On relations of dimensions of automorphic forms of Sp(2,R) and its compact twist Sp(2) (II)

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In this paper, we show some good global dimensional relations between automorphic forms of Sp(2,$\mathbb{R}$)(matrix size four) and its compact twist Sp(2). One of the author has already shown such relations when the p-adic completions(for a fixed prime p) of the discrete subgroups in question are maximal compact(See [24]). In this paper, we treat discrete subgroups whose p-adic completions are minimal parahoric. Our aim is a generalization of Eichler-Jacquet-Langlands correspondence between SL_2 and SU(2) to the symplectic case of higher degree. Such correspondence should be proved by comparison of the traces of all the Hecke operators. Our results mean that there exists good relations of traces at least for T(1) for some explicitly defined discrete subgroups of Sp(2,$\mathbb{R}$) and Sp(2)(§2 Main Theorem I). Besides, they give meaningful examples for Langlands philosophy on stable conjugacy classes(§2 Main Theorem II). Roughly speaking, such comparison is divided into character relations at infinite places(which are more or less known) and arithmetics

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at finite places. Our point is to execute the comparison of the arithmetical part explicitly. It seems that our Theorems are the first global results on such relations except for GL_n (cf. also [24]). In §1, after a brief introduction, we give a precise formulation on our problems between $Sp(n,\mathbb{R})$ and $Sp(n)$ for general n, e.g. on how to choose discrete subgroups explicitly. For automorphic forms with respect to these explicitly chosen discrete subgroups, we propose there two conjectures (which were first given in [21],[23]): coincidence of dimensions and existence of an isomorphism between new forms as Hecke algebra modules. For n = 1, these are nothing but the theorems by Eichler [10],[11], and the above conjectures are a natural generalization of his results. Langlands [34] has given a quite general philosophy on correspondence of automorphic forms of any reductive algebraic groups, but we understand that his philosophy does not give very detailed formulation at present for such typical and explicit cases as treated in this paper, and we believe that the above conjectures have its own interest. In §2, we state our Main Theorems, which assert that the first conjecture is true also for n = 2. The proof consists of explicit calculation of dimensions. Such explicit dimension formulae, given in §3 Th.3.2, 3.3, 3.4, 3.5, have their own value. Our Main Theorems are corollary to the results in §3 and [16],[19],[24].

The proof of the formulae in §3 starts from §4. In §4, for the convenience for the readers, we give expository review on the results in [15], [16], [19] how to calculate dimensions. In §5 and §6, we list up explicit data which are needed for the calculation of dimensions. In §7, we give a brief survey on some related topics.

§1. Conjectures

Let G and G' be two different reductive algebraic groups over algebraic number fields. For some good choice of G and G', sometimes we know that there exists a correspondence between automorphic representations $\pi = \bigotimes_v \pi_v$ of G_A and $\pi' = \bigotimes_v \pi'_v$ of G'_A which preserves L functions, where G_A or G'_A is the adelization of G or G'. Langlands [34] has given a general philosophy on such problems: he defined so called L groups ${}^LG$ or ${}^LG'$, and he conjectured that, if G is quasi split, and if there exists a L-Homomorphism $u : {}^LG' \longrightarrow {}^LG$, then there should exist "good" correspondence of automorphic representations. (As for more precise contents of this conjecture, see Langlands [34], or Borel [3].) For example, if G' is an inner twist of G, then ${}^LG \simeq {}^LG'$, and we can expect a good correspondence. One of the reason of this conjecture seems to be the fact that there exists a good character relations between π_∞ and π'_∞.

The basic example is GL(2). The first typical results on the relation between GL(2) and division quaternion algebras was due to Eichler [10], [11], and later completed by many mathematitians, notably, Shimizu [43], Jacquet-Langlands [29]. One obvious direction of generalizations of the GL(2)-case is GL(n), which has been studied also by various mathematitians.

Now, another direction is the symplectic groups, because we can regard GL(2) as the symplectic group of size two with similitudes. Let $Sp(n,\mathbb{R})$ be the symplectic group of size 2n, and Sp(n) be its compact twist: $Sp(n) = \{ g \in M_n(H);\ g\,{}^t\bar{g} = 1_n \}$, where H is the division quaternion algebra over R and $\bar{\ }$ is the canonical involution. When n = 2, for pairs of Q-forms of Sp(2,R) and Sp(2), Ihara [28] raised a conjectural problem on an existence of correspondence of automorphic forms(independent from and older than Langlands [34]). He clarified, among others, what should be the correspondence of weights(i.e. representations at infinity) of those forms by showing some character relations(unpublished). (As for some other works by him, see [28] or §7.) Later, Hina and Masumoto [20] gave character relations between some admissible representations of GSp(2,F)(size four) and its inner twist, when F is a non archimedean local field. But, there was no global result, and any global example had not been known before [21]. We would like to have some global(and rather classical) approach to this problem, and aim a generalization of the typical results of Eichler. Even if we restrict ourselves to such typical cases, the precise formulation had not been known before [21],[23]. Besides, such typical cases have their own fruitful structures. Our aim of this paper is to give good global dimensional relations in such cases. This can be regarded as the first step to the proof of such correspondence of automorphic forms.

Now, we explain our problem more precisely. Put

$$G = GSp(n,Q) = \left\{ g \in M_{2n}(Q);\ gJ{}^tg = n(g)J,\ n(g) \in Q^\times \right\},$$

where $J = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$ and 1_n is the unit matrix of size n.

On the other hand, let D be the definite quaternion algebra over Q with prime discriminant p. (We fix a prime number p.) Put

$$G' = \left\{ g \in M_n(D);\ g{}^t\bar{g} = n(g)1_n,\ n(g) \in Q^\times \right\}.$$

Then, G' is an inner twist of G. Let G_A (resp. G'_A) be the adelization of G(resp. G'), and for any place v of Q, let G_v (resp. G'_v) be the v-adic component of G_A (resp. G'_A).

We have $G_\infty = GSp(n,R)$ and $G'_\infty = GSp(n)$ (i.e. the group of symplectic similitudes). We note that $G'_v \cong G_v = GSp(n,Q_v)$, if $v \neq p, \infty$. We consider subgroups U_A (resp. U'_A) of G_A (resp. G'_A) of the following forms:

$$(1.1) \qquad U_A = G_\infty P \prod_{q \neq p} GSp(n,Z_q) \qquad \text{(resp.}$$

$$(1.2) \qquad U'_A = G'_\infty P' \prod_{q \neq p} GSp(n,Z_q) \qquad \text{)},$$

where P(resp. P') is an open compact subgroup of G_p (resp. G'_p), and, for any prime q,

$$GSp(n,Z_q) = \left\{ g \in GSp(n,Q_q);\ g,\ g^{-1} \in M_{2n}(Z_q) \right\}.$$

We define automorphic forms and Hecke operators.

Let H_n be the Siegel upper half space of degree n:

$$H_n = \{X+iY;\ X,\ Y \in M_n(R),\ {}^tX = X,\ {}^tY = Y,\ Y > 0, \text{i.e. } Y \text{ is positive definite}\}.$$

An element $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp(n,R)$ acts on H_n by:

$$Z \longrightarrow (AZ+B)(CZ+D)^{-1}.$$

Put

$$GSp(n,Q)^+ = \{g \in G;\ n(g) > 0\} \quad \text{and}$$

$$U = U_A \cap GSp(n,Q)^+.$$

Then, U acts on H_n discontinuously and vol$(U \backslash H_n)$ is finite.

The space $S_k(U)$ of cusp forms of weight k with respect to U is defined by:

$$S_k(U) = \Big\{ \text{holomorphic functions } f \text{ on } H_n \text{ such that}$$

$$(1)\quad f(\gamma Z) = f(Z)\det(CZ+D)^k \text{ for all } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U,$$

$$(2)\quad f(Z)(\det Y)^{k/2} \text{ is bounded on } H_n \Big\}.$$

For any natural integer m $(p \nmid m)$, the action of the Hecke operator $T(m)$ on $S_k(U)$ is defined as follows: Put

$T(m) = \bigcup_g UgU$, where g runs through elements of

$$\Delta_m = \{g \in G \cap M_{2n}(Z);\ gJ{}^tg = mJ\}.$$

We take a coset decomposition $T(m) = \coprod_{i=1}^{d} Ug_i$ (disjoint).

For any $f \in S_k(U)$, we define $f|T(m)$ by:

$$(f|T(m))(Z) = m^{nk-n(n+1)/2} \sum_{i=1}^{d} f(g_i Z)\det(C_i Z+D_i)^{-k},$$

where $g_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$.

On the other hand, let (ρ, V) be an irreducible representation of G'. We regard ρ as a representation of G_A by composing it with projection: $\rho : G'_A \longrightarrow G \longrightarrow GL(V)$.

The space $M_\rho(U'_A)$ of automorphic forms on G'_A of weight ρ with respect to U'_A is defined by:

$$M_\rho(U'_A) = \{ f : G'_A \longrightarrow V;\ f(uga) = \rho(u)f(g) \text{ for all } a \in G',\ u \in U'_A, \text{ and } g \in G'_A \}.$$

As well known, we can realize V in a space of some spherical functions. The strong approximation theorem does not hold for G' and the 'class number' of U'_A is not one in general.

A 'classical' interpretation of $M_\rho(U'_A)$ is given as follows:

Take a double coset decomposition $G'_A = \coprod_{i=1}^{h} U'_A g_i G'$ (disjoint),

and put

(1.3) $\quad \Gamma_i = g_i^{-1} U'_A g_i \cap G'$.

Put

$$V^{\Gamma_i} = \{ v \in V;\ \rho(\gamma)v = v \text{ for all } \gamma \in \Gamma_i \}.$$

then, we have

(1.4) $$M_\rho(U'_A) = \bigoplus_{i=1}^{h} V^{\Gamma_i}$$

where the isomorphism is given by $f \longrightarrow (\rho(g_i^{-1})f(g_i))_{i=1\ldots h}$.

Let ρ_ν be the representation of Sp(n) which corresponds with the Young diagram $\left.\begin{array}{|c|c|}\hline 1 \ \cdots & \nu \\ \hline 1 \ \cdots & \nu \\ \hline \cdots & \cdot \\ \hline 1 \ \cdots & \nu \\ \hline \end{array}\right\} n$.

We extend it by putting $\rho_\nu(a 1_n) = a^{n\nu}$ for $a \in R^\times$, $a > 0$.

We write $M_{\rho_\nu}(U'_A) = M_\nu(U'_A)$. If $-1 \in U'_A$, then $M_\nu(U'_A) = 0$, unless $(-1)^{n\nu} = 1$. We put $T'(m) = \bigcup_g U_A' g U'_A$ $(p \nmid m)$, where g runs through elements of

$$\Delta'_m = \{ g = (g_v) \in G'_A;\ g_v \in M_{2n}(Z_v) \text{ and } n(g_v) \in mZ_v^\times \text{ for all finite } v \} .$$

Take a coset decomposition

$$T'(m) = \coprod_{i=1}^{d'} g'_i U'_A \text{ (disjoint)}.$$

For any $f \in M_\nu(U'_A)$, $f|T'(m)$ is defined by:

$$(f|T'(m))(g) = \sum_{i=1}^{d'} \rho_\nu(g_i) f(g_i'^{-1} g) , \quad g \in G'_A.$$

The (abstract) Hecke algebra spanned by $T(m)$ $(p \nmid m)$ is isomorphic to the one spanned by $T'(m)$ $(p \nmid m)$. We sometimes denote $T'(m)$ by $T(m)$. For a common eigen form $f \in S_k(U)$ or $M_\nu(U'_A)$ of all the Hecke operators $T(m)$ $(p \nmid m)$, the L function of f is defined (up to the p-Euler factors) by:

$$L(s,F) = \text{the denominator of } \sum_{p \nmid m} \lambda(m) m^{-s}; \text{ where } T(m) f = \lambda(m) f.$$

Now, we review a typical case of Eichler's results on GL(2). Let O be a maximal order of D and O_p be its p-adic completion. Put $P' = O_p$ in (1.2). On the other hand, put

$$P = \{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,Z_p);\ c \equiv 0 \text{ mod.} p \}.$$

In the usual notation, $U = \Gamma_0(p)$ in this case. We write $U'_A = O_A^\times$ in this case.

<u>Theorem</u> 1.5 (Eichler [10], [11]) <u>If we denote by</u> $S_k^0(\Gamma_0(p))$ <u>the space of new forms of</u> $S_k(\Gamma_0(p))$, <u>then for</u> $k \geq 2$, <u>we have</u>: $M_{k-2}(O_A^\times) = S_k^0(\Gamma_0(p))$ ($\oplus$ C, if k = 2), <u>as modules over Hecke algebras</u> (i.e. <u>this isomorphism preserves</u> L-<u>functions</u>).

The new forms $S_k^0(\Gamma_0(p))$ are actually defined as the orthogonal complement of $S_k(SL_2(Z)) \oplus S_k(\rho\, SL_2(Z)\rho^{-1})$ in $S_k(\Gamma_0(p))$ with respect to the Petersson inner metric, where $\rho = \begin{pmatrix} 0 & -1 \\ p & 0 \end{pmatrix}$. So, we get

<u>Corollary</u> 1.6 (Eichler, loc.cit.) <u>For</u> $k \geq 2$, <u>we have</u>

$\dim M_{k-2}(O_A) = \dim S_k(\Gamma_0(p)) - 2 \dim S_k(SL_2(Z)) + \delta$,

<u>where</u> $\delta = 1$, <u>if</u> $k = 2$, <u>and</u> $\delta = 0$, <u>otherwise</u>.

This Cor.1.6 will be extended for n = 2 in this paper. But, before we state our Main Theorem, we would like tc propose general formulations and conjectures. If we want to generalize such a simple and beautiful typical results, several natural questions arise:

(1) What are the corresponding weights in the general case?

(2) What kind of U or U'_A should be taken instead of $\Gamma_0(p)$ and O_A ?

(3) What are new forms?

The answer to the question (1) for n = 2 is given by Ihara. The general case seems more or less known: If we take Siegel cusp forms of degree n with weight $k \geq n+1$, the corresponding weight of automorphic forms on G'_A should be ρ_{k-n-1}.

To questions (2) and (3), a hypothetical answer has been given in Ibukiyama [21],[23],[24]: First of all, as far as we take U_A or U'_A as in (1.1) or (1.2), this question is a local problem how to choose P or P'. Secondly, it is known that every reductive algebraic group over a non archimedean local field has the unique minimal parahoric subgroup up to conjugation (Tits [46]). Roughly speaking, the minimal parahoric subgroup is a group such that its reduction mod.p is the Borel subgroup. For example, P or P' chosen in Th.1.5 is minimal parahoric. So, it is natural to choose the minimal parahoric subgroup B of G_p or B' of G'_p as the first candidate for P or P', respectively. (As for another kinds of candidates, see [23],[24].) To obtain new forms, we should subtract automorphic forms belonging to U_A or U'_A with $P \supsetneq B$ or $P' \supsetneq B'$. To explain more precisely, we review briefly the Bruhat-Tits theory. The extended Dynkin diagram of G_p is the Coxeter graph of the affine Weyl group W of G_p, and the set S of summits of this graph can be regarded as a set of generators of W as a Coxeter system. We fix a minimal parahoric subgroup B of G_p. By the Bruhat decomposition, there is a one to one correspondence between the set of all subsets θ of S and the set of all subgroups of G_p containing B. More precisely, for each $w \in W$, there is a good representative of w in G_p, which we denote also by w. For a subset θ of S, put

$$P_\theta = \{\text{the group generated by all double cosets } BwB \text{ such that } w \in \theta\}.$$

Such groups are called standard parahoric subgroups. Then, we have $P_\theta \supset B$, and $P_\theta = P_{\theta'}$, if and only if $\theta = \theta'$. Besides, every group P which contains B is obtained in this way. For example, $P_\phi = B$ and $P_S = G_p$. For $\theta \subset S$, we put

$$(1.7) \qquad U_\theta(p)_A = G_\infty P_\theta \prod_{q \neq p} GSp(n, Z_q), \quad \text{and}$$

$$U_\theta(p) = GSp(n,Q)^+ \cap U_\theta(p)_A.$$

The above theory is completely the same also for G'_p, and we denote by S' the set of generators of the affine Weyl group of G'_p. We denote by P' the standard parahoric subgroup defined by $\theta' \subset S'$. We put

$$(1.8) \qquad U'_{\theta'}(p)_A = G'_\infty P'_{\theta'} \prod_{q \neq p} GSp(n, Z_q).$$

We often omit the suffix A in this case, because we do not treat 'global' disrete subgroups. We put $U_\phi(p) = B(p)$ and $U'_\phi(p) = B'(p)$. The second named author gave the following conjectures([21],[22]):

<u>Conjecture</u> 1.9 <u>For any integer</u> n , $\nu \geq 1$, <u>we should have</u>:

$$(1.10) \quad \sum_{\substack{\theta \subset S \\ \theta \neq S}} (-1)^{\#(\theta)} \dim S_{\nu+n+1}(U_\theta(p))$$

$$= \sum_{\substack{\theta' \subset S' \\ \theta' \neq S'}} (-1)^{\#(\theta)} \dim M_\nu(U'_{\theta'}(p)).$$

<u>If</u> $\nu = 0$, <u>we should add one to the right hand side</u>.

