Stable pairs on curves and surfaces

Daniel Huybrechts
Manfred Lehn

Manfred Lehn
Mathematisches Institut
Universität Zürich
Rämistraße 74
8001 Zürich

Schweiz

Daniel Huybrechts
Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI / 92-93

Stable pairs on curves and surfaces

D. Huybrechts M. Lehn

November 1992

Introduction

During the last years there has been growing interest in vector bundles with additional structures, e.g. parabolic and level structures. This paper results from an attempt to construct quasi-projective moduli spaces for framed bundles, i.e. bundles together with an isomorphism to a fixed bundle on a divisor as introduced in [Do], [L1] and [Lü]. More generally one can ask for bundles with a homomorphism to a fixed sheaf $\mathcal{E}_0$. We use techniques of geometric invariant theory to construct projective moduli spaces. This leads to natural stability conditions. In contrast to the pure bundle case an extra parameter appears in the definition of stability.

A pair $(\mathcal{E}, \alpha)$ consisting of a coherent sheaf $\mathcal{E}$ on a smooth, projective variety and a homomorphism α from $\mathcal{E}$ to $\mathcal{E}_0$ is called *stable* with respect to a polynomial δ if and only if the following conditions are satisfied.

i) $\chi_{\mathcal{G}} < (\mathrm{rk}\mathcal{G}/\mathrm{rk}\mathcal{E})\chi_{\mathcal{E}} - (\mathrm{rk}\mathcal{G}/\mathrm{rk}\mathcal{E})\delta$ for all subsheaves $\mathcal{G} \subset \mathrm{Ker}\alpha$.

ii) $\chi_{\mathcal{G}} < (\mathrm{rk}\mathcal{G}/\mathrm{rk}\mathcal{E})\chi_{\mathcal{E}} + \delta(\mathrm{rk}\mathcal{E} - \mathrm{rk}\mathcal{G})/\mathrm{rk}\mathcal{E}$ for all subsheaves $\mathcal{G} \subsetneqq \mathcal{E}$.

Here χ denotes the Hilbert polynomial and the inequalities must hold for large arguments. In §1 we prove

Theorem: *For a smooth, projective variety X of dimension one or two there is a fine quasi-projective moduli space of stable pairs $(\mathcal{E}, \alpha : \mathcal{E} \to \mathcal{E}_0)$ with respect to δ.*

Moreover, we will prove that this space can be naturally compactified (For a precise statement see 1.21).

In particular, this theorem proves the quasi-projectivity of many of the moduli spaces of framed bundles, which in [L1] were constructed only as algebraic spaces (2.24). In §2 we study two special cases for $\mathcal{E}_0$, where $\mathcal{E}_0$ is the structure sheaf $\mathcal{O}_X$ or a vector bundle on an effective divisor.

The case $\mathcal{E}_0 \cong \mathcal{O}_X$ leads to the definition of Higgs pairs, i.e. solution of the vortex equation as considered in [Br], [Be], [Ga], [Th]. A Higgs pair is a vector bundle $\mathcal{E}$ together with a global section φ satisfying certain stability conditions. The corresponding moduli spaces of rank two vector bundles on a curve were constructed by M. Thaddeus and A. Bertram. Dualizing the situation one gets a vector bundle $\mathcal{E}^\vee$ together with a homomorphism $\alpha = \varphi^\vee : \mathcal{E}^\vee \longrightarrow \mathcal{O}_X$. The stability conditions for Higgs pairs translate into *i)* and *ii)* above. This dual point of view allows us to compactify the moduli space in the surface case, too, by adding pairs with torsionfree sheaves. Instead of one moduli space M. Thaddeus consideres the whole series of moduli spaces, which result from changing the stability parameter in order to 'approximate' the usual moduli space of semistable bundles. We generalize this method for bundles on a surface and describe the 'limit' of this series. As a generalization of Bogomolov's result we prove a theorem about the restriction of stable pairs to curves of high degree (2.17).

The case of $\mathcal{E}_0$ being a vector bundle on a divisor leads to the concept of bundles with level structure ([Se]) and to the concept of framed bundles ([L1]) in dimension one and two, resp.

1 Moduli spaces of stable pairs

Throughout this paper we fix the following notations: X is an irreducible, nonsingular, projective variety of dimension e over an algebraically closed field k of characteristic zero, embedded by a very ample line bundle $\mathcal{O}_X(1)$. The canonical line bundle is denoted by $\mathcal{K}_X$. If $\mathcal{E}$ is a coherent $\mathcal{O}_X$-module, then $\chi_\mathcal{E}(n) := \chi(\mathcal{E}\otimes\mathcal{O}_X(n))$ denotes its Hilbert polynomial, $\mathrm{T}(\mathcal{E})$ its torsion submodule and det $\mathcal{E}$ its determinant line bundle. The degree of $\mathcal{E}$, deg $\mathcal{E}$, is the integral number $c_1(\det \mathcal{E}).H^{e-1}$, where $H \in |\mathcal{O}_X(1)|$ is a hyperplane section.

χ will always be a polynomial with rational coefficients which has the form

$$\chi(z) = \deg X \cdot r \cdot \frac{z^e}{e!} + (d - \frac{\deg \mathcal{K}_X}{2}\cdot r)\cdot z^{e-1} + \text{Terms of lower order in } z.$$

If $\chi = \chi_\mathcal{E}$, then $r = \mathrm{rk}\mathcal{E}$ and $d = \deg\mathcal{E}$. Finally, let $\mathcal{E}_0$ be a fixed coherent $\mathcal{O}_X$-module. By a *pair* we will always mean a pair $(\mathcal{E},\alpha)$ consisting of a coherent $\mathcal{O}_X$-module $\mathcal{E}$ with Hilbert polynomial $\chi_\mathcal{E} = \chi$ and a nontrivial homomorphism $\alpha : \mathcal{E} \rightarrow \mathcal{E}_0$. We write $\mathcal{E}_\alpha$ for $\mathrm{Ker}\,\alpha$.

In the next section we define the notion of semistability for such pairs with respect to an additional parameter δ. To simplify the notations and to be able to treat stability and semistability simultaneously, we employ the following short-hand: Whenever in a statement the word *(semi)stable* occurs together with a relation symbol in brackets, say $(\leq)$, the latter should be read as $\leq$ in the semistable case and as $<$ in the stable case. An inequality $p\,(\leq)\,p'$ between polynomials means, that $p(n)\,(\leq)\,p'(n)$ for large integers n. If p is a polynomial then $\Delta p(n) := p(n) - p(n-1)$ is the difference polynomial.

We proceed as follows: In section 1.1 we define semistability for pairs and formulate the moduli problem. In section 1.2 boundedness results for semistable pairs on curves

and surfaces are obtained. Moreover, a close relation between semistability and sectional semistability is established. The notion of sectional stability naturally appears by way of constructing moduli spaces for pairs. This is done in section 1.3 leading to the existence theorem 1.21. Section 1.4 is devoted to an invariant theoretical analysis of the construction in 1.3 and the proof of the main technical proposition 1.18.
The reader who is familiar with the papers of Gieseker and Maruyama ([Gi], [Ma]) will notice that many of our arguments are generalizations of their techniques.

1.1 Stable pairs and the moduli problem

Let δ be a polynomial with rational coefficients such that $\delta > 0$, i. e. $\delta(n) > 0$ for all $n \gg 0$. We write $\delta(z) = \sum_\nu \delta_{e-\nu} z^\nu$.

Definition 1.1. A pair $(\mathcal{E}, \alpha)$ is called *(semi)stable* (with respect to δ), if the following two conditions are satisfied:

(1) $\mathrm{rk}\mathcal{E}\cdot\chi_\mathcal{G}\,(\leq)\,\mathrm{rk}\mathcal{G}\cdot(\chi_\mathcal{E} - \delta)$ for all nontrivial submodules $\mathcal{G} \subsetneq \mathcal{E}_\alpha$.

(2) $\mathrm{rk}\mathcal{E}\cdot\chi_\mathcal{G}\,(\leq)\,\mathrm{rk}\mathcal{G}\cdot(\chi_\mathcal{E} - \delta) + \mathrm{rk}\mathcal{E}\cdot\delta$ for all nontrivial submodules $\mathcal{G}\,(\subseteq)\,\mathcal{E}$.

If no confusion can arise, we omit δ in the notations. Note that a stable pair a fortiori is semistable.

Lemma 1.2. *Suppose $(\mathcal{E}, \alpha)$ is a semistable pair, then:*

i) $\mathcal{E}_\alpha$ is torsion free. $h^0(\mathcal{G}) \leq h^0(\mathrm{T}(\mathcal{E}_0))$ for all submodules $\mathcal{G} \subseteq \mathrm{T}(\mathcal{E})$.

ii) Unless α is injective, δ is a polynomial of degree smaller than d.

Proof: ad i): If $\mathcal{G} \subset \mathcal{E}_\alpha$ is torsion, then $\mathrm{rk}\mathcal{G} = 0$. Condition (1) then shows $\chi_\mathcal{G} = 0$, hence $\mathcal{G} = 0$. Thus α embeds the torsion of $\mathcal{E}$ into the torsion of $\mathcal{E}_0$. This gives the second assertion. ad ii): Assume $\mathcal{E}_\alpha$ is nontrivial. By i) $\mathcal{E}_\alpha$ is torsion free of positive rank, and condition (1) implies $\delta/\mathrm{rk}\mathcal{E} \leq (\chi_\mathcal{E}/\mathrm{rk}\mathcal{E} - \chi_{\mathcal{E}_\alpha}/\mathrm{rk}\mathcal{E}_\alpha)$. The two fractions in the brackets are polynomials with the same leading coefficients. This shows $\deg\delta < e$. □

Thus if $\deg\delta \geq e$, then α must needs be an injective homomorphism, and isomorphism classes of semistable pairs correspond to submodules of $\mathcal{E}_0$ with fixed Hilbert polynomial. Note that condition (2) of the definition above is automatically satisfied. So in this case all pairs are in fact stable and parametrized by the projective quotient scheme $\mathrm{Quot}_{X/\mathcal{E}_0}^{\chi_{\mathcal{E}_0} - \chi_\mathcal{E}}$. For that reason we assume henceforth that δ has the form

$$\delta(z) = \delta_1 z^{e-1} + \delta_2 z^{e-2} + \cdots + \delta_e.$$

Definition 1.3. A pair $(\mathcal{E},\alpha)$ is called μ-(semi)stable (with respect to δ_1), if the following two conditions are satisfied:

(1) $\mathrm{rk}\mathcal{E}\cdot \deg\mathcal{G}\,(\leq)\,\mathrm{rk}\mathcal{G}\cdot(\deg\mathcal{E}-\delta_1)$ for all nontrivial submodules $\mathcal{G}\subseteq\mathcal{E}_\alpha$.

(2) $\mathrm{rk}\mathcal{E}\cdot \deg\mathcal{G}\,(\leq)\,\mathrm{rk}\mathcal{G}\cdot(\deg\mathcal{E}-\delta_1)+\mathrm{rk}\mathcal{E}\cdot\delta_1$ for all nontrivial submodules $\mathcal{G}\subseteq\mathcal{E}$ with $\mathrm{rk}\mathcal{G}<\mathrm{rk}\mathcal{E}$.

As in the theory of stable sheaves there are immediate implications for pairs $(\mathcal{E},\alpha)$:

$$\mu\text{-stable}\Rightarrow\text{stable}\Rightarrow\text{semistable}\Rightarrow\mu\text{-semistable}$$

A family of pairs parametrized by a noetherian scheme T consists of a coherent $\mathcal{O}_{T\times X}$-module $\mathcal{E}$, which is flat over T, and a homomorphism $\alpha:\mathcal{E}\to p_X^*\mathcal{E}_0$. If t is a point of T, let X_t denote the fibre $X\times\operatorname{Spec}k(t)$, $\mathcal{E}_t$ and α_t the restrictions of $\mathcal{E}$ and α to X_t. A homomorphism of pairs $\Phi:(\mathcal{E},\alpha)\to(\mathcal{E}',\alpha')$ is a homomorphism $\Phi:\mathcal{E}\to\mathcal{E}'$ which commutes with α and α', i. e. $\alpha'\circ\Phi=\alpha$. The correspondence

$$T\mapsto\{\text{Isomorphism classes of families of (semi)stable pairs parametrized by } T\}$$

defines a setvalued contravariant functor $\underline{\mathcal{M}}_\delta^{(s)s}(\chi,\mathcal{E}_0)$ on the category of noetherian k-schemes of finite type. We will prove that for $\dim X\leq 2$ there is a fine moduli space for $\underline{\mathcal{M}}_\delta^s(\chi,\mathcal{E}_0)$. It is compactified by equivalence classes of semistable pairs (1.21).

1.2 Boundedness and sectional stability

In section 1.3 we will construct moduli spaces of stable pairs by means of geometric invariant theory. The stability property needed in this construction differs slightly from the one given in 1.1 in refering to the number of global sections rather than to the Euler characteristic of a submodule of $\mathcal{E}$. In this section we compare the different notions and prove that semistable pairs form bounded families, if the variety X is a curve or a surface.

Definition 1.4. Let $\bar\delta$ be a positive rational number. A pair $(\mathcal{E},\alpha)$ is called *sectional (semi)stable* (with respect to $\bar\delta$), if $\mathcal{E}_\alpha$ is torsionfree and there is a subspace $V\subseteq H^0(\mathcal{E})$ of dimension $\chi(\mathcal{E})$ such that the following conditions are satisfied:

(1) $\mathrm{rk}\mathcal{E}\cdot\dim(H^0(\mathcal{G})\cap V)\,(\leq)\,\mathrm{rk}\mathcal{G}\cdot(\chi(\mathcal{E})-\bar\delta)$ for all nontrivial submodules $\mathcal{G}\subseteq\mathcal{E}_\alpha$.

(2) $\mathrm{rk}\mathcal{E}\cdot\dim(H^0(\mathcal{G})\cap V)\,(\leq)\,\mathrm{rk}\mathcal{G}\cdot(\chi(\mathcal{E})-\bar\delta)+\mathrm{rk}\mathcal{E}\cdot\bar\delta$ for all nontrivial submodules $\mathcal{G}\,(\subseteq)\,\mathcal{E}$.

We begin with the case of a curve. In this case δ is a rational number, and the Hilbert polynomial of any $\mathcal{O}_X$-module $\mathcal{G}$ depends on $\mathrm{rk}\mathcal{G}$ and $\deg \mathcal{G}$ only. Moreover, the polynomials occuring in the inequalities of definition 1.1 are linear and have the same leading coefficients. Therefore the Hilbert polynomials $\chi_{\mathcal{G}}$ can throughout be replaced by the Euler characteristics $\chi(\mathcal{G})$ without changing the essence of the definition.

Theorem 1.5. *Let X be a smooth curve of genus g. Assume that $d > r\cdot(2g-1)+\delta$.*

i) If $(\mathcal{E},\alpha)$ is semistable or sectional semistable, then $\mathcal{E}$ is globally generated and $h^1(\mathcal{E})=0$.

ii) $(\mathcal{E},\alpha)$ is a (semi)stable pair if and only if it is sectional (semi)stable.

Proof: ad i): On a smooth curve X there is a split short exact sequence

$$0 \longrightarrow \mathrm{T}(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow \bar{\mathcal{E}} \longrightarrow 0$$

with locally free $\bar{\mathcal{E}}$ for any coherent $\mathcal{O}_X$-module $\mathcal{E}$. Now $H^1(\mathcal{E}) = H^1(\bar{\mathcal{E}})$, and $\mathcal{E}$ is globally generated if and only if $\bar{\mathcal{E}}$ is globally generated. A glance at the short exact sequence

$$0 \longrightarrow \bar{\mathcal{E}}(-x) \longrightarrow \bar{\mathcal{E}} \longrightarrow \bar{\mathcal{E}}\otimes \mathcal{O}_x \longrightarrow 0$$

for some closed point $x\in X$ shows that the vanishing of $H^1(\bar{\mathcal{E}}(-x))$ for all $x \in X$ is a sufficient criterion for both $H^1(\mathcal{E})=0$ and the global generation of $\mathcal{E}$. If $H^1(\bar{\mathcal{E}}(-x)) \neq 0$, then there is a nontrivial homomorphism $\varphi : \bar{\mathcal{E}} \to \mathcal{K}_X(x)$. Let $\mathcal{G} := \mathrm{T}(\mathcal{E}) + \mathrm{Ker}\varphi$, so that there is a short exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow \mathcal{K}_X(x-C) \longrightarrow 0$$

with some effective divisor C on X. From this sequence we get

$$\chi(\mathcal{G}) \geq \chi(\mathcal{E}) - \chi(\mathcal{K}_X(x)) \quad \text{and} \quad h^0(\mathcal{G}) \geq h^0(\mathcal{E}) - h^0(\mathcal{K}_X(x)).$$

On the other hand,

$$\chi(\mathcal{G}) \leq \frac{\mathrm{rk}\mathcal{E}-1}{\mathrm{rk}\mathcal{E}}\chi(\mathcal{E}) + \frac{\delta}{\mathrm{rk}\mathcal{E}},$$

if $(\mathcal{E},\alpha)$ is semistable, and

$$\dim(V\cap H^0(\mathcal{G})) \leq \frac{\mathrm{rk}\mathcal{E}-1}{\mathrm{rk}\mathcal{E}}\chi(\mathcal{E}) + \frac{\delta}{\mathrm{rk}\mathcal{E}}$$

for some vector space $V \subseteq H^0(\mathcal{E})$ of dimension $\chi(\mathcal{E})$, if $(\mathcal{E},\alpha)$ is sectional semistable. In the first case we get $\chi(\mathcal{E}) \leq \mathrm{rk}\mathcal{E}\cdot\chi(\mathcal{K}_X(x)) + \delta$. And in the second case one has

$$\begin{aligned} h^0(\mathcal{E}) - h^0(\mathcal{K}_X(x)) \leq h^0(\mathcal{G}) &\leq \dim(H^0(\mathcal{G})\cap V) + (h^0(\mathcal{E}) - \dim V) \\ &\leq \frac{\mathrm{rk}\mathcal{E}-1}{\mathrm{rk}\mathcal{E}}\cdot\chi(\mathcal{E}) + \frac{\delta}{\mathrm{rk}\mathcal{E}} + (h^0(\mathcal{E}) - \chi(\mathcal{E})) \end{aligned}$$

