}$ *are defined respectively by* $f_5 := y^2w - z^2$, $g_5 := y(yw+1) - z^2$, $h_5 := (yw+1)^2 - z^2w$, z *and* w. *Furthermore, the base locus* $\mathrm{Bs}\,\mathcal{H}_{X_5} = L_5'$ *is defined by an ideal* $\mathfrak{a}_5 := (z,w) \subset \Gamma(\mathcal{O}_{U_5}) = \mathbf{C}[y,z,w]$ *on* U_5.

The following result about the Sarkisov degree of the pair $(X_5, \mathcal{H}_{X_5})$ is obtained by the similar argument as in the proof of Lemma 3.16, so we shall omit the detail.

Lemma 3.20.
(1) *The quasi-effective threshold* μ_5 *of* $(X_5, \mathcal{H}_{X_5})$ *equals* $1/2$ *and* $K_{X_5} + (1/\mu_5)\mathcal{H}_{X_5}$ *is not canonical;*
(2) *The canonical threshold* c_5 *of* $(X_5, \mathcal{H}_{X_5})$ *equals* 1 *and the maximal center is* L_5'.

Let $p_5 : E_5 \subset Z_5 \to L_5' \subset X_5$ be the blow-up along L_5' with an exceptional divisor $E_5 \cong \mathbf{F}_2$. The configuration of the boundary surfaces $E_{1\,Z_5}$, $E_{4\,Z_5}$ and E_5 is drawn as in Figure 10, where we use the same notations for the curves C_2, L_2, L_5, C_5 to represent their proper transforms on Z_5. Note that the curves C_6 and L_6' are respectively a fiber and the minimal section (resp. the minimal section and a fiber) on $E_{4\,Z_5} \cong \mathbf{F}_2$ (resp. on $E_5 \cong \mathbf{F}_2$). The proper transform $\mathcal{H}_{Z_5}$ of $\mathcal{H}_{X_5}$ is spanned by S_{Z_5}, $T_{Z_5} + E_{4\,Z_5}$, $H_{Z_5} + E_{4\,Z_5}$ and $E_{1\,Z_5} + E_{4\,Z_5} + E_5$. The following result holds true. We shall omit the proof because it is almost similar to that for Lemma 3.2.

Lemma 3.21.
(1) $K_{Z_5} \sim -2E_{1\,Z_5} - 2E_{4\,Z_5} - 3E_5$*;*
(2) ${E_1}_{Z_5}^2 \sim 2M_1 + l_1 - e_1$, ${E_4}_{Z_5}^2 \sim -M_2 - l_2$ *and* ${E_5}^2 \sim -M_2 - l_2$ *(see the argument before Lemma* 3.17*);*
(3) $S_{Z_5}|_{E_{1\,Z_5}} \sim T_{Z_5}|_{E_{1\,Z_5}} \sim H_{Z_5}|_{E_{1\,Z_5}} \sim 2M_1 + l_1$*;*
(4) $S_{Z_5}|_{E_{4\,Z_5}} \sim l_2$ *and* $T_{Z_5}|_{E_{4\,Z_5}} \sim H_{Z_5}|_{E_{4\,Z_5}} \sim M_2 + 2l_2$*;*
(5) $S_{Z_5}|_{E_5} \sim M_2 + 2l_2$ *and* $T_{Z_5}|_{E_5} \sim H_{Z_5}|_{E_5} \sim l_2$.

We, furthermore, determine the concrete configurations:

Lemma 3.22.
(1) $S_{Z_5}|_{E_{1\,Z_5}} = 2C_2$, $T_{Z_5}|_{E_{1\,Z_5}} = C_2 + L_2$ *and* $H_{Z_5}|_{E_{1\,Z_5}} = 2L_2 + L_5$*;*
(2) $S_{Z_5}|_{E_{4\,Z_5}} = C_6$, $T_{Z_5}|_{E_{4\,Z_5}} = C_5$ *and* $H_{Z_5}|_{E_{4\,Z_5}} = L_6' + 2L_5$*;*
(3) $S_{Z_5}|_{E_5} = C_6 + 2L_6$, $T_{Z_5}|_{E_5} = L_6$ *and* $H_{Z_5}|_{E_5} = L_6'$ *(cf. Figure 10).*