We define the space $S^0_k(B(p))$ of new forms of $S_k(B(p))$ as the orthogonal complement of $\sum_{\substack{\#(\theta)=1 \\ \theta \subset S}} S_k(U_\theta(p))$ (summation as $\mathbb{C}$-vector space) in $S_k(B(p))$ with respect to the Petersson inner metric. We define $M^0_\nu(B'(p))$ completely in the same way. These definitions mean that the p-adic admissible representation attached to a new form is the special representation (cf.[4]).

<u>Conjecture</u> 1.11 <u>For any integer</u> $n \geq 1$ <u>and</u> $\nu \geq 1$, <u>we have</u>:

$$(1.12) \qquad S^0_{\nu+n+1}(B(p)) = M^0_\nu(B'(p)),$$

<u>as modules over the Hecke algebra spanned by</u> $T(m)\ (p \nmid m)$.

For $n = 1$, these conjectures are nothing but Th.1.5 and Cor.1.6 by Eichler. For $n = 2$, Conjecture 1.9 is true (at least) for $\nu \geqq 2$ and $p \nmid 3$. This is our Main Theorem. For $n = 2$ and $p = 2$, there has been given some explicit examples $f \in S^0_{\nu+3}(B(p))$ and $f' \in M^0_{\nu}(B'(p))$ in Ibukiyama [21] such that Euler 3-factors of $L(s,f)$ and $L(s,f')$ coincide with each other and satisfy the Ramanujan Conjecture at 3 (i.e. these cannot be obtained as 'liftings' of one dimensional automorphic forms). For general n, the both sides of (1.10) are expressed as a sum of contributions of conjugacy classes of elements of G or G'. For some conjugacy classes, we can show the equality of contributions. For example, the main terms(i.e. the contribution of the unit elements) of both sides of (1.10) coincide with each other and given by:

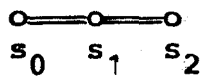
$$\frac{2(\nu+1)(\nu+2)\ldots(\nu+n)}{n!} \prod_{1\leq i\leq j\leq n} \frac{2\nu+i+j}{i+j} \frac{\prod_{i=1}^{n} \zeta(2i)}{(2\pi)^{n(n+1)}}$$

$$\times (p-1)(p^3-1)\ldots(p^{2n-1}-1).$$

As for this kind of relations for another algebraic groups which are not symplectic, see Ibukiyama [23].
We have some results also for some kind of unipotents elements of G or G'.

§2. Main Theorem

In this section, we explain our Main Theorem more in detail. For n = 2, the extended Dynkin diagrams of G_p and G'_p are given as follows:

G_p 

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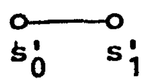
G'_p 

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where $\{s_0, s_1, s_2\}$ or $\{s'_0, s'_1\}$ is the set of generators of the affine Weyl group W or W' of G_p or G'_p, respectively. We can take the minimal parahoric subgroup B of G_p as follows:

$$B = GSp(2, Z_p) \cap \begin{pmatrix} * & * & * & * \\ p* & * & * & * \\ p* & p* & * & p* \\ p* & p* & * & * \end{pmatrix},$$

where * runs through all the p-adic integers..
We can fix representatives of $s_i (i=0,1,2)$ in G_p as follows:

$$s_0 = \begin{pmatrix} 0 & 0 & -p^{-1} & 0 \\ 0 & 1 & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Put $\rho = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & p & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}$.

Then, it is easy to show that

$$P_{\{s_1\}} = B \cup Bs_1B = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp(2, Z_p) ;\ C \equiv 0 \text{ mod.} p \right\} ,$$

$$P_{\{s_2\}} = B \cup Bs_2B = GSp(2, Z_p) \cap \begin{pmatrix} * & * & * & * \\ p* & * & * & * \\ p* & p* & * & p* \\ p* & * & * & * \end{pmatrix} ,$$

$$P_{\{s_0\}} = B \cup Bs_0B = \rho\, P_{\{s_2\}}\, \rho^{-1}$$

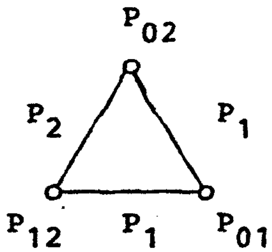
$$= GSp(2, Q_p) \cap \begin{pmatrix} * & * & p^{-1}* & * \\ p* & * & * & * \\ p* & p* & * & p* \\ p* & p* & * & * \end{pmatrix} ,$$

$$P_{\{s_0, s_2\}} = GSp(2, Q_p) \cap \begin{pmatrix} * & * & p^{-1}* & * \\ p* & * & * & * \\ p* & p* & * & p* \\ p* & * & * & * \end{pmatrix} ,$$

$$P_{\{s_1, s_2\}} = GSp(2, Z_p) \text{ , and } P_{\{s_0, s_1\}} = \rho\, GSp(2, Z_p) \rho^{-1} ,$$

where * runs through all the p-adic integers. For the sake of simplicity, we write $P_{\{s_0\}} = P_0$, $P_{\{s_0, s_2\}} = P_{02}$, etc.

The relations of the standard parahoric subgroups of G_p are illustrated as follows:



which means that every face is the intersection of its boundaries and every summit is spanned by simlicies containing it. For example, $P_2 = P_{02} \cap P_{12}$ and P_{12} is generated by P_1 and P_2 etc.

To explain the parahoric subgroups of G_p', it is convenient to take an another model. Put $D_p = D \underset{Q}{\otimes} Q_p$ and

$$G_p^* = \left\{ g \in M_2(D_p);\ \ g\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t\bar{g} = n(g)1_2,\ n(g) \in Q_p^\times \right\} .$$

Then, $G_p^* \simeq G_p$. We fix such an isomorphism once and for all.

Let π be a prime element of $O_p = O \underset{Z}{\otimes} Z_p$. We can take a minimal parahoric subgroup B' of G_p^* as follows:

$$B' = \begin{pmatrix} O_p & O_p \\ \pi O_p & O_p \end{pmatrix}^\times \cap G_p^* .$$

We can put

$$s_0' = \begin{pmatrix} 0 & \bar{\pi}^{-1} \\ \pi & 0 \end{pmatrix} , \quad \text{and} \quad s_1' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Then,

$$P'_0 = P_{\{s'_0\}} = \begin{pmatrix} O_p & \pi^{-1}O_p \\ \pi O_p & O_p \end{pmatrix} \cap G^*_p, \quad \text{and}$$

$$P'_1 = P_{\{s'_1\}} = M_2(O_p) \cap G^*_p.$$

We can illustrate these groups as follows:

$$P'_0 \ \circ\!\underset{B'}{\rule[0.5ex]{4em}{0.4pt}}\!\circ\ P'_1$$

We define $U_\theta(p)$ or $U'_\theta(p)$ as in (1.1), and put $U_0(p) = U_{\{s_0\}}(p)$, etc. In the notation of [21],[23],[24].

$U_1(p) = \Gamma_0(p)$, $U_2(p) = \Gamma'_0(p)$, $U_0(p) = \Gamma''_0(p)$, and $U_{02}(p) = K(p)$. You can get an expression of $U_\theta(p)$ or $B(p)$, if you replace p-adic integers or numbers by rational integers or numbers in the expression of P or B, and take $n(g)$ to be one. We have no standard global expression of $U'_\theta(p)$ or $B'(p)$, partly because the 'class number' of G' is not one. You can find some explicit examples of Γ_i defined in (1.3) in [21],[26].

Main Theorem I For $n = 2$, any integer $k \geqq 5$, and any prime $p \nmid 3$, the conjecture 1.9 is true, i.e., we have the following equality:

(2.1)

$$\dim S_k(B(p)) - \dim S_k(U_0(p)) - \dim S_k(U_1(p)) - \dim S_k(U_2(p))$$
$$+ 2 \dim S_k(Sp(2,Z)) + \dim S_k(U_{02}(p))$$
$$= \dim M_{k-3}(B'(p)) - \dim M_{k-3}(U_1'(p)) - \dim M_{k-3}(U_0'(p)).$$

As we shall explain in §4, dimension formulae are expressed as summations over the contributions of conjugacy classes of elements (with $n(g) = 1$) of G or G' of various types. Any elements of G' (with $n(g) = 1$) are semi simple, because they are embedded into the compact group Sp(2). We have $G \otimes \mathbb{C} = G' \otimes \mathbb{C} = GSp(2,\mathbb{C})$. Let C be a conjugacy class of some semi simple elements of $GSp(2,\mathbb{C})$. It is well known that C is determined only by the principal polynomial $f(x)$ of all elements of C. Let $T(f)$ (resp. $H(f)$) be the set of all G-(resp.G'-) conjugacy classes contained in C.

Main Theorem II The contribution of non-elliptic (i.e. non torsion) conjugacy classes to the left hand side of (2.1) is zero. For any polynomial $f(x)$ which is the principal polynomial of some elements of G or G' of finite order, the contribution of $T(f)$ to the left hand side of (2.1) is equal to the contribution of $H(f)$ to the right hand side.

Remark 2.2 This Main Theorem II can be regarded as an evidence for the philosophy by Langlands[36] on stable conjugacy classes.

Remark 2.3 The proof of the Main Theorem consists of an explicit calculation of each dimension. Some of the above mentioned dimensions have been already known: $\dim S_k(Sp(2,\mathbb{Z}))$ by Igusa[27] (cf. also [16]), $\dim S_k(U_1(p))$ by Hashimoto [16], $\dim S_k(U_{02}(p))$ by Ibukiyama[24], and $\dim M_{k-3}(U_i'(p))$ $(i=0,1)$ by Hashimoto and Ibukiyama[19]. So, we shall calculate $\dim S_k(B(p))$ explicitly in the following sections. We note that (2.1) has been known for $p = 2$ with $k \geq 3$, although we must add one to the right hand side, if $k = 3$ (cf. [24]). For general p, we need the trace formula, but it does not work well at present, unless $k \geq 5$. We assumed $p \neq 3$, because it sometimes makes computation easier. By virtue of the above mentioned results, we can also assume that $p \neq 2$.

§3. Explicit dimension formulae

In this section, we give explicit formulae for the dimensions of $S_k(B(p))$, $S_k(U_2(p))$, and $M_\nu(B'(p))$. The proof will be found in the following sections. First, we treat $B(p)$ and $U_2(p)$. The dimensions are expressed as sums of contributions of $B(p)$- or $U_2(p)$-conjugacy classes. But, by definition, we have $B(p), U_2(p) \subset Sp(2,Z)$.

So, it is convenient to group together those $B(p)$- or $U_2(p)$-conjugacy classes that are contained in a $Sp(2,Z)$-conjugacy class. Representatives of the $Sp(2,Z)$- conjugacy classes of elements of finite order were given by $\alpha_i (i=0,\ldots,22)$, $\beta_i (i=1,\ldots,6)$, $\gamma_i (i=1,2,3)$, or $\delta_i (i=1,2)$, up to sign, according to the notation of [16]. The non-semi-simple $Sp(2,Z)$-conjugacy classes are divided into various types as in [16], and those which have a contribution to the dimension formulae are of type $\pm \hat{\beta}_i(n) (i,\ldots,10)$, $\hat{\hat{\delta}}_i(m,n) (i=1,\ldots,4)$, $\pm \hat{\delta}_i(n) (i=1,2)$, $\hat{\gamma}_i(n) (i=1,\ldots,4)$, $\pm \hat{\gamma}_i(n) (i=5,6,7)$, or $\pm \mathcal{E}_i(S) (i=1,\ldots,4)$, according to the notation in [16]. We use above notations to denote the set of conjugacy classes of that "type". For example, α_1 (resp. $\hat{\beta}_1$) denotes the set of $Sp(2,Z)$-conjugacy classes which contain α_1 or $-\alpha_1$ (resp. $\hat{\beta}_1(n)$ or $-\hat{\beta}_1(n)$ for some $n \in Z$, $n \neq 0$). For $U = B(p)$, or $U_2(p)$, and a set C

of some Sp(2,Z)-conjugacy classes, we denote by t(U,C,k) the total sum of the contributions to dim $S_k(U)$ of U-conjugacy classes contained in C. We sometimes omit U and denote it by t(C,k), if no confusion is likely.
The principal polynomial of the above Sp(2,Z)-conjugacy classes are given as follows:

(3.1)

$f_1(x) = (x-1)^4$, or $f_1(-x)$ for $\pm\alpha_0$, $\pm\varepsilon_i (i=1,\dots,4)$,

$f_2(x) = (x-1)^2(x+1)^2$ for $\delta_i (i=1,2)$, $\hat{\hat{\delta}}_i (i=1,\dots,4)$, $\pm\hat{\delta}_i (i=1,2)$,

$f_3(x) = (x-1)^2(x^2+1)$ for $\pm\beta_i (i=5,6)$, $\pm\hat{\beta}_i (i=7,\dots,10)$,

$f_4(x) = (x-1)^2(x^2+x+1)$, or $f_4(-x)$ for $\pm\beta_i (i=1,2)$, $\pm\hat{\beta}_i (i=3,\dots,6)$,

$f_5(x) = (x-1)^2(x^2-x+1)$, or $f_5(-x)$ for $\pm\beta_i (i=3,4)$, $\pm\hat{\beta}_i (1,2)$,

$f_6(x) = (x^2+1)^2$ for $\pm\alpha_1$, $\gamma_i (i=1,2)$, $\hat{\gamma}_i (i=1,\dots,4)$,

$f_7(x) = (x^2+x+1)^2$ or $f_7(-x)$ for $\pm\alpha_i (i=2,3)$, $\pm\hat{\gamma}_i (i=5,6,7)$,

$f_8(x) = (x^2+1)(x^2+x+1)$ or $f_8(-x)$ for $\pm\alpha_i (i=19,20,21,22)$,

$f_9(x) = (x^2+x+1)(x^2-x+1)$ for $\pm\alpha_i (i=7,8)$, $\alpha_i (i=9,10,11,12)$,

$f_{10}(x) = x^4+x^3+x^2+x+1$ or $f_{10}(-x)$ for $\pm\alpha_i (i=15,16,17,18)$,

$f_{11}(x) = x^4+1$ for $\alpha_i (i=4,5)$ $\cdot\pm\alpha_6$,

$f_{12}(x) = x^4-x^2+1$ for $\alpha_i (i=13,14)$.

<u>Theorem</u> 3.2 <u>For a natural integer</u> $k \geq 5$ <u>and a prime</u> $p \neq 2,3$, $\dim S_k(B(p))$ <u>is given by the summation of the following quantities</u>:

$$t(\alpha_0,k) = (p+1)^2(p^2+1)(2k-2)(2k-3)(2k-4)/2^9 3^3 5,$$

$$t(\alpha_1,k) = (p+1)(1+(\tfrac{-1}{p}))(-1)^k/2^6,$$

$$t(\alpha_2,k)+t(\alpha_3,k) = -(p+1)(1+(\tfrac{-3}{p}))/2\cdot 3^3 \times [0,1,-1;\ 3],$$

$$\sum_{i=4}^{6} t(\alpha_i,k) = [1,\ 0,\ 0,\ -1;\ 4]\begin{cases} 1 \ \dots \text{ if } p \equiv 1 \text{ mod.}8, \\ 0 \ \dots \text{ if } p \not\equiv 1 \text{ mod.}8, \end{cases}$$

$$\sum_{i=7}^{12} t(\alpha_i,k) = [1,0,0,-1,0,0;\ 6](1+(\tfrac{-3}{p}))\ \frac{4}{9}$$

$$t(\alpha_{13},k)+t(\alpha_{14},k) = \frac{2}{3}[0,1,-1;\ 3]\begin{cases} 1 \ \dots \text{ if } p \equiv 1 \text{ mod.}12 \\ 0 \qquad \text{otherwise}, \end{cases}$$

$$\sum_{i=15}^{18} t(\alpha_i,k) = \frac{1}{5}[1,0,0,-1,0;\ 5]\begin{cases} 8 \ \dots \text{ if } p \equiv 1 \text{ mod.}5 \\ 1 \ \dots \text{ if } p = 5 \\ 0 \ \dots \text{ otherweise}, \end{cases}$$

$$\sum_{i=19}^{22} t(\alpha_i;k) = \frac{1}{6}(1+(\tfrac{-1}{p}))(1+(\tfrac{-3}{p}))[1,0,0,-1,-1,-1,-1,0,0,1,1,1;12],$$

$$t(\beta_1,k)+t(\beta_2,k) = (p+1)(1+(\tfrac{-3}{p}))[2k-3,-k+1,-k+2;\ 3]/2^2 3^3,$$

$$t(\beta_3,k)+t(\beta_4,k) = (p+1)(1+(\tfrac{-3}{p}))/2^2 3^2$$
$$\times [-1,-k+1,-k+2,1,k-1,k-2;\ 6],$$

$$t(\beta_5,k)+t(\beta_6,k) = (p+1)\left(1+\left(\frac{-1}{p}\right)\right)[k-2,-k+1,-k+2,-k-1;\ 4]/2^4 3,$$

$$t(\gamma_1,k)+t(\gamma_2,k) = 5(p+1)\left(1+\left(\frac{-1}{p}\right)\right)(2k-3)/2^6 3,$$

$$t(\gamma_3,k) = (p+1)\left(1+\left(\frac{-3}{p}\right)\right)(2k-3)/3^3,$$

$$t(\delta_1,k)+t(\delta_2,k) = 7(p+1)^2(-1)^k(2k-2)(2k-4)/2^8 3^2,$$

$$t(\hat{\beta}_1,k)+t(\hat{\beta}_4,k) = \left(1+\left(\frac{-3}{p}\right)\right)[0,1,1,0,-1,-1;\ 6]/3,$$

$$t(\hat{\beta}_3,k)+t(\hat{\beta}_4,k) = -\left(1+\left(\frac{-3}{p}\right)\right)[2,-1,-1;\ 3]/3^2,$$

$$t(\hat{\beta}_5,k)+t(\hat{\beta}_6,k) = -\left(1+\left(\frac{-1}{p}\right)\right)[1,-1,0;\ 3]\frac{4}{9}$$

$$t(\hat{\beta}_7,k)+t(\hat{\beta}_8,k) = -\left(1+\left(\frac{-1}{p}\right)\right)[1,-1,-1,1;\ 4]/4,$$

$$t(\hat{\beta}_9,k)+t(\hat{\beta}_{10},k) = -\left(1+\left(\frac{-1}{p}\right)\right)[1,-1,-1,1;\ 4]/4,$$

$$t(\hat{\hat{\delta}}_1,k)+t(\hat{\hat{\delta}}_2,k) = (-1)^k/2,$$

$$t(\hat{\hat{\delta}}_3,k)+t(\hat{\hat{\delta}}_4,k) = \left(3-\left(\frac{-1}{p}\right)\right)/2^3,$$

$$t(\hat{\delta}_1,k)+t(\hat{\delta}_2,k) = -(p+1)(-1)^k(2k-3)/2^2 3,$$

$$t(\varepsilon_1,k) = (p+1)/6,$$

$t(\varepsilon_2,k) = 0,$

$t(\varepsilon_3,k) = -(p+1)/2^2 3, \quad t(\varepsilon_4,k) = -(p+1)^2(2k-3)/2^4 3^2,$