So in any case we end up with $\deg\mathcal{E} \leq \text{rk}\mathcal{E}\cdot(2g-1)+\delta$ contradicting the assumption of the theorem.

ad ii): By part i) we have $\chi(\mathcal{E}) = h^0(\mathcal{E})$, $V = H^0(\mathcal{E})$ and, of course, $\chi(\mathcal{G}) \leq h^0(\mathcal{G})$ for any submodule $\mathcal{G} \subsetneq \mathcal{E}$. Hence sectional (semi)stability implies (semi)stability at once. Conversely, assume that $(\mathcal{E},\alpha)$ is a (semi)stable pair. If for a submodule $\mathcal{G}$ we have $h^1(\mathcal{G}) = 0$, then $h^0(\mathcal{G}) = \chi(\mathcal{G})$ and there is nothing to show. (This applies in particular when $\text{rk}\mathcal{G} = 0$). Hence assume $h^1(\mathcal{G}) \neq 0$. As above this leads to a short exact sequence

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{K}_X(-C) \longrightarrow 0$$

with $\text{rk}\mathcal{G}' = \text{rk}\mathcal{G} - 1$ and some effective divisor C on X, so that $h^0(\mathcal{G}') \geq h^0(\mathcal{G}) - g$. By induction we may assume that

$$h^0(\mathcal{G}')\,(\leq)\,\frac{\text{rk}\mathcal{G}-1}{\text{rk}\mathcal{E}}(h^0(\mathcal{E})-\delta)+\varepsilon\cdot\delta$$

with $\varepsilon = 0$ if $\mathcal{G} \subseteq \mathcal{E}_\alpha$ and $\varepsilon = 1$ if $\mathcal{G}\,(\subseteq)\,\mathcal{E}$. Combining these inequalities we get

$$h^0(\mathcal{G})\,(\leq)\,\frac{\text{rk}\mathcal{G}}{\text{rk}\mathcal{E}}(h^0(\mathcal{E})-\delta)+\varepsilon\cdot\delta+(g-\frac{h^0(\mathcal{E})}{\text{rk}\mathcal{E}}+\frac{\delta}{\text{rk}\mathcal{E}}).$$

Since $h^0(\mathcal{E}) = \chi(\mathcal{E}) = \deg\mathcal{E} + (1-g)\text{rk}\mathcal{E} > g\cdot\text{rk}\mathcal{E} + \delta$, we are done. □

Corollary 1.6. *Suppose X is a curve. The set of isomorphism classes of $\mathcal{O}_X$-modules occuring in semistable pairs is bounded.* □

Before we pass on to surfaces recall the following criterion due to Kleiman which we will use several times:

Theorem 1.7 (Boundedness criterion of Kleiman). *Suppose χ is a polynomial and K an integer. If T is a set of $\mathcal{O}_X$-modules $\mathcal{F}$ such that $\chi_\mathcal{F} = \chi$ and*

$$h^0(X, \mathcal{F}|_{H_1\cap\ldots\cap H_i}) \leq K \qquad \forall i = 0,\ldots,e,$$

for a $\mathcal{F}$-regular sequence of hyperplane sections $H_1,\ldots,H_e$, then T is bounded.

Proof: [Kl, Thm 1.13] □

We introduce the following notation: For integers ρ and ε let $P(\rho,\varepsilon)$ be the polynomial

$$P(\rho,\varepsilon,z) := \frac{\rho}{r}(\chi(z)-\delta(z))+\varepsilon\cdot\delta(z).$$

If $\mathcal{G} \subseteq \mathcal{E}$ is a submodule, let $\varepsilon(\mathcal{G}) = 0$ or 1 depending on wether $\mathcal{G} \subseteq \mathcal{E}_\alpha$ or not. Then the stability conditions can be conveniently reformulated:

- $(\mathcal{E}, \alpha)$ is (semi)stable if and only if $\chi_{\mathcal{G}} \,(\leq)\, P(\text{rk}\mathcal{G}, \varepsilon(\mathcal{G}))$ for all nontrivial submodules $\mathcal{G} \,(\subseteq)\, \mathcal{E}$.

- $(\mathcal{E}, \alpha)$ is μ-(semi)stable if and only if $\Delta\chi_{\mathcal{G}} \,(\leq)\, \Delta P(\text{rk}\mathcal{G}, \varepsilon(\mathcal{G}))$ for all nontrivial submodules $\mathcal{G} \subseteq \mathcal{E}$ with $\text{rk}\mathcal{G} < \text{rk}\mathcal{E}$.

- $(\mathcal{E}, \alpha)$ is sectional (semi)stable if and only if $\text{T}(\mathcal{E}_\alpha) = 0$ and there is a subspace $V \subseteq H^0(\mathcal{E})$ of dimension $\chi(\mathcal{E})$ such that $\dim(V \cap H^0(\mathcal{G})) \,(\leq)\, P(\text{rk}\mathcal{G}, \varepsilon(\mathcal{G}), 0)$ for all nontrivial submodules $\mathcal{G} \,(\subseteq)\, \mathcal{E}$.

Lemma 1.8. *Suppose X is a surface. There is an integer $n_0 < 0$, depending on X, $\mathcal{O}_X(1)$ and P only, such that $\Delta\chi_{\mathcal{O}_X(-n_0)} > \Delta P(1, \varepsilon)$ for $\varepsilon = 0, 1$.*

Proof: As polynomials in ν the expressions $\Delta\chi_{\mathcal{O}_X}(\nu - n)$ and $\Delta P(1, \varepsilon, \nu)$ are both linear and have the same positive leading coefficient. Hence for very negative numbers n one has $\Delta\chi_{\mathcal{O}_X}(\nu - n) > \Delta P(1, \varepsilon, \nu)$. □

The following technical lemma is an adaptation of [Gi, Lemma 1.2]. Unfortunately, we cannot apply Gieseker's lemma directly because it treats torsion free modules only, even though the necessary modifications are minor.

Lemma 1.9. *Suppose X is a surface. Let Q be a positive integer. Then there are integers N and M, depending on $X, \mathcal{O}_X(1), P$ and Q, such that if $\varepsilon \in \{0, 1\}$ and if $\mathcal{F}$ is an $\mathcal{O}_X$-module of rank $r' \leq r$ with the properties $h^0(\text{T}(\mathcal{F})) \leq Q$ and $\Delta\chi_{\mathcal{G}} \leq \Delta P(\text{rk}\mathcal{G}, \varepsilon)$ for all nontrivial submodules $\mathcal{G} \subseteq \mathcal{F}$, then either*

$$h^0(\mathcal{F}(n)) < P(r', \varepsilon, n) \text{ for all } n \geq N,$$

or the following assertions hold:

(1) $\Delta\chi_{\mathcal{F}} = \Delta P(r', \varepsilon)$,

(2) $h^2(\mathcal{F}(n)) = 0$ *for all* $n \geq N$,

(3) $h^0(\mathcal{F}(n_0)|_H) \leq M$ *for some $\mathcal{F}$-regular hyperplane section H,*

(4) if $h^1(\mathcal{F}(n_0)) \leq Q$, *then* $h^1(\mathcal{F}(n)) = 0$ *for all* $n \geq N$.

Proof: Let $\mathcal{F}$ be an $\mathcal{O}_X$-module satisfying the assumptions of the lemma. For every integer n let $\mathcal{H}'_n$ denote the image of the evaluation map $H^0(\mathcal{F}(n)) \otimes \mathcal{O}_X \to \mathcal{F}(n)$ and $\mathcal{S}'_n$ the quotient $\mathcal{F}(n)/\mathcal{H}'_n$. Let $\mathcal{H}_n$ be the kernel of the epimorphism

$$\mathcal{F}(n) \longrightarrow\!\!\!\!\!\rightarrow (\mathcal{S}'_n/\text{T}(\mathcal{S}'_n)) =: \mathcal{S}_n.$$

Then $\mathcal{H}_n$ is characterized by the following properties: $H^0(\mathcal{H}_n) = H^0(\mathcal{F}(n))$, $\mathcal{F}(n)/\mathcal{H}_n$ is torsion free and $\mathcal{H}_n$ is minimal with these two properties. Obviously $\mathcal{H}'_{n-1}(1) \subseteq \mathcal{H}'_n$ and therefore also $\mathcal{H}_{n-1}(1) \subseteq \mathcal{H}_n$. Moreover, being a submodule of the torsion free module $\mathcal{F}(n-1)/\mathcal{H}_{n-1}$ the quotient $\mathcal{H}_n(-1)/\mathcal{H}_{n-1}$ is itself torsion free. In particular either $\mathcal{H}_{n-1} = \mathcal{H}_n(-1)$ or $\mathrm{rk}\mathcal{H}_{n-1} < \mathrm{rk}\mathcal{H}_n$. Let $n_1 < \ldots < n_k$ be the indices with $\mathrm{rk}\mathcal{H}_{n_i-1} < \mathrm{rk}\mathcal{H}_{n_i}$. (If $\mathcal{F}$ is torsion, then $\mathcal{H}_n = \mathcal{F}(n)$ for all n. Let $k = 0$ in this case). By Serre's Theorem $\mathcal{H}_{n_k} = \mathcal{F}(n_k)$ and $k \le r'$.

Let $s \in H^0\mathcal{F}(n)$ be a nonzero section. Then either s is a torsion element or induces an injection $\mathcal{O}_X(-n) \to \mathcal{F}$. In the latter case one has $\Delta\chi_{\mathcal{O}_X(-n)} \le \Delta P(1,\varepsilon)$. This is impossible for $n \le n_0$. It follows that

$$h^0(\mathcal{F}(n_0)) = h^0(\mathrm{T}(\mathcal{F})(n_0)) \le h^0(\mathrm{T}(\mathcal{F})) \le Q$$

and that $\mathcal{H}_{n_0} = \mathrm{T}(\mathcal{F})(n_0)$. In particular $n_0 < n_1$ if $r' > 0$.

A generic hyperplane section $H \in |\mathcal{O}_X(1)|$ has the following properties:

a) H is a smooth curve (of genus $g = 1 + \deg \mathcal{K}_X/2$).

b) H is $\mathcal{H}_n$-regular for all integers n.

c) $\mathcal{H}_n|_H$ is globally generated at the generic point of H for all integers n.

(a) is just Bertini's Theorem. For (b) it is enough to consider the sheaves $\mathcal{H}_{n_i}$, $i = 0, \ldots, k$. H must not contain any of the finitely many associated points of the modules $\mathcal{H}_{n_i}$ in the scheme X. But this is an open condition. $\mathcal{H}_n$ is globally generated outside the support of $\mathrm{T}(\mathcal{S}_n)$, so for (c) it is sufficient that in addition H should not contain any of the associated points of the $\mathrm{T}(\mathcal{S}_{n_i})$. Hence for a generic hyperplane section H there are short exact sequences

$$0 \longrightarrow \mathcal{H}_n(-1) \longrightarrow \mathcal{H}_n \longrightarrow \mathcal{H}_n|_H \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_H^{r_n} \longrightarrow \mathcal{H}_n|_H \longrightarrow Q_n \longrightarrow 0,$$

where $r_n = \mathrm{rk}\mathcal{H}_n$ and Q_n is an $\mathcal{O}_H$-torsion module. From the second sequence one deduces estimates

$$h^1(\mathcal{H}_n|_H) \le r_n \cdot g \qquad \text{and} \qquad h^1(\mathcal{H}_n(\ell)|_H) = 0,$$

if $\deg(K_H - \ell H) < 0$, i. e. if $\ell > (2g-2)/H^2$. In particular we get for all integers n with $n_i + (2g-2)/H^2 < n < n_{i+1}$:

$$h^1(\mathcal{H}_n|_H) = h^1(\mathcal{H}_{n_i}(n-n_i)|_H) = 0.$$

This leads to the inequalities

$$\begin{aligned} h^0(\mathcal{F}(n)) - h^0(\mathcal{F}(n-1)) &= h^0(\mathcal{H}_n) - h^0(\mathcal{H}_n(-1)) \\ &\le h^0(\mathcal{H}_n|_H) = \chi(\mathcal{H}_n|_H) + h^1(\mathcal{H}_n|_H) \end{aligned}$$

and, summing up,

$$h^0(\mathcal{F}(n)) - h^0(\mathcal{F}(n_0) \leq \sum_{\nu=n_0+1}^{n} \chi(\mathcal{H}_\nu|_H) + \sum_{\rho=1}^{r_n} \rho g((2g-2)/H^2+1).$$

Let $K := Q + \binom{r'+1}{2} g((2g-2)/H^2+1)$. Then

$$h^0(\mathcal{F}(n)) \leq K + \sum_{\nu=n_0+1}^{n} \chi(\mathcal{H}_\nu|_H)$$

for all integers $n \geq n_0$. Suppose $n_0 \leq \nu < n_k$. Then $r_\nu < r'$. Since $\mathcal{H}_\nu(-\nu)$ is a submodule of $\mathcal{F}$,

$$\chi(\mathcal{H}_\nu|_H) = \Delta\chi_{\mathcal{H}_\nu(-\nu)}(\nu) \leq \Delta P(r_\nu, \varepsilon, \nu)$$

Now

$$\begin{aligned} \Delta P(r_\nu, \varepsilon, \nu) - \Delta P(r', \varepsilon, \nu) &= (r_\nu - r')\cdot(\deg X\cdot\nu + d/r + (1-g) - \delta_1) \\ &\leq -(\deg X\cdot\nu + C), \end{aligned}$$

where C is a constant depending on r, d, n_0, $\deg X$ and g. For $\nu \geq n_k$ one has $\mathcal{H}_\nu = \mathcal{F}(\nu)$ so that $\chi(\mathcal{H}_\nu|_H) = \Delta\chi_{\mathcal{F}}(\nu)$. Let $m(n) = \min\{n, n_k - 1\}$. Then the following inequality holds for all $n \geq n_0$:

$$\begin{aligned} h^0(\mathcal{F}(n)) - \sum_{\nu=n_0+1}^{n} \Delta P(r', \varepsilon, \nu) &\leq K - \sum_{\nu=n_0+1}^{m(n)} \{\deg X\cdot\nu + C\} \\ &\quad - \sum_{\nu=m(n)+1}^{n} \{\Delta P(r', \varepsilon, \nu) - \Delta\chi_{\mathcal{F}}(\nu)\}. \end{aligned}$$

Note that the summands of the second sum of the right hand side are all equal to some nonnegative constant C', (and that by convention the sum is 0 if $n < n_k$). Let f be the polynomial

$$f(z) := \deg X(\binom{z+1}{2} - \binom{n_0+1}{2}) + C\cdot(z - n_0) - K - P(r', \varepsilon, n_0).$$

Then for $n \geq n_0$:

$$h^0(\mathcal{F}(n)) - P(r', \varepsilon, n) \leq -f(m(n)) - C'\cdot(n - m(n))$$

There is an integer $N_1 > n_0$ such that $f(\nu) > 0$ for all $\nu \geq N_1$. Assume $N > N_1$. If $n_k - 1 \geq N_1$ then for all $n \geq N$ one has $m(n) \geq N_1$, hence $f(m(n)) > 0$ and $h^0(\mathcal{F}(n)) < P(r', \varepsilon, n)$. Hence we can restrict to the case that n_k is uniformly bounded by N_1. Let $G := \max\{-f(n) | n_0 \leq n \leq N_1\}$. Suppose $C' > 0$. There are positive integers T, T' with T' depending on X, P and r only, such that $C' = T/T'$. Choose an integer $N_2 > \max\{N_1, G\cdot T' + N_1\}$. Assume $N > N_2$. Then for all $n \geq N$

$$h^0(\mathcal{F}(n)) - P(r', \varepsilon, n) \leq -f(n_k - 1) - (n + 1 - n_k)\cdot C' \leq G - (N_2 - N_1)\cdot C' < 0.$$

Again we can restrict to the case $C' = 0$. But this gives (1).

Let $N_3 = \lceil N_2 + (2g-2)/H^2 + 1 \rceil$ and assume $N > N_3$. Then for all $n \geq N$, one has $n > n_k + (2g-2)/H^2$ so that $h^1(\mathcal{H}_n|_H) = h^1(\mathcal{F}(n)|_H) = 0$. In particular

$$H^2(\mathcal{F}(n)) = H^2(\mathcal{F}(n+1)) = H^2(\mathcal{F}(n+2)) = \dots,$$

and these cohomology groups must vanish for $n \gg 0$, hence already for $n \geq N$. This is assertion (2). Moreover,

$$h^0(\mathcal{F}(n_0)|_H) \leq h^0(\mathcal{F}(N_3)|_H) = \chi(\mathcal{F}(N_3)|_H) = \Delta\chi_{\mathcal{F}}(N_3) = \Delta P(r', \varepsilon, N_3)$$

according to (1). Let $M := \max\{\lceil \Delta P(r', \varepsilon, N_3) \rceil | 0 \leq r' \leq r\}$. Then (3) holds.

It remains to prove (4). Since $\mathcal{F}(N_3) = \mathcal{H}_{N_3}$, there are short exact sequences

$$0 \longrightarrow \mathcal{O}_H(\nu - N_3)^{\oplus r'} \longrightarrow \mathcal{F}(\nu)|_H \longrightarrow Q_{N_3} \longrightarrow 0$$

for all $\nu = n_0, \dots, N_3$. Hence

$$h^1(\mathcal{F}(\nu)|_H) \leq r' \cdot h^1(\mathcal{O}_H(\nu - N_3)) \leq r' \cdot h^1(\mathcal{O}_H(n_0 - N_3))$$

and

$$h^2(\mathcal{F}(n_0)) \leq h^2(\mathcal{F}(N_3)) + \sum_{\nu=n_0+1}^{N_3} h^1(\mathcal{F}(\nu)|_H) \leq r(N_3 - n_0) \cdot h^1(\mathcal{O}_X(n_0 - N_3))$$

is uniformly bounded. Since by assumption $h^1(\mathcal{F}(n_0)) \leq Q$ and $h^0(\mathcal{F}(n_0)) \leq Q$, the Euler characteristic $\chi(\mathcal{F}(n_0))$ lies in a finite set of integers. By (1) $\Delta\chi_{\mathcal{F}}$ is given. Hence $\chi_{\mathcal{F}}$ lies in a finite set of polynomials. Using (3) and criterion 1.7 we conclude that the set of modules $\mathcal{F}$ we are left with is bounded. Therefore there is a constant $N_4 > N_3$ such that $h^1(\mathcal{F}(n)) = 0$ if $n \geq N_4$. The lemma holds, if we choose any $N > N_4$. □

An immediate consequence of this lemma is the following boundedness result:

Corollary 1.10. *Suppose X is a surface. The set of isomorphism classes of $\mathcal{O}_X$-modules $\mathcal{E}$ which occur in μ-semistable pairs $(\mathcal{E}, \alpha)$ with $\mathrm{T}(\mathcal{E}_\alpha) = 0$ is bounded.*

Proof: Apply lemma 1.9 with $Q = h^0(\mathcal{E}_0)$. The proof of the lemma shows that $h^0(\mathcal{E}(n_0)) \leq Q$. By Serre's theorem $h^0(\mathcal{E}(n)) = \chi_{\mathcal{E}}(n) = P(r, 1, n))$ for all large enough numbers n, so the second alternative of the lemma holds. Part (3) then states: $h^0(\mathcal{E}(n_0)|_H) \leq M$ for some $\mathcal{E}$-regular hyperplane section H and some constant M which is independent of $\mathcal{E}$. Therefore the Kleiman criterion applies to the set of modules $\mathcal{E}(n_0)$ with the constant $K := \max\{Q, M, r \cdot \deg X\}$. □

As a consequence of the corollary there is an integer $\hat{N}$ such that $\mathcal{E}(n)$ is globally generated and $h^i(\mathcal{E}(n)) = 0$ for all $i > 0$, $n \geq \hat{N}$ and for all $\mathcal{O}_X$-modules $\mathcal{E}$ satisfying the hypotheses of the corollary. Note that according to lemma 1.2 among these all the modules occuring in semistable pairs can be found.

