Reflective modular forms in algebraic geometry

V. Gritsenko

Abstract

We prove that the existence of a strongly reflective modular form of a large weight implies that the Kodaira dimension of the corresponding modular variety is negative or, in some special case, it is equal to zero. Using the Jacobi lifting we construct three towers of strongly reflective modular forms with the simplest possible divisor. In particular we obtain a Jacobi lifting construction of the Borcherds-Enriques modular form Φ_4 and Jacobi liftings of automorphic discriminants of the Kähler moduli of Del Pezzo surfaces constructed recently by Yoshikawa. We obtain also three modular varieties of dimension 4, 6 and 7 of Kodaira dimension 0.

0 Introduction

A reflective modular form is a modular form on an orthogonal group of type $\mathrm{O}(2,n)$ whose divisor is determined by reflections. A strongly reflective form that vanishes of order one along the reflective divisors is the denominator function of a Lorentzian Kac-Moody (super) Lie algebra of Borcherds type. For example the famous Borcherds form Φ_{12} in 26 variables (see [B1]) defines the Fake Monster Lie algebra.

Reflective modular forms are very rare. Some of them have geometric interpretation as automorphic discriminants of some moduli spaces, for example of moduli spaces of lattice polarised K3 surfaces (see [GN5]). The first such example was the Borcherds–Enriques form Φ_4 (see [B2]). This strongly reflective form is the automorphic discriminant of the moduli space of Enriques surfaces and it is the denominator function of the fake monster superalgebra. In 2009 K.-I. Yoshikawa constructed the automorphic discriminant Φ_V of the Kähler moduli of a Del Pezzo surface V of $1 \le \deg V \le 9$. These functions are also related to the analytic torsion of special Calabi–Yau threefolds (see [Y]). The corresponding Borcherds superalgebras were predicted in the conjecture of Harvey–Moore (see [HM, §7]). We note that the generators and relations of Lorentzian Kac–Moody (super) Lie algebras of Borcherds type are defined by the Fourier expansion of a reflective modular form at a zero-dimensional cusp (see [GN1] and [GN3]). All modular forms mentioned above were constructed as Borcherds automorphic products which gives us the multiplicities of the positive roots.

The quasi pull-backs of the strongly reflective form Φ_{12} help us to prove the general type of some modular varieties of orthogonal type. See [GHS1] where we proved that the moduli space of polarised K3 surfaces of degree $2d$ is of the maximal Kodaira dimension if $d > 61$. In §1 of this paper we give a new geometric definition based on the results of [GHS1] of the reflective modular forms as modular forms with a small divisor. It gives us a new interesting application of reflective modular forms which is quite opposite to the results of [GHS1]–[GHS2]. In Theorem 1.5 we prove that the existence of a strongly reflective modular form of a large weight implies that the Kodaira dimension of the corresponding modular variety is negative or, in some special case, it is equal to zero. In §2 we give three new examples of modular varieties of orthogonal type of dimension 4, 6 and 7 of Kodaira dimension 0 (varieties of Calabi–Yau type) and we hope to consider more examples in the near future.

The geometric examples of §2 are based on the three towers of strongly reflective modular forms which we construct in §3–§5 with the help of Jacobi lifting. It is a rather surprising fact that we obtain very simple Jacobi lifting constructions of the Borcherds form Φ_4 and of the Yoshikawa functions Φ_V. These modular forms constitute the D_8-tower of the Jacobi liftings (see §3). In particular we obtain simple formulae for the Fourier expansions of Φ_V, i.e. the explicit generating formulae for the imaginary simple (super) roots of the corresponding Borcherds superalgebras. In §4 and §5 we present the towers of the strongly reflective modular forms based on the modular forms of singular weight for the root systems $3A_2$ and $4A_1$. The D_8-, $3A_2$- and $4A_1$-towers of the Jacobi liftings give us $15 = 8 + 3 + 4$ strongly reflective modular forms. We note that the last function in the $4A_1$-tower is the Siegel modular form Δ_5 which is the square root of the Igusa modular form of weight 10.

In the conclusion we formulate a conjecture about strongly reflective modular forms similar to the modular forms considered in this paper.

1 Modular varieties of orthogonal type and reflective modular forms

We start with the general set-up. Let L be an even integral lattice with a quadratic form of signature $(2, n)$ and let

$$\mathcal{D}(L) = \{[Z] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (Z, Z) = 0,\ (Z, \overline{Z}) > 0\}^+$$

be the associated n-dimensional classical Hermitian domain of type IV (here $+$ denotes one of its two connected components). We denote by $\mathrm{O}^+(L)$ the index 2 subgroup of the integral orthogonal group $\mathrm{O}(L)$ preserving $\mathcal{D}(L)$.

For any $v \in L \otimes \mathbb{Q}$ such that $v^2 = (v, v) < 0$ we define the *rational quadratic divisor*

$$\mathcal{D}_v = \mathcal{D}_v(L) = \{[Z] \in \mathcal{D}(L) \mid (Z, v) = 0\} \cong \mathcal{D}(v_L^\perp)$$

where $v_L^\perp$ is an even integral lattice of signature $(2, n-1)$. If $\Gamma < \mathrm{O}^+(L)$ is of finite index we define the corresponding *modular variety*

$$\mathcal{F}_L(\Gamma) = \Gamma \backslash \mathcal{D}(L),$$

which is a quasi-projective variety of dimension n.

The important examples of modular varieties of orthogonal type are
a) the moduli spaces of polarised K3 surfaces;
b) the moduli spaces of lattice polarised K3 surfaces (the dimension of such a moduli space is smaller than 19);
c) the moduli spaces of polarised Abelian or Kummer surfaces;
d) the moduli space of Enriques surfaces;
e) the periodic domains of polarised irreducible symplectic varieties (the dimension of a modular variety of this type is equal to 4, 5, 20 or 21).

One of the main tools in the study of the geometry of modular varieties is the theory of modular forms with respect to an orthogonal group. In the next definition we bear in mind Koecher's principle (see [B1], [Bai]).

Definition 1.1 *Let* $\operatorname{sign}(L) = (2, n)$ *with* $n \geq 3$. *A modular form of weight* k *and character* $\chi\colon \Gamma \to \mathbb{C}^*$ *with respect to* Γ *is a holomorphic function* $F\colon \mathcal{D}(L)^\bullet \to \mathbb{C}$ *on the affine cone* $\mathcal{D}(L)^\bullet$ *over* $\mathcal{D}(L)$ *such that*

$$F(tZ) = t^{-k} F(Z) \quad \forall\, t \in \mathbb{C}^*,$$

$$F(gZ) = \chi(g) F(Z) \quad \forall\, g \in \Gamma.$$

A modular form is called a cusp form if it vanishes at every cusp (a boundary component of the Baily–Borel compactification of $\mathcal{F}_L(\Gamma)$*).*

We denote the linear spaces of modular and cusp forms of weight k and character (of finite order) χ by $M_k(\Gamma, \chi)$ and $S_k(\Gamma, \chi)$ respectively. If $M_k(\Gamma, \chi)$ is nonzero then one knows that $k \geq (n-2)/2$ (see [G1]–[G2]). The minimal weight $k = (n-2)/2$ is called *singular*. The weight $k = n = \dim(\mathcal{F}_L(\Gamma))$ is called *canonical* because according to Freitag's criterion

$$S_n(\Gamma, \det) \cong H^0\big(\overline{\mathcal{F}}_L(\Gamma), \Omega(\overline{\mathcal{F}}_L(\Gamma))\big),$$

where $\overline{\mathcal{F}}_L(\Gamma))$ is a smooth compact model of the modular variety $\overline{\mathcal{F}}_L(\Gamma)$ and $\Omega(\overline{\mathcal{F}}_L(\Gamma))$ is the sheaf of canonical differential forms (see [F, Hilfssatz 2.1, Kap. 3]). We say that

$$\text{the weight } \ k \ \text{ is small if } \ k < n \ \text{ or big if } \ k \geq n. \tag{1}$$

For applications, the most important subgroups of $\mathrm{O}^+(L)$ are the stable orthogonal groups

$$\widetilde{\mathrm{O}}^+(L) = \{g \in \mathrm{O}^+(L) \mid g|_{L^\vee/L} = \mathrm{id}\}, \quad \widetilde{\mathrm{SO}}^+(L) = \mathrm{SO}(L) \cap \widetilde{\mathrm{O}}^+(L)$$

where $L^\vee$ is the dual lattice of L. If the lattice L contains two orthogonal copies of the hyperbolic plane $U \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (the even unimodular lattice of signature $(1,1)$) and its reduction modulo 2 (resp. 3) is of rank at least 6 (resp. 5) then $\widetilde{\mathrm{O}}^+(L)$ has only one non-trivial character det (see [GHS3]).

For any non isotropic $r \in L$ we denote by σ_r reflection with respect to r

$$\sigma_r(l) = l - \frac{2(l,r)}{(r,r)} r \in \mathrm{O}(L \otimes \mathbb{Q}).$$

This is an element of $\mathrm{O}^+(L \otimes \mathbb{Q})$ if and only if $(r,r) < 0$. If $r^2 = -2$, then $\sigma_r(l) \in \widetilde{\mathrm{O}}^+(L)$. In general, the ramification divisor of $\mathcal{F}_L(\widetilde{\mathrm{O}}^+(L))$ is larger than the union of the rational quadratic divisors $\mathcal{D}_r(L)$ defined by (-2)-roots in L.

Definition 1.2 *A modular form $F \in M_k(\Gamma, \chi)$ is called* **reflective** *if*

$$\mathrm{supp}(\mathrm{div}\, F) \subset \bigcup_{\substack{\pm r \in L \\ r \text{ is primitive} \\ \sigma_r \in \Gamma \text{ or } -\sigma_r \in \Gamma}} \mathcal{D}_r(L) = \mathrm{R.div}(\pi_\Gamma). \tag{2}$$

We call F **strongly reflective** *if the multiplicity of any irreducible component of* $\mathrm{div}\, F$ *is equal to one.*

We note that $\mathcal{D}_r(L) = \mathcal{D}_{-r}(L)$. In the definition of reflective modular forms in [GN4] only the first condition $\sigma_r \in \Gamma$ was considered. The present definition is explained by the following result proved in [GHS1, Corollary 2.13]

Proposition 1.3 *The ramification divisors of the modular projection*

$$\pi_\Gamma : \mathcal{D}(L) \to \Gamma \backslash \mathcal{D}(L)$$

are induced by elements $g \in \Gamma$ such that g or $-g$ is a reflection with respect to a vector in L.