$t(\hat{\gamma}_1,k)+t(\hat{\gamma}_2,k) = t(\hat{\gamma}_3,k)+t(\hat{\gamma}_4,k) = -(1+(\frac{-1}{p}))/2^2,$

$t(\hat{\gamma}_5,k)+t(\hat{\gamma}_6,k)+t(\hat{\gamma}_7,k) = -\frac{2}{3}(1+(\frac{-3}{p})),$

<u>where</u> $(\frac{*}{p})$ <u>is the Legendre symbol and</u>

$[t_0,\ldots,t_{r-1}; r]$ <u>means that we take the value</u> t_i <u>if</u>

$k \equiv i$ mod.r.

<u>Theorem</u> 3.3 <u>For a natural integer</u> $k \geq 5$ <u>and a prime number</u> $p \neq 2,3$, $\dim S_k(U_2(p)) = \dim S_k(U_0(p))$ <u>is given by the summation of the following quantities:</u>

$t(\alpha_0,k) = (p+1)(p^2+1)(2k-2)(2k-3)(2k-4)/2^9 3^3 5,$

$t(\alpha_1,k) = (p+1)(1+(\frac{-1}{p}))(-1)^k/2^7,$

$t(\alpha_2,k)+t(\alpha_3,k) = -(p+1)(1+(\frac{-3}{p}))\,[0,1,-1;\ 3]/2^2 3^3,$

$$\sum_{i=4}^{6} t(\alpha_i,k) = \frac{1}{2}[1,0,0,-1;4]\begin{cases} 1\ldots\text{if } p \equiv 1 \text{ mod.8} \\ 0\ldots\text{otherwise,} \end{cases}$$

$$\sum_{i=7}^{12} t(\alpha_i,k) = 2(1+(\frac{-3}{p}))[1,0,0,-1,0,0;6]/3^2,$$

$$t(\alpha_{13},k)+t(\alpha_{14},k) = \frac{1}{3}[0,1,-1;\ 3]\begin{cases} 1 \ \dots \ \text{if } p \equiv 1 \text{ mod.}12 \\ 0 \ \dots \ \text{otherwise,} \end{cases}$$

$$\sum_{i=15}^{18} t(\alpha_i,k) = \frac{1}{5}[1,0,0,-1,0;\ 5]\begin{cases} 4 \ \dots \ \text{if } p \equiv 1 \text{ mod.}5, \\ 1 \ \dots \ \text{if } p = 5, \\ 0 \ \dots \ \text{otherwise,} \end{cases}$$

$$\sum_{i=19}^{22} t(\alpha_i,k) = (2+(\tfrac{-1}{p})+(\tfrac{-3}{p}))[1,0,0,-1,-1,-1,-1,0,0,1,1,1;12]/2^2 3,$$

$$t(\beta_1,k)+t(\beta_2,k) = (p+2+(\tfrac{-3}{p}))[2k-3,-k+1,-k+2;3]/2^3 3^3,$$

$$t(\beta_3,k)+t(\beta_4,k) = (p+2+(\tfrac{-3}{p}))[-1,-k+1,-k+2,1,k-1,k-2;6]/2^3 3^2,$$

$$t(\beta_5,k)+t(\beta_6,k) = (p+2+(\tfrac{-1}{p}))[k-2,-k+1,-k+2,k-1;4]/2^5 3,$$

$$t(\gamma_1,k)+t(\gamma_2,k) = 5(p+1)(1+(\tfrac{-1}{p}))(2k-3)/2^7 3,$$

$$t(\gamma_3,k) = (p+1)(1+(\tfrac{-3}{p}))(2k-3)/2\cdot 3^3,$$

$$t(\delta_1,k)+t(\delta_2,k) = 7(p+1)(-1)^k(2k-2)(2k-4)/2^8 3^2,$$

$$t(\hat{\beta}_1,k)+t(\hat{\beta}_2,k) = (3+(\tfrac{-3}{p}))[0,1,1,0,-1,-1;\ 6]\ /2^2 3,$$

$$t(\hat{\beta}_3,k)+t(\hat{\beta}_4,k) = -(3+(\tfrac{-3}{p}))[2,-1,-1;\ 3]\ /2^2 3^2,$$

$$t(\hat{\beta}_5,k)+t(\hat{\beta}_6,k) = \begin{cases} -4\,[1,-1,0;\ 3]/3^2 & \text{... if } p \equiv 1 \text{ mod.3,} \\ -1\,[2,-1,-1;3]\,/3^2 & \text{... if } p \equiv 2 \text{ mod.3} \end{cases}$$

$$t(\hat{\beta}_7,k)+t(\hat{\beta}_8,k) = -(3+(\frac{-1}{p}))[1,-1,-1,1;4]\,/2^4,$$

$$t(\hat{\beta}_9,k)+t(\hat{\beta}_{10},k) = -(3+(\frac{-1}{p}))[1,-1,-1,1;\ 4]/2^4,$$

$$t(\hat{\hat{\delta}}_1,k)+t(\hat{\hat{\delta}}_2,k) = (-1)^k/2^2,$$

$$t(\hat{\hat{\delta}}_3,k)+t(\hat{\hat{\delta}}_4,k) = (3-(\frac{-1}{p}))/2^4,$$

$$t(\hat{\delta}_1,k)+t(\hat{\delta}_2,k) = -(p+3)(-1)^k(2k-3)/2^4 3,$$

$$t(\varepsilon_1,k) = (p+1)/2^2 3,$$

$$t(\varepsilon_2,k) = o,$$

$$t(\varepsilon_3,k) = -(p+3)/2^4 3,$$

$$t(\varepsilon_4,k) = -(p+1)^2(2k-3)/2^5 3^2,$$

$$t(\hat{\gamma}_1,k)+t(\hat{\gamma}_2,k) = t(\hat{\gamma}_3,k)+t(\hat{\gamma}_4,k) = -(1+(\frac{-1}{p}))/2^3,$$

$$t(\hat{\gamma}_5,k)+t(\hat{\gamma}_6,k)+t(\hat{\gamma}_7,k) = -(1+(\frac{-3}{p}))/3,$$

<u>where the notations are same as in</u> Th.3.2.

<u>Numerical</u> <u>examples</u> <u>of</u> dim $S_k(B(p))$ <u>and</u> dim $S_k(U_2(p))$
<u>for</u> <u>small</u> k <u>and</u> p.

In the following tables, we write dim $S_k(B(p))$ in the second row, and dim $S_k(U_2(p))$ in the third row.

(i) p = 5

k	5	6	7	8	9	10	11	12	13	14	15	16
B(p)	2	15	10	43	27	90	64	166	116	267	203	412
$U_2(p)$	1	2	2	6	6	15	13	27	20	42	37	68

(ii) p = 7

k	5	6	7	8	9	10	11	12	13	14	15	16
B(p)	11	45	43	125	123	277	263	505	471	825	791	1281
$U_2(p)$	2	5	7	15	17	34	37	63	61	100	104	160

(iii) p = 11

k	5	6	7	8	9	10	11	12	13	14	15	16
B(p)	66	202	283	603	756	1340	1581	2501	2854	4190	4679	6503
$U_2(p)$	5	12	21	42	60	103	130	198	229	331	338	528

(iv) p = 13

k	5	6	7	8	9	10	11	12	13	14	15	16
B(p)	141	387	578	1140	1507	2521	3120	4710	5557	7855	9094	12236
$U_2(p)$	12	27	45	80	113	180	232	337	403	556	662	875

<u>Theorem</u> 3.4. <u>For a natural integer</u> $k \geq 5$ and $p \neq 2,3$, <u>we have</u>

$$\dim S_k(B(p)) - \dim S_k(U_0(p)) - \dim S_k(U_1(p)) - \dim S_k(U_2(p))$$

$$+ \dim S_k(U_{02}(p)) + 2 \dim S_k(Sp(2,Z))$$

$$= \sum_{i=1}^{12} T_i,$$

<u>where</u> T_i <u>is the contribution of semi-simple conjugacy classes whose principal polynomial is</u> $f_i(x)$ <u>or</u> $f_i(-x)$ <u>in</u> (3.1). <u>Non-elliptic conjugacy classes</u> (i.e. <u>of infinite order</u>) <u>has no contribution</u>. T_i $(i=1,\ldots,12)$ <u>are given explicitly as follows</u>:

$$T_1 = (p-1)(p^3-1)(2k-2)(2k-3)(2k-4)/2^9 3^3 5,$$

$$T_2 = 7(p-1)^2(-1)^k(k-1)(k-2)/2^7 3^2,$$

$$T_3 = -(p-1)(1-(\tfrac{-1}{p}))[k-2,-k+1,-k+2,k-1;\ 4]/2^5 3,$$

$$T_4 = -(p-1)(1-(\tfrac{-3}{p}))[2k-3,\ -k+1,\ -k+2;\ 3]/2^3 3^3,$$

$$T_5 = -(p-1)(1-(\tfrac{-3}{p}))[-1,-k+1,-k+2,1,k-1,k-2;\ 6]/2^3 3^2,$$

$$T_6 = (p-1)(1-(\tfrac{-1}{p}))[-k+1,-k+2;\ 2]/2^6 3,$$

$$T_7 = (p-1)(1-(\tfrac{-3}{p}))[-2k+3,-2k+2,-2k+4;\ 3]/2^3 3^3,$$

$$T_8 = (1-(\tfrac{-1}{p}))(1-(\tfrac{-3}{p}))[1,0,0,-1,-1,-1,-1,0,0,1,1;\ 12]/2^2 3,$$

$$T_9 = (1-(\tfrac{-3}{p}))^2 [1,0,0,-1,0,0;\ 6]/3^2,$$

$$T_{10} = \frac{1}{5}[1,0,0,-1,0;\ 5]\begin{cases} 1 \ \dots \ \text{if } p = 5, \\ 2 \ \dots \ \text{if } p \equiv 2,3 \text{ mod.}5, \\ 4 \ \dots \ \text{if } p \equiv 4 \text{ mod.}5, \\ 0 \ \dots \ \text{if } p \equiv 1 \text{ mod.}5, \end{cases}$$

$$T_{11} = \frac{1}{2}[1,0,0,-1;\ 4]\begin{cases} 1 \ \dots \ \text{if } p \equiv 7 \text{ mod.}8 \\ 0 \ \dots \ \text{otherwise}, \end{cases}$$

$$T_{12} = \frac{1}{6}[1,0,0,-1,2,-2;\ 6]\begin{cases} 1 \ \dots \ \text{if } p \equiv 11 \text{ mod.}12 \\ 0 \ \dots \ \text{otherwise}, \end{cases}$$

where the notations are same as in Th.3.2.

Proof. One can get this Th.3.4, by straight forward calculation, using Th.3.2, 3.3 in this paper, Th.6.2, 7.1 in[16], and Th.4 in[24]. q.e.d.

Next, we treat $B'(p)$. In this case, every element of G' is semi simple, and if it is of finite order, then its principal polynomial is one of $f_i(x)$ or $f_i(-x)$ in (3.1). For any open compact subgroup U of G'_A, we denote by $H_i(U)$ the contribution to $\dim M_0(U)$ of elements of G' whose principal polynomial is $f_i(x)$ or $f_i(-x)$. For $g \in Sp(2)$, it is well known that $\mathrm{tr}\,\rho_\nu(g)$ depends only on the principal polynomial of g and $\mathrm{tr}\,\rho_\nu(g) = \mathrm{tr}\,\rho_\nu(-g)$. We fix an element $g_i \in Sp(2)$ whose principal polynomial is $f_i(x)$. Now, we state our results.

<u>Theorem</u> 3.5 <u>For any</u> U <u>as above and any integer</u> $\nu \geqq 0$, <u>we have</u>

$$\dim M_\nu(U) = \sum_{i=1}^{12} H_i(U)\,\mathrm{tr}\,\rho_\nu(g_i).$$

<u>For any prime</u> $p \neq 2,3$, <u>and</u> $U = U_1'(p)$, $U_0'(p)$, or $B'(p)$, $H_i(U)$ <u>is given as follows</u>:

$$H_1(B'(p)) = (p^4-1)/2^6 3^2 5,$$

$$H_1(U_1'(p)) = (p-1)(p^2+1)/2^6 3^2 5,$$

$$H_1(U_0'(p)) = (p^2-1)/2^6 3^2 5,$$

$$H_2(B'(p)) = H_2(U_2'(p)) = 0,$$

$$H_2(U_1'(p)) = 7(p-1)^2/2^6 3^2,$$

$$H_3(B'(p)) = H_3(U_2'(p)) = 0,$$

$$H_3(U_1'(p)) = (p-1)\left(1-\left(\frac{-1}{p}\right)\right)/2^6 3^2,$$

$$H_4(B'(p)) = H_4(U_0'(p)) = 0,$$

$$H_4(U_1'(p)) = (p-1)(1-(\tfrac{-3}{p}))/2^3 3^2,$$

$$H_5(B'(p)) = H_5(U_0'(p)) = 0,$$

$$H_5(U_1'(p)) = (p-1)(1-(\tfrac{-3}{p}))/2^3 3^2,$$

$$H_6(B'(p)) = (p+1)(1-(\tfrac{-1}{p}))/2^5 + 5(p-1)(1+(\tfrac{-1}{p}))/2^5 3,$$

$$H_6(U_1'(p)) = 5(p-1)/2^5 3 + (1-(\tfrac{-1}{p}))/2^5,$$

$$H_6(U_0'(p)) = (p+1)(1-(\tfrac{-1}{p}))/2^6 + 5(p-1)(1+(\tfrac{-1}{p}))/2^6 3,$$

$$H_7(B'(p)) = (p+1)(1-(\tfrac{-3}{p}))/2^2 3^2 + (p-1)(1+(\tfrac{-3}{p}))/2\cdot 3^2,$$

$$H_7(U_1'(p)) = (p-1)/2\cdot 3^2 + (1-(\tfrac{-3}{p}))/2^2 3^2,$$

$$H_7(U_0'(p)) = (p+1)(1-(\tfrac{-3}{p}))/2^3 3^2 + (p-1)(1+(\tfrac{-3}{p}))/2^2 3^2,$$

$$H_8(B'(p)) = H_8(U_0'(p)) = 0,$$

$$H_8(U_1'(p)) = (1-(\tfrac{-1}{p}))(1-(\tfrac{-3}{p}))/2^2 3,$$

$$H_9(B'(p)) = H_9(U_0'(p)) = 0,$$

$$H_9(U_1'(p)) = (1-(\tfrac{-3}{p}))^2/3^2,$$

$$H_{10}(B'(p)) = \begin{cases} 1/5 & \ldots \text{ if } p = 5, \\ 0 & \ldots \text{ otherwise}, \end{cases}$$

$$H_{10}(U_1'(p)) = \begin{cases} 1/5 & \ldots \text{ if } p = 5, \\ 4/5 & \ldots \text{ if } p \equiv 4 \text{ mod.} 5, \\ 0 & \ldots \text{ otherwise}, \end{cases}$$

$$H_{10}(U_0'(p)) = \begin{cases} 1/5 & \ldots \text{ if } p = 5, \\ 2/5 & \ldots \text{ if } p \equiv 2,3 \text{ mod.} 5, \\ 0 & \ldots \text{ otherwise}, \end{cases}$$

$$H_{11}(B'(p)) = (1-(\frac{2}{p}))/2^2,$$

$$H_{11}(U_1'(p)) = \begin{cases} 0 & \ldots \text{ if } p \equiv 1 \text{ mod.} 8, \\ 1/4 & \ldots \text{ if } p \equiv 3,5 \text{ mod.} 8, \\ 1/2 & \ldots \text{ if } p \equiv 7 \text{ mod.} 8, \end{cases}$$

$$H_{11}(U_0'(p)) = (1-(\frac{2}{p}))/2^3,$$

$$H_{12}(B'(p)) = \begin{cases} 1/3 & \ldots \text{ if } p \equiv 5 \text{ mod.} 12, \\ 0 & \ldots \text{ otherwise}, \end{cases}$$

$$H_{12}(U_1'(p)) = \begin{cases} 1/6 & \ldots \text{ if } p \equiv 5 \text{ mod.} 6, \\ 0 & \ldots \text{ otherwise}, \end{cases}$$

$$H_{12}(U_0'(p)) = \begin{cases} 1/6 & \ldots \text{ if } p \equiv 5 \text{ mod.} 12, \\ 0 & \ldots \text{ otherwise}. \end{cases}$$

Remark The above results for $U_1'(p)$ and $U_0'(p)$ has been already given in [19], including the case where $p = 2, 3$. We reproduced them here for the convenience of the readers. The Weyl character formula gives explicit values of $\mathrm{tr}\,\rho_\nu(g_i)$, which has been calculated in [19](I) Th.3(p.596).