Proof. For the proof we need to describe the blow-up $p_5 : Z_5 \to X_5$ concretely over the open affine subset $U_5\ (y,z,w)$. Note that the center L_5' of p_5 is defined by an ideal $\mathfrak{a}_5 = (z,w) \subset \Gamma(\mathcal{O}_{U_5}) = \mathbf{C}[y,z,w]$ on U_5. Hence $p_5^{-1}(U_5) \cong \mathrm{Proj}\big(\bigoplus_{d\geq 0}\mathfrak{a}_5^d\big)$ which is covered by two affine 3-spaces, say $U_5^{(1)}\ (y,z,w)$ and $U_5^{(2)}\ (y,z,w)$. We may and shall assume that a morphism $U_5^{(1)} \to U_5$ (resp. $U_5^{(2)} \to U_5$) is associated to the algebra homomorphism $y \mapsto y$, $z \mapsto zw$, $w \mapsto w$ (resp. $y \mapsto y$, $z \mapsto z$, $w \mapsto zw$). We then can verify that the surfaces S_{Z_5}, T_{Z_5}, H_{Z_5} and E_5 are defined by $f_5^{(1)} := y^2 - z^2w$, $g_5^{(1)} := y(yw+1) - z^2w^2$, $h_5^{(1)} := (yw+1)^2 - z^2w^3$ and w on $U_5^{(1)}$ (resp. $f_5^{(2)} := y^2w - z$, $g_5^{(2)} := y(yzw+1) - z^2$, $h_5^{(2)} := (yzw+1)^2 - z^3w$ and z on $U_5^{(2)}$). Then we have the assertion readily in combination of Lemma 3.21. Q.E.D.

After the blow-up $p_5 : Z_5 \to X_5$, we run $\big(K_{Z_5} + c_5\mathcal{H}_{Z_5}\big)$-MMP over B_5. The cone of curves $\overline{\mathrm{NE}}\,(Z_5/B_5)$ is generated by two rays one of which comes from the rulings on the exceptional divisor $E_5 \cong \mathbf{F}_2$ and the other comes from those on $E_{4\,Z_5} \cong \mathbf{F}_2$. The latter ray is $\big(K_{Z_5} + c_5\mathcal{H}_{Z_5}\big)$-negative. We contract $E_{4\,Z_5}$ along the fibers on it, say $q_5 : Z_5 \to X_6$, to get a Del Pezzo fibration $\phi_5 : X_6 \to B_6 = B_5 \cong \mathbf{P}^1$ with $E_{5\,X_6} \cong \mathbf{Q}_0^2$ as a singular fiber. The composite $\Phi_6 := q_5 \circ p_5^{-1} : X_5/B_5 \cdots \to X_6/B_6$ is an elementary link of type **II**. The configuration of the boundary surfaces $E_{1\,X_6}$ and $E_{5\,X_6}$ is drawn as in Figure 11, where we use the same notations for curves C_2, L_2, L_6, L_6' to represent their proper transforms on X_6. Note that $E_{1\,X_6} \cong \mathbf{P}^1 \times \mathbf{P}^1$. It is readily seen that $K_{X_6} \sim -2E_{1\,X_6} - 3E_{5\,X_6}$ and the proper transform $\mathcal{H}_{X_6}$ of $\mathcal{H}_{Z_5}$ is spanned by S_{X_6}, T_{X_6}, H_{X_6} and $E_{1\,X_6} + E_{5\,X_6}$. We put $U_6 := q_5(U_5^{(1)})\ (y,z,w) \subset X_6$. We can obtain the following result in consideration of the previous lemma. We shall omit the detail.

Lemma 3.23.
(1) $S_{X_6}|_{E_{1\,X_6}} = 2C_2$, $T_{X_6}|_{E_{1\,X_6}} = C_2 + L_2$, $H_{X_6}|_{E_{1\,X_6}} = 2L_2$, $S_{X_6}|_{E_{5\,X_6}} = 2L_6$, $T_{X_6}|_{E_{5\,X_6}} = L_6 + L_6'$ *and* $H_{X_6}|_{E_{5\,X_6}} = 2L_6'$ *(cf. Figure 11);*
(2) $\mathrm{Bs}\,\mathcal{H}_{X_6} = \{P_6\}$*, where we put* $P_6 := L_6 \cap L_6'$*;*
(3) *On the open affine subset* $U_6\ (y,z,w)$*, the surfaces* S_{X_6}, T_{X_6}, H_{X_6}, $E_{1\,X_6}$ *and* $E_{5\,X_6}$ *are defined respectively by* $f_6 := y^2 - z^2w$, $g_6 := y(yw+1) - z^2w^2$, $h_6 := (yw+1)^2 - z^2w^3$, z *and* w.