According to the last proposition the union of the rational quadratic divisors in the right hand side of (2) is the ramification divisor $\mathrm{R.div}(\pi_\Gamma)$ of the modular projection π_Γ.

Example 1 *The Borcherds modular form Φ_{12}.* The most famous example of a strongly reflective modular form is $\Phi_{12} \in M_{12}(O^+(II_{2,26}), \det)$ (see [B1]). This is the unique modular form of singular weight 12 with character det with respect to the orthogonal group $O^+(II_{2,26})$ of the even unimodular lattice of signature $(2,26)$. The form Φ_{12} is the Kac-Weyl-Borcherds denominator function of the Fake Monster Lie algebra. We expect only finite number of reflective modular forms (see [GN3]). The Borcherds automorphic products give us some number of interesting examples of strongly reflective modular forms (see [B1]–[B4]). Note that for a large class of integral lattices any reflective modular form has a Borcherds product according to [Br].

As we mentioned above, if the rank of the quadratic lattice is smaller or equal to 19 then one can interpreted the modular varieties of orthogonal type as moduli spaces of lattice polarised K3 surfaces. The stable locus of the reflections of the integral orthogonal group is related to the special singular K3 surfaces. It gives us an interpretation of reflective modular forms as *automorphic discriminants* of these moduli spaces (see [GN5]). Moreover, if the modular form is strongly reflective, then the Lorentzian Kac–Moody algebra determined by the automorphic discriminant can be considered as a variant of the arithmetic mirror symmetry for these K3 surfaces (see [GN6]). We remark also that the reflective modular forms of type Φ_{12} of singular weight with respect to congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ were classified by N. Scheithauer (see [Sch]).

Example 2 *Igusa modular forms.* We can apply Definition 1.2 to Siegel modular forms of genus 2 because $\mathrm{PSp}_2(\mathbb{Z})$ is isomorphic to $\mathrm{SO}^+(L(A_1))$ where $L(A_1) = 2U \oplus A_1(-1) = 2U \oplus \langle -2\rangle$. The Siegel modular form of odd weight $\Delta_{35} \in S_{35}(\mathrm{Sp}_2(\mathbb{Z}))$ and the product of the ten even theta-constants $\Delta_5 \in S_5(\mathrm{Sp}_2(\mathbb{Z}), \chi_2)$ (χ_2 is a character of order 2) are strongly reflective (see [GN1]–[GN2]). One more classical example is the "most odd" even Siegel theta-constant $\Delta_{1/2}$ which is a modular form of weight $1/2$ with respect to the paramodular group Γ_4. These examples are part of the classification of all reflective forms for the maximal lattices of signature $(2,3)$ in [GN4]. Moreover, Δ_5 and $\Delta_{1/2}$ are examples of modular forms with the simplest divisor (see [CG]).

Remark 1. *Modular forms of canonical weight.* Let $\mathrm{sign}(L) = (2,n)$. We consider $F \in M_n(\Gamma, \det)$. If $\sigma_r \in \Gamma$, then $F(\sigma_r(Z)) = -F(Z)$. Hence F vanishes along $\mathcal{D}_r$. If $-\sigma_r \in \Gamma$, then

$$(-1)^n F(\sigma_r(Z)) = F((-\sigma_r)(Z)) = \det(-\sigma_r)F(Z) = (-1)^{n+1}F(Z)$$

and F also vanishes along $\mathcal{D}_r$. Therefore any Γ-modular form of canonical weight with character det vanishes along $\mathrm{R.div}(\pi_\Gamma)$.

Remark 2. *Modular forms with small or big divisor.* According to the definition above a modular form $F \in M_k(\Gamma, \chi)$ is strongly reflective if and only if

$$\operatorname{div} F \le \mathrm{R.div}(\pi_\Gamma),$$

i.e. *the divisor of a strongly reflective modular form is small.* We say that the divisor of a modular form $F \in M_k(\Gamma, \chi)$ is *big* if

$$\operatorname{div} F \geq \mathrm{R.div}(\pi_\Gamma).$$

The role of modular forms of small weight (see (1)) with a big divisor in the birational geometry of moduli spaces was clarified in [GHS1]: the modular variety $\mathcal{F}_L(\Gamma)$ is of general type if there exists a cusp form of small weight with a big divisor. More exactly, we proved the following theorem called *low weight cusp form trick*:

Theorem 1.4 *Let $n \geq 9$. The modular variety $\mathcal{F}_L(\Gamma)$ is of general type if there exists $F \in S_k(\Gamma, \det^\varepsilon)$ ($\varepsilon = 0$ or 1) of small weight $k < n$ such that* $\operatorname{div} F \geq \mathrm{R.div}(\pi_\Gamma)$.

This is a particular case of [GHS1, Theorem 1.1]. We applied this theorem in order to prove that the moduli spaces of polarised K3 surfaces and the moduli spaces of polarised symplectic varieties deformationally equivalent to $\mathrm{Hilb}^2(\mathrm{K3})$ or to 10-dimensional symplectic O'Grady varieties are of general type (see [GHS1]–[GHS2]).

In this paper we give a new application of strongly reflective modular forms, which is quite opposite to Theorem 1.4. Namely, the Kodaira dimension of the modular variety $\mathcal{F}_L(\Gamma)$ is equal to $-\infty$ if there exists a modular form of big weight with a small divisor. More exactly we have

Theorem 1.5 *Let* $\operatorname{sign}(L) = (2, n)$ *and* $n \geq 3$. *Let* $F_k \in M_k(\Gamma, \chi)$ *be a strongly reflective modular form of weight k and character χ where $\Gamma < \mathrm{O}^+(L)$ is of finite index. By $\kappa(X)$ we denote the Kodaira dimension of X. Then*

$$\kappa(\Gamma \backslash \mathcal{D}(L)) = -\infty$$

if $k > n$, or $k = n$ and F_k is not a cusp form. If $k = n$ and F is a cusp form then

$$\kappa(\Gamma_\chi \backslash \mathcal{D}(L)) = 0,$$

where $\Gamma_\chi = \ker(\chi \cdot \det)$ *is a subgroup of* Γ.

Proof. To prove the first identity of the theorem we have to show that there are no pluricanonical differential forms on $\overline{\mathcal{F}}_L(\Gamma)$. Any such differential form can be obtained using a modular form (see [AMRT] where weight 1 corresponds to weight n in our definition of modular forms). Suppose that $F_{nm} \in M_{nm}(\Gamma, \det^m)$. We may realize $\mathcal{D}(L)$ as a tube domain by choosing a 0-dimensional cusp. In the corresponding affine coordinates of this tube domain we take a holomorphic volume element dZ on $\mathcal{D}(L)$. Then the differential form $F_{nm}\,(dZ)^m$ is Γ-invariant. Therefore it determines a section of the pluricanonical bundle $\Omega(\overline{\mathcal{F}}_L(\Gamma))^{\otimes m}$ over a smooth open part of the modular variety away from the branch locus of $\pi\colon \mathcal{D}(L) \to \mathcal{F}_L(\Gamma)$

and the cusps (see [AMRT, Chapter 4] and [F]). There are three kinds of obstructions to extending $F_{nm}\,(dZ)^m$ to a global section of $\Omega(\overline{\mathcal{F}}_L(\Gamma))^{\otimes m}$, namely, there are elliptic obstructions, arising because of singularities given by elliptic fixed points of the action of Γ; cusp obstructions, arising from divisors at infinity; and reflective obstructions, arising from the ramification divisor in $\mathcal{D}(L)$. The ramification divisor is defined by $\pm$ reflections in Γ according to Proposition 1.3. Therefore if F_{nm} determines a global section then F_{nm} has zeroes of order at least m on $\mathrm{R.div}(\pi_\Gamma)$. The modular form $F_k \in M_k(\Gamma,\chi)$ is strongly reflective of weight $k \ge n$ hence F_{nm}/F_k^m is a holomorphic modular form of weight $m(n-k) \le 0$. According to Koecher's principle (see [Bai] and [F]) this function is constant. We have that $F_{nm} \equiv 0$ if $k > n$ or $F_{nm} = C \cdot F_n^m$ if $k = n$. If the strongly reflective form F_n is non cuspidal of weight n, then $F_n^m\,(dZ)^{\otimes m}$ cannot be extended to the compact model due to cusp obstructions (F_n^m should have zeroes of order at least m along the boundary). If F_n is a cusp form of weight $k = n$ then we can consider F_n as a cusp form with respect to the subgroup

$$\Gamma_\chi = \ker(\chi \cdot \det) < \Gamma, \qquad F_n \in S_n(\Gamma,\,\chi) < S_n(\Gamma_\chi,\,\det).$$

Then $F_n\,dZ$ is Γ_χ-invariant and, according to Freitag's criterion, it can be extended to a global section of the canonical bundle $\Omega(\overline{\mathcal{F}}_L(\Gamma_\chi))$ for any smooth compact model $\overline{\mathcal{F}}_L(\Gamma_\chi)$ of $\mathcal{F}_L(\Gamma_\chi)$. Moreover the above consideration with Koecher's principle shows that any m-pluricanonical form is equal, up to a constant, to $(F_n\,dZ)^{\otimes m}$. Therefore in the last case of the theorem the strongly reflective cusp form of canonical weight determines essentially the unique m-pluricanonical differential form on $\mathcal{F}_L(\Gamma_\chi)$. $\square$

Some applications of this theorem will be given in the next section.

2 Modular varieties of Calabi-Yau type

The problem of constructing a strongly reflective cusp form of canonical weight (see the second case of Theorem 1.5) is far from trivial. Note that any reflective modular form has a Borcherds product expansion if the quadratic lattice is not very complicated (see [Br]) but it is rather difficult to construct Borcherds products of a fixed weight. See, for example [Ko] and [GHS1] where cusp forms of canonical weight were constructed for the moduli spaces of polarised K3-surfaces. Between those cusp forms there are no reflective modular forms.

