STATIONARY SETS FOR THE WAVE EQUATION IN CRYSTALLOGRAPHIC DOMAINS

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Abstract. Let W be a crystallographic group in $\mathbb{R}^n$ generated by reflections and let Ω be the fundamental domain of W. We characterize stationary sets for the wave equation in Ω when the initial data is supported in the interior of Ω. The stationary sets are the sets of time-invariant zeros of nontrivial solutions which are identically zero at $t = 0$.

We show that, for these initial data, the $(n-1)$-dimensional part of the stationary sets consists of hyperplanes that are mirrors of a crystallographic group $\tilde{W}$, $W < \tilde{W}$. This part comes from a corresponding odd symmetry of the initial data.

In physical language, the result is that if the initial source is localized strictly inside of the crystalline Ω then unmovable interference hypersurfaces can be only faces of a crystalline substructure of the original one.

1. Formulation of the problem and main results

Let Ω be a convex bounded polyhedron in $\mathbb{R}^n$, which is the fundamental domain of a crystallographic group, W, that is generated by orthogonal reflections around finite number of hyperplanes.

Let us consider the following initial boundary value problem for the wave equation in Ω with Dirichlet boundary conditions:

$$\begin{aligned}
&(\partial_t^2 - \Delta)u = 0 \quad \text{in} \quad \Omega, \quad u = u(x,t),\ x \in \Omega,\ t > 0,\\
&u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0,\\
&u(x,0) = 0, \quad x \in \Omega,\\
&\partial_t u(x,0) = f(x),\ x \in \Omega.
\end{aligned} \tag{1.1}$$

Define the stationary set $S_\Omega[f]$ as

$$S_\Omega[f] = \{x \in \Omega : u(x,t) = 0, \quad \forall\, t > 0\}, \quad u \not\equiv 0. \tag{1.2}$$

Thus, $S_\Omega[f]$ is the set of all points in Ω where the solution $u(x,t)$ to (1.1) vanishes for all time $t > 0$ assuming u is identically zero at $t = 0$.

For a separable solution to (1.1), $u(x,t) = \sin \lambda t \cdot \varphi_\lambda(t)$, where φ_λ is the eigenfunction of the Laplacian, $\Delta\varphi_\lambda = -\lambda^2\varphi_\lambda$, the stationary set $S_\Omega[f]$ coincides with the nodal set $\varphi_\lambda^{-1}(0)$. Characterizing nodal sets is a well-known and difficult problem. There is a rich literature on this and we will mention here only some results relating to our subject. The analytic structure of nodal sets is quite well understood: they consist of $(n-1)$-dimensional smooth (real-analytic) submanifolds and closed analytic sets of lower dimension (see, e.g. [6, 11, 13, 14, 7, 15] and others).

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On the other hand, understanding global geometry of a single nodal set is a very difficult problem (see, e.g. [16] where the question about existence of closed nodal lines for the second eigenvalue is answered). Nevertheless, and this is the main point of our paper, *joint* nodal sets in certain situations can be completely described geometrically.

More precisely, we study the case when the stationary sets are intersections of infinitely many nodal sets. This happens, in particular, when the initial velocity f in the wave equation has support strictly in the interior of the domain. It might be expected that in this case the $(n-1)$-dimensional component, if it is nonempty, has a precisely determined geometry.

Indeed, our main result states that the stationary hypersurfaces in crystallographic domains $\Omega = \mathbb{R}^n/W$, where the W is a crystallographic group generated by reflections, are hyperplanes constituting a crystallographic substructure in Ω. Here is a more precise formulation. We will say $A \Subset B$ if and only if A is a compact subset of the interior of B, $\operatorname{int} B$.

Theorem 1.1. *Let Ω be a crystallographic domain in $\mathbb{R}^n$ generated by the reflection group, W, and assume $f \in C(\Omega)$, $f \neq 0$, and* $\operatorname{supp} f$ *is a compact subset of* $\operatorname{int}\Omega$, $\operatorname{supp} f \Subset \Omega$. *Then the stationary set $S_\Omega[f]$* (1.2) *is the union*

$$S_\Omega[f] = S_0 \cup V,$$

where S_0, assuming $S_0 \neq \emptyset$, is the union of cross-sections of Ω by hyperplanes and V is an analytic set in Ω, $\dim V < n-1$.

Moreover, there exists a crystallographic group $\tilde{W}$ generated by reflections, W is a subgroup (of finite index) of $\tilde{W}$, such that S_0 defines a partition of Ω into a finite number of fundamental polyhedra of the group $\tilde{W}$.

In this case, the initial data f must be skew-symmetric with respect to reflections through the hyperplanes in S_0.

Let us briefly comment on the result. Brünning [9] characterized the compact sets in the plane with piecewise smooth boundary with the following property: there are an infinite number of eigenfunctions of the Laplacian on the set with common nodal curves that are line segments. His result shows that the only planar polygons with this property are rectangles, equilateral triangles and triangles with the angles $\pi/4, \pi/4, \pi/2$ and $\pi/6, \pi/3, \pi/2$ (and sets made up from them). Note that the listed polygons are just fundamental domains of crystallographic groups generated by reflections. Our result for $n = 2$ states that *common* nodal lines for infinite families of eigenfunctions in these polygons are *always* straight lines, provided the eigenfunctions appear as projections on eigenspaces of a function f with $\operatorname{supp} f \Subset \operatorname{int}\Omega$. Note that, in general, for these polygons, nodal curves for individual eigenfunctions can be very complicated with many nonlinear curves, including closed curves.