After these preparations we can prove the equivalent to theorem 1.5 in the surface case:

Theorem 1.11. *Suppose X is a surface. There is an integer N depending on X, $\mathcal{O}_X(1)$, $h^0(\mathcal{E}_0)$ and P, such that*

i) if $(\mathcal{E},\alpha)$ is (semi)stable (with respect to δ) then $(\mathcal{E}(n),\alpha(n))$ is sectional (semi)-stable (with respect to $\delta(n)$) for all $n \geq N$, and

ii) if $(\mathcal{E}(n),\alpha(n))$ is sectional (semi)stable for some $n \geq N$, then $(\mathcal{E},\alpha)$ is (semi)stable.

Proof: By the boundedness result 1.10 the dimension of $H^1(\mathcal{E}(n_0))$ is uniformly bounded for all $\mathcal{E}$ satisfying the hypotheses of the corollary. Let $Q := h^0(\mathcal{E}_0) + \max\{h^1(\mathcal{E}(n_0))\}$. Let N be the number obtained by applying lemma 1.9. Without loss of generality $N > \hat{N}$.

ad i): Suppose $(\mathcal{E},\alpha)$ is (semi)stable. Apply lemma 1.9 to $\mathcal{E}$. Since by Serre's theorem $h^0(\mathcal{E}(n)) = \chi(\mathcal{E}(n)) = P(r,1,n)$ for all sufficiently large n, the second alternative of the lemma holds and shows $h^1(\mathcal{E}(n)) = h^2(\mathcal{E}(n)) = 0$ for $n \geq N$. Hence $V := H^0(\mathcal{E}(n))$ has dimension $\chi(n)$. Now let $\mathcal{F}$ be a submodule of $\mathcal{E}$. Then either $h^0(\mathcal{F}(n)) < P(\mathrm{rk}\mathcal{F},\varepsilon(\mathcal{F}),n)$ for all $n \geq N$, in which case we are done, or we have $\Delta\chi_{\mathcal{F}} = \Delta(\mathrm{rk}\mathcal{F},\varepsilon(\mathcal{F}))$. Let $\mathcal{E}' = \mathcal{E}_\alpha$ if $\mathcal{F} \subseteq \mathcal{E}_\alpha$ and $\mathcal{E}' = \mathcal{E}$ else. Let $\mathcal{S} := \mathcal{E}'/\mathcal{F}$, $\bar{\mathcal{S}} := \mathcal{S}/\mathrm{T}(\mathcal{S})$ and let $\bar{\mathcal{F}}$ be the kernel of the epimorphism $\mathcal{E}' \to \bar{\mathcal{S}}$. Then $\mathrm{rk}\mathcal{F} = \mathrm{rk}\bar{\mathcal{F}}$, $\varepsilon(\mathcal{F}) = \varepsilon(\bar{\mathcal{F}})$, and we must have $\Delta\chi_{\bar{\mathcal{F}}} = \Delta P(\mathrm{rk}\mathcal{F},\varepsilon(\mathcal{F})) = \Delta\chi_{\mathcal{F}}$. Hence $\bar{\mathcal{F}}/\mathcal{F} = \mathrm{T}(\mathcal{S})$ has zero-dimensional support. There is a short exact sequence

$$0 \longrightarrow \bar{\mathcal{F}}(n_0) \longrightarrow \mathcal{E}'(n_0) \longrightarrow \bar{\mathcal{S}}(n_0) \longrightarrow 0.$$

Now $\bar{\mathcal{S}}(n_0)$ cannot have global sections. For otherwise there is a submodule in $\bar{\mathcal{S}}$ isomorphic to $\mathcal{O}_X(-n_0)$. Let $\mathcal{G}$ be its preimage in $\mathcal{E}'$. Then

$$\Delta\chi_{\mathcal{G}} = \Delta\chi_{\bar{\mathcal{F}}} + \Delta\chi_{\mathcal{O}_X(-n_0)} \leq \Delta P(\mathrm{rk}\mathcal{F}+1,\varepsilon(\mathcal{F})) = \Delta P(\mathrm{rk}\mathcal{F},\varepsilon(\mathcal{F})) + \Delta P(1,0)$$

contradicting lemma 1.8. But this shows that

$$\begin{aligned} h^1(\bar{\mathcal{F}}(n_0)) \leq h^1(\mathcal{E}'(n_0)) &\leq h^1(\mathcal{E}(n_0)) + h^0(\mathcal{E}(n_0)/\mathcal{E}'(n_0)) \\ &\leq h^1(\mathcal{E}(n_0)) + h^0(\mathcal{E}_0(n_0)) \leq Q. \end{aligned}$$

By part (4) of lemma 1.9 we now conclude that

$$h^0(\mathcal{F}(n)) \leq h^0(\bar{\mathcal{F}}(n)) = \chi(\bar{\mathcal{F}}(n))\,(\leq)\,P(\mathrm{rk}\mathcal{F},\varepsilon(\mathcal{F}))$$

for all $n \geq N$ if $\bar{\mathcal{F}}\,(\subseteq)\,\mathcal{E}$. Only the case $\bar{\mathcal{F}} = \mathcal{E}$ for stable pairs needs special attention: In this case one has $h^0(\mathcal{F}(n)) < h^0(\mathcal{E}(n))$, because $\mathcal{F}$ is a proper submodule of $\mathcal{E}$ and $\mathcal{E}(n)$ is globally generated for all $n \geq N$. Hence (semi)stability implies sectional (semi)stability for all $n \geq N$.

ad ii) Suppose $(\mathcal{E}(n), \alpha(n))$ is sectional (semi)stable for some $n \geq N$. Assume that there exists a submodule $\mathcal{F} \subseteq \mathcal{E}$ with $\Delta\chi_{\mathcal{F}} > \Delta P(\mathrm{rk}\mathcal{F}, \varepsilon(\mathcal{F}))$. If such a module exists at all, we may assume that it is maximal with this property among the submodules of $\mathcal{E}$. Let $\mathcal{S} = \mathcal{E}/\mathcal{F}$. The maximality of $\mathcal{F}$ implies that $\mathcal{S}$ is torsion free if $\varepsilon(\mathcal{F}) = 1$ and that α embeds $\mathrm{T}(\mathcal{S})$ into $\mathrm{T}(\mathcal{E}_0)$ if $\varepsilon(\mathcal{F}) = 0$. Hence $h^0(\mathrm{T}(\mathcal{S})) \leq Q$. Suppose $\mathcal{G}$ is any submodule of $\mathcal{S}$. Let $\mathcal{F}'$ be the preimage of $\mathcal{G}$ under the map $\mathcal{E} \to \mathcal{S}$. Then

$$\Delta\chi_{\mathcal{G}} + \Delta\chi_{\mathcal{F}} = \Delta\chi_{\mathcal{F}'} \leq \Delta P(\mathrm{rk}\mathcal{F} + \mathrm{rk}\mathcal{G}, 1) = \Delta P(\mathrm{rk}\mathcal{F}, \varepsilon(\mathcal{F})) + \Delta(\mathrm{rk}\mathcal{G}, 1 - \varepsilon(\mathcal{F})).$$

The inequality in the middle of this line is infered from the maximality of $\mathcal{F}$. Hence

$$\Delta\chi_{\mathcal{G}} \leq \Delta(\mathrm{rk}\mathcal{G}, 1 - \varepsilon(\mathcal{F})) + \{\Delta P(\mathrm{rk}\mathcal{F}, \varepsilon(\mathcal{F})) - \Delta\chi_{\mathcal{F}}\} < \Delta P(\mathrm{rk}\mathcal{G}, 1 - \varepsilon(\mathcal{F})).$$

Therefore we can apply lemma 1.9 to the module $\mathcal{S}$ with $\varepsilon = 1 - \varepsilon(\mathcal{F})$. But we did assume that $(\mathcal{E}(n), \alpha(n))$ was sectional semistable. Hence there exists a vector space $V \subseteq H^0(\mathcal{E}(n))$ of dimension $\chi(n)$ such that

$$\dim(V \cap H^0(\mathcal{F}(n))) \leq P(\mathrm{rk}\mathcal{F}, \varepsilon(\mathcal{F}), n)$$

and

$$h^0(\mathcal{S}(n)) \geq \dim V - \dim(V \cap H^0(\mathcal{F}(n))) \geq \chi(n) - P(\mathrm{rk}\mathcal{F}, \varepsilon(\mathcal{F}), n) = P(\mathrm{rk}\mathcal{S}, 1 - \varepsilon(\mathcal{F}), n)$$

This excludes the first alternative of the lemma, and we get $\Delta\chi_{\mathcal{S}} = \Delta(\mathrm{rk}\mathcal{S}, 1 - \varepsilon(\mathcal{F}))$ and equivalently $\Delta\chi_{\mathcal{F}} = \Delta(\mathrm{rk}\mathcal{F}, \varepsilon(\mathcal{F}))$, which contradicts the original assumption. Thus we have proven that $\Delta\chi_{\mathcal{F}} \leq \Delta P(\mathrm{rk}\mathcal{F}, \varepsilon(\mathcal{F}))$. But this means that $\mathcal{E}$ satisfies the hypotheses of corollary 1.10. By the remark following the corollary we have $h^0(\mathcal{E}(\nu)) = \chi(\mathcal{E}(\nu))$ for all $\nu \geq N$ since $N \geq \hat{N}$, so that necessarily $V = H^0(\mathcal{E}(n))$. Applying lemma 1.9 to $\mathcal{F}$ we see that either

$$h^0(\mathcal{F}(\nu)) < P(\mathrm{rk}\mathcal{F}, \varepsilon(\mathcal{F}), \nu) \text{ for all } \nu \geq N, \text{ in particular } \chi_{\mathcal{F}} < P(\mathrm{rk}\mathcal{F}, \varepsilon(\mathcal{F})),$$

or

> $h^2(\mathcal{F}(\nu)) = 0$ for all $\nu \geq N$ and hence
>
> $$\chi_{\mathcal{F}}(n) = h^0(\mathcal{F}(n)) - h^1(\mathcal{F}(n)) \leq h^0(\mathcal{F}(n))\,(\leq)\,P(\mathrm{rk}\mathcal{F}, \varepsilon(\mathcal{F}), n),$$
>
> which together with $\Delta\chi_{\mathcal{F}} = \Delta P(\mathrm{rk}\mathcal{F}, \varepsilon(\mathcal{F}))$ implies $\chi_{\mathcal{F}}\,(\leq)\,P(\mathrm{rk}\mathcal{F}, \varepsilon(\mathcal{F}))$.

This finishes the proof. □

1.3 The basic construction

Let X be a curve or a surface. By the results of the previous section the set of modules $\mathcal{E}$ with fixed Hilbert polynomial χ that occur in semistable pairs is bounded. In particular, there is a projective open and closed part A of the Picard scheme $\mathrm{Pic}(X)$ such that $[\det \mathcal{E}] \in A$ for all $\mathcal{E}$ in semistable pairs. Let $\mathcal{L} \in \mathrm{Pic}(A \times X)$ be a universal line bundle. Then there is an integer N such that for all $n \geq N$ the following conditions are simultaneously satisfied:

- $0 < \delta(n) < \chi(n)$.
- $\mathcal{E}$ is globally generated and $h^i(\mathcal{E}(n)) = 0$ for all $i > 0$ and for all $\mathcal{E}$ in semistable pairs.
- $(\mathcal{E}, \alpha)$ is (semi)stable (with respect to δ) if and only if $(\mathcal{E}(n), \alpha(n))$ is sectional (semi)stable (with respect to $\delta(n)$).
- If p_A, p_X denote the projection maps from $A \times X$ to A and X, respectively, then $R^i p_{A*}(\mathcal{L} \otimes p_X^* \mathcal{O}_X(n)) = 0$ for all $i > 0$, $\mathcal{U}_n := p_{A*}(\mathcal{L} \otimes p_X^* \mathcal{O}_X(n))$ is locally free and $p_A^*(\mathcal{U}_n) \otimes p_X^* \mathcal{O}_X(-n) \to \mathcal{L}$ is surjective.

By twisting the pairs $(\mathcal{E}, \alpha)$ with $\mathcal{O}_X(n)$ for sufficiently large n we can always assume that the assertions above hold for $N = 0$. We make this assumption for the rest of this section and write $p := \chi(0)$ and $\bar{\delta} := \delta(0)$.

Let V be a vector space of dimension p and let $V_X = V \otimes_k \mathcal{O}_X$. Quotient modules of V_X with Hilbert polynomial χ are parametrized by a projective scheme $\mathrm{Quot}^{\chi}_{X/V_X}$ ([Gr, 3.1.]). On the product $\mathrm{Quot}^{\chi}_{X/V_X} \times X$ there is a universal quotient $\tilde{q} : p_X^* V_X \longrightarrow\!\!\!\!\rightarrow \tilde{\mathcal{E}}$. Forming the determinant bundle of $\tilde{\mathcal{E}}$ induces a morphism

$$det : \mathrm{Quot}^{\chi}_{X/V_X} \longrightarrow \mathrm{Pic}(X)$$

so that $\det \tilde{\mathcal{E}} = (det \times \mathrm{id}_X)^*(\mathcal{L}) \otimes p_{Quot}^*(\mathcal{M})$ for some line bundle $\mathcal{M} \in \mathrm{Pic}(\mathrm{Quot}^{\chi}_{X/X_V})$. Let Q denote the preimage of A under the map det. We use the same symbols for the universal quotient and its restriction to $Q \times X$.

Further let $P := \mathbb{P}(\mathrm{Hom}(V, H^0(\mathcal{E}_0))^{\vee})$. Again there is a universal homomorphism $\tilde{a} : (V \otimes_k H^0(\mathcal{E}_0)^{\vee}) \otimes \mathcal{O}_P \longrightarrow\!\!\!\!\rightarrow \mathcal{O}_P(1)$. For sufficiently high n the direct image sheaf $\mathcal{H} := p_{Q*}(\mathrm{Ker}\, \tilde{q} \otimes p_X^* \mathcal{O}_X(n))$ is locally free and the canonical homomorphism

$$\beta : p_Q^* \mathcal{H} \to \mathrm{Ker}\, \tilde{q} \otimes p_X^* \mathcal{O}_X(n)$$

is surjective, so that there is an exact sequence

$$p_Q^* \mathcal{H} \otimes p_X^* \mathcal{O}_X(-n) \xrightarrow{\tilde{\beta}} p_X^* V_X \xrightarrow{\tilde{q}} \tilde{\mathcal{E}} \longrightarrow 0.$$

$\tilde{\beta}$ induces a homomorphism of $\mathcal{O}_Q$-modules

$$\gamma : \mathcal{H} \otimes_k H^0(\mathcal{E}_0(n))^\vee \to \mathcal{O}_Q \otimes_k (V \otimes_k H^0(\mathcal{E}_0)^\vee).$$

Let $\mathcal{I}$ be the ideal in the symmetric algebra $\mathcal{S}^*(V \otimes_k H^0(\mathcal{E}_0)^\vee) \otimes_k \mathcal{O}_Q$ which is generated by the image of γ and let $B \subset P \times Q$ be the corresponding closed subscheme. Let $\pi_P : B \to P$ and $\pi_Q : B \to Q$ be the projection maps and let $\mathcal{O}_B(1) := \pi_P^* \mathcal{O}_P(1)$. This scheme B is the starting point for the construction of the moduli space for semistable pairs. We introduce the following notations: Let

$$q_B := (\pi_Q \times \mathrm{id}_X)^* \tilde{q} : V \otimes \mathcal{O}_{B \times X} \to \mathcal{E}_B := (\pi_Q \times \mathrm{id}_X)^* \tilde{\mathcal{E}}$$

and

$$a_B := \pi_P^* \tilde{a} : (V \otimes H^0(\mathcal{E}_0)^\vee) \otimes \mathcal{O}_B \to \mathcal{O}_B(1).$$

By definition of B an arbitrary morphism $h : T \to P \times Q$ factors through the closed immersion $B \to P \times Q$ if and only if the pull-back under h of the composition $p_P^* \tilde{a} \circ p_Q^* \gamma$ is the zero map. This is equivalent to saying that the pull-back under $h \times \mathrm{id}_X$ of the induced homomorphism $V \otimes \mathcal{O}_{P \times Q \times X} \to p_P^* \mathcal{O}_P(1) \otimes p_X^* \mathcal{E}_0$ factors through $V \times \mathcal{O}_{T \times X} \to (h \times \mathrm{id}_X)^* \tilde{\mathcal{E}}$. This applies in particular to B itself. Let $\alpha_B : \mathcal{E}_B \to p_B^* \mathcal{O}_B(1) \otimes p_X^* \mathcal{E}_0$ be the induced homomorphism.