We shall check the Sarkisov degree of the pair $(X_6, \mathcal{H}_{X_6})$.

Lemma 3.24.
(1) *The quasi-effective threshold* μ_6 *of* $(X_6, \mathcal{H}_{X_6})$ *equals* $1/2$*. Furthermore,* $K_{X_6} + (1/\mu_6)\mathcal{H}_{X_6}$ *is canonical but not nef;*

(2) *The canonical threshold* c_6 *of* $(X_6, \mathcal{H}_{X_6})$ *equals* 2 *and* P_6 *is the maximal center of* $(X_6, \mathcal{H}_{X_6})$. *More precisely, such a discrete valuation* E *of* $\mathbf{C}(X_6)$ *with discrepancy* $a(E; X_6, c_6\mathcal{H}_{X_6}) = 0$ *is obtained by the (ordinary) blow-up at* P_6.

Proof. We can readily see that $\mu_6 = 1/2$ in consideration of $K_{X_6} \sim -2E_{1X_6} - 3E_{5X_6}$ and $\mathcal{H}_{X_6} \equiv E_{1X_6} + E_{5X_6}$. Now we shall verify that the canonical threshold c_6 coincides with $1/\mu_6 = 2$. By virtue of Lemma 3.23 (2), the possibility of the maximal center is the point P_6. We claim the following:

Claim. The members of $\mathcal{H}_{X_6}$ *except for* $E_{1X_6} + E_{5X_6}$ *is smooth at* P_6.

Proof of Claim. Note that $E_{1X_6}^2 \sim -M_0 + 2l_0$, so it can be contracted along the fibers l_0 onto a curve. Now assume to the contrary that a member $G \in \mathcal{H}_{X_6}$ with $G \neq E_{1X_6} + E_{5X_6}$ is not smooth at P_6. Let $m := \text{mult}_{P_6} G \geq 2$. After the contraction of E_{1X_6}, we blow-up at P_6 with an exceptional divisor E. Then the proper transform of E_{5X_6} is isomorphic to $\mathbf{F}_2$ with the self-intersection $\sim -M_2 + 2l_2$, so we can contract it along the fibers on it. Then we reach the projective 3-space $\mathbf{P}^3$ and we have the linearly equivalent relation $\overline{G} \sim -(m-2)\overline{E}$, where we denote by $\overline{G}$ (resp. $\overline{E}$) the proper transform on $\mathbf{P}^3$ of G (resp. E). This is absurd. Q.E.D.

In consideration of Theorem 2.2, let $E(a, b, c)$ be the discrete valuation of $\mathbf{C}(X_6)$ which is obtained by a weighted blow-up at P_6 of weight $\text{wt}(u, v, w) = (a, b, c)$, where (u, v, w) is a suitable system of local coordinates at the point $P_6 \in X_6$, and one of a, b, c equals 1 and the other two are pairwise coprime positive integers. By virtue of Claim, we can readily see that $a(E(1, 1, 1); X_6, 2\mathcal{H}_{X_6}) = 0$ and $a(E(a, b, c); X_6, 2\mathcal{H}_{X_6}) > 0$ if $(a, b, c) \neq (1, 1, 1)$. Thus $K_{X_6} + (1/\mu_6)\mathcal{H}_{X_6}$ is canonical and c_6 coincides with $1/\mu_6 = 2$. Since $K_{X_6} + (1/\mu_6)\mathcal{H}_{X_6} \sim -E_{5X_6}$, it is not nef. Thus we complete the proof. Q.E.D.

