ON LOEWY LENGTHS OF PROJECTIVE MODULES FOR p-SOLVABLE GROUPS

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Let G be a p-solvable group of order $p^a m$, $(p,m)=1$, and let $t=t_p(G)$ denote the nilpotence index of the Jacobson radical $J(FG)$ of the group algebra FG, where F denotes a field of characteristic p. It is well-known and easy to see that $t \geq a(p-1)+1$ (this follows e.g. from Lemma 1.1 below) and that equality holds if the Sylow p-subgroups of G are elementary abelian. The converse need not be true: the first known counterexample was $G=S_4$, the symmetric group on four letters, with $p=2$ [6], and later counterexamples were constructed for each prime p by Motose [5]. In this note we prove the following result which contains all examples constructed so far (Theorem 2.7):

Assume $G = N \rtimes H$ is a semidirect product with N a p-group and $H = Q \rtimes M$ a Frobenius group with kernel Q a p'-group and M a p-group. Then $t_p(G) = t_p(N) + t_p(M) - 1$ holds if and only if $\bigoplus_{i \geq 0} (\omega N)^i/(\omega N)^{i+1}$ is a semisimple FH-module under the conjugation action of H on N.

Here, ωN denotes the augmentation ideal of FN.

Notations and Conventions. Throughout this note, G will be a finite group and F will be a field of characteristic $p > 0$. All FG-modules are assumed to be finitely generated right modules, and I denotes the trivial one-dimensional FG-module. $J(FG)$ and ωG denote the Jacobson radical, resp. the augmentation ideal, of FG. For any FG-module V, $\ell(V)$ is the Loewy length of V, i.e. the smallest integer ℓ such that $V \cdot J(FG)^{\ell} = 0$. Furthermore, $P_G(V)$ and $\Omega_G(V)$ will be the projective cover, resp. the Heller module of V. Thus $\Omega_G(V) \subseteq P_G(V) \cdot J(FG)$ and there is an exact sequence $0 \longrightarrow \Omega_G(V) \longrightarrow P_G(V) \longrightarrow V \longrightarrow 0$. Finally, omitting reference to p which is fixed in the following, we set $t(G) = \ell(FG)$, the nilpotence index of $J(FG)$. The remaining notation is as in [7].

§ 1. Normal Subgroups

In this section, we study the situation where V is an FG-module and N is a normal subgroup of G acting trivially on V. Thus V can be viewed as either a G-module or a G/N-module, and we compare the Loewy lengths of the corresponding projective covers.

Our first lemma extends [9, Lemma 3.4].

Lemma 1.1. Let N be a normal subgroup of G and let V be an FG-module with $N \leq \ker_G(V)$. Then

i. $P_{G/N}(V) \cong P_G(V)/P_G(V)\cdot \omega N$;

ii. $\ell(P_G(V)) \geq \ell(P_{G/N}(V)) + \ell(P_N(I)) - 1$.

Proof. Set $P = P_G(V)$, $H = G/N$ and let $\bar{\ }: FG \longrightarrow FH$ denote the canonical map with kernel $(\omega N)FG$. Note that $\overline{J(FG)} = J(FH)$. (Images of semisimple Artinian rings are semisimple Artinian.) Since $P\cdot J(FG) \supseteq \Omega_G(V) \supseteq P\cdot \omega N$, we have a map of FH-modules $P/P\cdot \omega N \twoheadrightarrow V$ whose kernel $\Omega_G(V)/P\cdot\omega N$ is contained in $(P/P\cdot\omega N)\cdot J(FG) = (P/P\cdot\omega N)\cdot \cdot J(FH)$. As $P/P\cdot\omega N$ is projective over FH, we obtain the isomorphism $P_H(V) \cong P/P\cdot\omega N$, which proves (i).

Now write $\bar{\ell} = \ell(P_H(V))$ and $\ell_N = \ell(P_N(I))$. Then it follows from the foregoing that $P\cdot J(FG)^{\bar{\ell}-1} \nsubseteq P\cdot \omega N$.