Theorem 3.6 For any integer $\nu \geqq 0$, put $k = \nu + 3$. For any prime $p \neq 2, 3$, and for the above k, define T_i $(i=1,\dots,12)$ as in Th.3.4(, although k might be 3 or 4). Then, we get

$$(H_i(B'(p)) - H_i(U_1'(p)) - H_i(U_0'(p)))\ \mathrm{tr}\,\rho_\nu(g_i) = T_i$$

for all $i = 1,\dots,12$.

Proof. This is obtained by straight forward calculation. q.e.d.

We see very easily that $M_\nu(U_1'(p)) \cap M_\nu(U_0'(p)) = 0$, unless $\nu = 0$, and $\dim(M_0(U_1'(p)) \cap M_0(U_0'(p))) = 1$, if $\nu = 0$, so the dimensions of new forms belonging to $B'(p)$ is given by:

$$\dim M_\nu^0(B'(p)) = \dim M_\nu(B'(p)) - \dim M_\nu(U_1'(p)) - \dim M_\nu(U_0'(p)) + \delta,$$

where $\delta = 0$, if $\nu \neq 0$, and $\delta = 1$, if $\nu = 0$.

Numerical examples of dimensions of $M_\nu(B'(p))$, $M_\nu(U_i(p))$ $(i=0,1)$, and new forms $M^0_\nu(B'(p))$.

(i) $p = 5$

ν	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$B'(p)$	2	1	5	8	15	22	34	47	67	87	115	146	184	225
$U'_1(p)$	2	0	3	0	6	0	14	3	23	6	33	10	53	21
$U'_0(p)$	1	0	1	1	2	2	3	3	5	5	7	8	10	11
new forms	0	1	1	7	7	20	17	41	39	76	75	128	121	193

(ii) $p = 7$

ν	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$B'(p)$	2	6	14	28	50	80	122	176	244	328	430	550	692	856
$U'_1(p)$	2	0	5	0	16	3	29	8	55	21	85	37	133	67
$U'_0(p)$	1	1	1	2	3	4	5	6	8	10	13	15	18	22
new forms	0	5	8	26	31	73	88	162	181	297	332	498	541	767

(iii) $p = 11$

ν	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$B'(p)$	7	27	74	156	285	467	718	1044	1457	1965	2582	3314	4175	5171
$U'_1(p)$	5	1	16	3	45	16	99	48	186	106	296	182	474	318
$U'_0(p)$	1	1	2	3	5	6	9	12	16	20	26	32	40	48
new forms	2	25	56	150	235	445	610	984	1255	1839	2260	3100	3661	4805

(iv) p = 13

ν	0	1	2	3	4	5	6	7	8	9	10	11	12	13
B' (p)	13	53	144	304	555	911	1400	2036	2841	3833	5036	6464	8143	10087
$U'_1(p)$	4	0	23	7	70	32	154	88	288	184	483	333	750	546
$U'_0(p)$	2	2	3	5	8	10	14	18	24	30	39	47	58	70
new forms	8	51	118	292	477	869	1232	1930	2529	3619	4514	6084	7335	9471

§ 4 Arithmetic General Formula for Dimensions

4-0. This section is mostly an exposition of [15], [16], and [19]. Our purpose is to describe the general "arithmetic" formulae for the dimension of our space $S_k(\Gamma)$ (, $M_k(U_A)$) of automorphic forms for arithmetic subgroups of $Sp(n,\mathbb{R})$, and $Sp(n)$. Here n is an arbitrary positive integer. These formulas, Theorem A in § 4-2 and Theorem B in § 4-4, enable us to compute explicitly the dimension of $S_k(\Gamma)$, $M_k(U_A)$ for the special groups considered in § 1, 2, and 3, as we shall carry out in § 5, 6, which lead us to our main results in this paper.

In the split case (i.e. for $Sp(n,\mathbb{R})$), our formula is based on Selberg's trace formula ; and the derivation of our formula from Selberg's formula consists of two main parts i.e.,

(i) evaluation of certain integrals (analytic part), and

(ii) classification of conjugacy classes in Γ and their centralizers (arithmetic part)

(ii)(bis) when the conjugacy classes in question are semi-simple, we need only G_A-conjugacy classes instead of Γ-conjugacy classes, and certain G-Maße (see Theorem (4.31)).

On the other hand, in the compact case (i.e. for $Sp(n)$), our formula can be obtained in quite elementary way as a special case of the trace formula for the Brandt matrices $B_{\rho}(n)$ (c.f. [15]), which generalizes the method of Eichler [9,10] and Shimizu [43]. Here the analytic part (i) is quite simple: it is nothing but the character computation of the finite dimensional representation ρ, which is now a classical result of H.Weyl [50]. Therefore, the essential part of the derivation of our formula consists of only (ii)(bis), although explicit comptations are not so easy.

Moreover, as we shall see, the arithmetic part (ii)(bis) can be handled in a unified manner in both $Sp(n,\mathbb{R})$ and $Sp(n)$ cases. So we first describe this part in the following paragraph, where certain arithmetic invariant $H(g, U_A)$ will be defined for a semi-simple conjugacy class of $G^1_{\mathbb{Q}}$, a $\mathbb{Q}$-form of $Sp(n,\mathbb{R})$ or $Sp(n)$, and a closed formula for it will be given. It would be convenient, however, to describe here the motivation to introducing such invariant by sketching the special meaning of it in the compact case.

In the compact case, our space $M_k(U_A)$ of automorphic forms is isomorphic to $\bigoplus_{i=1}^{H} V^{\Gamma_i}$ (c.f. § 1), so we have

$$\dim M_k(U_A) = \sum_{i=1}^{H} \dim V^{\Gamma_i}$$

$$(4.1) \qquad = \sum_{i=1}^{H} \frac{1}{\# \Gamma_i} \sum_{g \in \Gamma_i} tr(\rho_k(g))$$

$$= \sum_{g \in [f]} tr(\rho_k(g)) \sum_{i=1}^{H} \frac{\# [\Gamma_i \cap [f]]}{\# \Gamma_i}$$

where $[f]$ denotes the set of elements of $G^1_{\mathbb{Q}}$ (or $G^1_{\mathbb{R}} = Sp(n)$) which have the principal polynomial $f = f(x)$. Note that $tr(\rho_k(g))$ depends only on $[f]$ to which g belongs, and that the inner sum does not involve ρ_k. Thus the computation of $\dim M_k(U_A)$ reduc[es] to that of :

$$(4.2)\qquad H(f,\, U_A) := \sum_{i=1}^{H} \frac{\#\, \Gamma_i \cap [f]}{\#\, \Gamma_i}$$

In general, the set $[f]$ in G_Q consists of <u>infinitely</u> many G_Q-conjugacy classes, while obviously only a finite number of them make nontrivial contributions to $\dim M_k(U_A)$. This leads to the following

<u>Definition</u> 4.3 . A conjugacy class $\{g\}_{G_Q}$ in G_Q^1 is called "<u>locally integral</u>" (with respect to U_A) if $\Gamma_i \cap \{g\}_{G_Q} \neq \emptyset$ for some i $(1 \leq i \leq H)$.

For each G_Q-conjugacy class $\{g\}_{G_Q}$, we define an invariant similar as (4.2):

$$(4.4)\qquad H(g,\, U_A) := \sum_{i=1}^{H} \frac{\#[\Gamma_i \cap \{g\}_{G_Q}]}{\#\, \Gamma_i}$$

Clearly, $\{g\}_{G_Q}$ is locally integral if and only if $H(g,\, U_A) \neq 0$. Note also that this implies g is of finite order.

4-1. <u>A Formula for</u> $H(g,\, U_A)$

Let D be a quaternion algebra over Q (definite or indefinite), and let the group G_Q be defined by

$$(4.5)\qquad \begin{aligned} G_Q &=: \text{the group of similitudes of the hermitian space } (D^n,\, F),\ F(x,y) = x_1\bar{y}_1 + \dots + x_n\bar{y}_n\,, \\ &= \left\{ g \in GL_n(D);\ g\,{}^t\bar{g} = n(g)\cdot 1_n\,,\ n(g) \in Q_+^\times \right\}, \end{aligned}$$

where for $g = (g_{ij})$, we write ${}^t\bar{g} = (\bar{g}_{ji})$, $a \longmapsto \bar{a}$ being the canonical involution of D. We may regard it as Q-rational points of an algebraic group G defined over Q. G is reductive. We denote its semi-simple part by $G^1 =: \{g \in G;\ n(g) = 1\}$. In G_A, we consider an open subgroup U_A which, as we assume throughout this paper, is of the form

$$(4.6)\qquad \begin{aligned} U_A &= \underline{R}_A^\times \cap G_A \\ &= G_R \times \prod_p U_p \qquad (U_p = \underline{R}_p^\times \cap G_p) \end{aligned}$$

for some $\mathbb{Z}$-order $\underline{R}$ of the Q-algebra $M_n(D)$. Then G_A is decomposed into a disjoint union of finite unmber of U_A-G_Q double cosets:

$$(4.7)\qquad G_A = \coprod_{i=1}^{H} U_A\ g_i\ G_Q .$$

By an "<u>arithmetic subgroup</u> Γ" of G_Q, or G_Q^1, we mean a system of subgroups $(\Gamma_i)_{i=1}^{H}$, where

$$(4.8)\qquad \Gamma_i = G_Q \cap g_i^{-1} U_A g_i \quad (= G_Q^1 \cap g_i^{-1} U_A g_i).$$

It is this system of groups $(\Gamma_i)_{i=1}^{H}$ with respect to which our space of automorphic forms are defined.
If D is definite, then Γ_i are all finite groups, since they are contained in the discrete subgroup G_Q^1 and the compact group $g_i^{-1} U_A g_i$. On the other hand, if D is indefinite, we have a natural isomorphism

$$(4.9)\qquad G_R^1 \xrightarrow{\sim} Sp(n,\mathbb{R}),\quad g = (g_{ij}) \longmapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$
$$g_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ c_{ij} & d_{ij} \end{pmatrix},\ A = (a_{ij}),\ B = (b_{ij})\ \text{etc.}$$

where we identify $D_R = D \otimes_Q \mathbb{R}$ and $M_2(\mathbb{R})$;and Γ_i's are arithmetic discrete subgroups of $Sp(n,\mathbb{R})$, which act on the Siegel upper half plane H_n properly discontinuously in the usual manner.
In this case, if $\underline{R}$ is sufficiently large, we have $H = 1$ by the strong approximation theorem (c.f. Kneser [32]), which is the case for all arithmetic subgroups of $GSp(2,Q) = G_Q$ treated in this paper. However, to treat uniformly the two cases (D = definite, or indefinite), we do not assume that $H = 1$.

We take and fix, once for all, an open subgroup U_A and a semi-simple conjugacy class $\{g\}_{G_Q}$ contained in G_Q^1. Put

$$\begin{aligned} Z(g) &=: \text{commutor algebra of } g \text{ in } M_n(D) \\ &= \{ z \in M_n(D);\ zg = gz \}, \end{aligned}$$

$$Z_G(g) = Z(g)^{\times} \cap G_Q \quad (\text{=the centralizer of } g \text{ in } G_Q)$$

Then $Z(g)$ is a semi-simple algebra over Q, and $Z_G(g)$ is an algebraic group, reductive, over Q. In the set $\{\Lambda\}$ of $\mathbb{Z}$-orders of $Z(g)$, we define two equivalence relations

$$\Lambda_1 \sim \Lambda_2 \iff \Lambda_2 = a\Lambda_1 a^{-1} \quad \text{for some } a \in Z_G(g)$$

$$\begin{aligned} (4.10) \qquad \Lambda_1 \approx \Lambda_2 \iff \Lambda_2 &= a\Lambda_1 a^{-1} \\ &=: \bigcap_p (a_p \Lambda_{1p} a_p^{-1} \cap Z(g)) \\ &\text{for some } a = (a_p) \in Z_G(g)_A . \end{aligned}$$

An equivalence class in the second relation is usually called a G-$\underline{\text{genus}}$, which we denote by $L_G(\Lambda)$ if it contains Λ ; it consists of finitely many classes with respect to the first equivalence relation. We have a disjoint decomposition of $\Gamma_i \cap \{g\}_{G_Q}$ for each i :

$$(4.11) \qquad \Gamma_i \cap \{g\}_{G_Q} = \coprod_{\Lambda/\sim} C(g,\Lambda,\ i) \cap \Gamma_i$$

where we put

$$(4.12) \qquad C(g,\Lambda,\ i) =: \{x^{-1}gx \ ;\ x \in G_Q,\ Z(g) \cap x\underline{R}_i x^{-1} \sim \Lambda\},$$

$$\underline{R}_i = g_i^{-1}\underline{R}\ g_i =: \bigcap_p (g_{ip}^{-1}\underline{R}_p\ g_{ip} \cap M_n(D))$$

and the union is extended over the (actually a finite) set of $\mathbb{Z}$-orders Λ of $Z(g)$, modulo the equivalence $\sim$.
Note that the set $C(g,\Lambda,\ i)$ is stable under the conjugation by Γ_i , and it consists of a finite number of Γ_i-conjugacy classes.
Now we define our arithmetic invariants

$$H(g,\Lambda,U_A) := \sum_{i=1}^{H} \#[C(g,\Lambda,\ i) \cap \Gamma_i / \widetilde{\Gamma_i}] ,$$

(4.13)

$$H(g,\ U_A) := \sum_{\Lambda/\sim} \mathrm{vol}\left(\Lambda^\times \cap G_Q^1 \backslash G_R^1 \cap Z_G(g)_{\mathbb{R}}\right) \cdot H(g,\Lambda,U_A) .$$

<u>Remark</u> 4.14. Note that these are invariants of the G_Q-conjugacy class $\{g\}_{G_Q}$. Since $H(g,\Lambda,U_A) \neq 0$ for only a finite number of classes of Λ , the last sum is actually a finite sum. Here the volume of the quotient $\Lambda^\times \cap G_Q^1 \backslash G_R^1 \cap Z_G(g)_{\mathbb{R}}$ is measured by a suitably normalized (fixed) Haar measure of $G_R^1 \cap Z_G(g)_{\mathbb{R}}$. In the case $G_R^1 = Sp(n)$ (i.e. D = definite), we may take the measure so normalized that $\mathrm{vol}(G_R^1 \cap Z_G(g)_{\mathbb{R}}) = 1$. Then we see that our invariant

$H(g, U_A)$ coincides with the one given by (4.4), since we have the following

<u>Lemma</u> 4.15. (D = <u>definite</u> <u>or</u> <u>indefinite</u>)
<u>If</u> $a \in C(g, \Lambda, i) \cap \Gamma_i$, <u>then</u> <u>we</u> <u>have</u>

$$\begin{aligned} C(a;\Gamma_i) &=: \text{centralizor of } a \text{ in } \Gamma_i \\ &= \Lambda^{\times} \cap G_Q^1 \qquad (= \text{independent of } a, i \;!). \end{aligned}$$

For the proof, see [15], Lemma 4. Thus we see that our invariant $H(g, U_A)$ is the weighted average of the number of elements in Γ_i which are conjugate in G_Q to g; and $H(g,\Lambda,U_A)$ is a refinement of it. We want to (indeed we should) give them some expressions which do not involve H, the class number of U_A. For this purpose, we put

$$\begin{aligned} &M(g,\Gamma_i,\Lambda) = \{x \in G_Q;\ x^{-1}gx \in \Gamma_i,\ Z(g) \cap x\underline{R}_i x^{-1} \sim \Lambda\} \\ (4.16)\quad &M_A(g,U_A,\Lambda) = \{x \in G_A;\ x^{-1}gx \in U_A,\ Z(g) \cap x\underline{R}x^{-1} \sim \Lambda\} \\ &M_p(g,U_p,\Lambda) = \{x \in G_p;\ x^{-1}gx \in U_p,\ Z(g)_p \cap x\underline{R}_p x^{-1} \sim \Lambda_p\}. \end{aligned}$$