Lemma 1.12. *(i) There is an open subscheme Q^0 of Q such that u is a point in Q^0 if and only if $h^i(\tilde{\mathcal{E}}_u) = 0$ for all $i > 0$ and the homomorphism $\tilde{q}_u : V_X \otimes k(u) \to \tilde{\mathcal{E}}_u$ induces an isomorphism on the spaces of global sections.*

(ii) Let $(\mathcal{E}, \alpha)$ be a flat family of pairs parametrized by a noetherian k-scheme T. Then there is an open subscheme $S \subseteq T$ such that $\mathrm{Ker}(\alpha_t)$ *is torsionfree for a geometric point t of T if and only if t is a point of S.*

Proof: (i) By semicontinuity of h^i there is an open subscheme of Q of points u for which the higher cohomology groups of $\tilde{\mathcal{E}}_u$ vanish. For those points $h^0(\tilde{\mathcal{E}}_u) = p$ and hence $H^0(q_u)$ is an isomorphism if and only if $h^0(\mathrm{Ker}\, q_u) = 0$, which again is an open condition for u.

(ii) For n large enough there is a locally free $\mathcal{O}_T$-module $\mathcal{G}$ and a surjection $\mathcal{G} \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E}^\vee$ and dually an inclusion $\beta' : \mathcal{E}^{\vee\vee} \to \mathcal{G}^\vee \otimes \mathcal{O}_X(n)$. Note that there is an open subscheme O of $T \times X$ which meets every fibre X_t and for which the restriction $\mathcal{E}|_O$ is locally free, so that in particular $\vartheta : \mathcal{E} \to \mathcal{E}^{\vee\vee}$ is an isomorphism when restricted to O. If we let $\beta = \beta' \circ \vartheta$, then the kernel of $\beta_t : \mathcal{E}_t \to \mathcal{G}^\vee(t) \otimes \mathcal{O}_X(n)$ is precisely the torsion part of $\mathcal{E}_t$. Hence the kernel of $\gamma_t := (\alpha_t, \beta_t) : \mathcal{E}_t \to \mathcal{E}_0 \oplus (\mathcal{G}^\vee(t) \otimes \mathcal{O}_X(n))$ is the torsion submodule of $\mathrm{Ker}(\alpha_t)$. It is enough to show that the points t with $\mathrm{Ker}(\gamma_t) = 0$ form an open set. But this is [EGA, Cor IV 11.1.2]. □

Let S be the open subscheme of B which according to the lemma belongs to the family $(\mathcal{E}_B, \alpha_B)$, and let $B^0 = S \cap (P \times Q^0)$. The algebraic group $\mathrm{SL}(V)$ acts naturally on Q and P from the right. On closed points $[q : V \otimes \mathcal{O}_X \to \mathcal{E}]$ and $[a : V \to H^0(\mathcal{E}_0)]$ this action is given by $[q] \cdot g = [q \circ (g \otimes \mathrm{id}_{\mathcal{O}_X})]$ and $[a] \cdot g = [a \circ g]$.

Lemma 1.13. *B^0 is invariant under the diagonal action of $\mathrm{SL}(V)$ on $P \times Q$.*

Proof: This is clear from the characterization of B as the subscheme of points $([q], [a])$ for which there is a commuting diagram

$$\begin{array}{ccc} V \otimes \mathcal{O}_X & \xrightarrow{q} & \mathcal{E} \\ a\downarrow & & \alpha\downarrow \\ H^0(\mathcal{E}_0) \otimes \mathcal{O}_X & \xrightarrow{ev} & \mathcal{E}_0. \end{array}$$

□

B^0 has the following local universal property:

Lemma 1.14. *Suppose T is a noetherian k-scheme parametrizing a flat family $(\mathcal{E}, \alpha)$ of semistable pairs on X. Then there is an open covering $T = \bigcup T_i$ and for each T_i a morphism $h_i : T_i \to B^0$ and a nowhere vanishing section s_i in $h_i^* \mathcal{O}_B(1)$ such that the pair $(\mathcal{E}, \alpha)|_{T_i}$ is isomorphic to the pair $((f_i \times \mathrm{id}_X)^* \mathcal{E}_B, (f_i \times \mathrm{id}_X)^*(\alpha_B)/s_i)$.*

Proof: Let T be a noetherian scheme and $(\mathcal{E}, \alpha)$ a flat family of semistable pairs on X parametrized by T. According to the remarks in the first paragraph of this section the direct image sheaf $p_{T*}\mathcal{E}$ is locally free of rank p [Ha, Thm 12.8]. Hence locally on T there are trivializations $V \otimes \mathcal{O}_T \to p_{T*}\mathcal{E}$, which lead to quotient maps $q : V \otimes \mathcal{O}_{T \times X} \to \mathcal{E}$. By the universal property of Q there is a k-morphism $f : T \to Q$ and a uniquely determined isomorphism $\Phi : (f \times \mathrm{id}_X)^* \tilde{\mathcal{E}} \to \mathcal{E}$ such that $\Phi \circ (f \times \mathrm{id}_X)^* \tilde{q} = q$. Moreover, the composition

$$V \otimes \mathcal{O}_{T \times X} \xrightarrow{q} \mathcal{E} \xrightarrow{\alpha} p_X^* \mathcal{E}_0$$

determines a homomorphism $a : V \otimes \mathcal{O}_T \to H^0(\mathcal{E}_0) \otimes \mathcal{O}_T$. By the universal property of P there is a morphism $g : T \to P$ and a uniquely determined nowhere vanishing section s in $g^* \mathcal{O}_P(1)$ such that $a = g^* \tilde{a}/s$. It is clear from the construction that $h := (f, g) : T \to P \times Q$ factors through B^0. Φ^{-1} is an isomorphism from $\mathcal{E}$ to $(f \times \mathrm{id}_X)^* \tilde{\mathcal{E}} = (h \times \mathrm{id}_X)^* \mathcal{E}_B$, and $\alpha \circ \Phi = (h \times \mathrm{id}_X)^*(\alpha_B)/s$. □

If $h : T \to B$ and $g : T \to \mathrm{SL}(V)$ are morphisms let $h \cdot g$ denote the composition $T \xrightarrow{(h,g)} B \times \mathrm{SL}(V) \to B$, where the last map is the induced group action of $\mathrm{SL}(V)$ on B.

Lemma 1.15. *Suppose T is a noetherian k-scheme and $h = (f, g) : T \to B^0 \subset P \times Q$ a k-morphism. h induces (locally) isomorphism classes of families of pairs. If $g : T \to \mathrm{SL}(V)$ is a morphism, then the families induced by h and $h \cdot g$ are isomorphic.*

Conversely, if h_1 and h_2 induce isomorphic families parametrized by T, then there is an etale morphism $c : T' \to T$ and a morphism $g' : T' \to \mathrm{SL}(V)$ such that the morphisms $(h_1 \circ c) \cdot g'$ and $(h_2 \circ c)$ are equal.

Proof: Let $h : T \to B^0$ be a k-morphism. Applying $(h \times \mathrm{id}_X)^*$ to $\mathcal{E}_B$ and α_B induces a family $\mathcal{E}_T$ and a homomorphism $\alpha_T : \mathcal{E}_T \to h^*(\mathcal{O}_B(1)) \otimes p_X^*\mathcal{E}_0$. Locally there are nowhere vanishing sections in $h^*\mathcal{O}_B(1)$. Dividing α_T by any of these sections defines families of pairs. Two such sections differ by a section in $\mathcal{O}_T^*$. But this sheaf embeds into the sheaf of automorphisms of $\mathcal{E}_T$. Hence the families induced by different sections are isomorphic. The second statement is clear. For the third assume that h_1 and h_2 are morphisms such that for $i = 1, 2$ there are nowhere vanishing sections $s_i \in H^0(T, h_i^*\mathcal{O}_B(1))$. Let

$$\mathcal{E}_i := (h_i \times \mathrm{id}_X)^*\mathcal{E}_B, \quad q_i := (h_i \times \mathrm{id}_X)^*q_B \quad \text{and} \quad \alpha_i := (h_i \times \mathrm{id}_X)^*\alpha_B/s_i.$$

Assume that there is an isomorphism $\Phi : (\mathcal{E}_1, \alpha_1) \to (\mathcal{E}_2, \alpha_2)$ of pairs. The quotient maps q_i induce isomorphisms $\bar{q}_i : V \otimes \mathcal{O}_T \to p_{T*}\mathcal{E}_i$ because of the definition of B^0 ([Ha, Thm 12.11]). The composition $\bar{q}_2^{-1} \circ p_{T*}\Phi \circ \bar{q}_1$ corresponds to a morphism $g : T \to \mathrm{GL}(V)$. Define morphisms c and ℓ by the fibre product diagram

$$\begin{array}{ccc} T' & \xrightarrow{\ c\ } & T \\ {\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle \det(g)} \\ \mathbf{G}_m & \xrightarrow{p^{th}\ power} & \mathbf{G}_m \end{array}$$

and let $g' := (g \circ c)/\ell : T' \to \mathrm{SL}(V)$. It is easy to check that $(h_1 \circ c)\cdot g' = (h_2 \circ c)$. □

$\tilde{q}$ induces a homomorphism $\Lambda^r(\mathcal{O}_Q \otimes V_X) \to \det \tilde{\mathcal{E}} = (det \times \mathrm{id})^*(\mathcal{L}) \otimes p_Q^*\mathcal{M}$ and hence a homomorphism $\Lambda^r V \otimes_k \mathcal{M}^\vee \to det^*\mathcal{U}_0 = det^*p_{A*}\mathcal{L}$ ([Ma]). This finally leads to morphisms $T : Q \to P' := \mathbb{P}(\mathcal{H}om(\Lambda^r V, \mathcal{U}_0)^\vee)$ and $\tau := (\pi_P, T) : B \to P \times P'$.

Lemma 1.16. $\mathrm{SL}(V)$ *acts naturally on P' from the right, T and τ are equivariant morphisms with respect to this action.* □

We can choose a very ample line bundle $\mathcal{N}$ on A such that $\mathcal{N}' := \mathcal{O}_{P'}(1) \otimes p_A^*\mathcal{N}$ is very ample on P'. For any positive numbers ν, ν' the line bundle $\mathcal{O}_P(\nu) \otimes (\mathcal{N}')^{\otimes \nu'}$ is very ample on $P \times P'$ and inherits a canonical linearization with respect to the $\mathrm{SL}(V)$-action [MF, 1.4,1.6]. Choose ν and ν'such that $\nu/\nu' = r\bar{\delta}/(p - \bar{\delta})$. Let $Z^{(s)s} \subseteq P \times P'$ be the open subscheme of (semi)stable points with respect to this linearization. Here *stable* means *properly stable* in the sense of Mumford.

Theorem 1.17. *The open subscheme $B^{(s)s} = B^0 \cap \tau^{-1}(Z^{(s)s})$ of B has the following property: A morphism $h : T \to B^0$ induces families of (semi)stable pairs in the sense of lemma 1.15 if and only if h factors through $B^{(s)s}$. The restriction of τ to B^{ss} is a finite morphism $\tau^{ss} : B^{ss} \to Z^{ss}$.*

For the proof we need a stability criterion for $\tau([a],[q])$, and we need it in slightly greater generality. But before this, note that if $q : V_X \to \mathcal{E}$ defines a point $[q]$ in $Q(k)$, then the fibre of the projective bundle P' through the point $T([q])$ is isomorphic to $P'' := \mathbb{P}(Hom(\Lambda^r V, H^0(\det \mathcal{E}))^\vee)$, and $\tau([a],[q])$ is a (semi)stable point in $P \times P'$ if and only if it is (semi)stable point in $P \times P''$ with respect to the canonical linearization of $\mathcal{O}_P(\nu) \otimes \mathcal{O}_{P''}(\nu')$ ([Ma, 4.12]). In particular, the choice of $\mathcal{N}$ is of no consequence for the definition of $Z^{(s)s}$.

Proposition 1.18. *Let $(\mathcal{E}, \alpha)$ be a pair with $\det \mathcal{E} \in A$ and torsionfree $\mathcal{E}_\alpha$. Suppose there is a generically surjective homomorphism $q : V_X \to \mathcal{E}$ such that $q \circ \alpha \neq 0$. Let $T : \Lambda^r V \to H^0(\det \mathcal{E})$ and $a : V \to H^0(\mathcal{E}_0)$ be the derived homomorphisms. Then $([a],[T])$ is a (semi)stable point in $P \times P''$ with respect to the given linearization if and only if q injects V into $H^0(\mathcal{E})$ and $(\mathcal{E}, \alpha)$ is sectional (semi)stable with respect to $\bar{\delta}$.*

The proof of this proposition is postponed to the next section.

Proof of theorem 1.17: Pairs $(\mathcal{E}, \alpha)$ that correspond to points $([a],[q])$ in B^0 satisfy the hypotheses of the proposition, for q is surjective, H^0q isomorphic and $\mathcal{E}_\alpha$ torsionfree. Hence by proposition 1.18 and theorems 1.5 and 1.11 $(\mathcal{E}, \alpha)$ is (semi)stable if and only if $\tau([a],[q])$ is a (semi)stable point. This proves the first assertion of the theorem. In order to show that $\tau^{ss} := \tau|_{B^{ss}}$ is a finite morphism it is enough to show that τ^{ss} is proper and injective ([EGA, IV 8.11.1]). This will be done in two steps:

Proposition 1.19. *τ^{ss} is a proper morphism.*

Proof: Using the valuation criterion it suffices to show the following: Let $\bar{C} = Spec\, R$ be a nonsingular affine curve, $c_0 \in \bar{C}$ a closed point defined by a local parameter $t \in R$ and C the open complement of c_0. Suppose we are given a commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad h \quad} & B^{ss} \\ \iota\downarrow & & \tau^{ss}\downarrow \\ \bar{C} & \xrightarrow{\quad m \quad} & Z^{ss}. \end{array}$$

We must show that (at least locally near c_0) there is a lift $\bar{h} : \bar{C} \to B^{ss}$ of m extending $\bar{h}$. Making C smaller if necessary we may assume that h induces homomorphisms

$$\mathcal{O}_C \otimes V_X \xrightarrow{\quad q \quad} \mathcal{F} \xrightarrow{\quad \alpha \quad} \mathcal{O}_C \otimes \mathcal{E}_0,$$

so that $(\mathcal{F}, \alpha)$ is a flat family of semistable pairs. Using Serre's theorem one can find a locally free $\mathcal{O}_X$-module $\mathcal{H}$ and an epimorphism $\mathcal{O}_C \otimes \mathcal{H}^\vee \twoheadrightarrow \mathcal{F}^\vee$. The kernel of the dual homomorphism $\beta : \mathcal{F} \to \mathcal{O}_C \otimes \mathcal{H}$ is the torsion submodule $\mathrm{T}(\mathcal{F})$. Since $\operatorname{Ker} \alpha$ and

and $\operatorname{Im}\alpha$ are C-flat, $(\operatorname{Ker}\alpha)_c \subset (\operatorname{Ker}\alpha_c)$. Since the kernel of the restriction of α to any fibre $X \times c$, $c \in C$, is torsion free by lemma 1.2, $\operatorname{Ker}\alpha$ is also torsion free. Therefore

$$(\alpha, \beta) : \mathcal{F} \to \mathcal{O}_C \otimes (\mathcal{E}_0 \oplus \mathcal{H})$$

is injective. There are integers a, b such that the composition

$$\mathcal{O}_C \otimes V_X \xrightarrow{\ q\ } \mathcal{F} \xrightarrow{(t^a\alpha, t^b\beta)} \mathcal{O}_C \otimes (\mathcal{E}_0 \oplus \mathcal{H})$$

extends to a homomorphism

$$\lambda : \mathcal{O}_{\bar{C}} \otimes V_X \longrightarrow \mathcal{O}_{\bar{C}} \otimes (\mathcal{E}_0 \oplus \mathcal{H})$$

which is nontrivial in each component when restricted to the special fibre X_{c_0}. Let $\bar{\mathcal{F}}$ be the maximal submodule of $\mathcal{O}_{\bar{C}} \otimes (\mathcal{E}_0 \oplus \mathcal{H})$ with the properties

$$\bar{\mathcal{F}}|_{C \times X} = \mathcal{F},\ \operatorname{Im}\lambda \subseteq \bar{\mathcal{F}} \text{ and } \dim \operatorname{Supp}(\bar{\mathcal{F}}/\operatorname{Im}\lambda) < e;$$

and let $\bar{\alpha} : \bar{\mathcal{F}} \to \mathcal{O}_{\bar{C}} \otimes \mathcal{E}_0$ be the projection map. Then $\bar{\mathcal{F}}$ is $\bar{C}$-flat, $(\bar{\mathcal{F}}, \bar{\alpha})|_{C \times X} \cong (\mathcal{F}, \alpha)$ and $q_{c_0} : V_X \to \mathcal{F}_{c_0}$ is generically surjective. Moreover $\bar{\alpha}_{c_0}$ is nonzero and $\operatorname{Ker}\bar{\alpha}_{c_0}$ is torsion free. For assume that $\mathrm{T}(\operatorname{Ker}\bar{\alpha}_{c_0}) \neq 0$ and let $\tilde{\mathcal{F}}$ be the kernel of the composite epimorphism

$$\bar{\mathcal{F}} \twoheadrightarrow \mathcal{F}_{c_0} \twoheadrightarrow \mathcal{F}_{c_0}/\mathrm{T}(\operatorname{Ker}\bar{\alpha}_{c_0}).$$

Then there is a short exact sequence

$$0 \longrightarrow \bar{\mathcal{F}} \longrightarrow \tilde{\mathcal{F}} \longrightarrow \mathrm{T}(\operatorname{Ker}\bar{\alpha}_{c_0}) \longrightarrow 0.$$

By construction $\bar{\alpha}$ extends to $\tilde{\alpha} : \tilde{\mathcal{F}} \to \mathcal{O}_{\bar{C}} \otimes \mathcal{E}_0$. Since $\mathcal{H}$ is normal and the codimension of $\operatorname{Supp}\mathrm{T}(\operatorname{Ker}\alpha)$ in $\bar{C} \times X$ is greater than 1, $\bar{\beta}$ also extends to a homomorphism $\tilde{\beta} : \tilde{\mathcal{F}} \to \mathcal{O}_{\bar{C}} \otimes \mathcal{H}$. Finally $(\tilde{\alpha}, \tilde{\beta}) : \tilde{\mathcal{F}} \to \mathcal{O}_{\bar{C}} \otimes (\mathcal{E}_0 \oplus \mathcal{H})$ is injective, contradicting the maximality of $\bar{\mathcal{F}}$. Hence indeed $\mathrm{T}(\operatorname{Ker}\bar{\alpha}_{c_0}) = 0$. Since q_{c_0} is generically surjective, $\operatorname{Ker}\bar{\alpha}_{c_0}$ torsionfree and $\bar{\alpha}_{c_0} \circ q_{c_0} \neq 0$, we can apply proposition 1.18 to the pair $(\bar{\mathcal{F}}_{c_0}, \bar{\alpha}_{c_0})$. By assumption on the map m the induced point in $P \times P'$ is semistable, hence $H^0 q_{c_0}$ is injective and $(\mathcal{F}_0, \alpha_0)$ sectional semistable. But then necessarily $\mathcal{F}_{c_0}$ is globally generated, $H^0 q_{c_0}$ isomorphic and q_{c_0} surjective. This shows that h extends to a morphism $\bar{h} : \bar{C} \to B$ with $\bar{h}(c_0) \in B^{ss}$. □