But $P \cdot \omega N = \mathrm{ann}_P \hat{N}$, where $\hat{N} = \sum_{n \in N} n \in FN$. Indeed since P is projective over FN, this follows from the fact that $\omega N = \mathrm{ann}_{FN} \hat{N}$ [7, Lemma 3.1.2]. Note further that $\hat{N} \in J(FN)^{\ell_N - 1}$, since viewing $P_N(I)$ as a summand of FN we have $F \cdot \hat{N} = \text{socle } P_N(I) = P_N(I) \cdot J(FN)^{\ell_N - 1} \subseteq J(FN)^{\ell_N - 1}$. We deduce that $P \cdot J(FG)^{\ell + \ell_N - 2} \supseteq P \cdot J(FG)^{\bar{\ell} - 1} \cdot \hat{N} \neq 0$. This proves (ii). □

We remark that if N, or G/N, is a p'-group then the inequality in (ii) becomes an equality. More generally, if $J(FN) \cdot J(FG) = J(FG) \cdot J(FN)$ in the situation of Lemma 1.1, then we have

$$\ell(P_G(V)) \leq \ell(P_{G/N}(V)) \cdot \ell(P_N(I)).$$

For, part (i) above implies that $P_G(V) \cdot J(FG)^{\bar{\ell}} \subseteq$ $\subseteq P_G(V) \cdot \omega N = P_G(V) \cdot J(FN)$, where we have set $\bar{\ell} = \ell(P_{G/N}(V))$ and where the latter equality holds since $P_G(V)$, as an FN-module, is isomorphic to a direct sum of copies of $P_N(I)$.

If N is a p-group then Lemma 1.1 can be strengthened as follows. Recall that $t(G)$ denotes the nilpotence index of $J(FG)$.

Lemma 1.2. Let N be a normal p-subgroup of G and let V be an FG-module with $N \le \ker_G(V)$. View FN as an FG-module via conjugation of G on N.

i. For all $i \ge 0$ we have FG-isomorphisms

$$P_{G/N}(V) \otimes_F \frac{(\omega N)^i}{(\omega N)^{i+1}} \cong \frac{P_G(V)\cdot(\omega N)^i}{P_G(V)\cdot(\omega N)^{i+1}} ,$$

where $P_{G/N}(V)$ is viewed as an FG-module by letting N act trivially.

ii. $$\ell(P_G(V)) \ge t(N)-1 + \max_X \ell(P_{G/N}(V) \otimes_F X)$$

$$\ge t(N)-1 + \max_X \ell(P_{G/N}(V \otimes_F X)) ,$$

where X runs over the FG-composition factors of FN.

Proof. Let $P = P_G(V)$ and $H = G/N$. For each $i \ge 0$ we have an F-epimorphism $g_i : P \otimes_F (\omega N)^i \twoheadrightarrow$ $\twoheadrightarrow P \cdot (\omega N)^i$, $p \otimes \alpha \longmapsto p\alpha$, which is in fact FG-linear if G acts by conjugation on $(\omega N)^i$. Thus we obtain FG-epimorphisms

$$\bar{g}_i : P \otimes_F \frac{(\omega N)^i}{(\omega N)^{i+1}} \cong \frac{P \otimes_F (\omega N)^i}{P \otimes_F (\omega N)^{i+1}} \twoheadrightarrow \frac{P\cdot(\omega N)^i}{P\cdot(\omega N)^{i+1}} .$$

Since $\bar{g}_i$ annihilates $P \cdot \omega N \otimes_F \frac{(\omega N)^i}{(\omega N)^{i+1}}$ and $P_H(V) \cong P/P \cdot \omega N$, by Lemma 1.1, $\bar{g}_i$ defines an FG-epimorphism

$$f_i : P_H(V) \otimes_F \frac{(\omega N)^i}{(\omega N)^{i+1}} \twoheadrightarrow \frac{P \cdot (\omega N)^i}{P \cdot (\omega N)^{i+1}} .$$