This paper continues a series of works by the authors on stationary sets for the wave equation with compactly supported initial data. For the problem (1.1) with $\Omega = \mathbb{R}^2$ and $f \in C_c(\mathbb{R}^2)$ (continuous functions with compact support), the stationary sets were completely described in [2]. They have reflection symmetry and consist, up to finite sets, of straight lines constituting a Coxeter system (equiangular configuration of straight lines through one point). In higher dimensions, $n > 2$, the stationary sets $S[f]$ in $\mathbb{R}^n$ were described for the case of finitely supported distributions f [4], and the general case of compactly supported initial data f is still open, but we have conjectured [3] that the result is analogous to the plane case, namely, the $(n-1)-$dimensional parts of stationary sets are cones. Articles such as [1, 5] provide insight into the general case.

The general idea of the proof of Theorem 1.1 is related to results from [2, 3, 4] and is the following. The first step is to extend the problem to the whole space $\mathbb{R}^n$ by reflections around the faces of the fundamental polyhedron Ω which results in a tiling of $\mathbb{R}^n$. Then we relate the stationary sets to the kernel of a spherical Radon type transform. A crucial step is a local symmetry principle that we derive from the microlocal analysis of the spherical transform. This and arguments of irrational wrapping type, help us analyze W-orbits of tangent planes to stationary surfaces and prove that the hypersurfaces are hyperplanes. The final conclusion follows from the reflection principle for the wave equation.

2. Stationary sets and the spherical means

Consider the tiling of $\mathbb{R}^n$ resulting from reflections of the fundamental polyhedron Ω through its faces and extend f to $\mathbb{R}^n$ by odd symmetry: $f(\sigma_1 \dots \sigma_k x) = (-1)^k f(x)$, $x \in \Omega$, where each element $\sigma_i \in W$ is a reflection. It easily follows from the definition of fundamental domain that the extended function is well defined. Let us call the extended function the *W-extension of f*.

Using this extension, from now on, we assume that f is defined in $\mathbb{R}^n$ and is skew-symmetric with respect to any reflection $\sigma \in W$, $f \circ \sigma = -f$. Let us consider the following initial value problem in $\mathbb{R}^n$:

$$
\begin{aligned}
&(\partial_t^2 - \Delta)u = 0, \quad u = u(x,t),\ x \in \mathbb{R}^n,\ t > 0,\\
&u(x,0) = 0, \quad x \in \mathbb{R}^n,\\
&\partial_t u(x,0) = f(x), \quad x \in \mathbb{R}^n,
\end{aligned}
\tag{2.1}
$$

where $f \in C(\mathbb{R}^n)$ and f has the properties:

1) $C \cap \operatorname{supp} f \Subset \operatorname{int} C$ for any open cell C of the group W (i.e., $C = w(\Omega)$ for some $w \in W$).
2) $f \circ \sigma = -f$ for any reflection $\sigma \in W$.

By the symmetry principle for the wave equation, the solution $u(x,t)$ to (2.1) possesses the same symmetry $u(\sigma x, t) = -u(x,t)$, $x \in \mathbb{R}^n$, $t > 0$, for any reflection $\sigma \in W$. Moreover by 1) and 2), the restriction of $u(x,t)$ to the original polyhedron Ω solves the problem (1.1) with the initial data $f\big|_\Omega$.

Now introduce the spherical transform

$$
Rf(x,r) = \int_{S(x,r)} f(y)\,dA(y),
\tag{2.2}
$$

where $S(x,r) = \{y \in \mathbb{R}^n : \|y - x\| = r\}$ and $dA(y)$ is the normalized area measure on the sphere $S(x,r)$.

Lemma 2.1. *The following identity holds:*

$$
S[f] = \{x \in \mathbb{R}^n : Rf(x,r) = 0, \quad \forall\ r > 0\}.
$$

The proof of the lemma is given, for instance, in [3] and directly follows from the Kirchoff-Poisson formula for the solution $u(x,t)$ and from the invertibility of the resulting generalized Volterra-Abel integral equation.

Lemma 2.2. *Let* $f = \sum_{k=1}^{\infty} c_k \varphi_k$ *be the Fourier decomposition of* $f\big|_{\Omega}$ *in the space* $L^2(\Omega)$ *into the orthonormal basis of Dirichlet eigenfunctions* φ_k, $(\Delta + \lambda_k^2)\varphi_k = 0$, $\lambda_k \neq 0$. *Then* $S[f] = \bigcap_{k=0, c_k \neq 0}^{\infty} \tilde{\varphi}_k^{-1}(0)$, *where* $\tilde{\varphi}_k$ *is the* W*-extension of* φ_k *to* $\mathbb{R}^n$. *Furthermore, each* $\tilde{\varphi}_k$ *is real-analytic.*

Proof. Note that the extended functions $\tilde{\varphi}_k$ are (smooth) eigenfunctions for the Laplace operator in $\mathbb{R}^n$, because they vanish on the faces of the polyhedra $w\Omega, w \in W$ and the normal derivatives in adjacent polyhedra agree on the common faces.