Since $K_{X_6} + (1/\mu_6)\mathcal{H}_{X_6}$ is canonical and not nef, there exists a $(K_{X_6} + (1/\mu_6)\mathcal{H}_{X_6})$-negative extremal ray $\mathbf{P} \subset \overline{\text{NE}}(X_6)$. By noting that $K_{X_6} + (1/\mu_6)\mathcal{H}_{X_6} \sim -E_{5X_6}$ and $E_{1X_6}^2 \sim -M_0 + 2l_0$, we know that this ray $\mathbf{P}$ comes from the rulings l_0 on $E_{1X_6} \cong \mathbf{P}^1 \times \mathbf{P}^1$. Let $\mathbf{R} \subset \overline{\text{NE}}(X_6)$ be an extremal ray yielding a Del Pezzo fibration $\phi_6 : X_6 \to B_6$, and let $\mathbf{F} := \text{span} < \mathbf{P}, \mathbf{R} >$ be an extremal face spanned by $\mathbf{P}$ and $\mathbf{R}$. In fact, since $\overline{\text{NE}}(X_6) \subset \mathbf{R}^2$, we have $\mathbf{F} = \overline{\text{NE}}(X_6)$. Let $\phi_F : X_6 \to \{pt.\}$ be the contraction of the extremal face $\mathbf{F}$. Then, according to the Sarkisov Program, we shall run $(K_{X_6} + (1/\mu_6)\mathcal{H}_{X_6})$-MMP (relative to ϕ_F). Hence we contract $E_{1X_6} \cong \mathbf{P}^1 \times \mathbf{P}^1$ with $E_{1X_6}^2 \sim -M_0 + 2l_0$ along the direction of ruling l_0 to obtain $q_6 : X_6 \to X_7 \cong \mathbf{Q}^3$. It is clear that $\Phi_7 := q_6 : X_6/B_6 \to X_7/\{pt.\}$ is an elementary link of type **III**. It follows that $K_{X_7} \sim -3E_{5X_7}$, $E_{5X_7} \cong \mathbf{Q}_0^2$ with $E_{5X_7}^2 \sim \mathcal{O}_{\mathbf{P}^3}(1) \otimes \mathcal{O}_{\mathbf{Q}_0^2}$ and the proper transform $\mathcal{H}_{X_7}$ is spanned by S_{X_7}, T_{X_7}, H_{X_7} and E_{5X_7}. The following result is easy to see, so we shall omit the proof.

Lemma 3.25.

(1) $S_{X_7}|_{E_{5X_7}} = 2L_6$, $T_{X_7}|_{E_{5X_7}} = L_6 + L'_6$, $H_{X_7}|_{E_{5X_7}} = 2L'_6$, *where we use the same notations* L_6, L'_6 *to represent their proper transforms (cf. Figure 12);*

(2) $\mathrm{Bs}\,\mathcal{H}_{X_6} = \{P_7\}$, *where* $P_7 := L_6 \cap L'_6 = q_6(P_6)$*;*

(3) *The general members of* $\mathcal{H}_{X_7}$ *except for* E_{5X_7} *are smooth at the base point* P_6*.*

We shall observe the Sarkisov degree of the pair $(X_7, \mathcal{H}_{X_7})$.

Lemma 3.26.

(1) *The quasi-effective threshold* μ_7 *of* $(X_7, \mathcal{H}_{X_7})$ *equals* $1/3$*. Furthermore,* $K_{X_7} + (1/\mu_7)\mathcal{H}_{X_7}$ *is nef but not canonical;*

(2) *The canonical threshold* c_7 *of* $(X_7, \mathcal{H}_{X_7})$ *equals* 2 *and* P_7 *is the maximal center of* $(X_7, \mathcal{H}_{X_7})$*. More precisely, such a discrete valuation* E *of* $\mathbf{C}(X_7)$ *with discrepancy* $a(E; X_7, c_7\mathcal{H}_{X_7}) = 0$ *is obtained by the (ordinary) blow-up at* P_7*.*

Proof. We can readily see $\mu_7 = 1/3$. Then $K_{X_7} + (1/\mu_7)\mathcal{H}_{X_7} \sim 0$ is nef but not canonical. In fact, a general member of $\mathcal{H}_{X_7}$ distinct from E_{5X_7} is smooth at P_7 by the same argument as in Claim in Lemma 3.24. Hence $a(E_7; X_7, (1/\mu_7)\mathcal{H}_{X_7}) = -1$ for a discrete valuation E_7 of $\mathbf{C}(X_7)$ which is obtained by the blow-up at P_7. Hence $K_{X_7} + (1/\mu_7)\mathcal{H}_{X_7}$ is not canonical and $a(E_7; X_7, 2\mathcal{H}_{X_7}) = 0$. Hence $c_7 \leq 2$. Since $\mathrm{Bs}\,\mathcal{H}_{X_7} = \{P_7\}$, we need to consider the discrete valuations of $\mathbf{C}(X_7)$ which is obtained by the weighted blow-up at P_7 with a suitably chosen weights (cf. Theorem 2.2). By the straightforward computation we see $a(E; X_7, 2\mathcal{H}_{X_7}) > 0$ for any such a discrete valuation of $\mathbf{C}(X_7)$ distinct from E_7. Thus we obtain the assertion. Q.E.D.