Then we obviously have the following

<u>Lemma</u> 4.17. <u>The</u> <u>map</u> $x^{-1}gx \mapsto x$ <u>induces</u> <u>the</u> <u>following</u> <u>bijection</u> <u>for</u> <u>each</u> i $(1 \leq i \leq H)$:

$$C(g,\Lambda, i) \cap \Gamma_i / \underset{\Gamma_i}{\sim} \xrightarrow{\ \sim\ } Z_G(g) \backslash M(g,\Gamma_i,\Lambda)/\Gamma_i$$

(c.f. [15], Lemma 3)

The next lemma plays a key role in our problem, since it enables us to get rid of H from $H(g, U_A)$:

Lemma 4.18. For each double coset $G_Q\, g_i^{-1} U_A$ in (4.7), we have a bijection induced from the map $a\, g_i^{-1} u \longmapsto a$ $(a \in G_Q,\ u \in U_A)$:

$$Z_G(g) \backslash M_A(g, U_A, \Lambda) \cap G_Q g_i^{-1} U_A \,/\, U_A$$

$$\xrightarrow{\ \sim\ } \quad Z_G(g) \backslash M(g, \Gamma_i, \Lambda) / \Gamma_i$$

(loc.cit. ,Lemma 5)

Corollary 4.19. We have

$$H(g, \Lambda, U_A) = \sum_{i=1}^{H} \#\left[Z_G(g) \backslash M(g, \Gamma_i, \Lambda) / \Gamma_i\right]$$

$$= \#\left[Z_G(g) \backslash M_A(g, U_A, \Lambda) / U_A\right]$$

To proceed further, we note that $M_A(g,U_A,\Lambda)$ is not stable under the action of $Z_G(g)_A$ from the left, and therefore put

$$(4.20) \qquad M_A^*(g, U_A, \Lambda) =: \bigcup_{\Lambda' \in L_G(\Lambda)} M_A(g, U_A, \Lambda') .$$

Now consider the natural projection:

$$\phi:\ Z_G(g) \backslash M_A(g, U_A, \Lambda) / U_A \longrightarrow Z_G(g)_A \backslash M_A^*(g, U_A, \Lambda) / U_A$$

$$\cong$$

$$\prod_p \left[Z_G(g)_p \backslash M_p(g, U_p, \Lambda_p) / U_p \right] .$$

Lemma 4.21. The map ϕ above is $h_0(\Lambda; G)$-to-one, where $h_0(\Lambda;G)$ is the two-sided G-class number of Λ defined by the following formula

$$(4.22)\qquad h_0(\Lambda;G) =: \#\left[Z_G(g)\setminus Z_G(g)\cdot I(\Lambda)\ /(\Lambda_A^{\times}\cap G_A)\right]$$

$$I(\Lambda) = \left\{ z\in Z_G(g)_A;\quad z\Lambda z^{-1} = \Lambda \right\}\ .$$

(loc.cit. (18)).

Definition Let $Z_G(g)_A$ be decomposed into a disjoint union

$$(4.23)\qquad Z_G(g)_A = \coprod_{j=1}^{h} Z_G(g)\ y_j\ (\Lambda_A^{\times}\cap G_A)\ ,$$

and put

$$\Lambda_j =:\ y_j\Lambda y_j^{-1} = \bigcap_p (y_{jp}\ \Lambda_p\ y_{jp}^{-1}\cap Z(g)).$$

(i) The number $h = h(\Lambda;G)$ of cosets in (4.23) is called G-class number of Λ, or $L_G(\Lambda)$ (note that it depends only on the G-genus $L_G(\Lambda)$).

(ii) The invariant of $L_G(\Lambda)$ defined by

$$(4.24)\qquad M_G(\Lambda) =: \sum_{j=1}^{h} \mathrm{vol}(\Lambda_j^{\times}\cap G_Q\setminus G_R^1\cap Z_G(g)_R)$$

is called "G-Maß (or G-measure)" of Λ, or $L_G(\Lambda)$.

Note that these invariants do not depend on the choice of (y_j) in the decomposition (4.23). It is not difficult to prove the following

Lemma 4.25. We have

$$h(\Lambda;G) = \sum_{\Lambda^{(k)}\in L_G(\Lambda)/\sim} h_0(\Lambda^{(k)};G)$$

$$M_G(\Lambda) = \sum_{\Lambda^{(k)}\in L_G(\Lambda)/\sim} h_0(\Lambda^{(k)};G)\ \mathrm{vol}(\Lambda^{(k)\times}\cap G_Q\setminus G_R^1\cap Z_G(g)_R\ )$$

(loc.cit. , Lemma 7).

Combining these results, we finally get the following expression of our invariant $H(g,U_A)$

<u>Theorem</u> 4.26. <u>We have</u>

$$H(g,U_A) = \sum_{L_G(\Lambda)} m_G(\Lambda) \prod_p c_p(g,U_p, \Lambda_p) ,$$

<u>where</u>

$$c_p(g,U_p,\Lambda_p) = \# \left[Z_G(g)_p \backslash M_p(g,U_p,\Lambda_p)/ U_p\right] .$$

<u>Proof</u>. By (4.19), (4.21), we have

$$\begin{aligned} H(g,U_A) &= \sum_{\Lambda/\sim} h_o(\Lambda;G) \operatorname{vol}(\Lambda^\times \cap G_Q \backslash G_R^1 \cap Z_G(g)_R) \\ &\qquad \times \# \left[Z_G(g)_A \backslash M_A^*(g,U_A,\Lambda)/ U_A\right] \\ &= \sum_{L_G(\Lambda)} \sum_{\Lambda' \in L_G(\Lambda)/\sim} h_o(\Lambda';G) \cdot \operatorname{vol}(\Lambda'^\times \cap G_Q \backslash G_R^1 \cap Z_G(g)_R) \\ &\qquad \times \prod_p \# \left[Z_G(g)_p \backslash M_p(g,U_p,\Lambda_p)/ U_p\right] . \end{aligned}$$

Here we used the fact that $M_A^*(g,U_A,\Lambda)$ depends only on the G-genus $L_G(\Lambda)$. The assertion now follows from lemma (4.25). q.e.d.

We note that the sum in (4.26), which is seemingly extended over all G-genera of $\mathbb{Z}$-orders in $Z(g)$, is actually a finite sum (c.f. Remark (4.14)). Moreover, the products are always finite; thus we have

(i) If Λ is fixed, $c_p(g,U_p,\Lambda_p) = 0$, or 1 for all but finitely many p.

(ii) For a given p, $c_p(g,U_p,\Lambda_p) \neq 0$ only for finitely many classes $\Lambda_p/\sim$; moreover such $\Lambda_p/\sim$ is unique for all but finitely many p .

Remark 4.27. The G-Maß $m_G(\Lambda)$ can be evaluated in a well known manner by using theory of Tamagawa numbers (c.f. Weil [49], see also [19],(I), § 3).

It would be worth noting that, in the same way as Theorem (4.26), we can combine (4.11), (4.13), (4.19) and (4.21) to obtain a closed formula for the sum of the number of Γ_i-conjugacy classes in $\Gamma_i \cap \{g\}_{G_Q}$ $(1 \leq i \leq H)$:

Theorem 4.28. Notations being as above, we have, for a semi-simple element $g \in G_Q^1$:

$$\sum_{i=1}^{H} \#\left[\Gamma_i \cap \{g\}_{G_Q} / \widetilde{\Gamma_i}\right] = \sum_{L_G(\Lambda)} h(\Lambda;G) \prod_p c_p(g,U_p,\Lambda_p)$$

4-2. General Dimension Formula (Compact Case)

Assume that D is definite. Then our space $M_{\rho}(U_A)$ of automorphic forms of weight ρ for an open subgroup U_A of G_A is defined as in § 1 . Combining the results in the preceeding paragraph and (4.1), we immediately have the following general formula for $\dim M_{\rho}(U_A)$ (a special case $\sigma = 1, e = 1$ of [15]):

Theorem A. For a finite dimensional representation ρ of $Sp(n)$, we have

(4.30) $$\dim M_{\rho}(U_A) = \sum_{f} \sum_{g \in [f]/\widetilde{G_Q}} tr(\rho(g)) \sum_{L_G(\Lambda)} M_G(\Lambda) \prod_{p} c_p(g, U_p, \Lambda_p),$$

where the first sum is extended over the set of polynomials $f(x)$ of degree $2n$ which are products of some cyclotomic polynomials, and the second is over the set of locally integral G_Q-conjugacy classes belonging to $f(x)$, and the third is over the G-genera of $\mathbb{Z}$-orders $L_G(\Lambda)$ of $Z(g)$ for each representative g of $[f]/\widetilde{G_Q}$.

4-3. Parametrization of semi-simple Conjugacy Classes

In the actual calculation of dimensions using (4.30), or (4.40) in the next paragraph. a fundamental role is played by the following

Theorem 4.31 (Hasse Principle for conjugacy classes in G_Q, G_Q^1)
Two elements g_1, g_2 of G_Q (resp. G_Q^1) are G_Q- (resp. G_Q^1-) conjugate if and only if they are conjugate in G_p (resp. G_p^1) for all p . (c.f. Asai [2], and [19], § 2).

For each monic polynomial $f(x) \in Q[x]$ of degree $2n$ such that $x^{2n}f(x^{-1}) = f(x)$, we denote by $G[f]$ the set of semi-simple elements of G^1 whose principal polynomial is $f(x)$. Then the above theorem means that the following natural map induced by the inclusion map is injective:

$$(4.32) \qquad G[f] \,/\underset{G_Q}{\sim} \hookrightarrow G_A[f] \,/\underset{G_A}{\sim} \quad .$$

This reduces our problem to classify the G_Q-conjugacy classes to those for G_p-conjugacy classes, if we can determine the image of this map. The latters are much easier than the former, because there are only finitely many (≤ 4, if $n=2$) G_p-conjugacy classes in each $G_p[f]$, and we can choose a representative of classes in $G_p[f]$ to have a very simple form which enable us to compute $c_p(g,U_p,\Lambda_p)$.

If the map (4.32) is surjective (hence bijective), we need nothing more than just putting local data together. However, this is not always the case; so we shall describe here the image of this map, under the following conditions: $n = 2$, $f(x) = f_i(x)$ $(1 \leq i \leq 12)$ are as in § 3 (for details as well as the general case, see [19] , § 2).

<u>Proposition</u> 4.33.
(i) <u>If</u> $f(x)$ <u>is either one of</u> $f_1(\pm x)$, $f_2(x)$, $f_3(\pm x)$, $f_4(\pm x)$, $f_5(\pm x)$, $f_8(\pm x)$, or $f_{10}(\pm x)$, <u>then</u> (4.32) <u>is surjective.</u>
(ii) <u>If</u> $f(x) = f_6(x)$ <u>or</u> $f_7(\pm x)$, <u>then the centralizer of each element</u> $g \in G_Q[f]$ <u>is expressed as</u>

$$(4.34) \qquad Z_G(g) = Q(g)^{\times} \cdot Z_0(g)^{\times} \quad ,$$

where $Z_0(g)$ is a quaternion algebra over Q such that

$$Z_0(g) \otimes_Q F = D \otimes_Q F \qquad (F = Q[x] / \sqrt{f(x)} \cong Q(g)),$$

and the product formula $\prod_p \mathrm{inv}_p(Z_0(g)) = 1$ for the invariants of $(Z_0(g)_p)$ determines the image of (4.32) which has index 2 in $G_A[f] / \widetilde{G_A}$.

(iii) If $f(x) = f_9(x)$, $f_{11}(x)$ or $f_{12}(x)$, then for each element $g \in G_Q[f]$, g^2 belongs to either $f_6(x)$ or $f_7(\pm x)$, and the image of (4.32) is determined by $Z_0(g^2)$ as in (ii) above, which has also index 2 in $G_A[f] / \widetilde{G_A}$.

4-4. General Dimension Formula (Split Case)

Let Γ be an arithmetic subgroup of $G_Q^1 = Sp(n,Q)$, or a Q-form of $Sp(n,\mathbb{R})$. The dimension of $S_k(\Gamma)$ is first expressed by Godement [13] as an integral of an infinite series

$$\dim S_k(\Gamma) = \frac{a_n(k)}{\# Z(\Gamma)} \int_{\Gamma \backslash H_n} \sum_{\gamma \in \Gamma} H_\gamma(Z) \, dZ \,, \tag{4.35}$$

where $k > 2n$, and

$$a_n(k) = \frac{1}{2^n (2\pi)^{n(n+1)/2}} \prod_{j=1}^{n-1} \frac{\Gamma(k - \frac{n-1}{2} + \frac{j}{2})}{\Gamma(k-n+\frac{j}{2})} \,,$$

$$H_\gamma(Z) = \det\left(\frac{Z-\gamma\cdot\bar{Z}}{2i}\right)^{-k}\det(C\bar{Z}+D)^{-k}\det(Y)^{k} \qquad (\gamma = \begin{pmatrix} A & B \\ C & D\end{pmatrix}),$$

and $dZ = (\det Y)^{-n-1}\, dX\, dY$ $(Z = X+iY)$ is an invariant measure on H_n ; $Z(\Gamma)$ =: center of Γ .

Our purpose here is to sketch briefly how one reforms it to a more manageable formula, suitable for an explicit computation. This has been done in the case $n = 2$ by Christian [6], Morita [39], and Arakawa [1](Q-rank one case), for the special case of principal congruence subgroups $\Gamma = \Gamma(N)$, $N \geq 3$, and by the first named author for arbitrary Γ ([16],(I)).

The main idea of the reformulation is well known and a routine; we should exchange the integral and infinite sum in (4.35) and then combine the integrals in each conjugacy classes of Γ , to get a closed expression as a sum extended over the set of conjugacy classes of Γ. But this is allowed only if $\Gamma\backslash H_n$ is compact, which never occurs in our case with $n \geq 2$, since the Q-rank of G^1_Q is n or $[\frac{n}{2}]$, according as $D = M_2(Q)$ or not, while $\Gamma\backslash H_n$ is compact if and only if Q-rank of G_Q is o . However, we can save this difficulty by introducing certain dumping factors and replacing $H_\gamma(Z)$ by $H_\gamma(Z;s) = H_\gamma(Z)\times$(a dumping factor in s). In order to justify this argument we have to choose dumping factors and make estimations of sums of $H_\gamma(Z;s)$ to apply Lebesgue's theorem for various subsets of Γ . Substantial part of these estimations has been established by Christian [5]. We omit the details and refer to [16], § 2, where the case $n = 2$ was discussed using resuls of [5].

The second difficulty is the fact that $C(\gamma;\Gamma)$, the centralizer of γ in Γ, is not (always) a lattice of $C(\gamma;G^1_R)$ (see Example (4.37)).

This means that $\mathrm{vol}({}_{C(\gamma;\Gamma)}\backslash C(\gamma;G^1_{\mathbb{R}}))$ is not always finite. To save this point, we first observe:

Proposition 4.36. For any $\gamma \in \Gamma$, there exists a connected closed subgroup $C_0(\gamma;G^1_{\mathbb{R}})$ of $C(\gamma;G^1_{\mathbb{R}})$ which is characterized, modulo compact semi-direct factor, by the following properties

(i) $C_0(\gamma;\Gamma) =: C_0(\gamma;G^1_{\mathbb{R}}) \cap \Gamma$ is a lattice of $C_0(\gamma;G^1_{\mathbb{R}})$

(ii) $[C(\gamma;\Gamma): C_0(\gamma;\Gamma)] < \infty$.

Example 4.37. Let $\Gamma = Sp(2,\mathbb{Z})$ and $\gamma = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix} \in \Gamma$ with $S = {}^tS \in M_2(\mathbb{Z})$. Then $C(\gamma;G^1_{\mathbb{R}}) \cong O(S) \times \mathbb{R}^3$, and $C(\gamma;\Gamma)$ is a lattice of $C(\gamma;G^1_{\mathbb{R}})$ if and only if $O_{\mathbb{Z}}(S) = \{A \in GL_2(\mathbb{Z});\ AS{}^tA = S\}$ is a lattice of $O(S)$; it is easy to see that this is equivalent to that either S is definite, or $-\det(S) \in (\mathbb{Q}^\times)^2$. Thus we have, removing the compact factor $O(S)$ if S is definite,

$$C_0(\gamma;G^1_{\mathbb{R}}) = \begin{cases} C(\gamma;G^1_{\mathbb{R}}) & \text{if } S \text{ is indefinite, } -\det(S) \notin (\mathbb{Q}^\times)^2 \\ \left\{\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix};\ X = {}^tX\right\} & \text{if } S \text{ is definite, or } -\det(S) \in (\mathbb{Q}^\times)^2. \end{cases}$$

Definition 4.38. Two elements γ_1, γ_2 of Γ are said to belong to the same "family", if (i) $C_0(\gamma_1;G^1_{\mathbb{R}}) = C_0(\gamma_2;G^1_{\mathbb{R}})$ and (ii) $\gamma_{1s} = \gamma_{2s}$, where $\gamma_i = \gamma_{is}\gamma_{iu}$ $(i=1,2)$ is the Jordan decomposition.