The solution $u(x,t)$ to (2.1) is given by the series

$$u(x,t) = \sum_{k=0}^{\infty} c_k \frac{\sin \lambda_k t}{\lambda_k} \tilde{\varphi}_k(x) \tag{2.3}$$

which converges in L^2 on compact subsets of $\mathbb{R}^n$. It follows that $u(x,t) = 0$ for all $t > 0$ is equivalent to $\tilde{\varphi}_k(x) = 0$, $k \in \mathbb{N}$, $c_k \neq 0$. □

3. Local symmetry principle (the support theorem)

The symmetry principle for the wave equation says that if a solution $u(x,t)$ vanishes on a hyperplane S for all $t > 0$, then $u(x,t)$ and, in particular, the initial data $f(x) = \partial_t u(x,0)$ are odd with respect to the reflection through S.

If the zero hypersurface S is not a hyperplane, this strong symmetry principle does not hold. Nevertheless, we will show that a weaker symmetry principle holds in the nonlinear case as well.

Definition 3.1. Let Π be a hyperplane in $\mathbb{R}^n$ and $a \in \Pi$. We say that a set $K \subset \mathbb{R}^n$ is *locally symmetric* with respect to the pair (Π, a) (or (Π, a)-*locally symmetric*) if whichever a point $x_0 \in K$ gives a local minimum of the distance function $K \ni x \to \operatorname{dist}(x,a) = \|x - a\|$, the Π-mirror point x_0^* belongs to K.

Note that this definition of local symmetry does not imply K is symmetric about Π even near x_0, only that if x_0 is a boundary point of K and K is locally to one side of $S(a, \|x_0 - a\|)$ near x_0, then the reflection of x_0 in Π is also in K.

The following theorem, which is equivalent to Theorem 3.2 in [4], is the key microlocal result. Throughout the article, if S is a manifold and $a \in S$ then $T_a(S)$ and $N_a(S)$ will denote tangent and normal spaces, correspondingly to S at a, and $T_a^{\circ}(S) = a + T_a(S)$ will denote the affine tangent plane.

Theorem 3.2 (Theorem 3.2 [4]). *Let* S *be a real-analytic hypersurface in* $\mathbb{R}^n$ *and* $f \in \mathcal{D}'(\mathbb{R}^n)$ *be a distribution in* $\mathbb{R}^n$ *such that* $Rf(x,r) = 0$ *for all* $x \in S$ *and* $r > 0$. *Then, for each* $a \in S$ $\operatorname{supp} f$ *is* $(T_a^{\circ}(S), a)$*-locally symmetric.*

This theorem is a generalization of a theorem of Courant and Hilbert ([12], p. 699 *ff.*) which is equivalent, due to Lemma 2.1, to the reflection principle for the wave equation and states that if $S = \Pi$ is a hyperplane and $Rf(a,r) = 0$ for all $a \in \Pi$ and $r > 0$, then f is odd about Π and so $\operatorname{supp} f$ is *globally* symmetric about Π. For Theorem 3.2, S can be curved, and in this case $\operatorname{supp} f$ is only *locally* symmetric about the tangent planes to S in the weak sense of Definition 3.1.

Since the statement of this theorem is slightly different in [4], let us outline the main ideas of the proof.

The key point is the regularity property of the operator R which implies cancellation of analytic wave front sets at points where a sphere, centered at a point

$a \in S$, touches the support of f. In turn, the cancellation is possible only at mirror points with respect to the tangent plane $T_a^\circ(S)$ and this implies that the point which is mirror to the touching point must be in the support as well, and the local symmetry follows.

More precisely, let x_0 give a local minimum for the distance from a to $\operatorname{supp} f$ and let $r_0 = \|x_0 - a\|$. Let $\mathrm{WF_A}(f)$ be the analytic wavefront set of f [17] and let ξ_0 be a conormal covector to $S(a, r_0)$ at x_0. Then, locally near x_0, $\operatorname{supp} f$ lies outside $S(a, r_0)$. By a theorem of Kawai, Kashiwara, and Hörmander [17] about analytic wavefront at boundary points of $\operatorname{supp} f$, $(x_0, \xi_0) \in \mathrm{WF_A}(f)$. Let x_0^* be the reflection of x_0 in $T_a^\circ(S)$. By Theorem 3.3 in [4], x_0^* must be in $\operatorname{supp} f$ (if not, (x_0, ξ_0) could not be in $\mathrm{WF_A}(f)$). Therefore, $\operatorname{supp} f$ is $(T_a^\circ(S), a)$-locally symmetric.

4. W-orbits of the tangent bundle $T(S[f])$

In the sequel, S will denote a connected real-analytic open $(n-1)-$submanifold of $S[f]$. Our aim in this section is to analyze the behavior of the pairs $(T_a^\circ(S), a), a \in S$, under the $W-$ action.

4.1. **Preparations.** First, we use the Bieberbach theorem [10] (actually the definition in some books) that states that the crystallographic group W contains a lattice subgroup $\mathbb{Z}[h_1, \cdots, h_n]$ generated by n linearly independent vectors h_i:

$$\mathbb{Z}[h_1, ..., h_n] = \{\sum_{i=1}^{n} k_i h_i : k_i \in \mathbb{Z}\}.$$

From now on, the basis $h_1, \cdots, h_n$ is fixed.