There are only two examples of strongly reflective cusp forms of canonical weights in the literature. Both are Siegel modular forms of genus 2 (i.e., the orthogonal group is of type $(2,3)$). The first example is related to the strongly reflective modular form $\Delta_1 \in S_1(\Gamma_3,\chi_6)$ of weight 1 (see [GN4, Example 1.14]) with respect to the paramodular group Γ_3 (the Siegel modular threefold $\Gamma_3\backslash\mathbb{H}_2$ is the moduli space of the $(1,3)$-polarised Abelian surfaces). Then $\Delta_1(Z)^3dZ$ is the unique canonical differential form on the Siegel threefold $(\ker\chi_6^3)\setminus\mathbb{H}_2$ having a Calabi–Yau model (see [GH]). The second example is the strongly reflective form $\nabla_3 \in S_3(\Gamma_0^{(2)}(2),\chi_2)$ (see [CG]) where $\Gamma_0^{(2)}(2)$ is the Hecke congruence subgroup of $\mathrm{Sp}_2(\mathbb{Z})$ and $\chi_2 : \Gamma_0^{(2)}(2) \to \{\pm 1\}$ is a binary character. The Siegel cusp form ∇_3^2 was first constructed by T. Ibukiyama in [Ib]. A Jacobi lifting and a Borcherds automorphic product of ∇_3 were given in [CG], where it was also proved that the Kodaira dimension of the Siegel threefold $(\ker\chi_2)\backslash\mathbb{H}_2$ is equal to zero. A Calabi-Yau model of this modular variety was founded in [FS-M]. Note that $(\ker\chi_2)\backslash\mathbb{H}_2$ is a double cover of the rational Siegel threefold $\Gamma_0^{(2)}(2)\setminus\mathbb{H}_2$. One of the main purposes of this paper is to construct similar examples for dimension larger than 3.

Let S be a positive definite lattice. We put

$$L(S) = 2U \oplus S(-1), \qquad \mathrm{sign}(L(S)) = (2, 2+\operatorname{rank} S) = (2, 2+n_0)$$

where $S(-1)$ denotes the corresponding negative definite lattice. In the applications of this paper S will be D_n, mA_1 or mA_2 where $mA_n = A_n \oplus \cdots \oplus A_n$ (m times). We define two modular varieties

$$\mathcal{SM}(S) = \widetilde{\mathrm{SO}}^+(L(S)) \setminus \mathcal{D}(L(S)), \tag{3}$$

$$\mathcal{M}(S) = \widetilde{\mathrm{O}}^+(L(S)) \setminus \mathcal{D}(L(S)). \tag{4}$$

Theorem 1.4 shows that the main obstruction to continuing of the pluricanonical differential forms on a smooth compact model of a modular variety is its ramification divisor. In many case the ramification divisor of $\mathcal{SM}(S)$ is strictly smaller than the ramification divisor of $\mathcal{M}(S)$.

Lemma 2.1 *For odd $n \geq 3$ the ramification divisor of the projection*

$$\pi^+_{D_n} : \mathcal{D}(2U \oplus D_n) \to \mathcal{SM}(D_n)$$

is defined by the reflections with respect to (-4)-vectors in $2U \oplus D_n$ with divisor 2 where $\mathrm{div}_L(v)\mathbb{Z} = (v, L)$.

Proof. We recall that D_n is an even sublattice of the Euclidian lattice $\mathbb{Z}^n$

$$D_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \mid x_1 + \cdots + x_n \in 2\mathbb{Z}\}.$$

We have $|D_n^\vee/D_n| = 4$. The discriminant form is generated by the following four elements

$$D_n^\vee/D_n = \{\,0,\ e_n,\ (e_1+\cdots+e_n)/2,\ (e_1+\cdots+e_{n-1}-e_n)/2 \mod D_n\}.$$

Then

$$D_n^\vee/D_n \cong \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\quad (n\equiv 0 \bmod 2)\quad \text{or}\quad \mathbb{Z}/4\mathbb{Z}\quad (n\equiv 1 \bmod 2).$$

Note that $(e_1+\dots+e_n)/2)^2 = ((e_1+\dots+e_{n-1}-e_n)/2)^2 = n/4$ is the minimal norm of the elements in the corresponding classes modulo D_n.

For any even integral lattice L with two hyperbolic planes there is a simple description of the orbits of the primitive vectors. According to the *Eichler criterion* (see [G2, page 1195] or [GHS3]), the $\widetilde{\mathrm{SO}}^+(L)$-orbit of any primitive $v\in L$ depends only on $v^2=(v,v)$ and $v/\operatorname{div}(v) \mod L$. We note that $v^*=v/\operatorname{div}(v)$ is a primitive vector in the dual lattice $L^\vee$.

If $\sigma_v\in \mathrm{O}^+(L(D_n))$ then $v^2<0$ and $\operatorname{div}(v)\mid v^2\mid 2\operatorname{div}(v)$. Therefore, if n is impair then $\operatorname{div}(v)$ is a divisor of 4 because $4D_n^\vee<D_n$. If $\operatorname{div}(v)=1$ then $v^2=-2$ and $-\sigma_v$ induces $-\mathrm{id}$ on the discriminant group. (Note that $\mathrm{id}=-\mathrm{id}$ on $D_n^\vee/D_n$ for even n.) If $\operatorname{div}(v)=2$ then $v^2=4$. It follows that v belongs to the $\widetilde{\mathrm{SO}}^+(L(D_n))$-orbit of one of vectors of type $2e_i$ or $\pm e_{i_1}\pm e_{i_2}\pm e_{i_3}\pm e_{i_4}$ of square 4. Any vector of type $\pm e_{i_1}\pm e_{i_2}\pm e_{i_3}\pm e_{i_4}$ has divisor 1 in D_n for odd $n>4$. If $v=\pm 2e_i$ then $-\sigma_v$ induces identity on the discriminant group for odd n. If $\operatorname{div}(v)=4$ and $v^2=-4$ or -8 then $(v/4)^2=-1/4$ or $-1/2$. Both cases are impossible for odd n because $(v^*)^2\equiv 0$, 1 or $n/4 \mod 2\mathbb{Z}$ for any $v^*\in D_n^\vee$. $\square$

Theorem 2.2 *We have that $\mathcal{SM}(D_7)$ is of general type and*

$$\kappa\bigl(\mathcal{SM}(D_n)\bigr)=\begin{cases} 0 & \text{if } n=5,\\ -\infty & \text{if } n=3.\end{cases}$$

Moreover

$$\kappa\bigl(\mathcal{SM}(2A_2)\bigr)=0\quad\text{and}\quad \kappa\bigl(\mathcal{SM}^{(2)}(2A_1)\bigr)=0$$

where

$$\mathcal{SM}^{(2)}(2A_1)=\ker\chi_2\setminus\mathcal{D}(L(2A_1))$$

and $\chi_2:\widetilde{\mathrm{SO}}^+(2U\oplus 2A_1(-1))\to\{\pm1\}$ is the binary character of the cusp form $\Delta_{4,\,2A_1}$ from Theorem 5.1.

Proof. In the proof we use strongly reflective cusp forms which will be constructed in the next sections with the help of liftings of Jacobi modular forms.

1) The divisor of the cusp form $\mathrm{Lift}(\psi_{12-n,\,D_n})$ of weight $12-n$ (see Theorem 3.2 below) for $n=3$ and 5 is equal to the ramification divisor of $\pi^+_{D_n}$. Therefore we can apply Theorem 1.5. For $n=7$ the weight of $\mathrm{Lift}(\psi_{5,\,D_7})$ is small. Therefore we can apply Theorem 1.4.

2) The modular variety $\mathcal{SM}(2A_2)$ is of dimension 6. The strongly reflective cusp form $\mathrm{Lift}(\psi_{6,\,2D_2})$ (see Theorem 4.2 below) has canonical weight.

The strongly reflective cusp form $\mathrm{Lift}(\psi_{4,2A_1}) \in S_4(\widetilde{\mathrm{SO}}^+(L(2A_1)), \chi_2)$ (see Theorem 4.2) is of weight 4 which is the dimension of $\mathcal{SM}^{(2)}(2A_1)$. Therefore we can apply the second part of Theorem 1.5. $\square$

Remark 1. *Varieties of Calabi–Yau type.* We conjecture that each of the three varieties of Kodaira dimension zero in Theorem 2.2 have a Calabi–Yau model similar to the two examples mentioned in the beginning of §2.
Remark 2. *Kodaira dimension of $\mathcal{M}(S)$ in Theorem 2.2.* Note that

$$\mathcal{SM}(S) = \widetilde{\mathrm{SO}}^+(L(S)) \setminus \mathcal{D}(L(S)) \to \widetilde{\mathrm{O}}^+(L(S)) \setminus \mathcal{D}(L(S)) = \mathcal{M}(S)$$

is a covering of order 2. The ramification divisor of $\mathcal{M}(S)$ contains all divisors of type $\mathcal{D}_r(L(S))$ where r is any of the (-2)-vectors of $L(S)$. Analyzing results of [B4] or using the quasi pull-back of the Borcherds form Φ_{12} (see a forthcoming paper of B. Grandpierre and V. Gritsenko "*The baby functions of the Borcherds form* Φ_{12}") we can construct strongly reflective modular forms for $\widetilde{\mathrm{O}}^+(L(S))$ of big weight (see (1)) with respect to $S = D_3$, D_5, D_7, $2A_2$ and $2A_1$. Therefore using Theorem 1.5 we obtain that for all S from Theorem 2.2 the modular variety $\mathcal{M}(S)$ is of Kodaira dimension $-\infty$.

As we mentioned above $\widetilde{\mathrm{SO}}^+(L(S)) \setminus \mathcal{D}(L(S))$ is a double covering of the modular variety $\mathcal{M}(S)$ which is the moduli space of the lattice polarised K3-surfaces with transcendence lattice $T = L(S)$ (see [N], [Do]). Therefore $\mathcal{SM}(D_5)$ can be considered as the moduli space of the lattice polarised K3-surfaces with transcendence lattice $T = L(D_5)$ together with a spin structure (a choice of orientation in T). See [GHS1, §5] where the case of polarised K3 surfaces of degree $2d$ with a spin structure was considered. The Picard lattice $\mathrm{Pic}(X_D)$ of a generic member X_D of this moduli space is

$$\mathrm{Pic}(X_D) \cong (2U \oplus D_5)^{\perp}_{II_{3,19}} \cong U \oplus E_8(-1) \oplus A_3(-1)$$

where $II_{3,19} = 3U \oplus 2E_8(-1) \cong H^2(X, \mathbb{Z})$ is the K3-lattice. The cases of $L(2A_2)$ and $L(2A_1)$ are similar

$$\mathrm{Pic}(X_{2A_2}) \cong U \oplus E_6(-1) \oplus E_6(-1) \cong U \oplus E_8(-1) \oplus A_2(-1) \oplus A_2(-1),$$

$$\mathrm{Pic}(X_{2A_1}) \cong U \oplus E_7(-1) \oplus E_7(-1) \cong U \oplus E_8(-1) \oplus D_6(-1).$$

The only difference here is that $\mathcal{SM}^{(2)}(2A_1)$ is a double covering of the moduli spaces $\mathcal{SM}(2A_1)$ of the lattice polarised K3-surfaces with a spin structure.