To see that f_i is injective, note that, as FN-modules, $P|_N \cong P_H(V) \otimes_F FN$ with the regular action of FN on FN. Indeed, by Lemma 1.1(i),

$$P/P \cdot \omega N \cong P_H(V) \cong \frac{P_H(V) \otimes_F FN}{(P_H(V) \otimes_F FN) \cdot \omega N} ,$$

and hence $P|_N \cong P_H(V) \otimes_F FN$, since both sides are projective over FN, and $\omega N = J(FN)$. It follows that f_i is an isomorphism, and part (i) is proved.

For (ii), set

$$\ell_i = \ell\left(P_H(V) \otimes_F \frac{(\omega N)^i}{(\omega N)^{i+1}}\right) = \ell\left(\frac{P \cdot (\omega N)^i}{P \cdot (\omega N)^{i+1}}\right) .$$

If $m < \ell_i$ then $P(\omega N)^i \cdot J(FG)^m \not\subseteq P \cdot (\omega N)^{i+1} =$ $= \mathrm{ann}_P(\omega N)^{t(N)-i-1}$, where the latter equality follows from [2, p. 261], since P is free over FN. Thus we conclude that

$$P \cdot (\omega N)^i \cdot J(FG)^m \cdot (\omega N)^{t(N)-i-1} \neq 0$$

and so $\ell(P) > t(N)+m-1$. Therefore, $\ell(P) \geq t(N)+\ell-1$, where $\ell = \max_i \ell_i$. Finally, since $P_H(V)$ is projective over FH, we have

$$\bigoplus_i P_H(V) \otimes_F (\omega N)^i/(\omega N)^{i+1} \cong \bigoplus_X P_H(V) \otimes_F X ,$$

where X runs over the composition factors of FN. Hence $\ell = \max_X \ell(P_H(V)\otimes_F X)$. Since $P_H(V\otimes_F X)$ is a summand of $P_H(V)\otimes_F X$, we also have $\ell \geq \ell(P_H(V\otimes_F X))$. This completes the proof of (ii). □

The following example illustrates the difference between the estimates provided by Lemmas 1.1. and 1.2.

<u>Example 1.3.</u> Let $G = S_4$ be the symmetric group on four letters and let char $F = 2$. Then $G = V_4 \rtimes GL_2(2)$ and there are two irreducible FG-modules, namely I and the canonical 2-dimensional module for $H = GL_2(2)$, denoted by 2. We have

$$P_H(I) = \begin{smallmatrix} I \\ I \end{smallmatrix} \ , \ P_H(2) = 2 \ , \text{ and } \ FV_4 = \begin{smallmatrix} & I & \\ I & & I \\ & I & \end{smallmatrix} \ .$$

Thus Lemma 1.1 yields $\ell(P_G(I)) \geq 4$ and $\ell(P_G(2)) \geq 3$. However, as FG-module, $FV_4 \cong (I_H)^G = \begin{smallmatrix} I \\ 2 \\ I \end{smallmatrix}$ and so Lemma 1.2 implies

$$\ell(P_G(2)) \geq 3 + \ell(P_H(2 \otimes 2)) - 1 \ .$$

Since $2 \otimes 2 = 2 \oplus \begin{smallmatrix} I \\ I \end{smallmatrix}$, we obtain $\ell(P_G(2)) \geq 4$. We will see shortly that, in fact, $\ell(P_G(I)) = \ell(P_G(2)) = 4$, a result due to Motose and Ninomiya [6]. A detailed discussion of the Loewy and socle series of $P_G(I)$ and $P_G(2)$ can be found in [2, p. 214 - 218].

For simplicity of formulation, we restrict ourselves to the case of an irreducible FG-module V in the following lemma.