Lemma 4.1. *Assume that S is not contained in a hyperplane. Then there exists a point $a \in S$ and a vector $\tau \in T_a(S)$,*

$$\tau = \tau_1 h_1 + \cdots + \tau_n h_n, \tau_i \in \mathbb{R},$$

such that the coordinates τ_i are linearly independent over the field $\mathbb{Q}$ of rational numbers.

Proof. Pick an arbitrary point $a \in S$. If the conclusion of the lemma is not true then for each vector $\tau \in T_a(S)$, $\tau = \sum\limits_{i=1}^{n} \tau_i h_i$, there are integers r_i such that

$$\sum_{i=1}^{n} r_i \tau_i = 0.$$

Thus, the linear space $T_a(S)$ is covered by the countable family of subspaces

$$T_a(S) \cap \Big\{ \sum_{i=1}^{n} \tau_i h_i \in \mathbb{R}^n : \sum_{i=1}^{n} r_i \tau_i = 0 \Big\}, \quad r_i \in \mathbb{Z}.$$

and this is possible only if $T_a(S)$ coincides with one of these subspaces, i.e.

$$T_a(S) = \Big\{ \sum_{i=1}^{n} \tau_i h_i \in \mathbb{R}^n : \sum_{i=1}^{n} r_i \tau_i = 0 \Big\},$$

for some $(r_1, \cdots, r_n) \in \mathbb{Z}^n$.

There are only countable many such linear spaces, and since the tangent space $T_a(S)$ depends on $a \in S$ continuously and S is connected, then $T_a(S)$ is a fixed hyperplane for all $a \in S$ and S is contained in a hyperplane. $\square$

Proposition 4.2. *[Kronecker, cf. [8],Ch.7,1.3, Prop.7] Let $\theta_1, \cdots, \theta_{n-1} \in \mathbb{R}$. Then the vectors*

$$(k_1 + k_n\theta_1, \cdots, k_{n-1} + k_n\theta_{n-1}), \quad k_i \in \mathbb{Z} \tag{4.1}$$

are dense in $\mathbb{R}^{n-1}$ if and only if the numbers $\theta_1, \cdots \theta_{n-1}, 1$ are linearly independent over $\mathbb{Q}$.

In fact, we will need a bit stronger version of Kronecker's theorem:

Corollary 4.3. *If $\theta_1, \cdots, \theta_{n-1}, 1$ are linearly independent over $\mathbb{Q}$ then the vectors (4.1) are dense in $\mathbb{R}^{n-1}$ when $(k_1, \cdots, k_n) \in \mathbb{Z}^n$ and $k_n \geq 0$.*

Proof. . Denote V_+ the semigroup of all vectors (4.1) with $k_n \geq 0$ and let $V_- = -V_+$.

If V_+ is not dense in $\mathbb{R}^{n-1}$ then V_- is not dense, either, and there exists an open ball $B = B(b, R), \quad R > 0$ that contains no element from the closure $\operatorname{cl} V_-$. By Proposition 4.2, $\operatorname{cl} V_+ \cup \operatorname{cl} V_- = \mathbb{R}^{n-1}$ and therefore $B \subset \operatorname{cl} V_+$.

From the definition (4.1) we have $\mathbb{Z}^{n-1} + \operatorname{cl} V_+ \subset \operatorname{cl} V_+$, and in particular, $\mathbb{Z}^{n-1} \subset \operatorname{cl} V_+$. Also $B + \operatorname{cl} V_+ \subset \operatorname{cl} V_+ + \operatorname{cl} V_+ \subset \operatorname{cl} V_+$ because $\operatorname{cl} V_+$ is a semigroup. Therefore, for any $s \in \mathbb{N}$:

$$\mathbb{Z}^{n-1} + sb + B(0, sR) = \mathbb{Z}^{n-1} + sB = \mathbb{Z}^{n-1} + \underbrace{B + \cdots + B}_{\text{s times}} \subset \operatorname{cl} V_+$$

Write $sb = [sb] + \{sb\}$, where $[sb]$ and $\{sb\}$ are vectors of integer and fractional parts of the coordinates, respectively. Since $\mathbb{Z}^{n-1} + [sb] = \mathbb{Z}^{n-1}$, it follows from the above inclusion that

$$\{sb\} + B(0, sR) \subset \operatorname{cl} V_+$$

and, since $\{sb\} \in [0, 1]^{n-1}$ and $s \in \mathbb{N}$ is arbitrary, we see that $\operatorname{cl} V_+ = \mathbb{R}^{n-1}$. $\square$

Lemma 4.4. *Let X, Y be linear normed vector spaces, $\dim X = n, \quad \dim Y = n-1$, and*

$$P : X \to Y$$

be a linear operator with one-dimensional kernel

$$\ker P = \mathbb{R}\tau, \quad \tau \in X \setminus \{0\}.$$

Fix a basis $h_1, \cdots, h_n \in X$ and define the half-lattice

$$\mathbb{Z}_+[h] = \{\sum_{j=1}^n k_j h_j : k_j \in \mathbb{Z}, k_n \geq 0\}.$$

If the coordinates $\tau_1, \cdots, \tau_n$ of τ in the basis $\{h_i\}$ are linearly independent over $\mathbb{Q}$ then the image $P(\mathbb{Z}_+[h])$ is dense in Y.