3 Jacobi theta-series and the D_n-tower of strongly reflective modular forms

We use Jacobi modular forms in many variables, the corresponding Jacobi lifting ([G2]) and automorphic Borcherds products ([B1], [B3], [GN4]) in order to describe special strongly reflective modular forms. This will give us the proof of Theorem 2.2. We see below that Jacobi forms (specially Jacobi theta-series) are sometimes more convenient to use in our considerations than the corresponding vector-valued modular forms.

Let $L = 2U \oplus S(-1)$ be an integral quadratic lattice of signature $(2, n_0+2)$ where U is the hyperbolic plane and S is a positive definite integral lattice of rank n_0 (then $S(-1)$ is negative definite). The representation $2U \oplus S(-1)$ of L gives us a choice of a totally isotropic plane in L. It gives the following tube realization $\mathcal{H}_{2+n_0}$ of the type IV domain $\mathcal{D}(L)$

$$\mathcal{H}_{2+n_0} = \{Z = (\omega, \mathfrak{z}, \tau) \in \mathbb{H}^+ \times (S \otimes \mathbb{C}) \times \mathbb{H}^+ \mid (\operatorname{Im} Z, \operatorname{Im} Z)_{U \oplus S(-1)} > 0\}$$

with $(\operatorname{Im} Z, \operatorname{Im} Z) = 2 \operatorname{Im} \tau \operatorname{Im} \omega - (\operatorname{Im} \mathfrak{z}, \operatorname{Im} \mathfrak{z}) > 0$ where $(\operatorname{Im} \mathfrak{z}, \operatorname{Im} \mathfrak{z})$ is the positive definite scalar product on S. In the definition of Jacobi forms in many variables we follow [G2].

Definition A holomorphic (cusp or weak) Jacobi form of weight k and index m with respect to S ($k \in \mathbb{Z}$) is a holomorphic function

$$\phi : \mathbb{H}^+ \times (S \otimes \mathbb{C}) \to \mathbb{C}$$

satisfying the functional equations

$$\phi(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d}) = (c\tau + d)^k \exp\bigl(\pi i \frac{cm(\mathfrak{z}, \mathfrak{z})}{c\tau + d}\bigr)\phi(\tau, \mathfrak{z}), \tag{5}$$

$$\phi(\tau, \mathfrak{z} + \lambda\tau + \mu) = \exp\bigl(-2\pi i\bigl(\frac{m}{2}(\lambda, \lambda)\tau + m(\lambda, \mathfrak{z})\bigr)\bigr)\phi(\tau, \mathfrak{z}) \tag{6}$$

for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and any $\lambda, \mu \in S$ and having a Fourier expansion

$$\phi(\tau, \mathfrak{z}) = \sum_{n \in \mathbb{Z},\ \ell \in S^\vee} f(n, \ell) \exp\bigl(2\pi i(n\tau + (\ell, \mathfrak{z}))\bigr),$$

where $n \geq 0$ for a weak Jacobi form, $N_m(n, \ell) = 2nm - (\ell, \ell) \geq 0$ for a holomorphic Jacobi form and $N_m(n, \ell) > 0$ for a Jacobi cusp form.

We denote the space of all holomorphic Jacobi forms by $J_{k,m}(S)$. We use the notation $J_{k,m}^{(cusp)}(S)$ and $J_{k,m}^{(weak)}(S)$ for the space of cusp and weak Jacobi forms. If $J_{k,m}(S) \neq \{0\}$ then $k \geq \frac{1}{2} \operatorname{rank} S$ (see [G1]). The weight $k = \frac{1}{2} \operatorname{rank} S$ is called *singular*. It is known (see [G2, Lemma 2.1]) that $f(n, \ell)$ depends only the hyperbolic norm $N_m(n, \ell) = 2nm - (\ell, \ell)$ and the image of ℓ in the discriminant group $D(S(m)) = S^\vee/mS$. Moreover, $f(n, \ell) = (-1)^k f(n, -\ell)$.

Remark 1. *Fourier-Jacobi coefficients.* Let $F \in M_k(\widetilde{\mathrm{O}}^+(L))$. We consider its Fourier expansion in ω

$$F(Z) = f_0(\tau) + \sum_{m \geq 1} f_m(\tau, \mathfrak{z}) \exp(2\pi i m\omega).$$

where $f_0(\tau) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ and $f_m(\tau, \mathfrak{z}) \in J_{k,m}(S)$. The lifting construction of [G1]–[G2] defines a modular form with respect to $\widetilde{\mathrm{O}}^+(L)$ with trivial character by its first Fourier-Jacobi coefficient from $J_{k,1}(S)$.

We note that $J_{k,m}(S) = J_{k,1}(S(m))$ where $S(m)$ denotes the same lattice S with the quadratic form multiplied by m, and the space $J_{k,m}(S)$ depends essentially only on the discriminant form of $S(m)$. Any Jacobi form determines a vector valued modular form related to the corresponding Weil representation (see [G2, Lemma 2.3–2.4]). For $\phi \in J_{k,1}(S)$ we have

$$\phi(\tau, \mathfrak{z}) = \sum_{\substack{n\in\mathbb{Z},\ \ell\in S^\vee \\ 2n-(\ell,\ell)\ge 0}} f(n,\ell)\, \exp\bigl(2\pi i(n\tau + (\ell, \mathfrak{z})\bigr) = \sum_{h\in D(S)} \phi_h(\tau)\Theta_{S,h}(\tau, \mathfrak{z}),$$

where $\Theta_{S,h}(\tau, \mathfrak{z})$ is the Jacobi theta-series with characteristic h and the components of the vector valued modular forms $(\phi_h)_{D(S)}$ have the following Fourier expansions at infinity:

$$\phi_h(\tau) = \sum_{n\equiv -\frac{1}{2}(h,h) \mod \mathbb{Z}} f_h(r) \exp(2\pi i n\tau)$$

where $f_h(n) = f(n+\frac{1}{2}(h,h), h)$. This representation for a weak Jacobi form gives us the next lemma

Lemma 3.1 *Let $f(n,l)$ $(n \ge 0,\ \ell \in S^\vee)$ be a Fourier coefficient of a weak Jacobi form $\phi \in J_{k,1}^{(weak)}(S)$. Then*

$$f(n,\ell) \ne 0 \Rightarrow 2n - (\ell,\ell) \ge -\min_{v\in \ell+S} v^2.$$

If $S = A_1 = \langle 2\rangle$ then $J_{k,m}(A_1) = J_{k,m}$ is the space of classical holomorphic Jacobi modular forms studied in the book of M. Eichler and D. Zagier [EZ]. One more function, not mentioned in [EZ], is very important for our considerations. This is the Jacobi theta-series

$$\vartheta(\tau, z) = \vartheta_{11}(\tau, z) = \sum_{m\in\mathbb{Z}} \left(\frac{-4}{m}\right) q^{m^2/8}\, \zeta^{m/2} \in J_{\frac{1}{2},\frac{1}{2}}(v_\eta^3 \times v_H)$$

which is the Jacobi form of weight $\frac{1}{2}$ and index $\frac{1}{2}$ with multiplier system v_η^3 and the binary character v_H of the Heisenberg group (see [GN4, Example 1.5]). In the last formula we put $q = \exp(2\pi i\tau)$ and $\zeta = \exp(2\pi i z)$, the Kronecker symbol $\left(\frac{-4}{m}\right) = \pm 1$ if $m \equiv \pm 1 \mod 4$ and is equal to zero for even m, v_η is the multiplier system of the Dedekind eta-function. The functional equation related to the character v_H is

$$\vartheta(\tau, z + \lambda\tau + \mu) = (-1)^{\lambda+\mu} \exp(-\pi i(\lambda^2\tau + 2\lambda z))\, \vartheta(\tau, z) \quad (\lambda, \mu \in \mathbb{Z}). \quad (7)$$

The multiplier system of $\vartheta(\tau, z)$ is obtained from the relation

$$(2\pi i)^{-1}\frac{\partial\vartheta(\tau,z)}{\partial z}\Big|_{z=0}=\sum_{n>0}\Big(\frac{-4}{n}\Big)nq^{n^2/8}=\eta(\tau)^3.$$

Therefore for any $M=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$ we have

$$\vartheta\big(\frac{a\tau+b}{c\tau+d},\,\frac{z}{c\tau+d}\big)=v_\eta^3(M)(c\tau+d)^{1/2}\exp(-\pi i\frac{cz^2}{c\tau+d})\vartheta(\tau,\,z). \qquad (8)$$

Moreover

$$\vartheta(\tau,\,z)=-q^{1/8}\zeta^{-1/2}\prod_{n\ge 1}(1-q^{n-1}\zeta)(1-q^n\zeta^{-1})(1-q^n) \qquad (9)$$

and $\vartheta(\tau,\,z)=0$ if and only if $\tau=\lambda\tau+\mu$ $(\lambda,\mu\in\mathbb{Z})$ with multiplicity 1.

The Jacobi modular forms related to $\vartheta(\tau,\,z)$ were very important in [GN4] for the construction of reflective Siegel modular forms and the corresponding Lorentzian Kac-Moody superalgebras. The next example shows the role of $\vartheta(\tau,\,z)$ in the context of this paper.

Example. *Jacobi form of singular weight for D_8.* Let us put

$$\psi_{4,\,D_8}(\tau,(z_1,\dots,z_8))=\vartheta(\tau,\,z_1)\vartheta(\tau,\,z_2)\cdot\dots\cdot\vartheta(\tau,\,z_8)\in J_{4,1}(D_8).$$

This is a Jacobi form of singular weight for D_8. The functional equations (7) and (8) give us the equations (5)–(6). Using the Dedekind η-function we can define Jacobi forms for any D_k. Let $2\le k\le 8$. We put

$$\psi_{12-k,\,D_k}(\tau,\mathfrak{z}_k)=\eta(\tau)^{24-3k}\ \vartheta(\tau,\,z_1)\dots\vartheta(\tau,\,z_k)\in J_{12-k,1}(D_k). \qquad (10)$$

Similar Jacobi forms we have for any $k>8$. The construction depends only on k modulo 8. The function $\psi_{12-k,\,D_k}$ vanishes with order one for $z_i=0$ $(1\le i\le k)$. Using the Jacobi lifting from [G2] we obtain a modular form $\mathrm{Lift}(\psi_{12-k,\,D_k})$ of weight $12-k$ with respect to $\widetilde{\mathrm{O}}^+(2U\oplus D_k(-1))$. The form $\psi_{12-k,\,D_k}$ is the first Fourier-Jacobi coefficient of $\mathrm{Lift}(\psi_{12-k,\,D_k})$. The lifting preserves the divisor of $\psi_{12-k,\,D_k}$ but the lifted form has, usually, some additional divisors. Therefore we see that using a Jacobi form of type (10) we obtain a modular form with respect to an orthogonal group whose divisor contains the union of the translations of the rational quadratic divisors defined by equation $z_i=0$ $(1\le i\le k)$. This example gives a good illustration of why the language of Jacobi forms is very useful for our considerations.