This is of course no real loss, since for any V we have $P_G(V) \cong \bigoplus_X P_G(X)$ where X runs over the simple components of $V/V \cdot J(FG)$, with multiplicities. Also, it would be enough to assume that $1 \to U(FN) \to U(FN)G \to G \to 1$ splits, where $U(.)$ denotes the group of units, but for simplicity we assume G to be a semidirect product.

<u>Lemma 1.4.</u> Let N be a normal p-subgroup of G and assume that $G = N \rtimes H$ for some $H \leq G$. Let V be an irreducible FG-module.

i. $P_G(V) \cong (I_H)^G \otimes_F P_H(V)$, where N acts trivially on $P_H(V)$.

ii. $\ell(P_G(V)) \geq \ell(P_H(V)) + \ell((V_H)^G) - 1$.

iii. For each $i \geq 0$ let V_i denote the FG-module $V_i = V \otimes_F \frac{(\omega N)^i}{(\omega N)^{i+1}}$, where G acts by conjugation on $\frac{(\omega N)^i}{(\omega N)^{i+1}}$. Then

$$\ell((V_H)^G) \geq t(N) - 1 + \max_i \ell(V_i) .$$

In particular, $\ell((V_H)^G) = t(N)$ if and only if all V_i are semisimple.

Proof. Set $T=(I_H)^G \otimes_F P_H(V) \cong (P_H(V)|_H)^G$ and $J=J(FG)$. Then $T/T\cdot J \cong T/T\cdot\omega N \big/ T\cdot J/T\cdot\omega N \cong P_H(V)/P_H(V)\cdot J(FH) \cong V$. Since T is projective, it follows that $T \cong P_G(V)$ so that (i) holds.

If $\ell = \ell(P_H(V))$ then, by part (i),

$$P_G(V)\cdot J^{\ell-1} = (P_H(V)\otimes(I_H)^G)\cdot J^{\ell-1} \supseteq (P_H(V)\otimes I_H)\cdot J(FH)^{\ell-1}\cdot FN$$
$$= (P_H(V)J(FH)^{\ell-1}\otimes I_H)\cdot FN = (V\otimes I_H)\cdot FN \cong (V_H)^G .$$

Assertion (ii) follows. As to (iii), note that

$$V_i \cong \frac{(V_H)^G\cdot(\omega N)^i}{(V_H)^G\cdot(\omega N)^{i+1}} .$$

Thus if $m < \ell_i = \ell(V_i)$ then $(V_H)^G\cdot(\omega N)^i\cdot J(FG)^m \nsubseteq$ $\nsubseteq (V_H)^G\cdot(\omega N)^{i+1} = \mathrm{ann}_{(V_H)^G}\ (\omega N)^{t(N)-i-1}$. Here, the latter equality follows from [2, p. 261], since $(V_H)^G$ is free over FN. Therefore, $\ell((V_H)^G) \geq t(N)+\ell_i-1$ for all i. Since the last assertion is clear, the lemma is proved. □

Example 1.5. Set $G = (C_3\times C_3) \rtimes SL_2(3)$, with the canonical action of $H=SL_2(3)$ on $N=C_3\times C_3$, and let char $F = 3$. The irreducible FG-modules all come from FH and are : I , the canonical 2-dimensional FH-module, 2, and 3 which is induced to H from a non-trivial 1-dimensional module for the quaternion group $Q_8 \leq H$. One checks that $t(N) = 5$ and $t(H) = 3$, and so the lower bound for $\ell(P_G(V))$, $V \in \{I,2,3\}$, provided by Lemma 1.2 (ii) can at most be 7. However, for $V = 2$, Lemma 1.4 gives $\ell(P_G(2)) \geq 9$. Indeed,

$$\frac{(\omega N)^i}{(\omega N)^{i+1}} = \begin{cases} I & (i=0,4) \\ 2 & (i=1,3) \\ 3 & (i=2\ ) \end{cases}$$

Hence, in the notation of part (iii), we have $V_2 = 2 \otimes 3 = \begin{matrix} 2 \\ 2 \\ 2 \end{matrix}$ so that $\ell((2_H)^G) \geq 7$ and $\ell(P_G(2)) \geq 9$. Actually, equality holds here and even $t(G) = 9$ (see Example 2.5).