Now we divide the set Γ into disjoint union of three subsets $\Gamma^{(e)}$, $\Gamma^{(h)}$, and $\Gamma^{(p)}$:

(i) $\Gamma^{(e)}$ consists of elliptic elements and ± 1 .

(An element $\gamma \neq \pm 1$ of $G^1_{\mathbb{R}}$ is called elliptic, if it has a fixed point in H_n ; or equivalently (under the condition $\gamma \in \Gamma$), it is of finite order.)

(ii) $\Gamma^{(h)}$ consists of those elements $\gamma \in \Gamma$ which are of "hyperbolic" type i.e., γ has a real eigenvalue $\neq \pm 1$.

(iii) $\Gamma^{(p)}$ consists of "p-unipotent (or parabolic)" elements of Γ i.e., those elements $\gamma \in \Gamma - \Gamma^{(e)}$ whose semi-simple factors γ_s belong to $\Gamma^{(e)}$; equivalently, γ is p-unipotent if and only if some power of γ is unipotent, from which the name comes.

We denote the contributions to (4.35) of each of these subsets by $T_k(\Gamma^{(e)})$, $T_k(\Gamma^{(h)})$, and $T_k(\Gamma^{(p)})$ respectively.

<u>Proposition</u> 4.39. <u>For semi-simple element γ of Γ</u> , $C(\gamma;\Gamma)$ <u>is always a lattice of</u> $C(\gamma;G^1_{\mathbb{R}})$. <u>Moreover, for the subset</u> $\Gamma^{(e)}$, <u>the termwise integrability is valid without dumping factors</u>.

Note that, in the case $\gamma \in \Gamma^{(e)}$, the integral $I(\gamma) = I(\gamma,s)\big|_{s=0}$ depends only on the conjugacy class $\{g\}_{G_{\mathbb{R}}}$; and this has been evaluated by Langlands [33] in a more general context (see § 4-5 and Remark (4.55)). After these remarks, we can immediately apply results of § 4-1 to obtain the following

<u>Theorem</u> $B^{(e)}$ (Elliptic Contributions)

<u>With the notations of</u> § 4-1, <u>we have for</u> $k > 2n$

$$(4.40) \quad T_k(\Gamma^{(e)}) = \sum_f \sum_{g\in[f]/\widetilde{G}_Q} t_k(g) \sum_{L_G(\Lambda)} m_G(\Lambda) \prod_p c_p(g,U_p,\Lambda_p) \quad ,$$

<u>where we put</u>

$$t_k(g) = a_n(k)\cdot I(g)$$
$$= a_n(k) \int_{C(g;G^1_{\mathbb{R}})\backslash H_n} H_g(Z) \quad dZ \quad .$$

Note that the above formula for $T_k(\Gamma^{(e)})$ is completely analogous to the dimension formula (4.30) in Theorem A . In both cases, the factors $tr(\rho(g))$ and $t_k(g)$, being invariants of $G_{\mathbb{R}}$-conjugacy classes $\{g\}_{G_{\mathbb{R}}}$, may be regarded as an "archimedean local factor $c_\infty(g,U_\infty,\Lambda_\infty)$", with $U_\infty = G_{\mathbb{R}}$, $\Lambda_\infty = C(g,G_{\mathbb{R}}) = Z_G(g)_{\mathbb{R}}$. As for the explicit formulas for them, see § 4-5.

Let us next consider $\Gamma^{(h)}$ and $\Gamma^{(p)}$:

<u>Theorem</u> $B^{(h)}$

<u>For</u> $\Gamma^{(h)}$, <u>we have</u> $T_k(\Gamma^{(h)}) = o$, <u>since for any</u> $\gamma \in \Gamma^{(h)}$,

$$(4.41) \qquad I_o(\gamma;s) =: \int_{C_o(\gamma;G^1_{\mathbb{R}})\backslash H_n} H_\gamma(Z;s) \quad dZ \quad = \quad o \; .$$

This is known in general as the "Selberg's Principle" (c.f. Warner [48]).

<u>Theorem</u> $B^{(p)}$ (Parabolic Contributions)

<u>For</u> $\Gamma^{(p)}$, <u>we have</u>

$$(4.42) \qquad T_k(\Gamma^{(p)}) = \frac{1}{\# Z(\Gamma)} \sum_F v(F) \lim_{s \downarrow o} \zeta(s;F) \; ,$$

<u>where the sum is extended over a complete set of</u> Γ<u>-conjugacy classes of families</u> $F \subseteq \Gamma^{(p)}$, <u>and</u> $v(F) = vol(C_o(\gamma;\Gamma)\backslash C_o(\gamma;G^1_{\mathbb{R}}))$

for $\gamma \in F$. The zetafunction $\zeta(s;F)$ attached to the family F is given by

$$(4.43) \qquad \zeta(s;F) = \sum_{\gamma \in F/\widetilde{\Gamma}} \frac{a_n(k) \cdot I_0(\gamma,s)}{[C(\gamma;\Gamma) : \pm C_0(\gamma;\Gamma)]}$$

with $I_0(\gamma,s)$ as in (4.41).

Remark 4.44. From the finiteness of the number of cusps of Γ, it follows that the set of non-conjugate families in $\Gamma^{(p)}$ is finite, so that the sum in (4.42) is a finite sum. Roughly speaking, $\zeta(s;F)$ is a zetafunction corresponding to F which is (a part of) a lattice, not necessarily homogeneous, in a vector space contained in the unipotent radical of a parabolic subgroup. The typical cases (i.e., purely unipotent elements) have been treated by Shintani [45]. In general, however, it is not easy to evaluate $\lim_{s \downarrow 0} \zeta(s;F)$.

4-5. Formulas for $\mathrm{tr}\,\rho(g)$, $t_k(g)$ and their Relations

Here we shall describe the explicit formulas for "∞-factors" $\mathrm{tr}\,\rho_k(g)$ and $t_k(g)$ of our dimension formulas (4.30), (4.40), for semi-simple (elliptic) element g. In our group $G^1_{\mathbb{R}} = Sp(n)$ or $Sp(n,\mathbb{R})$, we take the standard compact Cartan subgroup

$$H = \left\{ g(\theta) = \begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_n} \end{pmatrix} \in Sp(n);\ \theta_1,\dots,\theta_n \in \mathbb{R} \right\}$$

(4.45)

$$H = \left\{ g(\theta) = \left(\begin{array}{ccc|ccc} \cos\theta_1 & & & \sin\theta_1 & & \\ & \ddots & & & \ddots & \\ & & \cos\theta_n & & & \sin\theta_n \\ \hline -\sin\theta_1 & & & \cos\theta_1 & & \\ & \ddots & & & \ddots & \\ & & -\sin\theta_n & & & \cos\theta_n \end{array} \right) \in Sp(n,\mathbb{R});\ \theta_j \in \mathbb{R} \right\}$$

Here in $Sp(n)$, we identify $\mathbb{C}$ with the subalgebra of $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}ij$ by $\sqrt{-1} \to i$. Note that any (resp. elliptic) element of $Sp(n)$ (resp. $Sp(n,\mathbb{R})$) is conjugate to an element of H. We first assume that $g = g(\theta)$ is regular i.e., $C(g;G^1_{\mathbb{R}}) = H$; equivalently, $\theta_i + \theta_j \notin 2\mathbb{Z}$ for any i,j. Then we have

<u>Theorem</u> (Weyl [50]) <u>The irreducible character of</u> $Sp(n)$ <u>which corresponds to the Young diagram</u>

$$\left.\begin{array}{|c|c|c|c|} \hline 1 & 2 & \cdots & k \\ \hline & & \cdots & \\ \hline 1 & 2 & \cdots & k \\ \hline \end{array}\right\} n$$

<u>takes the following value at the regular element</u> $g = g(\theta)$:

$$\text{(4.46)} \qquad \operatorname{tr} \rho_k(g(\theta)) = \frac{\det\left[\sin(k+n+1-j)\theta_i\right]}{\det\left[\sin(n+1-j)\theta_i\right]}$$

Its degree is given by

$$(4.47)\qquad a_n(k) = \prod_{i\leq j} \frac{(2k+2n+2-i-j)}{(2n+2-i-j)}$$

We note the relation:

$$(4.48)\qquad d_n(k) = c_n \cdot a_n(k+n+1) \quad ,$$

$$c_n = \frac{1}{2^{n(n+2)}\pi^{n(n+1)/2}} \prod_{i\leq j} (2n+2-i-j) \quad .$$

Theorem (Langlands [33], see also Harish-Chandra [14])
Assume that $k > 2n$, and $g = g(\theta) \in Sp(n,\mathbb{R})$ has an isolated fixed point on H_n, which is the case for regular element. Then the integral $t_k(g) = a_n(k)\cdot I(g)$ in (4.40) is given by

$$(4.49)\qquad t_k(g(\theta)) = \frac{\prod_{j=1}^{n} e^{-ik\theta_j}}{\prod_{j\leq \ell}(1-e^{-i(\theta_j+\theta_\ell)})} \qquad (i = \sqrt{-1}).$$

Here, in the integral (4.40), we are taking the measure of $C(g;G^1_{\mathbb{R}})$ such that its volume is equal to 1. (Note that the condition on the isolated fixed point implies that $C(g;G^1_{\mathbb{R}})$ is compact).

Assuming that g is regular, we note that there are 2^n conjugacy classes in $Sp(n,\mathbb{R})$, each represented by $g(\pm\theta_1,..,\pm\theta_n)$, which are conjugate to g in $Sp(n,\mathbb{C})$, the complexification of $Sp(n,\mathbb{R})$, while in $Sp(n)$, $g(\pm\theta_1,..,\pm\theta_n)$ are all conjugate to g. By comparing the above two formulas, it is easy to observe the following

Theorem (Character Relation; regular case)*)

(4.50) $\operatorname{tr}\rho_k(g(\theta_1,\ldots,\theta_n))$

$$= (-1)^{\frac{n(n+1)}{2}} \sum_{\varepsilon_i = \pm 1} t_{k+n+1}(g(\varepsilon_1\theta_1,\ldots,\varepsilon_n\theta_n)) \quad .$$

This kind of character relations seem to be more or less well known to the experts in more general context, as long as regular elements are concerned. It seems less known, however, that similar relation remains to hold also for singular elliptic elements, under a suitable formulation, e.g., normalization of Haar measures. In fact, the relation (4.48) may be viewed as giving such relation in the extremely singular case $g = \pm 1$.

Here we note that the relation (4.50), which has been noticed (as well as (4.49)) in the case $n = 2$ by Y.Ihara around 1962, was one of the motivations to his conjectural question in [28], which is our main problem in this paper.

For singular elements g , we need much more involved notations to state the formulas for $\operatorname{tr}\rho_k(g)$ and $t_k(g)$; therefore we shall only give them in the case $n = 2$ below. As for the character relation, we content ourselves with the following description.

Theorem (Character Relation; general case)
Under a suitable normalization of the Haar measures, the following relation holds for arbitrary elliptic element $g(\theta)$:

$$(4.51) \qquad \operatorname{tr}\rho_k(g(\theta)) = (-1)^{\frac{n(n+1)}{2}} \sum_{\varepsilon} (-1)^{b(\varepsilon\theta)} t_{k+n+1}(g(\varepsilon\theta)) \ ,$$

---

*) If g is regular, $t_k(g)$ is in fact a character of a representation belonging to discrete series (c.f. [14]).

where $\varepsilon = (\varepsilon_i)$ runs over all possible values in $(\pm 1)^n$ so that $g(\varepsilon\theta) = g(\varepsilon_1\theta_1, \ldots, \varepsilon_n\theta_n)$ are all non-conjugate, and $b(\varepsilon\theta)$ denotes the complex dimension of the fixed point set of $g(\varepsilon\theta)$ in H_n.

Remark 4.52. It is easy to see that $b(\theta)$ is given by

$$b(\theta) = \#\{(i,j);\ 1 \leq i \leq j \leq n,\ \theta_i + \theta_j \in 2\pi\mathbb{Z}\} \ .$$

Moreover, from Langlands' formula for $t_k(g)$ in [33], it is observed that $t_k(g(\theta))$ is a polynomial of k of degree $b(\theta)$, modulo some factors $e^{2\pi ik/m}$ $(m \in \mathbb{Z})$. This observation is used to get asymptotic formulas for $\dim S_k(\Gamma)$ as a function of k (c.f. [17]).

Now assume $n = 2$. The Weyl's formula for $\operatorname{tr}\rho_k(g(\theta))$ for singular element is derived from the formula (4.46) by taking limits ; we have

$$\operatorname{tr}\rho_k(g(0,\theta)) = \frac{(k+2)\sin(k+1)\theta - (k+1)\sin(k+2)\theta}{2\sin\theta(1-\cos\theta)}$$

$$(4.53)\qquad \operatorname{tr}\rho_k(g(\theta,\theta)) = \frac{\left[\begin{array}{l}(k+1)\cos(k+1)\theta\sin(k+2)\theta \\ \qquad -(k+2)\cos(k+2)\theta\sin(k+1)\theta\end{array}\right]}{2\sin^3\theta}$$

$$\operatorname{tr}\rho_k(g(0,\pi)) = \frac{(-1)^k}{2}(k+1)(k+2)$$

The basic idea is similar also in the split case; here, however, the limit should be taken as a distribution (i.e., the limit formula of Harish-Chandra, see [33],[14]). The results are as follows:

$$t_k(g(0,\theta)) = \frac{i\left[(k-1)e^{-i(k-2)\theta} - (k-2)e^{-i(k-1)\theta}\right]}{2^5\pi^2\sin\theta(1-\cos\theta)}$$

$$(4.54)\quad t_k(g(\theta,-\theta)) = \frac{(2k-3)}{2^5\pi^2\sin^2\theta}$$

$$t_k(g(0,\pi)) = \frac{(-1)^k(2k-2)(2k-4)}{2^9\,\pi^4}\quad .$$

<u>Remark</u> 4.55. In [16], the first named author has computed the integral $t_k(g)$ by completely elementary method and obtained the above results. The constants in the denominators are due to the usual normalization of the Haar measure of $C(g;G^1_{\mathbb{R}})$, which will be cancelled by multiplying $\mathrm{vol}(_{C(g;\Gamma)}\backslash C(g;G^1_{\mathbb{R}}))$ $(g\in\Gamma)$. Also, we note that the above result for $t_k(g(0,\pi))$ does not agree with Langlands' formula ([33], (2), p 101); this is because the factor $e^{\rho_\gamma(H)}$ is missing in the denominator of (2), [33].

4-6. P-<u>unipotent</u> (<u>Parabolic</u>) <u>Contributions</u> (n = 2)

We assume $n = 2$, and describe briefly the zetafunction $\zeta(s;F)$ attached to each family F of p-unipotent elements of Γ. There are seven cases to be distinguished according as the types of their zetafunctions.

(i) <u>elliptic/parabolic</u> After normalizing by $G_{\mathbb{R}}$-conjugation simultaneously, we may assume that

$$(4.56) \qquad \gamma = \hat{\beta}(\theta,t) = \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & t \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\sin\theta \neq 0,\ t \neq 0),$$

and the family $F = F(\gamma)$ is given by

$$F(\gamma) = \{\hat{\beta}(\theta,\ a+n);\ n \in \mathbb{Z},\ a+n \neq 0\} \qquad (0 \leq a < 1).$$

We have $C_0(\gamma; G^1_{\mathbb{R}}) = \{\hat{\beta}(0,u\ );\ u \in \mathbb{R}\}$.

<u>Theorem</u> P-1 ([16], Theorem I-5)

<u>Under these notations, we have</u>

$$(4.57) \qquad \zeta(s;F) = \frac{-e^{-i(k-3/2)\theta}}{2^3\pi \sin\theta \sin\frac{\theta}{2}} \times \left[e^{-i\pi(s+1)/2}\zeta(s+1,a) + e^{i\pi(s+1)/2}\zeta(s+1,1-a)\right],$$

$$\lim_{s\downarrow 0} \zeta(s;F) = \frac{1}{2^2 \sin\theta \sin\frac{\theta}{2}}\left[\cos(k-\tfrac{3}{2})\theta + \cot^*(\pi a)\sin(k-\tfrac{3}{2})\theta\right] .$$

Here, $\zeta(s,a) = \sum_{n=0}^{\infty}(n+a)^{-s}$ is the Hurwitz zetafunction, and

$$\cot^*(x) = \begin{cases} \cot(x) & \text{if } x \notin \mathbb{Z}\pi \\ 0 & \text{if } x \in \mathbb{Z}\pi. \end{cases}$$

(ii) <u>paraelliptic</u> The normalized form of an element of this type is

$$(4.58) \qquad \gamma = \hat{\gamma}(\theta,t) = \begin{pmatrix} \cos\theta & \sin\theta & t\cos\theta & t\sin\theta \\ -\sin\theta & \cos\theta & -t\sin\theta & t\cos\theta \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad (t, \sin\theta \neq 0),$$

and the family is given by

$$F(\gamma) = \{\hat{\gamma}(\theta,\ a+n);\ n \in \mathbb{Z},\ a+n \neq 0\} \qquad (0 \leq a < 1).$$

we have $\quad C_o(\gamma;G_R^1) = \left\{ \hat{\gamma}(o,u); \quad u \in \mathbb{R} \right\}$.