Proposition 1.20. *τ^{ss} is injective.*

Proof: Assume that for $i = 1, 2$ there are closed points $([a_i : V \to H^0(\mathcal{E}_0)], [q_i : V_X \to \mathcal{E}_i])$ with the same image under τ. We may assume that $a_1 = a_2$ and $\det \mathcal{E}_1 = \det \mathcal{E}_2$. Then there is an open subscheme $\emptyset \neq U \subset X$ such that $\mathcal{E}_{1|U}$, $\mathcal{E}_{2|U}$ are locally free and are in fact isomorphic as quotients of $V_{X|U}$. Then $\mathcal{E}_1/\mathrm{T}(\mathcal{E}_1)$ and $\mathcal{E}_2/\mathrm{T}(\mathcal{E}_2)$ are isomorphic as quotients of V_X via a map $\Phi : \mathcal{E}_1/\mathrm{T}(\mathcal{E}_1) \to \mathcal{E}_2/\mathrm{T}(\mathcal{E}_2)$ ([Ma], lemma 4.8). The kernels of the induced homomorphisms $\alpha_i : \mathcal{E}_i \to \mathcal{E}_0$ are torsionfree, so that the natural map $\mathcal{E}_i \to \mathcal{E}_0 \oplus \mathcal{E}_i/\mathrm{T}(\mathcal{E}_i)$ are injective. The diagram

$$\begin{array}{ccccc} V_X & \twoheadrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_0 \oplus \mathcal{E}_1/\mathrm{T}(\mathcal{E}_1) \\ \| & & & & {\scriptstyle \mathrm{id}+\Phi}\downarrow \\ V_X & \twoheadrightarrow & \mathcal{E}_2 & \longrightarrow & \mathcal{E}_0 \oplus \mathcal{E}_2/\mathrm{T}(\mathcal{E}_2) \end{array}$$

commutes and shows $\mathcal{E}_1$ and $\mathcal{E}_2$ are isomorphic as quotients of V_X. □

This completes the proof of theorem 1.17 up to the proof of proposition 1.18. □

Theorem 1.21. *Assume that X is a smooth projective variety of dimension one or two. Then there is a projective k-scheme $\mathcal{M}_\delta^{ss}(\chi, \mathcal{E}_0)$ and a natural transformation*

$$\varphi : \underline{\mathcal{M}}_\delta^{ss}(\chi, \mathcal{E}_0) \longrightarrow \mathrm{Hom}_{\mathrm{Spec}\,k}(\quad, \mathcal{M}_\delta^{ss}(\chi, \mathcal{E}_0)) \ ,$$

such that φ is surjective on rational points and $\mathcal{M}_\delta^{ss}(\chi, \mathcal{E}_0)$ is minimal with this property. Moreover, there is an open subscheme $\mathcal{M}_\delta^{s}(\chi, \mathcal{E}_0) \subset \mathcal{M}_\delta^{ss}(\chi, \mathcal{E}_0)$ such that φ induces an isomorphism of subfunctors

$$\underline{\mathcal{M}}_\delta^{s}(\chi, \mathcal{E}_0) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Spec}\,k}(\quad, \mathcal{M}_\delta^{s}(\chi, \mathcal{E}_0)) \ ,$$

i.e. $\mathcal{M}_\delta^{s}(\chi, \mathcal{E}_0)$ is a fine moduli space for all stable pairs.

Proof: By [MF, 1.10] and [Gi] there is a projective k-scheme $\mathcal{M}^{ss}$ and a morphism $\rho : B^{ss} \longrightarrow \mathcal{M}^{ss}$ which is a good quotient for the $\mathrm{SL}(V)$-action on B^{ss}. By lemma 1.15 and theorem 1.17 any family of semistable pairs parametrized by T induces morphisms $T_i \to B^{ss}$ for an appropriate open covering $T = \bigcup T_i$ such that the composition with ρ glue to a well-defined morphism $T \to \mathcal{M}^{ss}$. This establishes a natural transformation

$$\varphi : \underline{\mathcal{M}}_\delta^{ss}(\chi, \mathcal{E}_0) \longrightarrow \mathrm{Hom}_{\mathrm{Spec}\,k}(\quad, \mathcal{M}^{ss}) \ .$$

If $\psi : \underline{\mathcal{M}}^{ss} \longrightarrow \mathrm{Hom}(\quad, N)$ is a similar transformation, then the family $(\mathcal{E}_B, \alpha_B)|_{B^{ss}}$ induces an $\mathrm{SL}(V)$-invariant morphism $B^{ss} \longrightarrow N$, which therefore factors through a morphism $\mathcal{M}^{ss} \longrightarrow N$. Moreover there is an open subscheme $\mathcal{M}^{s} \subset \mathcal{M}^{ss}$ such that $B^s = \rho^{-1}(\mathcal{M}^s)$ and $\rho|_{B^s} : B^s \longrightarrow \mathcal{M}^s$ is a geometric quotient. In order to see that the family $(\mathcal{E}_B, \alpha_B)|_{B^s}$ descends to give a universal pair on $\mathcal{M}^s$ it is enough to show that the stable pairs have no automorphism besides the identity. But assume that $\Phi \neq \mathrm{id}$ is an automorphism of a stable pair $(\mathcal{E}, \alpha)$, i.e. $\Phi : \mathcal{E} \xrightarrow{\cong} \mathcal{E}$ and $\alpha \circ \Phi = \alpha$. Then $\psi = \Phi - \mathrm{id}$ is a nontrivial homomorphism from $\mathcal{E}$ to $\mathcal{E}_\alpha$. Apply the stability conditions to $\mathrm{Ker}\psi \subset \mathcal{E}$ and $\mathrm{Im}\psi \subset \mathcal{E}_\alpha$ to get

$$\mathrm{rk}\,\mathcal{E} \cdot \chi_{\mathrm{Ker}\psi} < \mathrm{rk}(\mathrm{Ker}\psi)(\chi_{\mathcal{E}} - \delta) + \delta \cdot \mathrm{rk}\,\mathcal{E}$$

and

$$\mathrm{rk}\,\mathcal{E} \cdot \chi_{\mathrm{Im}\psi} < \mathrm{rk}(\mathrm{Im}\psi)(\chi_{\mathcal{E}} - \delta) \ .$$

Summing up and using $\chi_{\mathrm{Im}\psi} + \chi_{\mathrm{Ker}\psi} = \chi_{\mathcal{E}}$ and $\mathrm{rk}(\mathrm{Im}\psi) + \mathrm{rk}(\mathrm{Ker}\psi) = \mathrm{rk}\mathcal{E}$ we get the contradiction $\chi_{\mathcal{E}} < \chi_{\mathcal{E}}$. □

1.4 Geometric stability conditions

In this section we prove proposition 1.18. Let $q : V_X \to \mathcal{E}$ and $\alpha : \mathcal{E} \to \mathcal{E}_0$ be homomorphisms of $\mathcal{O}_X$-modules. To these data we can associate vector space homomorphisms $T : \Lambda^r V \to H^0(\det \mathcal{E})$ and $a : V \to H^0(\mathcal{E}_0)$. If q is generically surjective, then T is nontrivial, and if $\alpha \circ q \neq 0$, then a is nontrivial. Let $([a],[T])$ denote the corresponding closed point in $P \times P''$ (notations as in section 1.3).

The group $\mathrm{SL}(V)$ acts on $P \times P''$ by

$$([a],[T]) \cdot g = ([a \circ g], [T \circ \Lambda^r g]).$$

We want to investigate the stability properties of $([a],[T])$ with respect to an $\mathrm{SL}(V)$-linearization of the very ample line bundle $\mathcal{O}_{P \times P''}(\nu, \nu')$, where ν, ν' are positive integers. These stability properties depend on the ratio $\eta := \nu/\nu'$ only. We will make use of the Hilbert criterion to decide about (semi)stability. Let $\lambda : \mathbf{G}_m \to \mathrm{SL}(V)$ be a 1-parameter subgroup, i. e. a nontrivial group homomorphism. There is a basis $v_1, \ldots, v_p$ of V such that $\mathbf{G}_m$ acts on V via λ with weights $\gamma_1, \ldots, \gamma_p \in \mathbb{Z}$:

$$\lambda(u) \cdot v_i = u^{\gamma_i} \cdot v_i \qquad \text{for all } u \in \mathbf{G}_m(k).$$

Reordering the v_i if necessary we may assume that $\gamma_1 \leq \ldots \leq \gamma_p$, $\sum \gamma_i = 0$, since $\det\lambda = 1$, and $\gamma_1 < \gamma_p$, since $\lambda \neq 1$.

For any multiindex $I = (i_1, \ldots, i_r)$ with $1 \leq i_1 < \ldots < i_r \leq p$ let $v_I = v_{i_1} \wedge \ldots \wedge v_{i_r}$ and $\gamma_I = \gamma_{i_1} + \cdots + \gamma_{i_r}$. The vectors v_I form a basis of $\Lambda^r V$, and $\mathrm{SL}(V)$ acts with weights γ_I with respect to this basis. $T(v_I) \neq 0$ if and only if the sections $q(v_{i_1}), \ldots, q(v_{i_r})$ are generically linearly independent, i. e. generate $\mathcal{E}$ generically. Now let

$$\mu = \mu([a], \lambda) := -\min\{\gamma_i | a(v_i) \neq 0\}.$$

$$\mu' = \mu([T], \lambda) := -\min\{\gamma_I | T(v_I) \neq 0\}$$

Lemma 1.22 (Hilbert criterion). *$([a],[T])$ is a (semi)stable point in $P \times P''$ with respect to $\mathcal{O}(\nu, \nu')$ if and only if $\hat{\mu} := \eta \cdot \mu + \mu' (\geq) 0$ for all 1-parameter subgroups λ.*

Proof: [MF, Thm 2.1.] □

For any linear subspace $W \subset V$ let $\mathcal{E}_{(W)} \subset \mathcal{E}$ be the submodule which is characterized by the properties : $\mathcal{E}/\mathcal{E}_{(W)}$ is torsionfree and $\mathcal{E}_{(W)}$ is generically generated by $q(W \otimes \mathcal{O}_X)$. In particular, let $\mathcal{E}_{(i)} = \mathcal{E}_{(\langle v_1, \ldots, v_i \rangle)}$, $i = 0, \ldots, p$ for a given basis $v_1, \ldots, v_p$. Then there is a filtration

$$\mathrm{T}(\mathcal{E}) = \mathcal{E}_{(0)} \subset \mathcal{E}_{(1)} \subset \ldots \subset \mathcal{E}_{(p-1)} \subset \mathcal{E}_{(p)} = \mathcal{E}.$$

Since $\mathcal{E}_{(i)}/\mathcal{E}_{(i-1)}$ is torsionfree, one has either $\mathcal{E}_{(i)} = \mathcal{E}_{(i-1)}$ or $\mathrm{rk}\mathcal{E}_{(i)} > \mathrm{rk}\mathcal{E}_{(i-1)}$. Consequently, there are integers $1 \leq k_1 < \ldots < k_r \leq p$ marking the points, where the rank jumps, i. e. k_ρ is minimal with $\mathrm{rk}\mathcal{E}_{(k_\rho)} = \rho$. Let K denote the multiindex $(k_1, \ldots, k_r)$. If I is any multiindex as above, let $i_0 = 0$ and $i_{r+1} = p + 1$ for notational convenience.

Lemma 1.23. $\mu' = -\gamma_K$.

Proof: By construction $T(v_K) \neq 0$. We must show that $\gamma_K \leq \gamma_I$ for every multiindex I with $T(v_I) \neq 0$. For any I and any $t \in \{1, \dots, r\}$ we let $\mathcal{E}_{I,t} = \mathcal{E}_{\langle v_{i_1}, \dots, v_{i_t} \rangle)}$. Now suppose $T(v_I) \neq 0$. Let $\ell = \max\{\lambda | k_t = i_t \quad \forall t < \lambda\}$. If $\ell \geq r+1$, then $I = K$ and we are done. We will procede by descending induction on ℓ. By definition of K, we have $k_\ell < i_\ell$. Define $\mathcal{E}'_{I,t} = \mathcal{E}_{(\langle v_{k_1}, \dots, v_{k_\ell}, v_{i_\ell}, \dots v_{i_t} \rangle)}$ for $t = \ell, \dots p$. Then $\mathcal{E}_{I,t} \subset \mathcal{E}'_{I,t}$, and $t \leq \operatorname{rk} \mathcal{E}'_{I,t} \leq t+1$. Let $m = \min\{t | \operatorname{rk} \mathcal{E}'_{I,t} = t, \ell \leq t \leq p\}$. Now define a multiindex

$$I' = (k_1, \dots, k_\ell, i_\ell, \dots, i_{m-1}, i_{m+1}, \dots, i_p).$$

(If $m = \ell$, drop the $i_\ell, \dots, i_{m-1}$ part; if $m = p$, drop the $i_{m+1}, \dots, i_p$ part.) Then we have $T(v_{I'}) \neq 0$, and $\gamma_{I'} \leq \gamma_I$ by monotony of I and γ. Moreover, I' and K agree at least in the first ℓ entries. Thus by induction $\gamma_K \leq \gamma_{I'} \leq \gamma_I$. □

Let $\ell := \min\{i | a(v_i) \neq 0\}$. Obviously $\mu = -\gamma_\ell$, so that $\hat{\mu} = -\gamma_K - \eta \cdot \gamma_\ell$. Now ℓ and K depend on the basis $v_1, \dots, v_p$ only, and μ is a linear function of γ for fixed ℓ and K. Using these notations, the Hilbert criterion can be expressed as follows:

Lemma 1.24. $([a],[T])$ *is a (semi)stable point if and only if*

$$\min_{\text{bases of } V} \min_{\gamma} -(\gamma_K + \eta \cdot \gamma_\ell) \quad (\geq) \quad 0.$$

□

We begin with minimizing over the set of all weight vectors γ. This is the cone spanned by the special weight vectors

$$\gamma^{(i)} = (\underbrace{i-p, \dots, i-p}_{i}, \underbrace{i, \dots, i}_{p-i})$$

for $i = 1, \dots, p-1$. For any weight vector γ can be expressed as $\gamma = \sum_{i=1}^{p-1} c_i \gamma^{(i)}$ with nonnegative rational coefficients $c_i = (\gamma_{i+1} - \gamma_i)/p$. In order to check (semi)stability for a given point it is enough to show $\hat{\mu}(\geq)0$ for each of these basis vectors. Let $\delta_i = 1$ or 0 if $\ell \leq i$ or $> i$, respectively. Evaluating $\hat{\mu}$ on $\gamma^{(i)}$ we get numbers

$$\mu^{(i)} = p \cdot (\max\{j | k_j \leq i\} + \eta \cdot \delta_i) - i \cdot (r + \eta).$$

If i increases, $\mu^{(i)}$ decreases unless i equals ℓ or any of the numbers k_j, in which case $\mu^{(i)}$ might jump. The critical values of i therefore are $\ell - 1$ and $k_j - 1$, $j = 1, \dots, r$, and the corresponding critical values of $\mu^{(i)}$ are:

$$\begin{array}{ll} p \cdot (j-1) - (k_j - 1) \cdot (r + \eta) & \text{if } 1 \leq j \leq r,\ 1 < k_j \leq \ell, \\ p \cdot (j-1) - (\ell - 1) \cdot (r + \eta) & \text{if } 1 \leq j \leq r+1,\ k_{j-1} < \ell \leq k_j,\ 1 < \ell, \\ p \cdot (j - 1 + \eta) - (k_j - 1) \cdot (r + \eta) & \text{if } 1 \leq j \leq r,\ \ell < k_j. \end{array}$$

If we put $\ell_j = \min\{k_j, \ell\}$, then the conditions imposed by these values of $\hat{\mu}$ can be comprised as follows:

$$\begin{array}{lll} (1) & 0\,(\leq)\,p{\cdot}(j-1)-(\ell_j-1){\cdot}(r+\eta) & \text{if } 1\leq j\leq r+1,\ 1<\ell_j \\ (2) & 0\,(\leq)\,p{\cdot}(j-1+\eta)-(k_j-1){\cdot}(r+\eta) & \text{if } 1\leq j\leq r. \end{array}$$

In the next step one should minimize these terms over all bases of V. But in fact, the relevant information is not the used basis itself but the flag of subspaces of V which it generates. The stability criterion takes the following form:

Lemma 1.25. $([a],[T])$ *is a (semi)stable point if and only if*

1) $\dim W{\cdot}(r+\eta)\,(\leq)\,p{\cdot}\mathrm{rk}\,\mathcal{E}_{(W)}$ *for all subspaces* $0\neq W\subseteq \mathrm{Ker}\,a$.

2) $\dim W{\cdot}(r+\eta)\,(\leq)\,p{\cdot}(\mathrm{rk}\,\mathcal{E}_{(W)}+\eta)$ *for all subspaces* $0\neq W\subseteq V$ *with* $\mathrm{rk}\,\mathcal{E}_{(W)}\,(\leq)\,r$.

□

We give the stability criterion still another form, shifting our attention from subspaces of V to submodules of $\mathcal{E}$:

Lemma 1.26. $([a],[T])$ *is a (semi)stable point if and only if*

(0) H^0q *is an injective map.*

(1) $V\cap H^0\mathcal{F}=0$ *or* $\dim(V\cap H^0\mathcal{F}){\cdot}(r+\eta)\,(\leq)\,p{\cdot}\mathrm{rk}\mathcal{F}$ *for all submodules* $\mathcal{F}\subseteq \mathrm{Ker}\alpha$.