By Lemma 3.26, we need to consider the blow-up $p_7 : E_7 \subset Z_7 \to P_7 \in X_7$ at P_7, where $E_7 \cong \mathbf{P}^2$ is an exceptional divisor. The proper transform $\mathcal{H}_{Z_7}$ is spanned by S_{Z_7}, T_{Z_7}, H_{Z_7} and $E_{5Z_7} + E_7$ because every member of $\mathcal{H}_{X_7}$ except for $E_{5X_7} + E_7$ is smooth at P_7 (see Lemma 3.25). We perform $\left(K_{Z_7} + c_7\mathcal{H}_{Z_7}\right)$-MMP. Then we contract $E_{5Z_7} \cong \mathbf{F}_2$ with $E_{5}{}_{Z_7}^{2} \sim -M_2 + 2l_2$ onto a smooth conic to obtain $\Phi_8 := q_7 \circ {p_7}^{-1} : Z_7 \to X_8 \cong \mathbf{P}^3$, which is an elementary link of type **III**. We can verify that $K_{X_8} \sim -4E_{7X_8}$, the proper transform $\mathcal{H}_{X_8}$ is spanned by S_{X_8}, T_{X_8}, H_{X_8} and E_{7X_8} and $\mathcal{H}_{X_8} = |\mathcal{O}_{\mathbf{P}^3}(1)|$. Therefore, the quasi-effective threshold μ_8 of $(X_8, \mathcal{H}_{X_8})$ is $\mu_8 = 1/4$ and $K_{X_8} + (1/\mu_8)\mathcal{H}_{X_8}$ is canonical and nef. Hence by Noether-Fano-Iskovskih inequality (cf. [Cor95, Cor00]) we obtain the factorization $\Phi_\sigma = \Phi_8 \circ \cdots \circ \Phi_1$ of Φ_σ into eight elementary links according to the Sarkisov Program.

Summarizing the argument performed in this section, we obtain the following result Theorem 3.1 (= Theorem 1.1).

Theorem 3.1. *Let* σ *be the Nagata automorphism on the affine 3-space* $\mathbf{A}^3$ *and let* $\Phi_\sigma : \mathbf{P}^3 \cdots \to \mathbf{P}^3$ *be a Cremona transformation which is an extension of* σ*. Then* Φ_σ *can be factorized into eight elementary links explicitly, say* $\Phi_\sigma = \Phi_8 \circ \cdots \circ \Phi_1$*. More precisely, each elementary link* Φ_j $(1 \leq j \leq 8)$ *is described as follows (here we shall follow the notations used in Section 3 and each number of type is the type of elementary link; see Section 2):*

(i) $\Phi_1 : \mathbf{P}^3 \overset{p_0}{\leftarrow} Z_0 \overset{q_0}{\rightarrow} X_1 \cong \mathbf{Q}^3$, *where* p_0 *is the blow-up along the conic* C_0 *with an exceptional divisor* $E_0 \cong \mathbf{F}_2$ *and* q_0 *is the contraction of* $H_{\infty Z_0} \cong \mathbf{P}^2$ *(cf. Figure 1) to a smooth point (Type II),*

(ii) $\Phi_2 : X_1 \overset{p_1}{\leftarrow} X_2$ *is the blow-up along the curve* C_1 *(cf. Figure 2) with an exceptional divisor* $E_1 \cong \mathbf{P}^1 \times \mathbf{P}^1$ *(Type I),*

(iii) $\Phi_3 : X_2 \overset{p_2}{\leftarrow} Z_2 \overset{q_2}{\rightarrow} X_3$, *where* p_2 *is the blow-up along the curve* L_1 *(cf. Figure 3) with an exceptional divisor* $E_2 \cong \mathbf{F}_2$ *and* q_2 *is the contraction of* $E_{0\,Z_2} \cong \mathbf{F}_2$ *(cf. Figure 4) onto a curve (Type II),*