Proof. Let $v_i = P(h_i), \quad i = 1, \cdots, n$. Then

$$\sum_{j=1}^n \tau_j v_j = P(\tau) = 0$$

and since all $\tau_i \neq 0$ we can write

$$v_n = \theta_1 v_1 + \cdots + \theta_{n-1} v_{n-1},$$

where $\theta_i = -\tau_i/\tau_n$.

The condition for τ_i implies that $\theta_1, \cdots \theta_{n-1}, 1$ are linearly independent over $\mathbb{Q}$ and by Corollary 4.3 the vectors

$$x = P(\sum_{j=1}^{n} k_j h_j) = \sum_{j=1}^{n} k_j v_j = \sum_{j=1}^{n-1} (k_j + k_n \theta_j) v_j,$$

are dense in Y, where k_j runs over $\mathbb{Z}$ and $k_n \geq 0$. More precisely, Corollary 4.3 gives denseness in $\mathbb{R}^{n-1}$ of vectors $(k_1 + k_n\theta_1, \ldots, k_{n-1} + k_n\theta_{n-1}) \in R^{n-1}$ and then the denseness of vectors x in Y follows because the system $v_1, \cdots, v_{n-1}$ is a basis in Y (indeed, if $\sum\limits_{i=1}^{n-1} \alpha v_i = 0$ for some real α_i then $\sum\limits_{i=1}^{n-1} \alpha_i h_i \in \ker P$, so $(\alpha_1, \cdots, \alpha_{n-1}, 0) = t(\tau_1, \cdots, \tau_n)$ for some $t \in \mathbb{R}$ and $\tau_n \neq 0$ implies all $\alpha_i = 0$). $\square$

4.2. **A geometric construction.** In what follows K is a fixed compact set in $\mathbb{R}^n$. Fix the point $a \in S$ and the vector $\tau \in T_a(S)$ in accordance with Lemma 4.1. Possibly using a rotation, we can normalize τ by

$$\|\tau\| = 1, \quad \tau_n > 0.$$

Choose a unit normal vector $e \in N_a(S)$.

We can assume that

$$\max_{x \in K} (x, \tau) = 0. \tag{4.2}$$

Denote

$$Q = \{x \in \mathbb{R}^n : (x, \tau) = 0.\}$$

the supporting hyperplane. Let $K_0 = K \cap Q$ be the portion of K in Q.

Choose a real number d_0 so that

$$\max_{x \in K_0} (x, e) < d_0 \tag{4.3}$$

and define the hyperplane

$$\Pi_0 = \{x \in \mathbb{R}^n : (x, e) = d_0\}. \tag{4.4}$$

Finally, chose two closest points $x_0 \in K_0$ and $p_0 \in \Pi_0$:

$$\|x_0 - p_0\| = \mathrm{dist}(K_0, \Pi_0) \tag{4.5}$$

Since the hyperplanes Q and Π_0 are orthogonal and $x_0 \in Q$ the point p_0 belongs to the both hyperplanes, $p_0 \in Q \cap \Pi_0$. Also

$$\mathrm{dist}(K_0, \Pi_0) = d_0 - \max_{x \in K_0} (x, e) > 0\,.$$

4.3. **The supporting plane Q as a limit of supporting spheres.** Let $P_Q : \mathbb{R}^n \to Q$ be the orthogonal projection. In this subsection we construct, using the notation and preparations made in subsections 4.1 and 4.2, a sequence of spheres having the following properties:

1) they support (touch) the prescribed compact set K,
2) they converge to the supporting hyperplane Q,
3) the P_Q-projections of centers of the spheres converge to the point $p_0 \in Q$, defined in (4.5).

Because $\ker P_Q = \mathbb{R}\tau$ and since the vector τ is chosen in accordance with Lemma 4.1, i.e., its coordinates τ_i with respect to the basis h_i are linearly independent over $\mathbb{Q}$, Lemma 4.4 applies to the operator P_Q and yields that P_Q takes the half-lattice $Z_+[h]$ onto a dense set in the hyperplane Q.

In particular, if one fixes a natural number M,

$$M > |(a)_n| \tag{4.6}$$

where $(a)_n$ is the n^{th} coordinate of the point of tangency, a, in the basis h_j , then there exists an infinite sequence $t_m \in \mathbb{Z}_+[h]$ such that

$$P_Q(t_m) \to p_0 - P_Q(a + Mh_n), \quad m \to \infty$$

where p_0 is defined in (4.5).

If we let

$$a_m = a + t_m + Mh_n \tag{4.7}$$

then

$$\lim_{m\to\infty} P_Q(a_m) = p_0 \,.$$

The formula (4.7) defines a translation $w_m : a \to a_m$ that belongs to the group W, so a_m is in $W-$ orbit of the point a.

The sequence a_m is unbounded because it is infinite and discrete, because it belongs to the lattice $Z_+[h]$, so passing to a subsequence we can assume that $\|a_m\| \to \infty$ as $m \to \infty$.