(2) $\dim(V\cap H^0\mathcal{F}){\cdot}(r+\eta)\,(\leq)\,p{\cdot}(\mathrm{rk}\mathcal{F}+\eta)$ *for all submodules* $\mathcal{F}\subseteq\mathcal{E}$ *with* $\mathrm{rk}\mathcal{F}\,(\leq)\,\mathrm{rk}\mathcal{E}$.

Proof: If $([a],[T])$ is semistable, let $W:=\mathrm{Ker}H^0q$. Then $W\subseteq \mathrm{Ker}\,a$. From the lemma above it follows that $\dim W\leq p/(r+\eta){\cdot}\mathrm{rk}\,\mathcal{E}_{(W)}=0$. Hence (0) is a necessary condition. It is to show that the conditions(1) and (2) of lemma 1.25 and of lemma 1.26 are equivalent. Suppose we are given a submodule $\mathcal{F}\subseteq\mathcal{E}$. Let $W:=V\cap H^0\mathcal{F}$. Then $q(W\otimes\mathcal{O}_X)\subseteq\mathcal{F}$ and $\mathrm{rk}\,\mathcal{F}=\mathrm{rk}\,\mathcal{E}_{(W)}$. Moreover, if $\mathcal{F}\subseteq\mathcal{E}_\alpha$, then $W\subseteq\mathrm{Ker}\,a$. Now either $W=0$ or 1.25 applies and gives 1.26. Conversely, if $W\subseteq V$ is given, let $\mathcal{F}:=q(W\otimes\mathcal{O}_X)$. Then $W\subseteq V\cap H^0\mathcal{F}$ and $\mathrm{rk}\,\mathcal{E}_{(W)}=\mathrm{rk}\,\mathcal{F}$. Again, if $W\subseteq\mathrm{Ker}\,a$, then $\mathcal{F}\subseteq\mathcal{E}_\alpha$. Hence 1.26 implies 1.25.

Finally, we replace η by a more suitable parameter:

$$\bar{\delta}=\frac{p{\cdot}\eta}{r+\eta}\qquad \eta=\frac{r{\cdot}\bar{\delta}}{p-\bar{\delta}}.$$

Since η was a positive rational number, $\bar{\delta}$ is confined to the open interval $(0,p)$, which of course tallies with the data of the previous section. The following theorem, which differs from proposition 1.18 only in the choice of words, summarizes the discussion of this section:

Theorem 1.27. *If in addition to the global assumptions of this section $\mathcal{E}_\alpha$ is torsionfree, then $([a],[T])$ is a (semi)stable point of $P \times P''$ if and only if the following conditions are satisfied:*

- *H^0q is an injective homomorphism.*
- *$(\mathcal{E},\alpha)$ is sectional stable with respect to $\bar{\delta}$.*

Proof: If $\mathcal{E}_\alpha$ is torsionfree then every nontrivial submodule of $\mathcal{E}_\alpha$ has positive rank. Hence condition (1) in 1.26 can be replaced by

(1') $dim(V \cap H^0\mathcal{F}) \cdot (r+\eta)(\leq) p \cdot \mathrm{rk}\,\mathcal{F}$ for all submodules $\mathcal{F} \subseteq \mathcal{E}_\alpha$.

As a result of replacing η by $\bar{\delta}$ in (1') and 1.26(0),(2) one obtains the defintion of sectional (semi)stability. □

2 Applications

This chapter is organized as follows. In 2.1 we show that the existence of semistable pairs gives an upper bound for δ. Rationality conditions on δ imply the equivalence of semistability and stability. If δ varies within certain regions the semistability conditions remain unchanged. This is formulated and specified for the rank two case.

2.2 deals with Higgs pairs. Again we concentrate on the rank two case. We make the first step to generalize the diagrams of Bertram and Thaddeus to algebraic surfaces. The restriction of $\mu-$stable vector bundles on an algebraic surface to a curve of high degree induces an immersion of the moduli space of vector bundles on the surface into the moduli space of vector bundles on the curve. The understanding of this process is important, e.g. for the computation of Donaldson polynomials and for the study of the geometry of the moduli space on the surface ([Ty]). With the help of a restriction theorem for $\mu-$stable pairs $(\mathcal{E}, \alpha : \mathcal{E} \to \mathcal{O})$ we construct an approximation of this immersion, which will hopefully shed some light on the relation between the original moduli spaces. It is remarkable that the limit of any approximation is independent of the polarization.

In 2.3 we first compare our stability for $\mathcal{E}_0 = \mathcal{O}_D^{\oplus r}$, where D is a divisor on a curve, with the notion of Seshadri of stable sheaves with level structure along a divisor([Se]). We will have a closer look at the moduli space of rank two sheaves of degree 0 with a level structure at a single point. Furthermore certain results from 2.2 are reconsidered in the case of $\mathcal{E}_0$ being a vector bundle on a divisor.

2.1 Numerical properties of δ

Let X be a smooth projective variety with an ample divisor H, $\mathcal{E}_0$ a coherent $\mathcal{O}_X$−module and δ a positive rational polynomial of degree $\dim X - 1$ with leading coefficient $\delta_1 \geq 0$.

Lemma 2.1. *Assume $(\mathcal{E}, \alpha)$ is a semistable pair such that $\mathcal{E}_\alpha \neq 0$. Then*

$$\delta (\leq) \chi_{\mathcal{E}} - \frac{\mathrm{rk}\mathcal{E}}{\mathrm{rk}\mathcal{E}_\alpha}(\chi_{\mathcal{E}} - \chi_{\mathcal{E}_0}) \ .$$

If $\mathcal{E}_0 \cong \mathcal{O}_X$ and $\mathrm{rk}\mathcal{E} > 1$, then

$$\delta (\leq) \frac{\mathrm{rk}\mathcal{E} \cdot \chi_{\mathcal{O}_X} - \chi_{\mathcal{E}}}{\mathrm{rk}\mathcal{E} - 1}$$

and in particular

$$\delta_1 (\leq) - \frac{\deg \mathcal{E}}{\mathrm{rk}\mathcal{E} - 1} \ .$$

If $\mathcal{E}_0$ is torsion, then

$$\delta (\leq) \chi_{\mathcal{E}_0}$$

and in particular $\delta_1 (\leq) \deg \mathcal{E}_0$.

Proof: The first inequality follows immediately from the stability condition i). If $\mathcal{E}_0 \cong \mathcal{O}_X$ use $\chi_{\mathcal{E}_\alpha} = \chi_{\mathcal{E}} - \chi_{\mathrm{Im}\alpha} \geq \chi_{\mathcal{E}} - \chi_{\mathcal{E}_0}$ and $\mathrm{rk}\mathcal{E}_\alpha = \mathrm{rk}\mathcal{E} - 1$. □

It is much more convenient to work with μ−stability only. In fact for the general δ one can achieve that every semistable pair is μ−stable.

Lemma 2.2. *There exists a discrete set of rationals $0 \leq ... < \eta_i < \eta_{i+1} < ...$ including 0, such that for $\delta_1 \in (\eta_i, \eta_{i+1})$ every semistable pair with respect to δ is in fact μ−stable and the μ−stability conditions depend only on i.*

Proof: Define $\{\eta_i\} := [0, -d/(r-1)) \cap \{(ar - sd)/(r-s) | a, s \in \mathbb{Z},\ 0 \leq s < r\}$. If $\delta_1 \in (\eta_i, \eta_{i+1})$, then the right hand sides of the μ− semistability conditions $\deg \mathcal{G} \leq sd/r - \delta_1 s/r$ and $\deg \mathcal{G} \leq sd/r + \delta_1 (r-s)/r$ are not integer ($s = \mathrm{rk}\mathcal{G}$). Therefore μ−semistability and μ−stability coincide. Moreover, the integral parts of the right hand sides depend only on i, i.e. for two different choices of δ_1 in the intervall (η_i, η_{i+1}) the μ−stability conditions are the same. □

More explicit results can be achieved in special cases:

Proposition 2.3. *For $r = 2$ and $\mathcal{E}_0 \in Pic(X)$ and $\delta_1 \in (\eta_i, \eta_i + 2)$, where $\eta_i := \max\{0, 2i + d\}$ with $i \in \mathbb{Z}$, every semistable pair is μ−stable. The stability in this region does not depend on δ.*

Proof: For $\mathcal{E}_0 \in Pic(X)$ all semistable pairs $(\mathcal{E}, \alpha)$ have torsionfree $\mathcal{E}$ and $\mathrm{rk}\mathcal{E}_\alpha = 1$. In particular the stability conditions concern rank one subsheaves only. Now $\delta_1 \in (\eta_i, \eta_i + 2)$ is equivalent to $-1 - i < d/2 - \delta_1/2 < -i$, $i + d - 1 < d/2 + \delta_1/2 < i + d$ and $\delta_1 > 0$. □

As the last numerical criterion we mention

Lemma 2.4. *Assume* $\delta_1 < \min_{0 \leq s < r}\{(r - sd)/(r - s) + r(r - s)[sd/r]\}$.

i) Then every sheaf $\mathcal{E}$ in a semistable pair $(\mathcal{E}, \alpha)$ without torsion in dimension zero is torsionfree and $\mu-$semistable.

ii) If $\mathcal{E}$ is torsionfree and $\mu-$semistable and $\alpha : \mathcal{E} \to \mathcal{E}_0$ a nontrivial homomorphism such that $\mathcal{E}_\alpha$ does not contain a destabilizing subsheaf, then $(\mathcal{E}, \alpha)$ is $\mu-$stable.

Proof: The condition on δ_1 is equivalent to either of the two conditions:

$[sd/r, sd/r + \delta_1(r - s)/r) \cap \mathbb{Z} = \emptyset$ for $0 \leq s < r$.

$[sd/r - \delta_1/r, sd/r) \cap \mathbb{Z} = \emptyset$ for $0 < s \leq r$. □

2.2 Higgs pairs in dimension one and two

A Higgs pair in this context is a vector bundle together with a global section. (This notion should not be confused with a Higgs field as a section $\theta \in H^0(\mathcal{E}nd\mathcal{E} \otimes \Omega^1_X)$ with $\theta \wedge \theta = 0$!) Instead of considering a global section we prefer to work with a homomorphism from the dualized bundle to the structure sheaf. These objects will be called pairs as in the general context.

First we remind of the situation in the curve case, which was motivation for us to go on.

Definition 2.5. Let C be a smooth curve. As introduced in 1.3 $\mathcal{M}^{ss}_\delta(d, 2, \mathcal{O})$(resp. $\mathcal{M}^{ss}_\delta(\mathcal{Q}, 2, \mathcal{O})$) denotes the moduli space of semistable pairs $(\mathcal{E}, \alpha : \mathcal{E} \to \mathcal{O})$ with respect to δ, where $\mathcal{E}$ is a rank two sheaf of degree d (with determinant $\mathcal{Q}$).

Remark 2.6. Notice, that δ is just a number and that a sheaf occuring in a semistable pair is always torsionfree and hence a vector bundle. Moreover the stability conditions reduce to $\deg(\mathcal{E}_\alpha) \leq d/2 - \delta/2$ and $\deg(\mathcal{G}) \leq d/2 + \delta/2$ for all line bundles $\mathcal{G} \subset \mathcal{E}$.

For the following we assume $d < 0$.

Definition 2.7. $U_{C,i}(d) := \mathcal{M}^{ss}_\delta(d, 2, \mathcal{O})$ and $SU_{C,i}(\mathcal{Q}) := \mathcal{M}^{ss}_\delta(\mathcal{Q}, 2, \mathcal{O})$, where $\delta \in (max\{0, 2i + d\}, 2i + d + 2)$.

Note that according to proposition 2.3 the spaces $U_{C,i}(d)$ and $SU_{C,i}(Q)$ do not depend on the choice of δ

Proposition 2.8. *(M. Thaddeus) $U_{C,i}(d)$ and $SU_{C,i}(Q)$ are projective fine moduli spaces. Every semistable pair is automatically stable.*

Proof: [Th] or 1.21 □

Proposition 2.9. *i) For $i \geq -d$ the moduli spaces $U_{C,i}(d)$ are empty.*

ii) For $i = \lfloor -d/2 - 1 \rfloor + 1$ there are morphisms

$$U_{C,i}(d) \longrightarrow U(d)$$

and

$$SU_{C,i}(Q) \longrightarrow SU(Q) \quad ,$$

where $U(d)$ and $SU(Q)$ are the moduli spaces of semistable vector bundles of degree d and determinante Q, resp. The fibre over a stable bundle $\mathcal{E}$ is isomorphic to $\mathbb{P}(H^0(\mathcal{E}^\vee)^\vee)$. In particular they are projective bundles for $0 \gg d \equiv 1(2)$.

iii) A pair $(\mathcal{E}, \alpha)$ lies in $SU_{C,-d-1}(Q)$ if and only if there is a nonsplitting exact sequence of the form

$$0 \longrightarrow Q \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{O} \longrightarrow 0 \ .$$

Thus $SU_{C,-d-1} \cong \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}, Q)^\vee)$.

Proof: i) and ii) follow from the general criteria. A similar result as iii) holds in the surface case. We give the proof there. □

The following picture illustrates the situation:

$$\begin{array}{llll} SU_{C,\lfloor -d/2-1\rfloor+1}(Q) & SU_{C,\lfloor -d/2-1\rfloor+2}(Q) & \dots & SU_{C,-d-1}(Q) \cong \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}, Q)^\vee) \\ \quad\downarrow & & & \\ SU(Q) & & & \end{array}$$

M. Thaddeus is able 'to resolve the picture' by a sequence of blowing ups and downs. In particular all the spaces $SU_{C,i}$ are rational. This process makes it possible to trace a generalized theta divisor on $SU_{C,i}$ to a certain section of $\mathcal{O}(k)$ on $\mathbb{P}(H^1(Q))$. This method is used in [Th] to give a proof of the Verlinde formula.

We go on to proceed in a similar way in the case of a surface.

Let X be an algebraic surface with an ample divisor H. Now $\mathcal{M}_\delta^{ss}(d, c_2, 2, \mathcal{O})$ ($\mathcal{M}_\delta^{ss}(Q, c_2, 2, \mathcal{O})$) denotes the moduli space of semistable pairs $(\mathcal{E}, \alpha : \mathcal{E} \to \mathcal{O}_X)$ with respect to δ, where $\mathcal{E}$ is a rank two sheaf of degree d ($:= c_1.H$) (with determinant Q)

and second Chern class c_2. For the existence of such pairs it is necessary that δ be linear with nonnegative leading coefficient δ_1. As in 2.6 a sheaf occuring in a semistable pair is torsionfree and the stability conditions are

$$\chi_{\mathcal{G}}(\leq)\frac{\chi_{\mathcal{E}}}{2}-\frac{\delta}{2}$$

for all rank one subsheaves $\mathcal{G}\subset\mathcal{E}_\alpha$ and

$$\chi_{\mathcal{G}}(\leq)\frac{\chi_{\mathcal{E}}}{2}+\frac{\delta}{2}$$

for all rank one subsheaves $\mathcal{G}\subset\mathcal{E}$.

Definition 2.10. For δ such that $\delta_1\in(\max\{0,2i+d\},2i+d+2)$ we define $U_i:=\mathcal{M}_\delta^{ss}(d,c_2,2,\mathcal{O})$ and $SU_i:=\mathcal{M}_\delta^{ss}(\mathcal{Q},c_2,2,\mathcal{O})$.

Again, note that according to 2.3 the definition does not depend on the choice of δ.

Corollary 2.11. *U_i and SU_i are projective fine moduli spaces. Every semistable pair is $\mu-$stable.*

Proof: It follows immediately fom 1.21 and section 2.1. □

Proposition 2.12. *If $(\mathcal{E},\alpha)$ is a $\mu-$semistable pair with respect to δ, then $4c_2(\mathcal{E})-c_1^2(\mathcal{E})\geq-\delta_1/(4H^2)$.*

Proof: If $(\mathcal{E},\alpha)$ is a $\mu-$semistable pair the homomorphism α can be extended to a homomorphism $\mathcal{E}^{\vee\vee}\to\mathcal{O}$ and the resulting pair is still $\mu-$semistable with $c_1(\mathcal{E}^{\vee\vee})=c_1(\mathcal{E})$ and $c_2(\mathcal{E}^{\vee\vee})\leq c_2(\mathcal{E})$. Thus it is enough to prove the inequality for locally free pairs. If $\mathcal{E}$ itself is a $\mu-$semistable bundle the Bogomolov inequality says $4c_2-c_1^2\geq 0$. If $\mathcal{E}$ is not $\mu-$semistable, then there is an exact sequence

$$0\longrightarrow\mathcal{L}_1\longrightarrow\mathcal{E}\longrightarrow\mathcal{L}_2\otimes I_Z\longrightarrow 0\ ,$$

where I_Z is the ideal sheaf of a zero dimensional subscheme and $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles with $\deg\mathcal{E}/2<\deg\mathcal{L}_1\leq\deg\mathcal{E}/2+(1/2)\delta_1$ and $\deg\mathcal{E}/2-(1/2)\delta_1\leq\deg\mathcal{L}_2<\deg\mathcal{E}/2$. Using $c_2(\mathcal{E})=c_1(\mathcal{L}_1)c_1(\mathcal{L}_2)+l(Z)\geq c_1(\mathcal{L}_1)c_1(\mathcal{L}_2)=(1/4)\{(c_1(\mathcal{L}_1)+c_1(\mathcal{L}_2))^2-(c_1(\mathcal{L}_1)-c_1(\mathcal{L}_2))^2\}=(1/4)c_1^2(\mathcal{E})-\frac{1}{4}(c_1(\mathcal{L}_1)-c_1(\mathcal{L}_2))^2$ and Hodge index theorem, which gives $(c_1(\mathcal{L}_1)-c_1(\mathcal{L}_2))^2\leq((\deg\mathcal{L}_1-\deg\mathcal{L}_2)^2)/H^2$ we infer the claimed inequality. Notice, that for $\delta_1\to 0$ the inequality converges to the usual Bogomolov inequality. □

Proposition 2.13. *i) For $i\geq -d$ the moduli spaces U_i and SU_i are empty.*

ii) If $i = \lfloor -d/2-1 \rfloor + 1$, then every pair $(\mathcal{E},\alpha) \in U_i$ has a μ−semistable $\mathcal{E}$. There is rational map $U_i \to U(c_1,c_2)$ (the moduli space of semistable, torsionfree sheaves), which is a morphism for $d \equiv 1(2)$. The image of the rational map contains all μ−stable sheaves $\mathcal{E}$ with $\mathrm{Hom}(\mathcal{E},\mathcal{O}) \neq 0$. The fibre over such a point is $\mathbb{P}(\mathrm{Hom}(\mathcal{E},\mathcal{O})^{\vee})$.

iii) Every pair $(\mathcal{E},\alpha) \in SU_{-d-1}$ sits in an nontrivial extension of the form

$$0 \longrightarrow I_{Z_1} \otimes \mathcal{Q} \longrightarrow \mathcal{E} \xrightarrow{\alpha} I_{Z_2} \longrightarrow 0 \quad ,$$

where I_{Z_i} are the ideal sheaves of certain zero dimensional subscheme. In the case $Z_1 = \emptyset$, e.g. $\mathcal{E}$ is locally free, every such extension gives in turn a stable pair $(\mathcal{E},\alpha) \in SU_{-d-1}$.