Theorem P-2 (loc.cit. Theorem I-6)

Under these notations, we have

$$\zeta(s;F) = \frac{1}{2^3\pi \sin^2\theta}$$

$$\times\left[e^{-i\pi(s+1/2)}\zeta(2s+1,a) + e^{i\pi(s+1/2)}\zeta(2s+1,1-a)\right] \tag{4.59}$$

$$\lim_{s\downarrow o}\zeta(s;F) = -\frac{1}{2^3\sin^2\theta}\left(1 + i\cot^*(\pi a)\right).$$

(iii) δ-parabolic (nondegenerate case)

$$\gamma = \hat{\delta}(s_1,s_2) = \begin{bmatrix} 1 & o & s_1 & o \\ o & -1 & o & s_2 \\ o & o & 1 & o \\ o & o & o & -1 \end{bmatrix} \quad (s_1,s_2 \neq o),$$

(4.60)

$$F(\gamma) = \left\{\hat{\delta}(m+c, am+bn+ac);\ m,n \in \mathbb{Z},\ m+c, am+bn+ac \neq o\right\}$$

$$(a,b \in \mathbb{Z},\ (a,b) = 1,\ b > o,\ o \leq c < 1).$$

We have

$$C_o(\gamma;G_{\mathbb{R}}^1) = \left\{ \begin{bmatrix} 1 & o & t_1 & o \\ o & 1 & o & t_2 \\ o & o & 1 & o \\ o & o & o & 1 \end{bmatrix} \ ;\ t_1,t_2 \in \mathbb{R} \right\}.$$

Theorem P-3 (loc.cit. Theorem I-7)

Under these notations, we have

$$\zeta(s;F) = \frac{(-1)^k}{2^3\pi^2b^2}\sum_{j=o}^{b-1}\left[e^{-i\pi(s+1)/2}\zeta\left(s+1,\frac{j+c}{b}\right) + e^{i\pi(s+1)/2}\zeta\left(s+1,\frac{b-j-c}{b}\right)\right]$$

$$\times\left[e^{i\pi(s+1)/2}\zeta\left(s+1,\frac{a(j+c)}{b}\right) + e^{-i\pi(s+1)/2}\zeta\left(s+1,\frac{b-a(j+c)}{b}\right)\right] \tag{4.61}$$

$$\lim_{s \downarrow 0} \zeta(s;F) = \frac{(-1)^k}{2^3 b^2} \sum_{j=0}^{b-1} \left[1 + i\cot^*\left(\frac{(j+c)\pi}{b}\right)\right]\left[1 - i\cot^*\left(\frac{a(j+c)\pi}{b}\right)\right]$$

(iv) δ-<u>parabolic</u> (<u>degenerate</u> <u>case</u>)

(4.62) $\gamma = \hat{\delta}(t,0)$, $(t \neq 0,\ \hat{\delta}$: as in (4.60)),

$F(\gamma) = \{\hat{\delta}(n,\ 0);\ n \in \mathbb{Z}\}$.

We have

$$C_0(\gamma; G^1_{\mathbb{R}}) = \left\{ \begin{bmatrix} 1 & 0 & u & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{bmatrix};\quad \begin{matrix} u,a,b,c,d \in \mathbb{R} \\ ad-bc = 1 \end{matrix} \right\} .$$

<u>Theorem</u> P-4 (loc.cit. Theorem I-8)

<u>Under the above notations, we have</u>

(4.63)
$$\zeta(s;F) = \frac{(-1)^k(2k-3)}{2^5\pi^3}\,\zeta(s+1)\cos\left(\frac{s+1}{2}\right)\pi,$$
$$\lim_{s\downarrow 0} \zeta(s;F) = -\frac{(-1)^k(2k-3)}{2^6\pi^2} .$$

(v) To describe the purely unipotent contributions, we need some preparations. We note first that, if $I_0(\gamma,s) \neq 0$ for a unipotent element of Γ, then γ is conjugate in $G_{\mathbb{Q}}$ to an element of the following form

(4.64) $\gamma = \gamma(S) = \begin{pmatrix} 1 & S \\ 0 & 1 \end{pmatrix}$ $\quad S = {}^tS,$

with either (i) det $S = 0$, (ii) $-\det S \in (\mathbb{Q}^\times)^2$, or $S \gtrless 0$ i.e., S = definite. This, in particular, means that such γ belongs to the unipotent radical P_U of a parabolic $\mathbb{Q}$-subgroup P of $G_{\mathbb{Q}}$ which corresponds to a point cusp that γ fixes. If det $S \neq 0$, we can associate in this way a lattice $L = P_U \cap \Gamma$, which we also regard as a lattice of $Sm_2(\mathbb{R})$, the 2x2 symmetric real matrices via a fixed isomorphism $P_U(\mathbb{R}) \cong SM_2(\mathbb{R})$. We have an action of a Levi-subgroup P_M of P on P_U, which may be assumed as

$$T \longrightarrow AT\,{}^tA \qquad (T \in SM_2(\mathbb{R}),\ A \in GL_2(\mathbb{R}))$$

under an isomorphism $P_M(\mathbb{R}) \cong GL_2(\mathbb{R})$. Moreover, for simplicity, we assume that

$$(4.65) \qquad P \cap \Gamma = (P_M \cap \Gamma)\cdot(P_U \cap \Gamma) .$$

We denote by $(P_M \cap \Gamma)_o$ the image of $P_M \cap \Gamma$ in $GL_2(\mathbb{R})$, and put $(P_M \cap \Gamma)_o^+ =: (P_M \cap \Gamma) \cap SL_2(\mathbb{R})$. Also put, for $\gamma = g\,\gamma(S)g^{-1}$ as above,

$$(4.66) \qquad O_\Gamma(S) = \left\{ A \in (P_M \cap \Gamma)_o;\ \ AS\,{}^tA = S \right\}.$$

If S is as in (ii), (4.64), the family F represented by γ is given by

$$F(\gamma) = g\left\{ \gamma(S');\ \ S' \in L,\ -\det(S') \in (Q^\times)^2 \text{ or } S \gtrless 0 \right\} .$$

We divide F into two parts $F^\pm$ and F^s according as S' satisfies $S' \gtrless 0$, or $-\det(S') \in (Q^\times)^2$.

<u>Theorem</u> P-5 (loc.cit. Theorem I-9)

<u>Notations being as above, we have</u>

$$(4.67) \qquad \zeta(s;F^\pm) = \frac{s}{2\pi} \sum_{S \in L^+ \bmod (P_M\cap\Gamma)_o} \frac{1}{\# O_\Gamma(S)(\det S)^{s+3/2}} \qquad (L^+ = \{S \in L;\ S > 0\})$$

$$\lim_{s \downarrow 0} \zeta(s;F^\pm) = \frac{1}{2^2\pi} \frac{\mathrm{vol}((P_M\cap\Gamma)_o^+ \backslash H_1)}{\left[(P_M\cap\Gamma)_o : (P_M\cap\Gamma)_o^+\right] \mathrm{vol}(L\backslash SM_2(\mathbb{R}))} .$$

Theorem 4-6 (loc.cit. Theorem 1-10)

(4.68)
$$\zeta(s;F^s) = -\frac{1}{2^4\pi^2}\sum_{j=1}^{t}\sum_{S\in L_j^s \bmod B_j}\frac{1}{|\det S|^{s+3/2}}$$

$$\lim_{s\downarrow 0}\zeta(s;F^s) = -\frac{1}{2^5 3}\sum_{j=1}^{t}\frac{c_j}{d_j^3},$$

Here notations are as follows: let $\beta_1,\dots,\beta_t$ be the set of non-equivalent cusps of $(P_M\cap\Gamma')_0^+$ in $\overline{H}_1$. Take $V\in SL_2(\mathbb{Q})$ such that $V\langle\beta_j\rangle=\infty$, and put $L_j^s =: V^{-1}\begin{pmatrix}* & * \\ * & 0\end{pmatrix}{}^tV^{-1}\cap L$. B_j is the parabolic subgroup of $(P_M\cap\Gamma')_0^+$ which stabilizes β_j. The module $VL_j^s\,{}^tV$ has a unique basis of the form

$$\begin{pmatrix}t_1^{(j)} & 0 \\ 0 & 0\end{pmatrix}, \begin{pmatrix}t_2^{(j)} & d_j \\ d_j & 0\end{pmatrix}; \quad d_j>0,\quad t_1^{(j)}>|t_2^{(j)}|\geq 0,$$

and c_j is defined by

$$B_j = \left\{\pm\begin{pmatrix}1 & (t_1^j c_j/2d_j)\mathbb{Z} \\ 0 & 1\end{pmatrix}\right\}.$$

Finally in the case (i) $\det S = 0$, the family F represented by $\gamma = g\,\gamma(S)g^{-1}$ may be assumed to be given by

$$F(\gamma) = g\left\{\begin{bmatrix}1 & 0 & dn & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix};\ n\in\mathbb{Z}-\{0\}\right\}g^{-1} \quad (d\in\mathbb{Q}_+^\times)$$

$$\leq g\begin{pmatrix}1 & L \\ 0 & 1\end{pmatrix}g^{-1},$$

and we hava $C_0(\gamma;G_{\mathbb{R}}^1) = C(\gamma;G_{\mathbb{R}}^1)$.

<u>Theorem</u> P-7 (loc.cit. Theorem 1-11)

<u>Notations being as above, we have</u>

$$\zeta(s;F) = -\frac{(2k-3)}{2^5\pi^4}\sum_{n=-\infty}^{\infty}\frac{1}{|dn|^{s+2}} ,$$

$$\lim_{s\downarrow 0}\zeta(s;F) = -\frac{2k-3}{2^3 3\pi^2 d^2} .$$

§ 5 Conjugacy Classes of $U_2(p)(=\Gamma_0'(p))$ and $B(p)$ (Proof of Theorem 3-2, 3-3)

5-1. In this section, we shall use the usual notation:

$$U_1(p) = \Gamma_0(p), \quad U_2(p) = \Gamma_0'(p), \text{ and } U_{12}(p) = Sp(2,\mathbb{Z}) .$$

We shall describe the conjugacy classes in $\Gamma_0'(p)$ and $B(p)$ of those elements (or families) which make nontrivial contributions to $\dim S_k(\Gamma_0'(p))$, $\dim S_k(B(p))$, in such a form that is sufficient to work out the explicit formulas for them, as presented in § 3, if we put all data given here to our general formulas (4.30), (4.40), and (4.42). Since $\Gamma_0'(p)$ (resp. $B(p)$) is a subgroup of $Sp(2,\mathbb{Z})$ (resp. $\Gamma_0(p)$), and the list of conjugacy classes of the latter groups has been given in [16], § 6, 7, we need not begin at the beginning. So, we mainly apply the global method (i.e., argument on Γ-conjugacy classes) also for semi-simple elements. Of course, in that case we can replace it by the local method described in Theorem $B^{(e)}$, as excuted in [24] for $U_{02}(p)(= K(p))$, and in [16], (II) for other arithmetic subgroups in Q-rank one case.

In general, for two lattices Γ_1, Γ_2 of $G^1_{\mathbb{R}}$ such that $\Gamma_1 \geq \Gamma_2$, $[\Gamma_1 : \Gamma_2] < \infty$, we have a bijection in the same way as (4.17)

$$(5.1) \qquad \{\gamma\}_{\Gamma_1} \cap \Gamma_2 / \widetilde{\Gamma_2} \quad \xrightarrow{\sim} \quad C(\gamma;\Gamma_1)\backslash M(\gamma,\Gamma_1,\Gamma_2)/\Gamma_2$$

for any $\gamma \in \Gamma_1$, where we put $M(\gamma,\Gamma_1,\Gamma_2) = \{x \in \Gamma_1;\ x^{-1}\gamma x \in \Gamma_2\}$. Let $\gamma_1,\dots,\gamma_d$ ($d = d(\gamma) = \#[C(\gamma;\Gamma_1)\backslash M(\gamma,\Gamma_1,\Gamma_2)/\Gamma_2]$) be a complete set of representatives of Γ_2-conjugacy classes in $\{\gamma\}_{\Gamma_1} \cap \Gamma_2$.

we define "<u>relative Maß</u>" of γ with respect to Γ_1/Γ_2, by

$$(5.2)\qquad m(\gamma;\Gamma_1/\Gamma_2) = \sum_{i=1}^{d}\left[C(\gamma_i;\Gamma_1):C(\gamma_i;\Gamma_2)\right].$$

Then the elliptic contributions to $\dim S_k(\Gamma_i)$ $(i=1,2)$ are related as follows: namely for $\gamma\in\Gamma_1^{(e)}$

$$(5.3)^{*)}\qquad T_k(\{\gamma\}_{\Gamma_1}\cap\Gamma_2^{(e)}) = m(\gamma;\Gamma_1/\Gamma_2)\, T_k(\{\gamma\}_{\Gamma_1})$$

Thus to compute the elliptic contributions for $\Gamma_2 = \Gamma_0'(p)$, $B(p)$, it suffices to calculate the relative Maß's for each conjugacy class $\{\gamma\}_{\Gamma_1}$ of $\Gamma_1 = Sp(2,\mathbb{Z})$, $\Gamma_0(p)$. On the other hand, the p-unipotent (parabolic) contributions require more careful treatment.

<u>Lemma</u> 5.4. <u>As a complete set of representatives of the coset space</u> $Sp(2,\mathbb{Z})/\Gamma_0'(p)$ (resp. $\Gamma_0(p)/B(p)$), <u>we can take the following</u> $[Sp(2,\mathbb{Z}):\Gamma_0'(p)] = (p+1)(p^2+1)$ (resp. $[\Gamma_0(p):B(p)] = p+1$) <u>elements</u>:

(i) $Sp(2,Z)/\Gamma_0'(p)$:

$$X_1(a,b,c):=\begin{pmatrix}1&0&0&0\\a&1&0&0\\b&c&1&-a\\c&0&0&1\end{pmatrix}\qquad X_2(a,b):=\begin{pmatrix}0&1&0&0\\1&0&0&0\\a&0&0&1\\b&a&1&0\end{pmatrix}$$

$$X_3(a):=\begin{pmatrix}0&-a&-1&0\\0&1&0&0\\1&0&0&0\\a&0&0&1\end{pmatrix}\qquad X_4:=\begin{pmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

(ii) $\Gamma_0(p)/B(p)$:

$$Z_1(t):=\begin{pmatrix}1&0&0&0\\t&1&0&0\\0&0&1&-t\\0&0&0&1\end{pmatrix}\qquad Z_2:=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}.$$

---

*) If Γ_1, Γ_2 are defined by U_{1A}, U_{2A} respectively as in (4.8), we have the following relation:

$$H(\gamma;U_{2A}) = m(\gamma;\Gamma_1/\Gamma_2)\cdot H(\gamma\cdot U_{1A}).$$

Here $a,b,c,$ and t runs over the integers modulo p.

By using this lemma and the list of conjugacy classes of $Sp(2,\mathbb{Z})$, $\Gamma_0(p)$ given in [16], we can find a complete set of representatives $x_1,\ldots,x_d$ of the double cosets of (5.1), where x_i are taken from the above set of representatives. This can be done in completely elementary way, and we omit the details of the calculations. In the following, we describe only the list of these x_i's with the invariants attached to each conjugacy classes such as $m(\gamma;\Gamma_1/\Gamma_2)$, which are necessary to obtain explicit formulas for $\dim S_k(\Gamma_0'(p))$, and $\dim S_k(B(p))$.

5-2. Conjugacy Classes of $\Gamma_0'(p)$. (p = prime, $\neq 2,3$)

We use the notations of [16], Theorem 6-1. However, for the convenience of readers, we reproduce here the matrix representatives of each conjugacy classes of $Sp(2,\mathbb{Z})$, and those of $Sp(2,\mathbb{R})$ taken in the standard Cartan subgroup H as in (4.45) for elliptic elements. The symbol $\pm\gamma$ means that $-\gamma$ should be added, though we write $+\gamma$ alone.

(5.5) $\quad \gamma = \pm\alpha_0\,,\quad \alpha_0 = 1_4 \sim g(0,0)$

$\qquad d(\gamma) = 1,\quad x = X_1(0,0,0)$

$\qquad m(\gamma;Sp(2,\mathbb{Z})/\Gamma_0'(p)) = [Sp(2,\mathbb{Z}):\Gamma_0'(p)] = (p+1)(p^2+1)$.

(5.6) $\gamma = \pm\alpha_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \sim g(\pi/2, \pi/2)$

$d(\gamma) = (p+1)(1+(\frac{-1}{p}))$

$x = X_1(a,b,ab),\ X_2(0,b,0)$ with $b^2+1 \equiv 0 \pmod p$

$m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p)) = (p+1)(1+(\frac{-1}{p}))$.