Now represent a_m as the direct sum

$$a_m = P_Q(a_m) \oplus (a_m, \tau)\tau \tag{4.8}$$

The first summand converges and hence is bounded, therefore $\|(a_m, \tau)\| \to \infty, \quad m \to \infty$. If $a_m = \|a_m\|\xi_m, \quad \|\xi_m\| = 1$, then dividing (4.8) by $\|a_m\| = (\|P_Q(a_m)\|^2 + |(a_m, \tau)|^2)^{1/2}$ and letting $m \to \infty$ we obtain

$$\xi_m \to \pm\tau, \quad m \to \infty.$$

However, $\xi_m \to -\tau$ is impossible because the n^{th} coordinates, in the basis h_i, of the vectors τ and ξ_m have opposite signs: $\tau_n > 0$ by construction, and $\|a_m\|(\xi_m)_n = (t_m)_n + (a)_n + M > (t_m)_n \geq 0$ because $t_m \in \mathbb{Z}_+[h]$ and by the choice of M, (4.6).

Thus,

$$a_m/\|a_m\| = \xi_m \to \tau, \quad m \to \infty \,. \tag{4.9}$$

Lemma 4.5. *Denote $r_m = \operatorname{dist}(a_m, K)$ and let $x_m \in K$ be such that $\|x_m - a_m\| = r_m$. Then the sequence x_m converges to the point x_0 defined in subsection 4.2.*

Proof. It suffices to proof that x_0 is the only limit point of the sequence x_m. Let $\tilde{x}_0$ be a limit point, $\tilde{x}_0 = \lim_{k\to\infty} x_{m_k}$. We can assume, by renumbering, that $x_m \to \tilde{x}_0$ as $m \to \infty$.

Clearly, $\tilde{x}_0 \in K$ because $x_{m_k} \in K$ and K is closed. We now show that $\tilde{x}_0 \in K \cap Q = K_0$. Indeed, for any $x \in K$,

$$\|x_m - a_m\|^2 = r_m^2 \le \|x - a_m\|^2$$

or, equivalently,

$$(x, a_m) \le (x_m, a_m) + (1/2)(\|x\|^2 - \|x_m\|^2).$$

Dividing both sides by $\|a_m\|$ and letting $m \to \infty$, so $\|a_m\| \to \infty$, we obtain from (4.9):

$$(x, \tau) \le (\tilde{x}_0, \tau).$$

Thus,

$$(\tilde{x}_0, \tau) = \max_{x\in K}(x, \tau) = 0$$

and hence $\tilde{x}_0 \in Q$ by the definition of the hyperplane Q. Geometrically, what we have checked is that the spheres $S(a_m, r_m)$, touching the compact K tend to the supporting hyperplane Q when the centers a_m go to infinity.

Note that $(x_m, \tau) \le 0$ by (4.2) because $x_m \in K$. On the other hand, $(a_m, \tau) \to +\infty, \quad m \to \infty$, and therefore $\exists m_0 \in \mathbb{N}$ such that $(a_m, \tau) > 0$ for $m \ge m_0$. Thus, for such m, x_m and a_m are on opposite sides of the hyperplane Q, and therefore the open ball $B(a_m, r_m)$ meets Q. Hence its intersection with Q coincides with nonempty intersection of Q and another open ball centered on Q:

$$B(a_m, r_m) \cap Q = B(p_m, s_m) \cap Q,$$

where $p_m = P_Q(a_m)$.

By minimality of r_m, no open ball $B(a_m, r_m)$ contains points from K, and since $B(p_m, s_m) \cap Q \subset B(a_m, r_m)$, the ball $B(p_m, s_m) \cap Q$ is disjoint from K. Therefore $\|x_0 - p_m\| \ge s_m, \quad \|\tilde{x}_0 - p_m\| \ge s_m$ because $x_0, \tilde{x}_0 \in K_0 \subset K$. In particular, the sequence s_m is bounded (because p_m converges to p_0) and passing to a subsequence we can assume that $s_m \to s_0, \quad m \to \infty$. Taking limits we obtain

$$\|x_0 - p_0\| \ge s_0, \quad \|\tilde{x}_0 - p_0\| \ge s_0 \tag{4.10}$$

On the other hand, we have for $m \ge m_0$:

$$\|P_Q(x_m) - a_m\|^2 = \|x_m - (x_m, \tau)\tau - a_m\|^2 = \|x_m - a_m\|^2 + (x_m, \tau)(2(a_m, \tau) - (x_m, \tau)) \le r_m^2,$$

because $\|x_m - a_m\| = r_m, \quad (x_m, \tau) \le 0$ and $(a_m, \tau) > 0$.

So, $P_Q(x_m)$ belongs to the closed ball $\overline{B}(a_m, r_m)$ and since it belongs to Q we have $P_Q(x_m) \in \overline{B}(p_m, s_m)$, i.e. $\|P_Q(x_m) - p_m\| \le s_m$. Letting $m \to \infty$ we conclude

$$\|\tilde{x}_0 - p_0\| \le s_0\,.$$

Here we have taken into account that $\tilde{x}_0 \in Q$ and therefore $P_Q(\tilde{x}_0) = \tilde{x}_0$. Together with (4.10) it gives

$$\|\tilde{x}_0 - p_0\| = s_0. \tag{4.11}$$

To complete the proof, recall that, by the construction in 4.3, the vector $p_0 - x_0$ is proportional to the vector e (normal to Π_0): $p_0 - x_0 = \alpha e$, where the coefficient $\alpha = \mathrm{dist}(K_0, \Pi_0)$. Write, based on (4.11), $p_0 - \tilde{x}_0 = s_0\eta, \quad \|\eta\| = 1$.