§ 2. Groups of p-Length 2

Our goal here is to show that, under certain circumstances, the inequality of Lemma 1.4(ii) does in fact become an equality. For example, this is always the case if H is p-nilpotent with elementary abelian Sylow p-subgroups (Corollary 2.4).

Lemma 2.1. Let V and W be FG-modules. Set $v = \ell(V)$, $w = \ell(W)$ and $T_{ij} = \frac{VJ^i}{VJ^{i+1}} \otimes_F \frac{WJ^j}{WJ^{j+1}}$ $(0 \le i \le v-1, 0 \le j \le w-1)$, where $J = J(FG)$. Then

$$\ell(V \otimes_F W) \le \sum_{\ell=0}^{v+w-2} \max\{\ell(T_{ij}) \mid i + j = \ell\} .$$

Proof. Set $U_{ij} = VJ^i \otimes_F WJ^j$. Then $U_{ij} \supseteq U_{i+1,j} + U_{i,j+1}$ and $U_{ij}/U_{i+1,j} + U_{i,j+1} \cong T_{ij}$.

Now let $U_\ell = \sum_{i+j=\ell} U_{ij}$ for $0 \le \ell \le v+w-1$. Then $0 = U_{v+w-1} \subseteq U_{v+w-2} \subseteq \ldots \subseteq U_0 = V \otimes_F W$, and the canonical map $\bigoplus_{i+j=\ell} U_{ij} \twoheadrightarrow U_\ell$ yields an epimorphism

$$\bigoplus_{i+j=\ell} T_{ij} \cong \bigoplus_{i+j=\ell} U_{ij}/U_{i+1,j} + U_{i,j+1} \twoheadrightarrow T_\ell/T_{\ell+1}.$$

Therefore, $\ell(T_\ell/T_{\ell+1}) \le \max\{\ell(T_{ij}) \mid i+j=\ell\}$ and the lemma follows. □

Corollary 2.2. Let U be a normal subgroup of G such that G/U is a p-group. Let W be an FG-module and set $V = (W|_U)^G \cong W \otimes_F (I_U)^G$. Then $\ell(V) \le t(G/U) + \ell(W)-1$.

Proof. Set $M=G/U$ and view FM as FG-module via $FM \cong (I_U)^G$. Then $(\omega M)^i = FM \cdot J^i$, where $J=J(FG)$, and $(\omega M)^i/(\omega M)^{i+1} \cong I_G^{(n_i)}$ for suitable integers n_i. In the notation of the preceding lemma, we therefore have

$$T_{ij} = \frac{(\omega M)^i}{(\omega M)^{i+1}} \otimes_F \frac{WJ^j}{WJ^{j+1}} \cong \left(\frac{W \cdot J^j}{W \cdot J^{j+1}}\right)^{(n_i)}$$

and so $\ell(T_{ij}) = 1$ for all i,j. Thus Lemma 2.1 yields $\ell(V) \leq \ell(FM) + \ell(W)-1$ which proves the corollary

□

The estimate given above does not hold for arbitrary induced modules. For example, if $U=C_2 \leq G=C_4$ and $W=FC_2$, where char $F=2$, then $\ell(W^G)=\ell(FC_4) > \ell(W)+2-1=3$.

Proposition 2.3. (F algebraically closed) Assume that $G=N \rtimes H$, where N is a p-group and H is p-nilpotent, say $H=Q \rtimes M$ with $p \nmid |Q|$ and M a p-group. Let V be an irreducible FG-module, let W be an irreducible component of $V|_Q$, and let T denote the inertia group of W in H. Then

i. $P_G(V) \cong W^G$;

ii. $\ell(P_G(V)) \geq t(T/Q) + \ell((V_H)^G)-1$. If $T \cap M$ has a normal complement in M, then equality holds.