Proof: *i)* and *ii)* follow again from 2.1 If $(\mathcal{E},\alpha) \in SU_{-d-1}$, then $\deg \mathcal{E}_\alpha < d + 1/2$, which is equivalent to $\deg(\mathrm{Im}\alpha) > -1/2$. Since $\mathrm{Im}\alpha \subset \mathcal{O}$ it follows $\mathrm{Im}\alpha = I_{Z_2}$. A splitting of the induced exact sequence would lead to the contradiction $0 \leq \deg I_{Z_2} \leq -1/2$. Let $(\mathcal{E},\alpha)$ be given by a sequence with $Z_1 = \emptyset$. For $\mathcal{G} \subset \mathcal{E}_\alpha$ one gets the required inequality $\deg \mathcal{G} \leq d < d + 1/2$. If $\mathcal{G} \subset \mathcal{E}$ and $\mathcal{G} \not\subset \mathcal{E}_\alpha$, the sheaf $\mathcal{G}$ has the form $\mathcal{G} = I_{Z_3} \subset I_{Z_2}$. Without restriction we can assume that $\mathcal{E}/\mathcal{G}$ is torsionfree. Since $\mathcal{E}/\mathcal{G}$ is an extension of I_{Z_2}/I_{Z_3} by $\mathcal{Q}$ and $\mathrm{Ext}^1(I_{Z_2}/I_{Z_3},\mathcal{Q}) = 0$, $\mathcal{G}$ in fact equals I_{Z_2} and therefore defines a splitting of the sequence. □

Corollary 2.14. *The set of all pairs $(\mathcal{E},\alpha) \in SU_{-d-1}$ with $\mathcal{E}_\alpha$ locally free, which in particular contains all locally free pairs, forms a projective scheme over $Hilb^{c_2}(X)$ with fibre over $[Z] \in Hilb^{c_2}(X)$ isomorphic to $\mathbb{P}(\mathrm{Ext}^1(I_Z,\mathcal{Q})^{\vee})$.*

Proof: If $(\mathcal{E},\alpha)$ is a universal family over $SU_{-d-1} \times X$, then the set of points $t \in SU_{-d-1}$ with $l((\mathrm{coker}\,\alpha)_t)$ maximal is closed. It is easy to see that $(\mathrm{coker}\,\alpha)_t \cong \mathrm{coker}(\alpha_t)$ and that $l(\mathrm{coker}(\alpha_t))$ is maximal, i.e. is equal to c_2 if $\mathrm{Ker}(\alpha_t)$ is locally free. Therefore the set of all pairs with locally free kernel $\mathcal{E}_\alpha$ is closed and $\mathcal{O}/\mathrm{Im}\alpha$ induces the claimed morphism to $Hilb^{c_2}(X)$. □

Corollary 2.15. *The moduli space of all locally free pairs $(\mathcal{E},\alpha) \in SU_{-d-1}$ does not depend on the polarization of X.*

Remark 2.16. *i)* Bradlow introduced in [Br] the notion of ϕ−stability with respect to a parameter τ. If we set $\delta_1 = -d + (\tau/2\pi)vol(X)$ (d is the degree of $\mathcal{E}$) both notions coincide, i.e. a pair $(\mathcal{E},\alpha : \mathcal{E} \to \mathcal{O})$ with a locally free $\mathcal{E}$ is μ−stable in our sense if and only if $(\mathcal{E}^{\vee}, \phi = \alpha^{\vee} \in H^0(\mathcal{E}^{\vee}))$ is ϕ−stable with respect to the parameter τ in Bradlow's sense. He proves a Kobayashi-Hitchin correspondence in this situation, i.e. he shows: $(\mathcal{E},\alpha)$ is μ−stable (or a sum of a μ−stable pair with μ−stable bundles) if and only if the vortex equation has a solution, i.e. there exists a hermitian metric H on $\mathcal{E}^{\vee}$, such that

$$\Lambda_\omega F_H + \tau\frac{i}{2}id = \frac{i}{2}\phi \otimes \phi^{*_H} \quad .$$

F_H is the curvature of the metric connection on $\mathcal{E}^{\vee}$, ω is a fixed Kähler form and Λ_ω is the adjoint of $\wedge\omega$. Now,if $(\mathcal{E},\alpha) \in SU_{-d-1}$ one can take δ near to $-d$. That corresponds to $\tau \to 0$. Although 2.15 shows that SU_{-d-1} is independent of the polarization H, i.e. of the Hodge metric, for us there is no obvious reason in the analytical equation.
ii) In [Rei] the space SU_{-d-1} is stratified and equipped with certain line bundles. These objects Reider calls Jacobians of rank two alluding to a Torelli kind theorem for algebraic surfaces.

In order to study the restriction of $\mu-$stable vector bundles to curves of high degree it could be usefull to study the restriction of $\mu-$stable pairs to those curves. As a generalization of a result of Bogomolov we prove

Theorem 2.17. *For fixed c_1, c_2, δ and H there exists a constant n_0, such that for $n \geq n_0$ and any smooth curve $C \in |nH|$ the restriction of every locally free, $\mu-$stable pair to C is a $\mu-$stable pair on the curve with respect to $n\delta_1$.*

Proof: If $\mathcal{E}$ is locally free the kernel $\mathcal{E}_\alpha$ is a line bundle. In particular the restriction of the injection $\mathcal{E}_\alpha \subset \mathcal{E}$ to a curve remains injective. Thus $(\mathcal{E}_\alpha)_C = \mathrm{Ker}(\alpha_C)$. Since $\deg(\mathcal{E}_\alpha)_C = n\deg\mathcal{E}_\alpha$, the two inequalities $\deg\mathcal{E}_\alpha < \deg\mathcal{E}/2 - \delta_1/2$ and $\deg(\mathcal{E}_\alpha)_C < \deg\mathcal{E}_C/2 - n\delta_1/2$ are equivalent. Thus the first of the stability conditions on C is always satisfied. In order to prove the second we proceed in two steps.
i) By Bogomolov's result ([Bo]) there is a constant n_0, such that the restriction of a $\mu-$stable vector bundle to a smooth curve $C \in |nH|$ for $n \geq n_0$ is stable. Since the inequality $\deg\mathcal{G} < \deg\mathcal{E}_C/2 + n\delta_1/2$ for a line bundle $\mathcal{G} \subset \mathcal{E}_C$ is weaker than the stability condition on $\mathcal{E}_C$, the theorem follows immediately from Bogomolov's result for all $\mu-$stable pairs $(\mathcal{E},\alpha)$, where $\mathcal{E}$ is a $\mu-$stable vector bundle.
ii) Therefore it remains to prove the theorem for pairs with $\mathcal{E}$ not $\mu-$stable. Any such vector bundle is an extension of $\mathcal{L}_2 \otimes I_Z$ by $\mathcal{L}_1$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles with $\deg\mathcal{E}/2 \leq \deg\mathcal{L}_1 < \deg\mathcal{E}/2 + (1/2)\delta_1$. I_Z is as usual the ideal sheaf of a zero dimensional subscheme. If $C \in |nH|$ is a curve with $C \cap Z = \emptyset$, then the restriction of the extension to C induces the exact sequence

$$0 \longrightarrow (\mathcal{L}_1)_C \longrightarrow \mathcal{E} \longrightarrow (\mathcal{L}_2)_C \longrightarrow 0 \quad .$$

If $\mathcal{G} \subset \mathcal{E}_C$ is a line bundle, then either $\mathcal{G} \subset (\mathcal{L}_1)_C$ or $\mathcal{G} \subset (\mathcal{L}_2)_C$. This implies $\deg\mathcal{G} \leq \deg(\mathcal{L}_1)_C = n\deg\mathcal{L}_1 < \deg\mathcal{E}_C/2 + (1/2)n\delta_1$ or $\deg\mathcal{G} \leq \deg(\mathcal{L}_2)_C = n\deg\mathcal{E} - n\deg\mathcal{L}_1 \leq \deg\mathcal{E}_C/2$. Hence $(\mathcal{E}_C, \alpha_C)$ is stable. If $C \cap Z \neq \emptyset$ we only get a sequence of the form

$$0 \longrightarrow (\mathcal{L}_1)_C(Z.C) \longrightarrow \mathcal{E}_C \longrightarrow (\mathcal{L}_2)_C(-Z.C) \longrightarrow 0$$

Notice, that $\mathcal{O}_C(-Z.C) \cong (I_Z \otimes \mathcal{O}_C)/\mathrm{T}(I_Z \otimes \mathcal{O}_C)$. As above $\deg \mathcal{G} \leq \deg(\mathcal{L}_1)_C + \deg(Z.C) \leq \deg(\mathcal{L}_1)_C + l(Z)$ or $\deg \mathcal{G} \leq \deg(\mathcal{E}_C)/2$ for every line bundle $\mathcal{G} \subset \mathcal{E}_C$. If $\deg \mathcal{L}_1 + l(Z)/n < \deg \mathcal{E}/2 + \delta_1/2$, then $(\mathcal{E}_C, \alpha_C)$ is stable. There exists a positive number ε depending only on the degree, δ and H, such that $\deg \mathcal{L}_1 \leq \deg \mathcal{E}/2 + \delta_1/2 - \varepsilon$. Thus it suffices to bound $l(Z)$ by $n_0\varepsilon$. That is done by the following computation. $l(Z) = c_2 - c_1(\mathcal{L}_1)c_1(\mathcal{L}_2) = c_2 - c_1^2/4 + (1/4)(c_1(\mathcal{L}_1) - c_1(\mathcal{L}_2))^2 \leq c_2 - c_1^2/4 + \delta_1^2/(4H^2)$. Thus $n_0 > (1/\varepsilon)(c_2 - c_1^2/4 + \delta_1^2/(4H^2))$ satisfies $l(Z) < n_0\varepsilon$. □

With the notation of 2.7 and 2.10 one proves

Corollary 2.18. *For fixed c_1, c_2 and H there exists a number n_0, such that for every smooth curve $C \in |nH|$ for $n_0 \leq n \equiv 1(2)$ and every i with $\lfloor -d/2 - 1 \rfloor + 1 \leq i \leq -d - 1$ the restriction of pairs gives an injective immersion, i.e. an injective morphism with injective tangent map:*

$$U_i^f \to U_{C, in+(n-1)/2}$$

(The superscript denotes the subset of all locally free pairs)
Proof: The technical problem here is, that the constant n_0 in the last theorem depends on δ and not only on i. Therefore we fix for every i a very special δ, namely $\delta_1 = 2i + d + 1$. Since we only consider finitely many i's there is an n_0, such that the restriction gives a morphism $U_i^f \to U_{C,in+(n-1)/2}$. Here we use $n \equiv 1(2)$. Since the occuring family of vector bundles is bounded one can choose n_0, such that $H^k(X, \mathcal{H}om(\mathcal{E}, \mathcal{E}')(-nH)) = 0$ $(k = 0, 1)$ and $H^0(\mathcal{E}^\vee(-nH)) = 0$ for $n \geq n_0$ and all vector bundles $\mathcal{E}$ and $\mathcal{E}'$ occuring in a pair in one of the moduli spaces U_i. Thus $(\mathcal{E}, \alpha)_C \cong (\mathcal{E}', \alpha')_C$ if and only if $\mathcal{E} \cong \mathcal{E}'$ and α maps to α' under this isomorphism, i.e. the restriction morphism is injective. A standard argument in deformation theory shows that the Zariski tangent space of U_i^f at $(\mathcal{E}, \alpha)$ is isomorphic to the hypercohomology $\mathbb{H}^1(\mathcal{E}nd\mathcal{E}^\vee \to \mathcal{E}^\vee)$ of the indicated complex which is given by $\varphi \mapsto \varphi(\alpha^\vee)$ ([We]). Analogously, the Zariski tangent space of $U_{C,j}$ at $(\mathcal{E}_C, \alpha_C)$ is isomorphic to the hypercohomology $\mathbb{H}^1(\mathcal{E}nd\mathcal{E}_C^\vee \to \mathcal{E}_C^\vee)$. The Zariski tangent map is described by the restriction of hypercohomology classes. Both hypercohomology groups sit in exact sequences of the form

$$\ldots \to H^0(\mathcal{E}^\vee) \to \mathbb{H}^1(\mathcal{E}nd\mathcal{E}^\vee \to \mathcal{E}^\vee) \to H^1(\mathcal{E}nd\mathcal{E}) \to \ldots$$

and

$$\ldots \to H^0(\mathcal{E}_C^\vee) \to \mathbb{H}^1(\mathcal{E}nd\mathcal{E}_C^\vee \to \mathcal{E}_C^\vee) \to H^1(\mathcal{E}nd\mathcal{E}_C) \to \ldots \quad ,$$

resp. By our assumptions the restrictions $H^0(\mathcal{E}^\vee) \to H^0(\mathcal{E}_C^\vee)$ and $H^1(\mathcal{E}nd\mathcal{E}^\vee) \to H^1(\mathcal{E}nd\mathcal{E}_C)$ are injective. Hence the Zariski tangent map of the restriction of stable pairs is injective, too. □

We remark that neither the starting nor the end point of the series of moduli spaces on the surface is sent to the corresponding point of the series moduli spaces on the curve. A slight generalization of the theorem allows to restrict μ-stable pairs to a stable pair on a curve $C \in |nH|$ with respect to the parameter $n\delta_1 + c$, where c is a constant depending only on δ_1, c_1, c_2 and H.

2.3 Framed bundles and level structures

In this paragraph we consider pairs of rank r, where $\mathcal{E}_0 \cong \mathcal{O}_D^{\oplus r}$ or more generally where $\mathcal{E}_0$ is a vector bundle of rank r on a divisor D.
We start with pairs on a curve. In this case D is a finite sum of points. As far as we know, Seshadri was the first to consider and to construct moduli spaces for such pairs. In [Se] they were called sheaves with a level structure. The general stability conditions as developped in this paper and specialized to this case present a slight generalization of Seshadris stability concept in terms of the parameter δ, which in [Se] is always $l(D)$. The geometric invariant theory which Seshadri used to construct the moduli spaces differs from the one in 1.3. In [Se] a point $[\mathcal{O}^{\oplus N} \twoheadrightarrow \mathcal{E}]$ of the Quotscheme is mapped to a point

$$([\mathcal{O}^{\oplus N}(x_1) \twoheadrightarrow \mathcal{E}(x_1)], ..., [\mathcal{O}^{\oplus N}(x_n) \twoheadrightarrow \mathcal{E}(x_n)])$$

in the product of Grassmannians (the x_i are sufficiently many generic points. The conditions for a point in this product to be semistable in the sense of geometric invariant theory translate into the semistability properties for pair. However, to generalize the construction to the higher dimensional case one has to map the Quotscheme into a different projective space as in 1.3 and study the stability conditions there.

Lemma 2.19. *If the genus of the curve is at least 2, there exists a semistable pair of rank r and degree d with respect to $(\mathcal{O}_D^{\oplus r}, \delta)$ if and only if $0 < \delta \leq r \cdot l(D)$.*

Proof: The 'only if' part was proven in 2.1, since $r \cdot l(D) = h^0(\mathcal{E}_0)$. For the 'if' direction we pick a stable vector bundle $\mathcal{E}$ of rank r and degree d and an isomorphism $\alpha : \mathcal{E}_D \cong \mathcal{O}_D^{\oplus r}$. The induced pair is semistable. □

Corollary 2.20. *The moduli spaces $\mathcal{M}_\delta^{ss}(d, r, \mathcal{O}_D^{\oplus r})$ of semistable pairs with $0 < \delta \leq r \cdot l(D)$ exist as projective schemes of generic dimension $r^2(g-1) + r^2 \cdot l(D)$*

(cp. [Se], III.5., there is a misprint in the dimension formula in [Se])
There are two new features in the theory of pairs compared with the moduli spaces of vector bundles. First, to compactify one really has to use sheaves with torsion supported on D. Secondly, the set of semistable pairs which are not stable may have only codimension 2, whereas the set of semistable vector bundles which are not stable is at least $2g - 3$ codimensional in the moduli space of all semistable vector bundles. To give an example we describe the moduli space $\mathcal{M}_1^{ss}(0, 2, k(P)^{\oplus 2})$ of sheaves of rank two

and degree zero with a level structure at a reduced point $P \in X$ with $\delta = 1$. Here we try to compute the S-equivalence in geometric terms, which is not clear to us in the general context.
The stability conditions say

i) $\deg \mathcal{G}(\leq) - \frac{1}{2}$ for all rank one subsheaves $\mathcal{G} \subset \mathcal{E}_\alpha = \mathrm{Ker}\alpha$.

ii) $\deg \mathcal{G}(\leq)\frac{1}{2}$ for all rank one subsheaves $\mathcal{F} \subset \mathcal{E}$.

iii) $l(\mathcal{E}/\mathcal{E}_\alpha)(\geq)1$

iv) $l(\mathrm{T}(\mathcal{E}))(\leq)1$

v) α is injective on the torsion $\mathrm{T}(\mathcal{E})$.