From the first inequality in (4.10), we have $s_0 \leq \alpha$. Then using the fact that $\tilde{x}_0 \in K_0$ and $p_o \in \Pi_0$ we obtain

$$s_0 \leq \alpha = \mathrm{dist}(K_0, \Pi_0) \leq \mathrm{dist}(\tilde{x}_0, \Pi_0) = (p_0 - \tilde{x}_0, e) = s_0(\eta, e) \leq s_0.$$

This yields $\eta = e$ and $\alpha = s_0$, i.e., $\tilde{x}_0 = x_0$. So, $x_m \to x_0$. □

4.4. **Dense $W-$ orbits and local symmetry.**

Lemma 4.6. *Let Ω be a fundamental domain of the group W and a compact set K lies strictly inside Ω, $K \Subset \Omega$. Define*

$$\tilde{K} = \bigcup_{w \in W} w(K).$$

Let S be a real-analytic connected hypersurface in $\mathbb{R}^n$. Suppose that $\tilde{K}$ is locally symmetric with respect to any pair $(T_a^\circ(S), a)$, $\quad s \in S$. Then $\tilde{K} = \emptyset$ unless S is a hyperplane.

Proof. Suppose that $\tilde{K} \neq \emptyset$ (and therefore $K \neq \emptyset$) and S is not a hyperplane. We apply the constructions and the results of the previous subsections 4.1-4.4 to the compact K. Let $a \in S$ and $\tau \in T_a(S)$ be as in Lemma 4.1, and $e \in N_a(S)$, $\quad Q$, $\quad \Pi_0$, $\quad x_m$, $\quad a_m$, $\quad x_0$, $\quad p_0$ are as in subsections 4.3 and 4.4.

First of all, applying translation, we can assume that the condition (4.2) is fulfilled for the compact K. As well as the condition (4.3) for the number d_0, we require:

$$\max_{x \in K_0}(x, e) < d_0 < \rho/2 + \max_{x \in K_0}(x, e), \tag{4.12}$$

where $\rho = \mathrm{dist}(K, \partial\Omega)$. Note that $\rho > 0$ because $K \Subset \Omega$.

The points $a_m = w_m(a)$ (4.7) belong to the $W-$orbit of the point a and since $\tilde{K}$ is $W-$invariant and $(T_a^\circ(S), a)-$ locally symmetric then $\tilde{K}$ is locally symmetric with respect to any pair (Π_m, a_m), where $\Pi_m = T_{a_m}^\circ(S_m)$ and $S_m = w_m(S)$.

Because K is open and closed in $\tilde{K}$, the function $\tilde{K} \ni x \to \|x - a_m\|$ attains local minimum r_m at the point x_m and, by local symmetry, Theorem 3.2, the $(\Pi_m, a_m)-$ mirror point x_m^* belongs to $\tilde{K}$.

It follows that the Π_0- symmetric point x_0^* is in $\tilde{K}$. Indeed, since P_Q is the orthogonal projection along the vector τ, orthogonal to e, then $(a_m, e) = (p_m, e)$ (recall that $p_m = P_Q(a_m)$) and therefore the hyperplane Π_m is defined by the equation $(x, e) = d_m$ where $d_m = (p_m, e)$.

The Π_m-mirror point $x_m^* = x_0 + 2(d_m - (x_0, e))e \in \tilde{K}$. By Lemma 4.5 $x_m \to x_0$, and also $p_m \to p_0$ so taking limit we have the Π_0- mirror point

$$x_0^* = x_0 + 2(d_0 - (x_0, e))e \in \tilde{K}, \quad d_0 = (p_0, e).$$

Furthermore the right inequality in (4.12) implies

$$\|x_0^* - x_0\| = 2|d_0 - (x_0, e)| < \rho$$

and since $x_0 \in K \subset \Omega$ then x_0^* must be inside Ω by the definition of ρ and so $x_0^* \in \Omega \cap \tilde{K} = K$.

We will now show x_0^* is not in K by showing x_0^* is above K_0. This will finish the proof.

Moreover, $x_0^* \in Q \cap K = K_0$ because $x_0 \in Q$, and the hyperplane Q is orthogonal to Π_0 and hence Q is invariant under reflection in Π_0. From this and from the left inequality in (4.12) we have

$$(x_0^*, e) \le \max_{x \in K_0} (x, e) < d_0.$$

However, the same inequality in (4.12) implies

$$(x_0^*, e) = (x_0, e) + 2(d_0 - (x_0, e)) = 2d_0 - (x_0, e) \ge 2d_0 - \max_{x \in K_0} (x, e) > d_0.$$

This contradiction proves that if S is not a hyperplane then $\tilde{K} = \emptyset$. □

5. Proof of Theorem 1.1

Recall that we deal with the wave equation (2.1) in the whole space $\mathbb{R}^n$ and the initial data f is obtained by the $W-$extension in section 1. Correspondingly, the stationary set $S[f]$ is obtained from $S_\Omega[f]$ (1.2) by applying the group W and adding hyperplanes that bound all the polyhedra $w(\Omega), w \in W$. We call these hyperplanes $W-$mirrors; they are obtained from the system of hyperplanes generating the group W by applying elements from W and each $W-$ mirror consists of fixed points of some reflection in W.