<u>Proof.</u> By [1, §3], we have $P_H(V) \cong W^H$ and $\ell(W^H) = t(T/Q)$. Therefore, Lemma 1.4 implies that $\ell(P_G(V)) \geq t(T/Q) + \ell((V_H)^G)-1$ and $P_G(V) \cong (I_H)^G \otimes_F \otimes_F W^H \cong W^G$. By [8], there exists a unique FT-module U such that $U|_Q \cong W$. The induced module U^H is irreducible, and $U^H \cong V_H$, since both have a common FQ-component. Now let M_1 be a normal complement for $T \cap M$ in M and set $S = \langle Q,M_1 \rangle \leq H$. Then S is normal in H and $V|_S \cong U^H|_S \cong W^S$. Hence

$$P_G(V) \cong (I_H)^G \otimes_F W^H \cong (I_H)^G \otimes_F (V|_S)^H \cong$$

$$\cong (I_H)^G \otimes_F (V_{\langle N,S\rangle})^G \cong \left((V_H)^G\Big|_{\langle N,S\rangle}\right)^G .$$

Corollary 2.2 implies that $\ell(P_G(V)) \leq t(G/\langle N,S\rangle) + \ell((V_H)^G)-1$. Since $G/\langle N,S\rangle \cong T/Q$, the proposition is proved. □

<u>Corollary 2.4.</u> In the situation of Proposition 2.3,

assume that M is elementary abelian. Then, for any irreducible FG-module V,

$$\ell(P_G(V)) = (\mathrm{rk} M - d)(p-1) + \ell((V_H)^G),$$

where p^d is the p-part of $\dim_F V$.

<u>Proof.</u> By assumption on M, $T \cap M$ has a normal complement in M, and $t(T/Q) = \mathrm{rk}\,(T \cap M)(p-1)+1$ [4]. Let U be as in the proof of Proposition 2.3 so that $U^H \cong V|_H$. Then $\dim_F U$ is not divisible by p and so the p-part of $\dim_F V$ equals $p^d = |H/T|$. Therefore, $\mathrm{rk}\,(T \cap M) = \mathrm{rk}\, M - d$ and the corollary follows. □

<u>Example 2.5.</u> Let $G = N \rtimes H$ be as in Example 1.5, with $N = C_3 \times C_3$ and $H = SL_2(3) = Q_8 \rtimes C_3$. Then $FN = (I_H)^G$ has Loewy series

$$(I_H)^G = \begin{matrix} I \\ 2 \\ 3 \\ 2 \\ I \end{matrix},$$

where $I, 2, 3$ denote the simple FH-modules as in Example 1.5, and so Corollary 2.4 yields $\ell(P_G(I)) = 2+5 = 7$.

Also, by Corollary 2.4, $\ell(P_G(2)) = 2+\ell((2_H)^G)$. Using $2 \otimes 2 \cong I \oplus 3$ and $2 \otimes 3 \cong \begin{smallmatrix}2\\2\\2\end{smallmatrix}$, we see that $(2_H)^G =$ $= 2 \otimes (I_H)^G$ has Loewy length at most 7. On the other hand, we already know that $\ell((2_H)^G) \geq 7$ (Example 1.5) and so we obtain $\ell(P_G(2)) = 9$.