Theorem 3.2 *For $2\le k\le 8$*

$$\Delta_{12-k,D_k}=\mathrm{Lift}(\psi_{12-k,\,D_k})\in M_{12-k}(\widetilde{\mathrm{O}}^+(L(D_k)))$$

is strongly reflective. More exactly, if $k \neq 4$ then

$$\operatorname{div}_{\mathcal{D}(L(D_k))} \operatorname{Lift}(\psi_{12-k,\,D_k}) = \bigcup_{\substack{\pm v \in L(D_k)\\ v^2=-4,\ \operatorname{div}(v)=2}} \mathcal{D}_v(L(D_k)),$$

where all vectors v in the last union belong to the same $\widetilde{\mathrm{O}}^+(L(D_k))$-orbit. The divisor for $k=4$ is defined by the orbit of $2e_1 \in D_4$. If $k<8$ then $\operatorname{Lift}(\psi_{12-k,\,D_k})$ *is a cusp form.*

Remark 1. $\operatorname{Lift}(\psi_{4,\,D_8})$ and $\operatorname{Lift}(\psi_{7,\,D_5})$ are strongly reflective modular form of singular and canonical weight respectively. The modular group of the lifting is, in fact, larger. For $k \neq 4$ we have

$$\operatorname{Lift}(\psi_{12-k,\,D_k}) \in M_{12-k}(\mathrm{O}^+(L(D_k)), \tilde{\chi}) \qquad (11)$$

where the binary character $\tilde{\chi}$ of $\mathrm{O}^+(L(D_k))$ is defined by the relation

$$\tilde{\chi}(g) = 1 \ \Leftrightarrow\ g|_{L(D_k)^\vee/L(D_k)} = \mathrm{id}\,.$$

If $k=4$ then the maximal modular group of $\operatorname{Lift}(\psi_{8,\,D_4})$ is a subgroup of order 3 in $\mathrm{O}^+(L(D_4))$ (see the proof of the theorem).

Proof. We described D_k and $D_k^\vee/D_k$ in the proof of Lemma 2.1. In particular we see that $2D_8^\vee < D_8$. We denote the "half-integral" part of $D_8^\vee$ by

$$D_8^\vee(1) = \{(e_1 + \dots + e_7 \pm e_8)/2 + D_8\}. \qquad (12)$$

The Fourier expansion of $\psi_{4,\,D_8}$ has the following form

$$\psi_{4,\,D_8}(\tau, \mathfrak{z}_8) = \sum_{\substack{n\in\mathbb{Z},\ \ell\in D_8^\vee\\ 2n-(\ell,\ell)=0}} f(n,\ell)\, \exp\bigl(2\pi i(n\tau + (\ell,\mathfrak{z}_8)\bigr) =$$

$$= \sum_{\ell \in D_8^\vee(1)} \left(\frac{-4}{2\ell}\right) \exp(\pi i\bigl((\ell,\ell)\tau + 2(\ell,\mathfrak{z}_8)\bigr))$$

where

$$\left(\frac{-4}{2\ell}\right) = \left(\frac{-4}{2l_1}\right)\cdots\left(\frac{-4}{2l_8}\right). \qquad (13)$$

According to [G2, Theorem 3.1]

$$\operatorname{Lift}(\psi_{12-k,D_k})(Z) \in M_{12-k}(\widetilde{\mathrm{O}}^+(L(D_k)))$$

is a modular form with trivial character. The lifting of a Jacobi form $\phi_k(\tau,\mathfrak{z}) \in J_{k,1}(S)$ (with $f(0,0)=0$) of weight k was defined in [G2] by the formula

$$\operatorname{Lift}(\phi_k)(\tau,\mathfrak{z}_n,\omega) = \sum_{m\geq 1} m^{-1}\bigl(\phi_k(\tau,\mathfrak{z}_n)e^{2\pi i m\omega}\bigr)\ |_k\ T_-(m)$$

$$= \sum_{m\ge 1} m^{-1} \sum_{\substack{ad=m\\ b \mod d}} a^k \phi_k\Big(\frac{a\tau+b}{d},\ a\mathfrak{z}_n\Big)\, e^{2\pi i m\omega}. \tag{14}$$

According to (14)

$$\mathrm{Lift}(\phi_k)(Z) = \sum_{\substack{n,m>0,\,\ell\in S^\vee\\ 2nm-(\ell,\ell)\ge 0}} \ \sum_{d|(n,\ell,m)} d^{k-1} f(\frac{nm}{d^2}, \frac{\ell}{d})\, e(n\tau + (\ell,\mathfrak{z}) + m\omega) \tag{15}$$

where $d|(n,\ell,m)$ denotes a positive integral divisor of the vector in $U \oplus S^\vee(-1)$. For example we can calculate the Fourier expansion of the modular form of singular weight

$$\mathrm{Lift}(\psi_{4,D_8})(Z) = \sum_{\substack{n,\,m\in\mathbb{Z}_{>0}\\ \ell=(l_1,\dots,l_8)\in D_8^\vee\\ 2nm-(\ell,\ell)=0}} \ \sum_{\substack{t=2^w\\ (n,\,\ell,\,m)/t\in U\oplus D_8^\vee(1)}} t^3\sigma_3((n/t,\ell/t,m/t))$$

$$\left(\frac{-4}{2\ell/t}\right)\exp(2\pi i(n\tau + (\ell,\mathfrak{z}_8) + m\omega)) \tag{16}$$

where $t = t(n,\ell,m)$ is the maximal common power of 2 in (n,ℓ,m), i.e. $t = 2^w$, n/t and m/t are integral, $\ell/t \in D_8^\vee(1)$. Then the greatest common divisor $d = (n/t,\ell/t,m/t)$ of a vector in $U \oplus D_8(-1)$ is an odd number and $\sigma_3(d) = \sum_{a|d} a^3$.

The maximal modular group of $\mathrm{Lift}(\psi_{12-k,\,D_k})$ is, in fact, larger than $\widetilde{\mathrm{O}}^+(L(D_k))$. The orthogonal group $\mathrm{O}(D_k^\vee/D_k)$ of the finite discriminant group is of order 2 for any $k \not\equiv 4 \mod 8$. If $k \equiv 4 \mod 8$ then $\mathrm{O}(D_k^\vee/D_k) \cong S_3$. The lifting is anti-invariant under the transformation $z_1 \to -z_1$ (the reflection σ_{e_1}) inducing the non-trivial automorphism of $\mathrm{O}(D_k^\vee/D_k)$ if $k \ne 4$. Therefore, if $k \ne 4$ then $\mathrm{Lift}(\psi_{12-k,\,D_k})$ is a modular form with respect to $\mathrm{O}^+(L(D_k))$ with the character $\tilde{\chi}$ defined in (11). If $k = 4$ then the permutation of the coordinates give us only the permutation of the classes of $(e_1+e_2+e_3+e_4)/2$ and $(e_1+e_2+e_3-e_4)/2$ in $D_4^\vee/D_4$. Therefore $\mathrm{Lift}(\psi_{8,D_4})$ is modular with a binary character for a subgroup of index 3 in $\mathrm{O}^+(L(D_4))$.

To show that $\mathrm{Lift}(\psi_{12-k,D_k})$ is strongly reflective we consider its Borcherds product expansion. We can construct the Borcherds product using a Jacobi form of weight 0 in a way similar to [GN4, Theorem 2.1]. We can obtain a weak Jacobi form of weight 0 for D_8 using the so-called “minus” Hecke operator $T_-(2)$ (see (14) and [G2], page 1193). We put

$$\phi_{0,\,D_8}(\tau,\mathfrak{z}) = \frac{2^{-1}\psi_{4,D_8}|_4\, T_-(2)}{\psi_{4,D_8}} =$$

$$8\prod_{i=1}^{8}\frac{\vartheta(2\tau,2z_i)}{\vartheta(\tau,z_i)} + \frac{1}{2}\prod_{i=1}^{8}\frac{\vartheta(\frac{\tau}{2},z_i)}{\vartheta(\tau,z_i)} + \frac{1}{2}\prod_{i=1}^{8}\frac{\vartheta(\frac{\tau+1}{2},z_i)}{\vartheta(\tau,z_i)}.$$

Then $\phi_{0,\,D_8}$ is a weak Jacobi form of weight 0 and index 1. Using the Jacobi product formula (9) we obtain

$$\phi_{0,\,D_8}(\tau,\mathfrak{z}_8)=\sum_{n\ge 0,\ \ell\in D_8^\vee} c(n,\ell)\,\exp\bigl(2\pi i(n\tau+(\ell,\mathfrak{z}_8)\bigr)=$$

$$\zeta_1^{\pm 1}+\dots+\zeta_8^{\pm 1}+8+q(\dots)\qquad \text{where}\quad \zeta_i=\exp(2\pi i z_i).$$