Denote $K = \Omega \cap \operatorname{supp} f$. Since $f \neq 0$, we have $K \neq \emptyset$. Because f is skew-symmetric with respect to reflections from the group W, $\tilde{K} = \operatorname{supp} f = \bigcup_{w \in W} wK$.

According to Lemma 2.2, the stationary set

$$S[f] = \bigcap_{k=0, c_k \neq 0}^{\infty} \tilde{\varphi}_k^{-1}(0);$$

and since each nodal set $\tilde{\varphi}_k(0)$ is a union of a smooth hypersurface and a lower dimensional analytic set (see Theorem 1.1), the representation

$$S[f] = S_0 \cap V$$

holds, where S_0 is a smooth hypersurface and V is an analytic set of dimension less than $n-1$.

Theorem 3.2 states that the set $\tilde{K} = \operatorname{supp} f$ is $(T_a^\circ(S_0), a)$-locally symmetric for any $a \in S_0$. Then Lemma 4.6 implies that any connected smooth hypersurface $S \subset S_0$ must be a hyperplane. Therefore S_0 is the union of hyperplanes.

Denote by Σ the system of these hyperplanes, so that

$$S_0 = \bigcup_{\pi \in \Sigma} \pi.$$

Since $S_0 \subset S[f]$, then by the reflection principle the initial data f is odd with respect to reflections through any hyperplane in Σ. Also all $W-$mirrors belong to Σ.

Let $\tilde{W}$ be the group generated by the reflections through the hyperplanes in Σ. Since Σ contains $W-$mirrors, $W \subset \tilde{W}$. By the above property of f, each set $S[f], S_0$ and V is $\tilde{W}-$ invariant and for this reason Σ is a $\tilde{W}-$ invariant system of hyperplanes.

We claim that the system Σ is locally finite. Indeed, if it is not then there exists a hyperplane π_0 in $\mathbb{R}^n$ that is a limit point of hyperplanes in Σ:

$$\pi_0 = \lim_{m \to \infty} \pi_m, \quad \pi_m \in \Sigma, \quad \pi_m \neq \pi_0.$$

By Lemma 2.2 each eigenfunction $\tilde{\varphi}_k$ in the Fourier decomposition (2.3) vanishes on $S[f]$ and hence vanishes on each hyperplane π_m. Then $\tilde{\varphi}_k$ vanishes on π_0 to infinite order and $\varphi_k = 0$ due to real analyticity. This would imply $f = 0$ which is not the case.

Now we want to check that the group $\tilde{W}$ is discrete. It suffices to prove that the unit e, the identity map, is an isolated point of $\tilde{W}$.

Suppose $w_m \to e$, where $w_m \in \tilde{W}$. The convergence is uniform on compact sets. Due to local finiteness of the system Σ, only a finite number of hyperplanes in Σ meet the fundamental polyhedron Ω, therefore $\Omega \setminus S_0$ consists of finite number of disjoint open polyhedra. Let Δ be one of them and $c \in \Delta$ be a generic point (not a fixed point of any nontrivial isometry of Δ). Then $w_m(c) \to c$ and hence $\exists m_0 \in \mathbb{N}$ such that $w_m(c) \in \Delta$ for $m \geq m_0$. Since for each $m > m_0$, w_m is a composition of $\Sigma-$ reflections and the polyhedron $w_m(\Delta)$ and Δ have a common point, $w_m(c)$, they must coincide: $w_m(\Delta) = \Delta, \quad m \geq m_0$.

However, the group of isometries of the polyhedron Δ is finite and hence the sequence w_m contains only finite number of distinct elements. Then, since c is generic, $w_m(c) \to c$ implies that $w_m = e$ when m is sufficiently large. This means that the group $\tilde{W}$ is discrete.

In addition, the factor-space $\mathbb{R}^n/\tilde{W}$ is compact because of the imbedding

$$\mathbb{R}^n/\tilde{W} \hookrightarrow \mathbb{R}^n/W \simeq \overline{\Omega}/W$$

and since $\overline{\Omega}$ is compact.

Thus the group $\tilde{W}$ is uniform (the factor space is compact) and discrete and therefore is crystallographic (see. e.g. [10], Def. 1.7). Any reflection from $\tilde{W}$ takes the polyhedron Δ to a polyhedron that is either inside Ω or outside $\overline{\Omega}$, the polyhedra obtained by subsequent reflections through the faces are mutually disjoint and cover $\mathbb{R}^n \setminus S_0$.

Therefore Δ serves as a fundamental polyhedron of the group $\tilde{W}$. By construction, the group W is a subgroup of the group $\tilde{W}$, of finite index $\#(\tilde{W}/W)$, which equals to the number of reflections of the fundamental polyhedron Δ through its faces needed to cover Ω. This completes the proof.

Remark 5.1. The proof is valid for more complicated domains, namely domains that tile $\mathbb{R}^n$ and that are made up of unions of fundamental domains of the same crystallographic group. For example, the hexagon Ω is a union of equilateral triangles, and it tiles the plane. Our theorem is true for the hexagon as long as f is zero near the boundary of Ω and f is oddly symmetric about each reflection in each diameter of Ω. The proof is as given above and the extra condition on f is used to ensure the extension of f to $\mathbb{R}^2$ by odd symmetry is consistent (see start of §2).

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