As to the remaining irreducible module, 3, recall that $3=1^G$, where 1 is a non-trivial 1-dimensional module for $U = < N, Q_8 > \leq G$. Thus, by Proposition 2.3, $P_G(3) = (1_{Q_8})^G = (3_H)^G = 3 \otimes (I_H)^G$. Clearly, $\ell((3_H)^G) \leq 2 + \ell(X)$ where

$$X = 3 \otimes \begin{matrix}2\\3\\2\end{matrix} = \left(1 \otimes \begin{matrix}2\\3\\2\end{matrix} \Big|_U \right)^G .$$

Since $J(FU) = (\omega N)FU$, the Loewy series of $1 \otimes \begin{smallmatrix}2\\3\\2\end{smallmatrix}\Big|_U$ is easy to compute:

$$1 \otimes \begin{matrix}2\\3\\2\end{matrix}\Big|_U = I_U \quad \begin{matrix}2_U\\1'1''\\2_U\end{matrix} ,$$

where 1' and 1" denote the G-conjugates of 1. In particular, $1 \otimes \begin{smallmatrix}2\\3\\2\end{smallmatrix}\Big|_U$ is a homomorphic image of $Y = P_U(2)/ P_U(2) \cdot (\omega N)^3$.

Now $P_U(2) = 2_U \otimes (I_Q)^U = (2 \otimes (I_H)^G)\big|_U$ and $P_U(2) \cdot (\omega N)^3 = (2 \otimes (I_H)^G \cdot (\omega N)^3)\big|_U$, hence

$$Y = \left(2 \otimes \frac{(I_H)^G}{(I_H)^G \cdot (\omega N)^3}\right)\bigg|_U = \left(2 \otimes \begin{smallmatrix} I \\ 2 \\ 3 \end{smallmatrix}\right)\bigg|_U$$

$$= \begin{array}{ccc} & 2 & \\ I & & 3 \\ & 2 & \\ & 2 & \\ & 2 & \end{array}\Bigg|_U .$$

Corollary 2.2 implies that $\ell\,(Y^G) \le 3 + 5 - 1 = 7$. Therefore, $\ell(X) \le 7$ and $\ell(P_G(3)) \le 9$. In particular, we obtain $t(G) = 9$.

In the following, we set

$$\operatorname{gr} FN = \bigoplus_{i \ge 0} \frac{(\omega N)^i}{(\omega N)^{i+1}} ,$$

and we view gr FN as FG-module by letting G act by conjugation.

<u>Corollary 2.6.</u> Let G be as in Proposition 2.3 and let V be an FG-module such that $V|_Q$ is irreducible. Then

$$\ell(P_G(V)) \geq t(M) + t(N)-1 \quad ,$$

and equality holds if and only if $V \otimes_F \mathrm{gr}\, FN$ is semisimple.

Proof. By Proposition 2.3, $\ell(P_G(V))=t(M)+\ell((V_H)^G-1$ and, by Lemma 1.4 (iii), $\ell((V_H)^G)\geq t(N)$ with equality occuring if and only if $V\otimes_F \mathrm{gr}\, FN$ is semisimple. □

Theorem 2.7. Assume that $G=N\rtimes H$ with N a p-group and $H=Q\rtimes M$ a Frobenius group with kernel Q a p'-group and M a p-group. Then $t(M)+t(N)-1=t(G)$ if and only if $\mathrm{gr}\, FN$ is semisimple.

Proof. The condition is clearly necessary in view of Corollary 2.6. Conversely, assume the condition is satisfied. Our assumption on H implies that $J(FM) = e\cdot \omega M$, where $e=|Q|^{-1} \Sigma_{q\in Q} q$ is a central idempotent of FH. Indeed, this follows from the fact that for any irreducible FQ-module $W \neq I$ the induced module W^H is irreducible [3, Lemma 15.15]. Thus the semisimplicity of $\dfrac{(\omega N)^i}{(\omega N)^{i+1}}$ just says that for all $\alpha \in (\omega N)^i$ and $m \in M$ we have $\Sigma_{q\in Q} \alpha^{qm} - \Sigma_{q\in Q} \alpha^q \in$ $\in (\omega N)^{i+1}$. It follows by a straightforward calculation that, for all $i \geq 0$,

$$e\cdot\omega M\cdot(\omega N)^i\cdot e + e(\omega N)^{i+1}e = e(\omega N)^i\omega Me + e(\omega N)^{i+1}e.$$