(iv) $\Phi_4 : X_3 \overset{p_3}{\leftarrow} Z_3 \overset{q_3}{\rightarrow} X_4$, *where* p_3 *is the blow-up along the curve* L_3 *(cf. Figure 5) with an exceptional divisor* $E_3 \cong \mathbf{F}_2$ *and* q_3 *is the contraction of* $E_{2\,Z_3} \cong \mathbf{F}_2$ *(cf. Figure 6) onto a curve (Type II),*

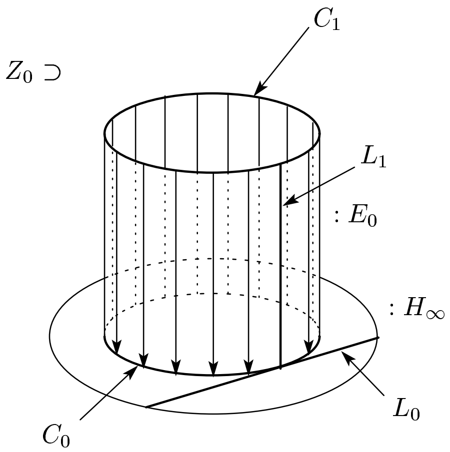
(v) $\Phi_5 : X_4 \overset{p_4}{\leftarrow} Z_4 \overset{q_4}{\rightarrow} X_5$, *where* p_4 *is the blow-up along the curve* L_4 *(cf. Figure 7) with an exceptional divisor* $E_4 \cong \mathbf{F}_2$ *and* q_4 *is the contraction of* $E_{3\,Z_4} \cong \mathbf{F}_2$ *(cf. Figure 8) onto a curve (Type II),*

(vi) $\Phi_6 : X_5 \overset{p_5}{\leftarrow} Z_5 \overset{q_5}{\rightarrow} X_6$, *where* p_5 *is the blow-up along the curve* L_5' *(cf. Figure 9) with an exceptional divisor* $E_5 \cong \mathbf{F}_2$ *and* q_5 *is the contraction of* $E_{4\,Z_5} \cong \mathbf{F}_2$ *(cf. Figure 10) onto a curve (Type II),*

(vii) $\Phi_7 : X_6 \overset{q_6}{\rightarrow} X_7 \cong \mathbf{Q}^3$ *is the contraction of* $E_{1\,X_6} \cong \mathbf{P}^1 \times \mathbf{P}^1$ *(cf. Figure 11) onto a curve (Type III),*

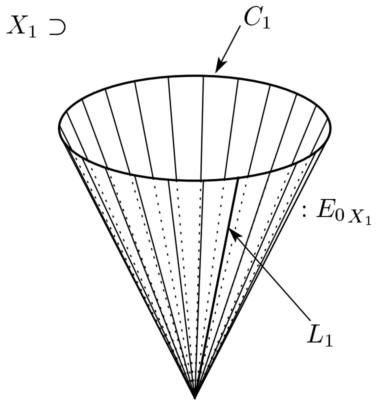
(viii) $\Phi_8 : X_7 \overset{p_7}{\leftarrow} Z_7 \overset{q_7}{\rightarrow} X_8 \cong \mathbf{P}^3$, *where* p_7 *is the blow-up at the point* P_7 *(cf. Figure 12) with an exceptional divisor* $E_7 \cong \mathbf{P}^2$ *and* q_7 *is the contraction of* $E_{5\,Z_7} \cong \mathbf{F}_2$ *onto a conic (Type II).*

Remark 1. Let $\Phi : X/B \cdots \to X'/B'$ be a given birational map between 2-dimensional or 3-dimensional Mori fiber spaces. As explained in Section 2, we fix a very ample complete linear system $\mathcal{H}'$ on X' and take its proper transform $\mathcal{H} := \Phi_*^{-1}(\mathcal{H}')$ on X. In dimension 2 and 3, it is guaranteed that such a birational map Φ can be factorized into a finite number of elementary links as explained in Section2 according to the Sarkisov Program. (In dimension 3, this is due to Sarkisov, Reid and Corti [Cor95]. In dimension 2, this result is classical.) In two-dimensional case, we can describe the factorization more explicitly. Indeed, in this case there are no log-flips and each divisorial blow-up is just the blow-up at the point at which general members of the proper transform of $\mathcal{H}$ has the highest multiplicity. On the other hand, the factorization of a birational map between 3-dimensional Mori fiber spaces is not unique in general. This fact results from the choice of the divisorial blow-ups along the maximal centers with respect to the proper transform of $\mathcal{H}$. For instance, in the decompostion of the Cremona transformation $\Phi_\sigma : \mathbf{P}^3 \cdots \to \mathbf{P}^3$ which is an extension of the Nagata automorphism, there are some possibilities for the divisorial blow-ups of the maximal centers of $(X_3, \mathcal{H}_{X_3})$ (cf. Lemma 3.12). More precisely, we are free to choose either the blow-up along the curve L_3 or the (ordinary) blow-up at the point Q_3. In our factorization of Φ_σ we chose the former blow-up.