Therefore the sheaves $\mathcal{E}$ occuring in semistable pairs in $\mathcal{M}_1^{ss}(0, 2, k(P)^{\oplus 2})$ are either locally free or of the form $\mathcal{F} \oplus k(P)$ with $\mathcal{F}$ locally free.
First we classify all pairs $(\mathcal{E}, \alpha)$ with locally free $\mathcal{E}$. By ii) such a bundle $\mathcal{E}$ has to be semistable as a bundle. If $\mathcal{E}$ is a stable bundle, then every pair $(\mathcal{E}, \alpha)$ with an arbitrary $\alpha \neq 0$ is semistable and is stable if and only if $\mathrm{rk}(\alpha) = 2$, i.e. $\alpha(P)$ is bijective. If $\mathcal{E}$ is only semistable there are two cases to consider: Either a) $\mathcal{E} \cong \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles of degree 0 or b) $\mathcal{E}$ is given as a nontrivial extension of two such line bundles.
a) If $\mathcal{L}_1 \cong \mathcal{L}_2$ then $(\mathcal{E}, \alpha)$ is semistable if and only if α is bijective. If $\mathcal{L}_1 \ncong \mathcal{L}_2$ then $(\mathcal{E}, \alpha)$ is semistable if and only if none of the restrictions $\alpha_{|\mathcal{L}_i(P)}$ is trivial.
If $\mathcal{L}_1 \cong \mathcal{L}_2$ and α bijective, the pair $(\mathcal{E}, \alpha)$ is in fact stable, since $l(\mathcal{E}/\mathcal{E}_\alpha) = 2 > 1$. If α is only of rank one we can always find an inclusion $\mathcal{L}_1 \subset \mathcal{L}_1 \oplus \mathcal{L}_1$ with $\mathcal{L}_1 = \mathrm{Ker}(\alpha_{|\mathcal{L}_1})$, which contradicts i). For $\mathcal{L}_1 \ncong \mathcal{L}_2$ one has to consider line bundles $\mathcal{L} \subset \mathcal{L}_1 \oplus \mathcal{L}_2$ of degree zero with $\mathcal{L} = \mathrm{Ker}(\alpha_{|\mathcal{L}})$, because that is the only possibility to contradict i). But such a line bundle has to be isomorphic to one of the summands with the natural inclusion. Therefore the stability condition is equivalent to $\alpha_{|\mathcal{L}_i} \neq 0$.
b) If $\mathcal{E}$ is a nonsplitting extension

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_2 \longrightarrow 0$$

a pair $(\mathcal{E}, \alpha)$ is semistable iff $\mathcal{L}_1 \neq \mathrm{Ker}(\alpha_{|\mathcal{L}_1})$. That is, since every line bundle $\mathcal{L} \subset \mathcal{E}$ of degree 0 either defines a splitting of the sequence or maps isomorphically to $\mathcal{L}_1$.
The next step is to determine all semistable pairs $(\mathcal{F} \oplus k(P), \alpha)$. Here we claim, that such a pair is semistable iff $\mathcal{F}$ is stable and $\alpha_{|k(P)}$ is injective. Let $(\mathcal{F} \oplus k(P), \alpha)$ be semistable and $\mathcal{L} \subset \mathcal{F}$ a line bundle. Then, since $\mathcal{L} \oplus k(P) \subset \mathcal{F} \oplus k(P)$, the semistability conditions for the pair give $\deg(\mathcal{L} \oplus k(P)) \leq 1/2$, i.e. $\deg \mathcal{L} \leq -1/2 = \deg \mathcal{F}/2$. Let now $\mathcal{F}$ be a stable bundle. If $\mathcal{L}$ is a rank one subsheaf of $\mathcal{F} \oplus k(P)$, then it either injectively injects into $\mathcal{F}$ or has torsion part $k(P)$ and therefore satisfies the required inequality.

Next we look at the isomorphism classes of stable pairs. If $\mathcal{E}$ is a stable bundle, two pairs $(\mathcal{E},\alpha)$ and $(\mathcal{E},\alpha')$ are isomorphic if and only if α and α' differ by a scalar. For $\mathcal{E}$ of the form $\mathcal{L}_1 \oplus \mathcal{L}_2$ the automorphism group of $\mathcal{E}$ is either $\mathbb{C}^* \times \mathbb{C}^*$ for $\mathcal{L}_1 \not\cong \mathcal{L}_2$ or GL(2) for $\mathcal{L}_1 \cong \mathcal{L}_2$. In the first case the set of isomorphism classes of stable pairs for fixed $\mathcal{E}$ is isomorphic to $\mathrm{PGL}(2)/\{\left(\begin{smallmatrix}\beta & 0\\ 0 & \gamma\end{smallmatrix}\right) | \beta,\gamma \in \mathbb{C}^*\}$. In the latter case all stable pairs are isomorphic for fixed $\mathcal{E}$, they all define the same point in the moduli space. If $\mathcal{E}$ is given by a nonsplitting exact sequence

$$0 \longrightarrow \mathcal{L}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{L}_2 \longrightarrow 0$$

the automorphism group is either $\mathbb{C}^*$ for $\mathcal{L}_1 \not\cong \mathcal{L}_2$ or $\{\left(\begin{smallmatrix}\beta & \gamma\\ 0 & \beta\end{smallmatrix}\right) | \beta \in \mathbb{C}^*, \gamma \in \mathbb{C}\}$ for $\mathcal{L}_1 \cong \mathcal{L}_2$. Therefore every such extension induces either a PGL(2)−family of stable pairs in the moduli space or a $\mathrm{PGL}(2)/\{\left(\begin{smallmatrix}\beta & \gamma\\ 0 & \beta\end{smallmatrix}\right) | \beta \in \mathbb{C}^*, \gamma \in \mathbb{C}\}$−family of stable pairs in the moduli space.

In order to describe the S-equivalence we claim, that the orbit of a pair $(\mathcal{E},\alpha)$ is closed if and only if either the pair is stable, i.e. $\mathcal{E}$ is a semistable vector bundle and α of rank two, or $\mathcal{E}$ is of the form $\mathcal{F} \oplus k(P)$ with a stable vector bundle $\mathcal{F}$ of degree -1 and $\alpha_{|\mathcal{F}} = 0$.

If $\mathcal{E}$ is locally free and α of rank one there is an extension of the form

$$0 \longrightarrow \mathcal{E}_\alpha \longrightarrow \mathcal{E} \longrightarrow k(P) \longrightarrow 0 \quad .$$

If $\psi \in \mathrm{Ext}^1(k(P),\mathcal{E}_\alpha)$ denotes the extension class one can easily construct a family of pairs over $\mathbb{C}\cdot\psi$, which gives the pair $(\mathcal{E},\alpha)$ outside 0 and $(\mathcal{E}_\alpha \oplus k(P), \alpha \cdot \mathrm{pr}_{k(P)})$ on the special fibre, where $\mathrm{pr}_{k(P)}$ is the projection to $k(P)$. Obviously this pair is again semistable. If $(\mathcal{F} \oplus k(P),\alpha)$ is a semistable pair with $\alpha = (\alpha_1,\alpha_2)$, the pair $(\mathcal{F} \oplus k(P), (t\cdot\alpha_1,\alpha_2))$ converges constantly to a pair with $\alpha_{|\mathcal{F}} = 0$ for $t \to 0$. In order to prove the claim it is therefore enough to show that the orbit of such a pair is closed. If there were a family parametrized by a curve with a point O, which outside O were isomorphic to a fixed semistable pair $(\mathcal{F} \oplus k(P),\alpha)$ with $\alpha_{|\mathcal{F}} = 0$ and over this point O isomorphic to another pair of this kind, the family of the kernels would give a family of stable bundles, which would be constant for all points except O. Since the stable bundles are separated, it has to be constant everywhere. Finally, using the constance of the images of the maps α outside the point O one concludes that the family of pairs is constant.

If $\mathcal{M}_1^s(0,2,k(P)^{\oplus 2})$ denotes the subset of all stable pairs we summarize the results in the following proposition

Proposition 2.21. *i)* $\mathcal{M}_1^{ss}(0,2,k(P)^{\oplus 2}) \setminus \mathcal{M}_1^s(0,2,k(P)^{\oplus 2}) \cong \mathbb{P}_1 \times U(-1,2)$, *where* $U(-1,2)$ *is the moduli space of stable rank two vector bundles of degree* -1.

ii) There is a morphism $\mathcal{M}_1^s(0,2,k(P)^{\oplus 2}) \to U(0,2)$, *which is a* PGL(2)−*fibre bundle over* $U(0,2)^s$ *and whose fibre over a point* $[\mathcal{L}_1 \oplus \mathcal{L}_2] \in U(0,2) \setminus U(0,2)^s$ *is* *isomorphic to*

$$\mathrm{PGL}(2)/\{\begin{pmatrix}\beta & 0\\ 0 & \gamma\end{pmatrix}|\beta,\gamma\in\mathbb{C}^*\}\cup\{\mathrm{PGL}(2)\times\mathbb{P}(\mathrm{Ext}^1(\mathcal{L}_1,\mathcal{L}_2)^\vee)\}$$

for $\mathcal{L}_1\not\cong\mathcal{L}_2$ *and isomorphic to*

$$\{pt\}\cup\{\mathrm{PGL}(2)/\{\begin{pmatrix}\beta & \gamma\\ 0 & \beta\end{pmatrix}|\beta\in\mathbb{C}^*,\gamma\in\mathbb{C}^*\}\times\mathbb{P}(\mathrm{Ext}^1(\mathcal{L}_1,\mathcal{L}_2)^\vee)\}$$

for $\mathcal{L}_1\cong\mathcal{L}_2$.

Proof: The isomorphism in *i)* is given by $(\mathcal{F}\oplus k(P),\alpha)\mapsto(\alpha(k(P)),\mathcal{F})$. The morphism in *ii)* is induced by the universality property of the moduli space. □

In particular the dimension of $\mathcal{M}_1^{ss}(0,2,k(P)^{\oplus 2})$ is $4g$ (g is the genus of the curve) and the dimension of $\mathcal{M}_1^{ss}(0,2,k(P)^{\oplus 2})\setminus\mathcal{M}_1^{s}(0,2,k(P)^{\oplus 2})$ is $4g-2$. Thus the codimension is two, independently of the genus.

Finally we want to study the situation in the two dimensional case. Let X be a surface with an effective divisor C and $\mathcal{E}_0$ be a vector bundle of rank r on C. A framing of a vector bundle $\mathcal{E}$ of rank r on X along C in the strong sense as introduced in [L1] is an isomorphism $\alpha:\mathcal{E}_C\cong\mathcal{E}_0$. In [L1] the question of the existence of moduli spaces for such pairs $(\mathcal{E},\alpha)$ was asked (α denotes the isomorphism as well as the composition of this isomorphism with the surjection $\mathcal{E}\twoheadrightarrow\mathcal{E}_C$). In fact, under additional conditions, fine moduli spaces for such framed bundles were constructed as algebraic spaces. These additional conditions are: C is good and $\mathcal{E}_0$ is simplifying. If $C=\sum b_iC_i$ with prime divisors C_i and $b_i>0$ C is called good if there exist nonnegative integers a_i, such that $\sum a_iC_i$ is big and nef. The vector bundle $\mathcal{E}_0$ is called simplifying if for two framed bundles $\mathcal{E}$ and $\mathcal{E}'$ the group $H^0(X,\mathcal{H}om(\mathcal{E},\mathcal{E}')(-C))$ vanishes. At the first glance it is surprising that there are no further stability conditions for such pairs. However, in many situations the general stability conditions of chapter one are hidden behind the concept of framed bundles.

Definition 2.22. For $0<s<r$ the number $\nu_s(\mathcal{E}_0,C_i)$ is defined as the maximum of $\deg(\mathcal{F})/s-\deg(\mathcal{E}_0|_{C_i})/r$, where $\mathcal{F}\subset\mathcal{E}_0|_{C_i}$ is a vector bundle of rank s.

In the following we assume, that there are nonnegative integers a_i, s.t. $H=\sum a_iC_i$ is ample. This is equivalent to saying that $X\setminus C$ is affine.

Proposition 2.23. *If* δ_1 *is positive with*

$$\max_{0<s<r}\{r\cdot s/(r-s)\sum a_i\nu_s(\mathcal{E}_0,C_i)\}<\delta_1<(r-1)(C.H)\ ,$$

then every vector bundle $\mathcal{E}$ *of rank* r *together with an isomorphism* $\alpha:\mathcal{E}_C\cong\mathcal{E}_0$ *forms a* μ*−stable pair* $(\mathcal{E},\alpha)$.

Proof: The μ-stability for such pairs is defined by the following two inequalities:
i) $\deg \mathcal{G}/\mathrm{rk}\mathcal{G} < \deg \mathcal{E}/r - \delta_1/r$ for every vector bundle $\mathcal{G} \subset \mathcal{E}_\alpha$ with $0 < \mathrm{rk}\mathcal{G} < r$ and
ii) $\deg \mathcal{G}/\mathrm{rk}\mathcal{G} < \deg \mathcal{E}/r + \delta_1(r - \mathrm{rk}\mathcal{G})/(r \cdot \mathrm{rk}\mathcal{G})$ for every vector bundle $\mathcal{G} \subset \mathcal{E}$ with $0 < \mathrm{rk}\mathcal{G} < r$.
We first check *ii)*. It is enough to consider vector bundles $\mathcal{G}$, s.t. the quotient $\mathcal{E}/\mathcal{G}$ is torsionfree. In particular we can assume, that $\mathcal{G}_{C_i} \longrightarrow \mathcal{E}_{C_i}$ is injective. Then we conclude $\deg \mathcal{G}/\mathrm{rk}\mathcal{G} = c_1(\mathcal{G}).H/\mathrm{rk}\mathcal{G} = \sum a_i \deg(\mathcal{G}_{C_i})/\mathrm{rk}\mathcal{G} \leq \sum a_i(\deg(\mathcal{E}_0)_{C_i}/r + \nu_{\mathrm{rk}\mathcal{G}}(\mathcal{E}_0, C_i)) = \deg \mathcal{E}/r + \sum a_i \nu_{\mathrm{rk}\mathcal{G}}(\mathcal{E}_0, C_i) < \deg \mathcal{E}/r + ((r - \mathrm{rk}\mathcal{G})/r \cdot \mathrm{rk}\mathcal{G})\delta_1$. To prove *i)* one uses $\mathcal{E}_\alpha = \mathcal{E}(-C)$ and *ii)*: For $\mathcal{G} \subset \mathcal{E}_\alpha$ the inequality *ii)* applied to $\mathcal{G}(C) \subset \mathcal{E}$ implies $\deg \mathcal{G}/\mathrm{rk}\mathcal{G} + C.H = \deg \mathcal{G}(C)/\mathrm{rk}\mathcal{G} < \deg \mathcal{E}/r + ((r - \mathrm{rk}\mathcal{G})/r \cdot \mathrm{rk}\mathcal{G})\delta_1$. Therefore $\delta_1 < (r-1)C.H$ suffices to give *i)*. □

Corollary 2.24. *For* $\max_{0<s<r}\{r \cdot s/(r-s) \sum a_i \nu_s(\mathcal{E}_0, C_i)\} < (r-1)(C.H)$ *and* C*, such that there exists an effective, ample divisor* H*, whose support is contained in* C*, the moduli spaces* $\mathcal{M}^{fr}_{X/C/\mathcal{E}_0/X}$ *of framed vector bundles are quasi-projective.*

Proof: These moduli spaces are in fact open subsets of the μ-stable part of the moduli space of all semistable pairs $(\mathcal{E}, \alpha)$. □

There is a special interest in the case $\mathcal{E}_0 \cong \mathcal{O}_C^{\oplus r}$, since the corresponding moduli spaces are in fact invariants of the affine surface $X \setminus C$ ([L2]). In this case all the numbers $\nu_s(\mathcal{E}_0, C_i)$ vanish. Therefore a trivially framed bundle gives a μ-stable pair $(\mathcal{E}, \alpha)$ with respect to every $\delta_1 < (r-1)C.H$.
In ([L1],2.1.5.) a sufficient condition for a bundle $\mathcal{E}_0$ to be simplifying is proven: If $\mathrm{Hom}(\mathcal{E}_0, \mathcal{E}_0(-kC)) = 0$ for all $k > 0$, then $\mathcal{E}_0$ is simplifying. We remark that at least in the rank two case this condition is closely related to the numerical condition we gave. It is possible to make the condition finer, because in the definition of the numbers ν_{C_i} it is sufficient to take the maximum over those bundles, which actually live on X.

References

[Be] Bertram, A. *? (quoted in [Th])*

[Bo] Bogomolov, F. A. *On stability of vector bundles on surfaces and curves.* Preprint (1991).

[Br] Bradlow, S. B. *Special metrics and stability for holomorphic bundles with global sections.* J. Diff. Geom. 33 (1991), 169-213

[Do] Donaldson, S. K. *Instantons and Geometric Invariant Theory.* Comm. Math. Phys. 93 (1984), 453-460.

[Ga] Garcia-Prada, O. *Invariant connections and vortices.* IHES-Preprint (1992).

[Gi] Gieseker, D. *On the moduli of vector bundles on an algebraic surface.* Ann. of Math. 106 (1977), 45-60.

[EGA] Grothendieck, A. Dieudonné, J. *Eléments de Géométrie Algébrique.* Publ. Math. de IHES, No 28 (1966).

[Gr] Grothendieck, A. *Techniques de construction et théorem d'existence en géometrie algébrique IV: Les schémas de Hilbert.* Sém. Bourbaki, 1960/61, 221

[Ha] Hartshorne, R. *Algebraic Geometry.* Graduate Texts in Mathematics 52, Springer Verlag, New York (1977).

[Kl] Kleiman, S. *Les théorèmes de finitude pour le foncteur de Picard.* SGA de Bois Marie, 1966/67, exp. XIII.

[L1] Lehn, M. *Modulräume gerahmter Vektorbündel.* Dissertation Bonn 1992, Bonner Math. Schriften 241 (1992).

[L2] Lehn, M. *Framed vector bundles and affine surfaces.* Preprint (1992).

[Lü] Lübke, M. *The analytic moduli space of framed vector bundles.* Preprint Leiden 1991.

[Ma] Maruyama, M. *Moduli of stable sheaves I.* J. Math. Kyoto Univ. 17 (1977), 91-126.

[MF] Mumford, D., Fogarty, J. *Geometric Invariant Theory.* Erg. d. Math. 34 (neue Folge), 2nd ed., Springer Verlag, Berlin (1982).

[Rei] Reider, I. *On Jacobians of higher rank (1).* Preprint (1991).

[Se] Seshadri, C. S. *Fibrés vectoriels sur les courbes algébriques.* Asterisque 96 (1982).

[Th] Thaddeus, M. *Talk at 'Journées de géométrie algébrique d'Orsay '.* (July 1992).

[Ty] Tyurin, A. N. *The moduli space of vector bundles on threefolds, surfaces and curves.* Preprint (1990).

[We] Welters, G. E. *Polarized abelian varities and the heat equation.* Comp. Math. 49 (1983), 173-194.