Set $\ell = t(N) + t(M)-1$, $X = (\omega N)FG$, and $Y = e\cdot\omega M$. Then $J(FG) = X + Y$, and we have to show that if $\alpha \in FG$ can be written as a product of ℓ factors each of which belongs to either X or Y then $\alpha = 0$. We argue by descending induction on the number $\ell_X = \ell_X(\alpha)$ of X-factors involved in α . If $\ell_X \geq t(N)$ then $\alpha \in X^{t(N)} = \{0\}$. So assume that $\ell_X < t(N)$. Then the number of Y-factors involved in α is at least $t(M)$. Let $n_Y = n_Y(\alpha)$ denote the length of the longest consecutive subproduct of α consisting entirely of Y-factors. Clearly, if $n_Y \geq t(M)$ then $\alpha = 0$. So assume that $n_Y < t(M)$. Then α contains a subproduct which either belongs to $YX^iY^{n_Y}$ or to $Y^{n_Y}X^iY$ $(i>0)$. We consider the first case, the second being entirely analogous. Now

$$YX^iY^{n_Y} = e\cdot\omega M\cdot(\omega N)^i\cdot e(\omega M)^{n_Y} \subseteq e(\omega N)^i(\omega M)e(\omega M)^{n_Y} + e(\omega N)^{i+1}e(\omega M)^{n_Y} \subseteq X^iY^{n_Y+1} + X^{i+1}Y^{n_Y} .$$

Thus we have $\alpha = \alpha_1 + \alpha_2$ with $\alpha_1, \alpha_2 \in J(FG)^{\ell}$, but $\ell_X(\alpha_2) > \ell_X(\alpha)$ and $\ell_X(\alpha_1) = \ell_X(\alpha)$, $n_Y(\alpha_1) > n_Y(\alpha)$. By induction, we conclude that $\alpha_1 = \alpha_2 = 0$ and so $\alpha = 0$. □

Certainly, $\operatorname{gr} FN$ is semisimple if FN is semisimple over FH. The converse, of course, need not be true. For example, if $G = N \rtimes H$ is as in Examples 1.5 and 2.5, then $\operatorname{gr} FN = I^{(2)} \oplus 2^{(2)} \oplus 3$ is semisimple but FN is not. To see the latter, let $z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $m = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in H = SL_2(3)$. Then $\alpha = (1-z)(1+m+m^2)$ belongs to $J(FH)$ and if $\langle a,b \rangle$ is the standard basis of $N = \mathbb{F}_3 \oplus \mathbb{F}_3$ then $b \cdot \alpha = (b-b^{-1}) \cdot (1+m+m^2) =$
$= (b-b^2) \cdot (1+m+m^2) = b-b^2 + ab-a^2b^2 + ab^2-a^2b \neq 0$. –
If $H = Q \rtimes M$ is Frobenius, as in the theorem, then it is easily seen that FN is semisimple over FH if and only if M stabilizes all Q-orbits in N.

<u>Example 2.8.</u> Let $G = S_4$ and use the notation of Example 1.3. Then $G = V_4 \rtimes H$ with $H = GL_2(2) =$ $= C_3 \rtimes C_2$ a Frobenius group. Also , C_2 stabilizes the C_3-orbits $\{1\}$ and $V_4 \setminus \{1\}$ in V_4 . Hence, by the above remark, $\operatorname{gr} FV_4$ is semisimple (in fact, $\operatorname{gr} FV_4 = I^{(2)} \oplus 2$) and we conclude that $t(G)=t(V_4) +$ $+ t(GL_2(2)) - 1 = 4$, a result due to Motose and Ninomiya[6]. In particular, $\ell(P_G(I)) = \ell(P_G(2)) = 4$.

Further examples of a similar form have been contructed by Motose [5] , for every prime p .

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