We noted above that the Fourier coefficient $c(n,\ell)$ of weak Jacobi form $\phi_{0,\,D_8}$ depends only on the hyperbolic norm $2n-\ell^2$ and the class $\ell \mod D_8^\vee$. Moreover, if $c(n,\ell)\ne 0$ then $2n-(\ell,\ell)\ge -2$ (see Lemma 3.1 and the representation of $D_8^\vee/D_8$ above). According to the Eichler criterion, the primitive vectors in $2U\oplus D_8^\vee$ with norm equal to -1 and -2 form three $\widetilde{\mathrm{SO}}^+(L(D_8))$-orbits represented by the elements of the minimal norms in $D_8^\vee$. Therefore the q^0-term in the Fourier expansion of $\phi_{0,\,D_8}$ contains all types of the Fourier coefficients $\phi_{0,\,D_8}$ with $2n-(\ell,\ell)<0$. Using the Borcherds product construction as in Theorem 2.1 of [GN4] we obtain a modular form

$$B(\phi_{0,\,D_8})(Z)=q(\zeta)^{(1/2,\dots,1/2)}s\prod_{\substack{n,m\ge 0\\(n,\ell,m)>0}}(1-q^n(\zeta)^\ell s^m)^{c(nm,\ell)}$$

$$=\bigl(\psi_{4,D_8}(\tau,\mathfrak{z}_8)e^{2\pi i\omega}\bigr)\exp\left(-\phi_{0,D_8}(\tau,\mathfrak{z}_8)|\sum_{m\ge 1}m^{-1}T_-(m)e^{2\pi i m\omega}\right)$$

in $M_4(\widetilde{\mathrm{O}}^+(L(D_8)))$ with the trivial character of $\widetilde{\mathrm{O}}^+(L(D_8))$ where $q=\exp(2\pi i\tau)$, $s=\exp(2\pi i\omega)$ and $(\zeta)^\ell=(\zeta_1^{l_1}\zeta_2^{l_2}\dots\zeta_8^{l_8})$. Its divisors are determined by the Fourier coefficients $\zeta_i^{\pm 1}$, i.e. by the vectors $\pm e_i\in D_8$ $(1\le i\le 8)$. According to the Eichler criterion the $\widetilde{\mathrm{SO}}^+(L(D_8))$-orbit of any vector $v\in 2U\oplus D_8(-1)$ with $v^2=-4$ and $\mathrm{div}(v)=2$ is defined by $v/2 \mod D_8$. Therefore any such vector belongs to the orbit of e_1. We proved that

$$\mathrm{div}_{\mathcal{D}(L(D_8))}\,B(\phi_{0,\,D_8})(Z)=\bigcup_{\substack{\pm v\in L(D_8)\\ v^2=-4,\ \mathrm{div}(v)=2}}\mathcal{D}_v.$$

The modular projection of this divisor on $\mathcal{SM}(D_8)$ is irreducible. The formula (14) shows that the lifting preserves the divisor of type $z_i=0$ of the Jacobi form. It follows that the divisor of $\mathrm{Lift}(\psi_{4,D_8})(Z)$ contains the divisor given in Theorem 3.2. According to Koecher's principle

$$\mathrm{Lift}(\psi_{4,\,D_8})(Z)=B(\phi_{0,\,D_8})(Z)$$

because they have the same first Fourier-Jacobi coefficient.

We can also use the weak Jacobi form ψ_{4,D_8} in order to construct Borcherds products for the lattices $2U\oplus D_k(-1)$ with $2\le k\le 8$. We put

$$\phi_{0,\,D_k}(\tau,\mathfrak{z}_k)=\phi_{0,\,D_8}(\tau,\mathfrak{z}_k,0,\dots,0)=\zeta_1^{\pm 1}+\dots\zeta_k^{\pm 1}+(24-2k)+q(\dots).$$

Then $\phi_{0,\,D_k}(\tau, \mathfrak{z}_k)$ is a weak Jacobi form of weight 0 for D_k. Using the same arguments as for D_8 above we obtain

$$\mathrm{Lift}(\psi_{12-k,\,D_k})(Z) = B(\phi_{0,\,D_k})(Z).$$

If $k = 4$ then the divisor of the last function is smaller than the divisor in Theorem 3.2. It is defined by the $\widetilde{\mathrm{O}}^+(L(D_4))$-orbit of $2e_1$ (see the proof of Lemma 2.1).

If $k < 8$ then the Jacobi form $\psi_{12-k,\,D_k}$ is cuspidal. Therefore its lifting is a cusp forms because D_k is a maximal even lattice. $\square$

Let $\mathfrak{G}(F)$ be a Lorentzian Kac–Moody (super) Lie algebra of Borcherds type determined by an automorphic form F. Note that the generators and relations of this algebra are defined by the Fourier expansion of F at a zero-dimensional cusp. The Borcherds product of F determines only the multiplicities of the positive roots of this algebra. Therefore for an explicit construction of $\mathfrak{G}(F)$ one has to find the Fourier expansion of F at a cusp. This explains the importance of explicit formulae for the Fourier coefficients. We give the Fourier expansion of $\Delta_{12-k,\,D_k}$ in Corollary 3.5 below (see also (16) and (17)).

In 1996 (see [B2]) Borcherds constructed the strongly reflective automorphic discriminant Φ_4 of the moduli space on Enriques surfaces

$$\Phi_4 \in M_4(\mathrm{O}^+(U \oplus U(2) \oplus E_8(-2)),\, \chi_2)$$

where χ_2 is a binary character. The Borcherds products of Φ_4 were given in two non-equivalent cusps (see [B3, Example 13.7]). This function is called sometimes Borcherds-Enriques form. See [HM] for its applications in string theory. In the next corollary we obtain a Jacobi lifting construction of Φ_4.

Corollary 3.3 *The form* $\mathrm{Lift}(\psi_{4,\,D_8})$ *is equal, up to a constant, to the Borcherds modular form* Φ_4.

Proof. The divisor of Φ_4 in $\mathcal{D}(L_E)$ with $L_E = U \oplus U(2) \oplus E_8(-2)$ is determined by the $\mathrm{O}^+(M_E)$-orbit of a (-2)-vector $v \in U$ (see [B2]). Note that a renormalization of L ($L \to L(n)$) does not change the orthogonal group and $\mathrm{O}(L) = \mathrm{O}(L^\vee)$ for any lattice L. Therefore

$$\mathrm{O}^+(L_E) = \mathrm{O}^+(L_E^\vee(2)) = \mathrm{O}^+(U(2) \oplus U \oplus E_8(-1)) \cong \mathrm{O}^+(L(D_8))$$

because

$$U(2) \oplus E_8(-1) \cong U \oplus D_8.$$

These two hyperbolic lattices correspond to the two different 0-dimensional cusps of the modular variety $\mathrm{O}^+(L_E) \setminus \mathcal{D}(L_E)$. An arbitrary (-2)-vector of L_E becomes a reflective vector of $L_E^\vee(2)$ of length -4. Therefore the

(-2)-reflective divisor of $\mathrm{O}^+(L_E)\setminus\mathcal{D}(L_E)$ corresponds to the (-4)-reflective divisor of $\mathrm{O}^+(L(D_8))\setminus\mathcal{D}(L(D_8))$. We see that the modular forms Φ_4 and $\mathrm{Lift}(\psi_{4,\,D_8})$ have the same divisor. Therefore they are equal, up to a constant, according to Kocher's principle. $\square$

The automorphic Borcherds products related to the quasi pull-backs of Φ_4 appear in the new paper [Y] of K.-I. Yoshikawa. These modular forms Φ_V are the automorphic discriminants of the Kähler moduli of a Del Pezzo surface V of degree $1\le n\le 9$ (compare with [GN5]). The function Φ_V determines also the analytic torsion of some exceptional Calabi–Yau threefolds of Borcea-Voisin type (see [Y, Theorem 1.1]).

Corollary 3.4 *Let V be a Del Pezzo surface of degree $1\le \deg V\le 6$. The modular form Φ_V of Yoshikawa is equal, up to a constant, to the modular form $\Delta_{4+\deg V,\,D_{8-\deg V}}=\mathrm{Lift}(\psi_{4+\deg V,\,D_{8-\deg V}})$ of Theorem 3.2.*

Proof. The proof of Theorem 3.2 shows that the singular modular form $\mathrm{Lift}(\psi_{4,\,D_8})(Z)$ is the generating function for the D_8-towers of the strongly reflective modular forms of Theorem 3.2. We have

$$(2\pi i)^{-1}\frac{\partial\psi_{4,\,D_8}(\tau,\mathfrak{z}_8)}{\partial z_8}\big|_{z_8=0}=\psi_{7,\,D_7}(\tau,\mathfrak{z}_7).$$

Therefore $\mathrm{Lift}(\psi_{5,D_7})$ is the quasi pull-back (see [B1, pp. 200–201], and [GHS1, §6]) of $\mathrm{Lift}(\psi_{4,\,D_8})$ along the divisor $z_8=0$. We can continue this process. Then $\mathrm{Lift}(\psi_{6,\,D_6})$ is the quasi pull-back of $\mathrm{Lift}(\psi_{5,\,D_7})$ along $z_7=0$ and so on till $\mathrm{Lift}(\psi_{10,\,D_2})$.

The Yoshikawa modular forms Φ_V for $\deg V\ge 1$ also constitute a similar tower with respect to the quasi pull-backs based on the Borcherds form Φ_4 (see [Y, §6]). To finish the proof we use again Koecher's principle. $\square$

Theorem 3.2 and (14) give the formula for the Fourier expansion of $\mathrm{Lift}(\psi_{12-k,\,D_k})$ and Φ_V. For any integral $3m>0$ we put

$$\eta(\tau)^{3m}=\sum_{n>0}\tau_{3m}(\frac{n}{8})\,q^{n/8}.$$

We put $\tau_0(n)=1$ if and only if $n=1$.

Corollary 3.5 *We have the following Fourier expansion*

$$\mathrm{Lift}(\psi_{12-k,\,D_k})(Z)=\sum_{\substack{n,\,m\in\mathbb{Z}_{>0}\\ \ell=(l_1,\dots,l_n)\in D_k^\vee}}\ \sum_{d\mid(n,\ell,m)}d^{11-k}\tau_{24-3k}(\frac{2nm-\ell^2}{2d^2})$$
$$\left(\frac{-4}{2\ell/d}\right)\exp(2\pi i(n\tau+(\ell,\mathfrak{z}_k)+m\omega))$$

where $d\mid(n,\,\ell,\,m)$ is an integral divisor in $U\oplus D_k(-1)^\vee$ and we use the notation (13).

Proof. In the notation above we have $\psi_{12-k,\,D_k}(\tau,\mathfrak{z}_k)=$

$$\sum_{n\in\mathbb{N},\,\ell\in D_k^\vee(1)} \tau_{24-3k}(\frac{2n-\ell^2}{2})\left(\frac{-4}{2\ell}\right)\exp(2\pi i(n\tau+(\ell,\mathfrak{z}_k))).$$

Using (15) we obtain the formula of the corollary. The Kronecker symbol $\left(\frac{-4}{2l_i/d}\right)=0$ if $2l_i/d$ is pair. Therefore the vector ℓ/d belongs in fact to the odd part of $D_k^\vee$ (see (12)) and we can divide any vector by its maximal common power of 2 in (n,ℓ,m). Then we make a summation on odd common divisors like in (16).