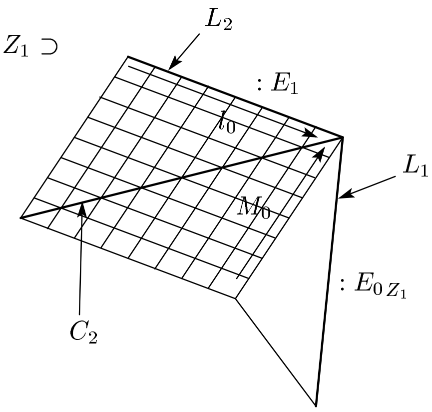
$E_0 \cong \mathbf{F_2}$, $H_\infty \cong \mathbf{P^2}$

Figure 1:



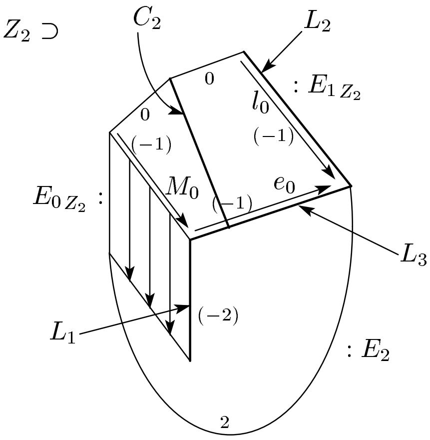
$E_{0\,X_1} \cong \mathbf{Q_0^2}$

Figure 2:



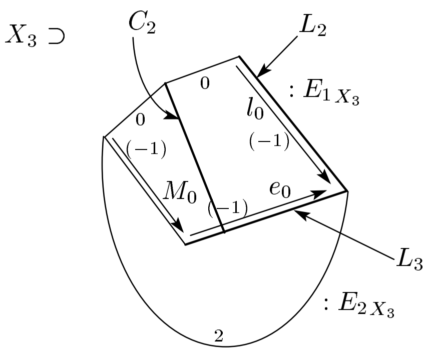
$E_1 \cong \mathbf{P^1} \times \mathbf{P^1}$, $E_{0\,Z_1} \cong \mathbf{Q_0^2}$

Figure 3:



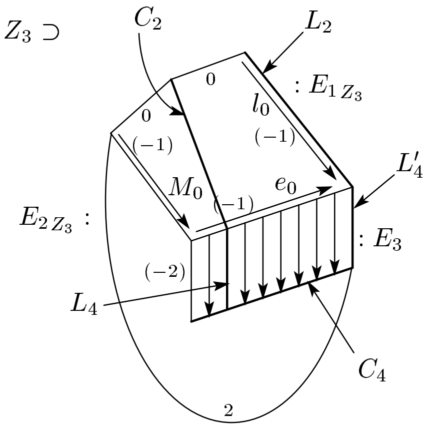
$E_{0\,Z_2} \cong \mathbf{F_2}$, $E_2 \cong \mathbf{F_2}$

Figure 4:



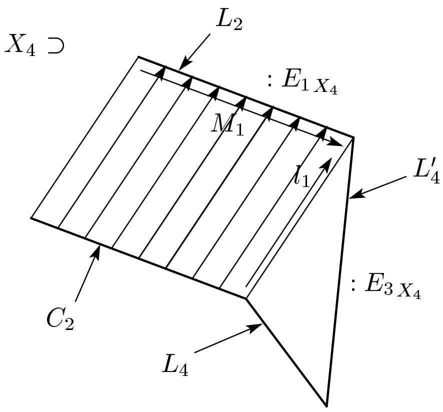
$E_{2\,X_3} \cong \mathbf{Q_0^2}$

Figure 5:



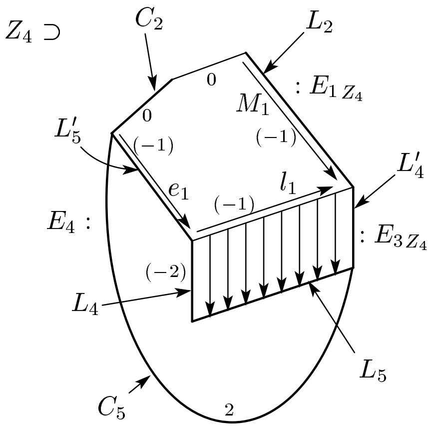
$E_{2\,Z_3} \cong \mathbf{F_2}$, $E_3 \cong \mathbf{F_2}$

Figure 6:



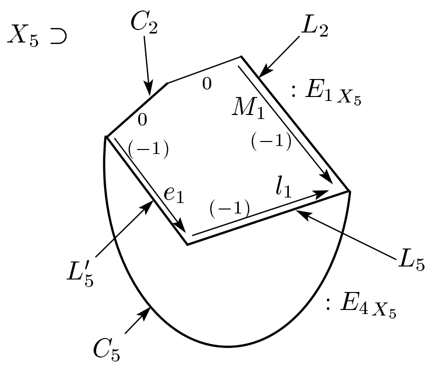
$E_{1\,X_4} \cong \mathbf{F_1}$, $E_{3\,X_4} \cong \mathbf{Q_0^2}$

Figure 7:



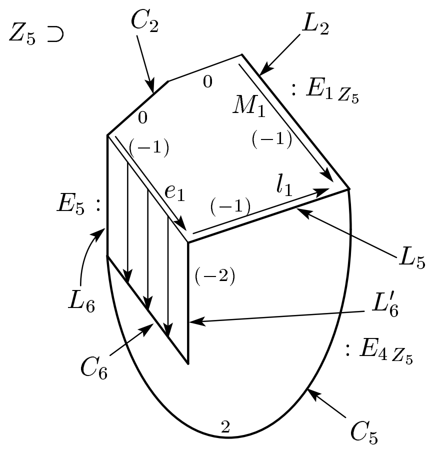
$E_{3\,Z_4} \cong \mathbf{F_2}$, $E_4 \cong \mathbf{F_2}$

Figure 8:



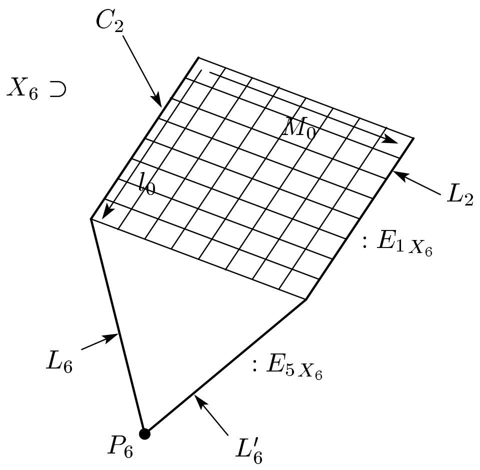
$E_{4\,X_5} \cong \mathbf{Q_0^2}$

Figure 9:



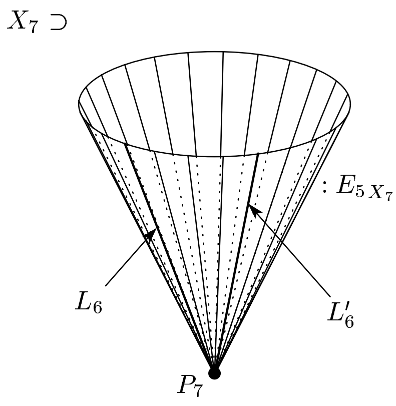
$E_5 \cong \mathbf{F_2}$, $E_{4\,Z_5} \cong \mathbf{F_2}$

Figure 10:



$E_{1\,X_6} \cong \mathbf{P^1} \times \mathbf{P^1}$, $E_{5\,X_6} \cong \mathbf{Q_0^2}$

Figure 11:



$E_{5\,X_7} \cong \mathbf{Q_0^2}$

Figure 12:

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