We note that $\eta(\tau)^3$ has elementary formula for its Fourier coefficients: $\tau_3(n/8)=\left(\frac{-4}{N}\right)N$ if and only if $n=N^2$. Therefore we have

$$\mathrm{Lift}(\phi_{5,\,D_7})(\tau,\mathfrak{z}_7,\omega)=\sum_{N\geq 1}\ \sum_{\substack{2nm-\ell^2=\frac{N}{4}\\ n,m\in\mathbb{N},\,\ell\in D_7^\vee}}\ \sum_{d|(n,\ell,m)}$$

$$N\left(\frac{-4}{2\ell/d}\right)\exp(2\pi i(n\tau+(\ell,\mathfrak{z}_7)+m\omega)). \tag{17}$$

□

The modular forms Φ_V determine some automorphic Lorentzian Kac-Moody super Lie algebras related to Del Pezzo surfaces. The formula of Corollary 3.5 gives us the generating function of the imaginary simple roots of these algebras.

Remark 2. The quasi pull-back of $\Delta_{10,\,D_2}=\Phi_V$ $(\deg V=6)$ is the Siegel modular form $\Delta_{11}\in S_{11}(\Gamma_2)$ (see [GN4, (3.11)]) where Γ_2 is the paramodular group of type $(1,2)$. Note that (see [GN4, Lemma 1.9])

$$\Gamma_2/\{\pm\,\mathrm{id}\}\cong\widetilde{\mathrm{SO}}^+(L(D_1))\qquad\text{where}\quad D_1=\langle 4\rangle.$$

Then $\Delta_{11}\left(\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right)=\mathrm{Lift}(\eta(\tau)^{21}\vartheta(\tau,2z))$ is strongly reflective (see [GN4, Example 1.15 and (3.11)]). Its divisor contains two irreducible components $\{z=0\}$ and $\{z=1/2\}$. The quasi pull-back of the Siegel modular form Δ_{11} along $z=0$ is equal to the product of the Ramanujan modular forms $\Delta_{12}(\tau)\Delta_{12}(2\omega)$ (in the Γ_2-coordinates).

4 A_2-tower of strongly reflective modular forms

We start with a useful general fact.

Lemma 4.1 *Let $\phi_i(\tau,\mathfrak{z}_i)\in J_{k_i,m}(S_i)$ where S_i is a positive definite integral lattice. Then*

$$\phi_1(\tau,\mathfrak{z}_1)\cdot\phi_2(\tau,\mathfrak{z}_2)\in J_{k_1+k_2,m}(S_1\oplus S_2).$$

The proof of the lemma follows directly from the definition.

In order to construct Jacobi modular forms for D_n we used the Jacobi form $\vartheta(\tau, z)$ which is the denominator function of the affine Lie algebra $\hat{A}_1$. In this section we use the denominator function of the affine Lie algebra $\hat{A}_2$. Let $A_2 = \mathbb{Z}a_1 + \mathbb{Z}a_2$ where a_1 and a_2 are the simple roots of A_2. We can rewrite this lattice in the Euclidian basis (e_1, e_2, e_3)

$$A_2 \otimes \mathbb{C} = z_1' e_1 + z_2' e_2 + z_3' e_3$$

where

$$z_1' = z_1, \quad z_2' = z_2 - z_1, \quad z_3' = -z_2.$$

The denominator function of the affine Kac–Moody algebra $\hat{A}_2$ is associated to the holomorphic Jacobi form of singular weight 1 with character v_η^8 of order 3

$$\Theta(\tau, z_1, z_2) = \frac{1}{\eta(\tau)}\vartheta(\tau, z_1)\vartheta(\tau, z_2 - z_1)\vartheta(\tau, z_2) \in J_{1,1}(A_2; v_\eta^8)$$

(see [Ka] and [De]). Therefore using Lemma 4.1 we can define three holomorphic Jacobi forms with trivial character

$$\begin{aligned}
\psi_{9,\,A_2}(\tau, z_1, z_2) &= \eta^{16}(\tau)\Theta(\tau, z_1, z_2) \in J_{9,1}^{(cusp)}(A_2),\\
\psi_{6,\,2A_2}(\tau, z_1, \ldots, z_4) &= \eta^{8}(\tau)\Theta(\tau, z_1, z_2)\Theta(\tau, z_3, z_4) \in J_{6,1}^{(cusp)}(2A_2),\\
\psi_{3,\,3A_2}(\tau, z_1, \ldots, z_6) &= \Theta(\tau, z_1, z_2)\Theta(\tau, z_3, z_4)\Theta(\tau, z_5, z_6) \in J_{3,1}(3A_2).
\end{aligned}$$

The Jacobi lifting construction gives three modular forms of orthogonal type. In particular we obtain

$$\mathrm{Lift}(\psi_{3,\,3A_2}) \in M_1(\widetilde{\mathrm{O}}^+(L(3A_2)))$$

of singular weight with trivial character. For $n = 2$ we obtain a cusp form of canonical weight. The divisor of the Jacobi form induces a divisor of the lifting. Note that $z_i = 0$ is the hyperplane of the reflection σ_{λ_i} where λ_i is a fundamental weight of A_2 and

$$A_2^\vee / A_2 = \{\, 0,\ \lambda_1,\ \lambda_2 \mod A_2\}$$

where the fundamental weights are vectors of minimal length $\lambda_i^2 = 2/3$ in the corresponding A_2-classes. Then $3\lambda_i \in A_2(-1)$ is reflective (-6)-vector, i.e. one of the G_2-roots of the lattice A_2.

Theorem 4.2 *Let $k = 1$, 2 or 3. Then*

$$\Delta_{12-3k,\,kA_2} = \mathrm{Lift}(\psi_{12-3k,\,kA_2}) \in M_{12-3k}(\widetilde{\mathrm{O}}^+(L(kA_2)))$$

is strongly reflective and

$$\operatorname{div}_{\mathcal{D}(L(kA_2))} \operatorname{Lift}(\psi_{12-3k,\,kA_2}) = \bigcup_{\substack{\pm v \in L(kA_2)\\ v^2=-6,\ \operatorname{div}(v)=3}} \mathcal{D}_v(L(kA_2)).$$

For $k = 1$ and 2 the lifting is a cusp form.

Proof. The proof is similar to the proof of Theorem 3.2. It is enough to find a Borcherds product for the singular modular form $\operatorname{Lift}(\psi_{3,\,3A_2})$. To construct a weak Jacobi form of weight 0 we again use the Hecke operator $T_-(2)$ of the Jacobi lifting. We put

$$\phi_{0,\,3A_2}(\tau, \mathfrak{z}) = \frac{2^{-1}\psi_{3,\,3A_2}|_3 T_-(2)}{\psi_{3,\,3A_2}}$$

where

$$2^{-1}\psi_{3,\,3A_2}|\,T_-(2) = 4\psi_{3,\,3A_2}(2\tau, 2\mathfrak{z}_6) + \frac{1}{2}\psi_{3,\,3A_2}(\frac{\tau}{2}, \mathfrak{z}_6) + \frac{1}{2}\psi_{3,\,3A_2}(\frac{\tau+1}{2}, \mathfrak{z}_6).$$

Analyzing the divisor of $\psi_{3,\,3A_2}$ we see that $\phi_{0,\,3A_2} \in J_{0,1}^{(week)}(3A_2)$. Moreover the direct calculation shows that

$$\phi_{0,\,3A_2}(\tau, \mathfrak{z}) = \sum_{\substack{n \ge 0\\ \ell \in U \oplus 3A_2(-1)^\vee}} c(n, \ell) \exp(2\pi i(n\tau + (\ell, \mathfrak{z}_6))) =$$

$$6 + \sum_{i=1,3,5} (\zeta_i^{\pm 1} + \zeta_{i+1}^{\pm 1} + (\zeta_i \zeta_{i+1}^{-1})^{\pm 1}) + q(\dots)$$

where $\zeta_i = \exp(2\pi i z_i) = \exp(2\pi i(\mathfrak{z}_6, \lambda_i))$ and λ_i are the fundamental weights of the corresponding copies of A_2. According to Lemma 3.1, in order to obtain all Fourier coefficients $c(n, \ell) \neq 0$ with $2n - \ell^2 < 0$ one has to check only coefficients with $2n - \ell^2 \geq -2$. The sum over $i \in \{1, 2, 3\}$ in the formula above contains all such coefficients. Therefore the Borcherds product $B(\phi_{0,\,3A_2})$ is of weight $c(0, 0)/2 = 3$ with divisors of order 1 along all $\widetilde{\mathrm{O}}^+(L(3A_2))$-orbits of the (-6)-vectors $\pm\lambda_i$, $\pm\lambda_{i+1}$ and $\pm(\lambda_i - \lambda_{i+1})$ $(i \in \{1, 2, 3\})$. Using Koecher's principle as in the proof of Theorem 3.2 we find that

$$\operatorname{Lift}(\psi_{3,\,3A_2}) = B(\phi_{0,\,3A_2}).$$

Therefore $\operatorname{Lift}(\psi_{3,\,3A_2})$ is strongly reflective. To find a weak Jacobi for A_2 and $2A_2$ we put

$$\phi_{0,\,2A_2}(\tau, \mathfrak{z}_4) = \phi_{0,\,3A_2}(\tau, \mathfrak{z}_4, 0, 0) = 12 + \sum_{i=1,3} (\zeta_i^{\pm 1} + \zeta_{i+1}^{\pm 1} + (\zeta_i \zeta_{i+1}^{-1})^{\pm 1}) + q(\dots)$$

and

$$\phi_{0,\,A_2}(\tau,\mathfrak{z}_2)=\phi_{0,\,3A_2}(\tau,\mathfrak{z}_2,0,0,0,0)=18+\zeta_1^{\pm 1}+\zeta_2^{\pm 1}+(\zeta_1\zeta_2^{-1})^{\pm 1}+q(\dots).$$

Therefore

$$\mathrm{Lift}(\psi_{6,\,2A_2})=B(\phi_{0,\,2A_2})\quad\text{and}\quad \mathrm{Lift}(\psi_{9,\,A_2})=B(\phi_{0,\,A_2}).$$

The last modular form of weight 9 was constructed in [De, Proposition 4.2]. The method of construction of $\phi_{0,\,A_2}$ in [De] was different. □

Remark. The strongly reflective form $\mathrm{Lift}(\psi_{12-3n,\,nA_2})$ (for $n=2,\ 3$) is invariant with respect to permutations of any two copies of A_2. A permutation does not change the branch divisor. It follows that the cusp form $\mathrm{Lift}(\psi_{6,\,2A_2})$ determines, in fact, two modular varieties of Kodaira dimension 0. These are the variety of Theorem 2.2 and the variety for the double extension of the stable orthogonal group $\widetilde{\mathrm{SO}}^+(2U\oplus 2A_2(-1))$ corresponding to the permutation of the two copies of $A_2(-1)$.

5 A_1-tower of strongly reflective modular forms

In this section we use the Jacobi theta-series as Jacobi modular forms of half-integral index.

Theorem 5.1 *There exist four strongly reflective modular forms for $L(nA_1)$ with $n=1,\ 2,\ 3$ and 4:*

$$\Delta_{6-n,\,nA_1}\in M_{6-n}(\widetilde{\mathrm{SO}}^+(L(nA_1)),\chi_2)$$

where $\chi_2:\widetilde{\mathrm{SO}}^+(L(nA_1))\to\{\pm 1\}$. Moreover

$$\mathrm{div}_{\mathcal{D}(L(nA_1))}\,\Delta_{6-n,\,nA_1}=\bigcup_{\substack{\pm v\in L(nA_1)\\ v^2=-2,\ \mathrm{div}(v)=2}}\mathcal{D}_v(L(nA_1))$$

and $\Delta_{6-n,\,nA_1}$ is a cusp form if $n<4$.

Proof. In the proof we construct these modular forms as Borcherds products. As in the proof of Theorem 3.2 and Theorem 4.2, the main function of this $4A_1$-tower is the modular form $\Delta_{2,\,4A_1}$ of singular weight. We put

$$\psi_{2,\,4A_1}(\tau,\mathfrak{z}_4)=\vartheta(\tau,z_1)\vartheta(\tau,z_2)\vartheta(\tau,z_3)\vartheta(\tau,z_4).$$

This is a Jacobi form of weight 2 and index $\frac{1}{2}$ with character $v_\eta^{12}\times v_H$ of order 2 where v_H is the binary character of the Heisenberg group $H(4A_1)$. We can define the following weak Jacobi form of weight 0

$$\phi_{0,\,4A_1}(\tau,\mathfrak{z}_4)=\frac{3^{-1}\psi_{2,\,4A_2}|_2T_-(3)}{\psi_{2,\,4A_1}}\in J_{0,1}^{(weak)}(4A_1)$$

where

$$3^{-1}\psi_{2,\,4A_1}|_2T_-(3) = 3\psi_{3,\,3A_2}(3\tau, 3\mathfrak{z}_4) + \frac{1}{3}\sum_{b=0}^{2}\psi_{2,\,4A_1}(\frac{\tau+b}{3}, \mathfrak{z}_4).$$

The straightforward calculation shows that

$$\phi_{0,\,4A_1}(\tau, \mathfrak{z}_4) = 4 + \sum_{i=1}^{4}\zeta_i^{\pm 1} + q(\dots).$$

We put $\tilde{\phi}_{0,\,4A_1}(Z) = \phi_{0,\,4A_1}(\tau, \mathfrak{z}_4)e^{2\pi i\omega}$. In terms of this Jacobi form the Borcherds product is given by the following formula (see [GN4, (2.7)])

$$B(\phi_{0,\,4A_1})(\tau, \mathfrak{z}_4, \omega) = \bigl(\psi_{2,\,4A_1}(\tau, \mathfrak{z}_4)e^{\pi i\omega}\bigr)\exp\Bigl(-\sum_{m\ge 1} m^{-1}\tilde{\phi}_{0,4A_1}|T_-(m)(Z)\Bigr).$$

This formula shows that $\Delta_{2,\,4A_1} = B(\phi_{0,\,4A_1})$ is a modular form of weight 2 with respect to $\widetilde{\mathrm{SO}}^+(L(4A_1))$ and with divisor described in the theorem. The coefficient before the exponent is the first Fourier-Jacobi coefficient of $B(\phi_{0,\,4A_1})$. Therefore the character of $B(\phi_{0,\,4A_1})$ is the binary character induced by the character of $\psi_{2,\,4A_1}(\tau, \mathfrak{z}_4)$. As in the proof of Theorem 3.2 and Theorem 4.2 of this section we put

$$\phi_{0,\,nA_1}(\tau, \mathfrak{z}_n) = \phi_{0,\,4A_1}(\tau, \mathfrak{z}_n, 0, \dots, 0) \qquad (1 \le n < 4).$$

It gives the three other strongly reflective modular forms. The last function of this $4A_1$-tower is the Siegel modular form Δ_5, which is the Borcherds product defined by the Jacobi form

$$\phi_{0,\,A_1}(\tau, z) = \phi_{0,1}(\tau, z) = \zeta + 10 + \zeta^{-1} + q(\dots) \in J_{0,1}^{(weak)}$$

(see [GN1]). □

It is possible to get a Jacobi lifting construction of the strongly reflective modular forms of the last theorem. We can prove that

$$\Delta_{6-n,\,nA_1} = \mathrm{Lift}\bigl(\eta(\tau)^{12-3n}\prod_{i=1}^{n}\vartheta(\tau, z_i)\bigr).$$

Here we take a Jacobi lifting with a character similar way to [GN4, Theorem 1.12]. This lifting gives the elementary formula for the Fourier coefficients of $\Delta_{2,\,4A_1}$ and $\Delta_{3,\,3A_1}$ similar to (16). For example we have the following Fourier expansion of the modular form of singular weight

$$\mathrm{Lift}(\psi_{2,\,4A_1})(Z) = \sum_{\substack{\ell=(l_1,\dots,l_4)\\ l_i\equiv\frac{1}{2} \bmod \mathbb{Z}}}$$

$$\sum_{\substack{n,\,m\in\mathbb{Z}_{>0}\\ n\equiv m\equiv 1 \bmod \mathbb{Z}\\ nm-(\ell,\ell)=0}} \sigma_1((n,\ell,m))\left(\frac{-4}{2l_1}\right)\dots\left(\frac{-4}{2l_4}\right)\exp(\pi i(n\tau+2(\ell,\mathfrak{z}_4)+m\omega)).$$

See details in the forthcoming paper of F. Cléry and V. Gritsenko "*Jacobi modular forms and root systems*".

Remark. The 14 strongly reflective modular forms constructed in Theorems 3.2, 4.2 and 5.1 determine Lorentzian Kac–Moody super Lie algebras of Borcherds type in a way described in [GN1]–[GN4]. The details of this construction and many other examples will appear in our forthcoming paper with V. Nikulin.

Conclusion. To finish this paper we would like to characterize the three series of the strongly reflective modular forms considered above and to formulate a conjecture on similar modular forms. To this aim we come back to the first two examples of §1. The divisor of the Borcherds form Φ_{12} is defined by all (-2)-roots in $II_{2,26}$. This is the irreducible reflective divisor in $\mathcal{F}_{II_{2,26}}(\mathrm{O}^+(II_{2,26}))$. For the Igusa modular forms the situation is different. The divisor of Δ_{35} is the branch divisor of the Siegel modular threefold

$$\pi:\mathbb{H}_2\to \mathrm{Sp}_2(\mathbb{Z})\setminus\mathbb{H}_2\cong \mathcal{F}_{2U\oplus A_1(-1)}(\mathrm{SO}^+(2U\oplus A_1(-1)))$$

containing two irreducible components. The first one $\pi(\mathcal{D}_{-2}(1))$ is defined by the (-2)-vectors with divisor 1. The second one $\pi(\mathcal{D}_{-2}(2))$ is generated by the (-2)-vectors with divisor 2. They are the Humbert modular surfaces of determinant 4 and 1 respectively. The divisor of the Igusa form $\Delta_5\left(\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right)=\mathrm{Lift}(\eta(\tau)^9\vartheta(\tau,z))$ coincides with $\pi(\mathcal{D}_{-2}(2))$. This is the simplest divisor of the Siegel threefold $\pi(\{z=0\})$. The modular form Δ_{35} is not a Jacobi lifting. Its first Fourier-Jacobi coefficient is zero and the second one is equal to $\eta(\tau)^{69}\vartheta(\tau,2z)$. See [GN2] where the Borcherds product of Δ_{35} was constructed. Moreover Δ_{35} can be considered as a "baby" function of Φ_{12} (see our forthcoming paper mentioned in Remark 2 of §3). We may say that the fourteen strongly reflective modular forms constructed in §3–§5 are similar to Δ_5. Each of them is the Jacobi lifting of its first Fourier–Jacobi coefficient and its divisor is the simplest divisor of the corresponding modular variety. For the modular forms of the $4A_1$-tower, the simplest divisor is generated by the (-2)-vectors with divisor 2. For the D_8-tower, the divisor is generated by the (-4)-reflective vectors, and for the $3A_2$-tower, it is generated by the (-6)-reflective vectors of divisor 3. All these divisors are complementary to the divisor defined by the (-2)-roots with divisor one.

We remind the following general fact. Let L be a non-degenerate even integral lattice and $h\in L$ be a primitive vector with $h^2=2d$. If L_h is the orthogonal complement of h in L then

$$|\det L_h|=\frac{|2d|\cdot|\det L|}{\mathrm{div}(h)^2}.$$

Therefore, $\det L_h$ for the reflective vectors considered above is smaller than $\det L_r$ for a (-2)-root r with $\mathrm{div}(r) = 1$.

There is the second explanation why these divisors are simpler. The divisor $\pi(\mathcal{D}_v)$, where π is a modular projection, is a modular variety of orthogonal type. For reflective vectors (σ_v or $-\sigma_v$ is in Γ) they form the reflective obstruction to extending of pluricanonical forms to a compact model (see the proof of Theorem 1.5). A numerical measure of this obstruction is given by the Hirzebruch–Mumford volume of $\pi(\mathcal{D}_v)$. This volume was calculated explicitly in [GHS4] for arbitrary indefinite lattice. We can say that the divisor $\pi(\mathcal{D}_v)$ is simpler if its Hirzebruch–Mumford volume is smaller.

We would like to formulate a conjecture related to the modular forms of this paper. Let L be a lattice of signature $(2, n)$ with $n \geq 3$. We assume that the branch divisor of the modular variety $\mathcal{F}_L(\Gamma)$ has several components. We suppose that there exists a strongly reflective modular forms F whose divisor is equal to the simplest reflective divisor. *We conjecture that F could be constructed as an additive (Jacobi) lifting.*

Note that the existence of a strongly reflective modular form implies a strong condition on the lattice (see [GN3]). The Weyl group W of the hyperbolic root system related to the simplest divisor should be arithmetic (elliptic or parabolic in the sense of [GN3]) and the root system admits a Weyl vector.

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V.A. Gritsenko
Université Lille 1
Laboratoire Paul Painlevé
F-59655 Villeneuve d'Ascq, Cedex
France
`Valery.Gritsenko@math.univ-lille1.fr`