(5.7) $\gamma = \pm\alpha_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{bmatrix} \sim g(2\pi/3, 2\pi/3)$; and $\alpha_3 = \alpha_2^{-1}$.

$d(\gamma) = (p+1)(1+(\frac{-3}{p}))$

$x = X_1(a,b,ab),\ X_2(0,b,0)$ with $b^2+b+1 \equiv 0 \pmod p$

$m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p)) = (p+1)(1+(\frac{-3}{p}))$.

(5.8) $\gamma = \alpha_4 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim g(-\pi/4, 3\pi/4)$; and $\alpha_5 = \alpha_4^3$.

$d(\gamma) = \begin{cases} 4 & \text{if } p \equiv 1 \pmod 8 \\ 0 & \text{otherwise} \end{cases}$

$x = X_1(a, a^2, -a^{-1})$ with $a^4+1 \equiv 0 \pmod p$

$m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p)) = d(\gamma)$.

Here, and throughout the following, we are confusing the integers mod p with elements of the finite field F_p, writing a^{-1} the integer $x \pmod p$ such that $ax \equiv 1 \pmod p$.

(5.9) $\gamma = \pm\alpha_6 = \begin{bmatrix} 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim g(\pi/4, 3\pi/4)$

$$d(\gamma) = \begin{cases} 4 & \text{if } p \equiv 1 \pmod 8 \\ 0 & \text{otherwise} \end{cases}$$

$x = X_1\left(\frac{1-b}{b^2-3b+3}, b, \frac{-(1-b)^2}{b^2-3b+3}\right)$ with $(b-1)^4+1 \equiv 0 \pmod p$

$m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p)) = d(\gamma)$.

(5.10) $\gamma = \pm\alpha_7 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim g(-\pi/3, -2\pi/3)$

$d(\gamma) = 2(1+(\frac{-3}{p}))$

$x = X_1(a,a,-1),\ X_1(a,-a,1)$ with $a^2-a+1 \equiv 0 \pmod p$

$m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p)) = 2(1+(\frac{-3}{p}))$

(5.11) $\gamma = \pm\alpha_8 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim g(-2\pi/3, -\pi/3)$

$d(\gamma) = 2(1+(\frac{-3}{p}))$

$x = X_1(0,b,0),\ X_2(0,b)$ with $b^2+b+1 \equiv 0 \pmod p$

$m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p)) = 2(1+(\frac{-3}{p}))$

(5.12) $\gamma = \alpha_9 = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim g(2\pi/3, -\pi/3)$; and $\alpha_{10} = \alpha_9^{-1}$.

$d(\gamma) = 2(1+(\frac{-3}{p}))$.

$$x = \begin{cases} X_1(a,a^2,1) & \ldots \text{ with } a^2+a+1 \equiv 0 \\ X_1(a,a^2,-1) & \ldots \text{ with } a^2-a+1 \equiv 0 \end{cases} \pmod p$$

$m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p)) = 2(1+(\frac{-3}{p}))$

(5.13) $\gamma = \alpha_{11} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \sim g(-2\pi/3, \pi/3)$; and $\alpha_{12} = \alpha_{11}^{-1}$

$d(\gamma) = 2(1+(\frac{-3}{p}))$

$x = X_1(0,b,0),\ X_2(0,b)$ with $b^2+b+1 \equiv 0 \pmod p$

$m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p)) = 2(1+(\frac{-3}{p}))$

(5.14) $\gamma = \alpha_{13} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim g(-\pi/6, 5\pi/6)$; and $\alpha_{14} = \alpha_{13}^{-1}$

$$d(\gamma) = \begin{cases} 4 & \text{if } p \equiv 1 \pmod{12} \\ 0 & \text{otherwise} \end{cases}$$

$x = X_1(a, -a^{-2}, -a^{-1})$ with $a^4 - a^2 + 1 \equiv 0 \pmod p$

$m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p)) = d(\gamma)$

(5.15) $\gamma = \pm\alpha_{15} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & -1 \end{bmatrix} \sim g(2\pi/5, -4\pi/5)$;

and $\alpha_{16} = \alpha_{15}^{2}$, $\alpha_{17} = \alpha_{15}^{3}$, $\alpha_{18} = \alpha_{15}^{4}$

$$d(\gamma) = \begin{cases} 4 & \text{if } p \equiv 1 \pmod 5 \\ 1 & \phantom{\text{if }} p = 5 \\ 0 & \text{otherwise} \end{cases}$$

If $p=5$, $x = X_1(2,2,2)$,

if $p \equiv 1 \pmod 5$, $x = X_1(a,b,c)$ with

$a^2+a-1 \equiv 0$, $b \equiv 1+(1+a)c$, $c^2+c+\frac{1}{a+2} \equiv 0 \pmod p$

$m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p)) = d(\gamma)$

(5.16) $\gamma = \pm\alpha_{19} = \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim g(-2\pi/3, -\pi/2)$:

and $\alpha_{20} = \alpha_{19}^{-1}$, $\alpha_{21} = \alpha_{19}^{7}$, $\alpha_{22} = \alpha_{19}^{5}$

$d(\gamma) = 2 + (\frac{-1}{p}) + (\frac{-3}{p})$

$x = \begin{cases} X_1(0,b,0) & \text{with } b^2+b+1 \equiv 0 \text{, and} \\ X_2(0,b) & \text{with } b^2+1 \equiv 0 \pmod p \end{cases}$

$m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p)) = 2 + (\frac{-1}{p}) + (\frac{-3}{p})$

(5.17) $\gamma = \pm B_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim g(-2\pi/3, 0)$; and $B_2 = B_1^{-1}$

$\gamma = \pm B_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim g(-\pi/3, 0)$; and $B_4 = B_3^{-1}$

$d(\gamma) = 2 + (\frac{-3}{p}) = m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p))$

$x = X_2(0,0)$ and $X_1(0,b,0)$ with $b^2+b+1 \equiv 0 \pmod p$

(5.18) $\gamma = \pm B_5 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim g(-\pi/2, 0)$; and $B_6 = B_5^{-1}$

$d(\gamma) = 2 + (\frac{-1}{p}) = m(\gamma; Sp(2,\mathbb{Z})/\Gamma_0'(p))$

$x = X_2(0,0)$ and $X_1(0,b,0)$ with $b^2+1 \equiv 0 \pmod p$

(5.19) $\gamma = \gamma_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim g(\pi/2, -\pi/2)$

$\gamma = \gamma_2 = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim g(\pi/2, -\pi/2)$

$d(\gamma) \;=\; 1 + (\frac{-1}{p})$

$x = X_1(a,o,o)$ with $a^2+1 \equiv o (\mathrm{mod}\ p)$

$m(\gamma; Sp(2,\mathbb{Z})/\Gamma_o'(p)) = (p+1)(1+(\frac{-1}{p}))$

(5.20) $\gamma = \pm\gamma_3 = \begin{bmatrix} o & -1 & o & o \\ 1 & -1 & o & o \\ o & o & -1 & -1 \\ o & o & 1 & o \end{bmatrix} \sim g(2\pi/3, -2\pi/3)$

$d(\gamma) \;=\; 1 + (\frac{-3}{p})$

$x = X_1(a,o,o)$ with $a^2-a+1 \equiv o$ (mod p)

$m(\gamma; Sp(2,\mathbb{Z})/\Gamma_o'(p\ )) = (p+1)(1+(\frac{-3}{p}))$

(5.21) $\gamma = \delta_1 = \begin{bmatrix} 1 & o & o & o \\ o & -1 & o & o \\ o & o & 1 & o \\ o & o & o & -1 \end{bmatrix}$ and $\delta_2 = \begin{bmatrix} 1 & o & o & -1 \\ o & -1 & 1 & o \\ o & o & 1 & o \\ o & o & o & -1 \end{bmatrix} \sim g(o,\pi)$

$d(\gamma) \;=\; 2$

$x = X_1(o,o,o),\ X_2(o,o)$

$m(\gamma; Sp(2,\mathbb{Z})/\Gamma_o'(p)) \;=\; 2(p+1)$

P-unipotent classes

We first note that $\Gamma_o'(p)$ has two point cusps and three one dimensional cusps, corresponding to the following parabolic subgroups:

Print cusps: $\quad P_o^{(j)} = x_j^{-1} \begin{pmatrix} * & * \\ o & * \end{pmatrix} x_j\ , \quad x_1 = 1_4,\ x_2 = X_1(o,1,o)$

One dimensional cusps : $p_1^{(j)} = x_j^{-1}\begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} x_j$,

$$x_1 = 1_4,\ x_2 = X_1(o,o,-1),\ x_3 = X_2(o,o)\ .$$

Each family of p-unipotent elements belong to (at least) one of these parabolic subgroups, up to $\Gamma_0'(p)$-conjugation. In the following, we give a typical element of each family of $Sp(2,\mathbb{Z})$ listed in [16], Theorem 6-1, and describe the decomposition of it into $\Gamma_0'(p)$-conjugacy classes. We put, for each class $x^{-1}\gamma x$ of $\Gamma_0'(p)$,

$$i_o(x) =: \left[C_o(x^{-1}\gamma x; Sp(2,\mathbb{Z})) : C_o(x^{-1}\gamma x; \Gamma_0'(p))\right] \quad .$$

(5.22) $\gamma = \pm\, \hat{B}_1(n) = \begin{bmatrix} o & o & 1 & o \\ o & 1 & o & n \\ -1 & o & 1 & o \\ o & o & o & 1 \end{bmatrix} \sim \hat{B}(\pi/3,n)$; and $\hat{B}_2(n) = \hat{B}_1^{-1}(-n)$

$n \in \mathbb{Z} - \{o\}$

$d(\gamma) = 3 + \left(\frac{-3}{p}\right)$

$$x = \begin{cases} X_2(o,o),\ X_1(o,b,o) & \text{with } b^2+b+1 \equiv o \pmod p) \\ & \ldots\quad n\text{: arbitrary} \\ X_4 & \ldots\quad n \equiv o \pmod p \end{cases}$$

$i_o(x) = 1$ for each x .

(5.23) $\gamma = \pm\, \hat{B}_3(n) = \begin{bmatrix} o & o & -1 & o \\ o & 1 & o & n \\ 1 & o & -1 & o \\ o & o & o & 1 \end{bmatrix} \sim B(4\pi/3,n)$; and $\hat{\beta}_4(n) = \hat{B}_3^{-1}(-n)$

$n \in \mathbb{Z} - \{o\}$

$d(\gamma) = 3 + \left(\frac{-3}{p}\right)$

$$x = \begin{cases} X_2(o,o),\ X_1(o,b,o) & \text{with } b^2+b+1 \equiv o \pmod p) \\ & \ldots\quad n\text{: arbitrary} \\ X_4 & \ldots\quad n \equiv o \pmod p \end{cases}$$

$i_o(x) = 1$ for each x.

$$(5.24)\quad \gamma = \pm\ \hat{B}_5(n) = \begin{bmatrix} -1 & o & 1 & 1 \\ -1 & 1 & o & n \\ -1 & o & o & o \\ o & o & o & 1 \end{bmatrix} \sim \hat{B}(2\pi/3, n-1/3);\quad \hat{B}_6(n) = \hat{B}_5^{-1}(-n) \qquad n \in \mathbb{Z}$$

$d(\gamma) = 3 + (\frac{-3}{p})$

$$x = \begin{cases} X_2(o,o),\ X_1(a,b,o) \ \text{ with } \ b^2-b+1 = o,\ a = (2-b)^{-1} \\ \qquad\qquad\qquad \ldots\ n\text{: arbitrary} \\ X_1(o,-1,3) \qquad\qquad \ldots\ 3n-1 \equiv o \pmod p \end{cases}$$

$i_o(x) = 1$ for each x .

$$(5.25)\quad \gamma = \pm\ \hat{B}_7(n) = \begin{bmatrix} o & o & -1 & o \\ o & 1 & o & n \\ 1 & o & o & o \\ o & o & o & 1 \end{bmatrix} \sim B(-\pi/2, n);\quad \hat{B}_8(n) = \hat{B}_7^{-1}(-n) \qquad n \in \mathbb{Z}-\{o\}$$

$d(\gamma) = 3 + (\frac{-1}{p})$

$$x = \begin{cases} X_2(o,o),\ X_1(o,b,o) \ \text{ with } \ b^2+1 \equiv o \pmod p \\ \qquad\qquad\qquad \ldots\ n\text{: arbitrary} \\ X_4 \qquad\qquad\qquad \ldots\ n \equiv o \pmod p \end{cases}$$

$i_o(x) = 1$ for each x .

$$(5.26)\quad \gamma = \pm\ \hat{B}_9(n) = \begin{bmatrix} o & o & 1 & -1 \\ 1 & 1 & o & n \\ -1 & o & o & o \\ o & o & o & 1 \end{bmatrix} \sim \hat{B}(\pi/2, n-1/2);\quad \hat{B}_{10}(n) = \hat{B}_9^{-1}(-n) \qquad n \in \mathbb{Z}$$

$d(\gamma) = 3 + (\frac{-1}{p})$

$$x = \begin{cases} X_2(o,o),\ X_1(\frac{-b}{b+1},b,o) \ \text{ with } \ b^2+1 \equiv o \pmod p \\ \qquad\qquad\qquad \ldots\ n\text{: arbitrary} \\ X_1(o,-1,-2) \qquad\qquad \ldots\ 2n-1 \equiv o \pmod p \end{cases}$$

$i_o(x) = 1$ for each x .

(5.27) $\gamma = \hat{\gamma}_1(n) = \begin{bmatrix} 0 & -1 & 0 & -n \\ 1 & 0 & n & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \hat{\gamma}(-\pi/2, n),\ n \in \mathbb{Z} - \{0\}$

$d(\gamma) = 2(1+(\frac{-1}{p}))$

$$x = \begin{cases} X_1(a,0,0) \text{ with } a^2+1 \equiv 0, \text{ n:arbitrary}, & i_0(x) = 1 \\ X_1(a,-a,1) \text{ with } a^2+1 \equiv 0, n \equiv 0 \pmod p, & i_0(x) = p \end{cases}$$

(5.28) $\gamma = \hat{\gamma}_2(n) = \begin{bmatrix} 0 & -1 & 0 & -n \\ 1 & 0 & n+1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \hat{\gamma}(-\pi/2, n+1/2),\ n \in \mathbb{Z}$

$d(\gamma) = 2(1+(\frac{-1}{p}))$

$$x = \begin{cases} X_1(a,0,0) \text{ with } a^2+1 \equiv 0, \text{ n:arbitrary}, & i_0(x) = 1 \\ X_1(0,-2,c) \text{ with } c^2+4 \equiv 0, 2n+1 \equiv 0 \pmod p, & i_0(x) = p \end{cases}$$

(5.29) $\gamma = \hat{\gamma}_3(n) = \begin{bmatrix} 0 & -1 & 1 & -n \\ 1 & 0 & n & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \hat{\gamma}(-\pi/2, n),\ n \in \mathbb{Z} - \{0\}$

$d(\gamma) = 2(1+(\frac{-1}{p}))$

$$x = \begin{cases} X_1(a,0,0) \text{ with } a^2+1 \equiv 0, \text{ n:arbitrary}, & i_0(x) = 1 \\ X_1(0,b,1) \text{ with } b^2+1 \equiv 0, n \equiv 0 \pmod p, & i_0(x) = p \end{cases}$$

(5.30) $\gamma = \hat{\gamma}_4(n) = \begin{bmatrix} 0 & -1 & 1 & -n \\ 1 & 0 & n+1 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \hat{\gamma}(-\pi/2, n+1/2),\ n \in \mathbb{Z}$

$d(\gamma) = 2(1+(\frac{-1}{p}))$

$$x = \begin{cases} X_1(a,0,0) \text{ with } a^2+1 \equiv 0, \text{ n:arbitrary}, & i_0(x) = 1 \\ X_1(1/2,b,1) \text{ with } b^2+1 \equiv 0, 2n+1 \equiv 0 \pmod p, & i_0(x) = p \end{cases}$$

$$(5.31)\quad \gamma = \pm\hat{\gamma}_5(n) = \begin{bmatrix} o & -1 & -n & -2n \\ 1 & -1 & n & -n \\ o & o & -1 & -1 \\ o & o & 1 & o \end{bmatrix} \sim \hat{\gamma}(2\pi/3,n) \qquad n\in\mathbb{Z}-\{o\}$$

$$d(\gamma) = 2(1+(\tfrac{-3}{p}))$$

$$x = \begin{cases} X_1(a,o,o) & \text{with } a^2-a+1 \equiv o,\ n\text{:arbitrary}, \quad i_o(x) = 1 \\ X_1(a,-a,1) & \text{with } a^2-a+1 \equiv o,\ n\equiv o \pmod p, \quad i_o(x) = p \end{cases}$$

$$(5.32)\quad \gamma = \pm\hat{\gamma}_6(n) = \begin{bmatrix} o & -1 & -n & -2n \\ 1 & -1 & n+1 & -n \\ o & o & -1 & -1 \\ o & o & 1 & o \end{bmatrix} \sim \hat{\gamma}(2\pi/3,n+1/3), \quad n\in\mathbb{Z}$$

$$d(\gamma) = 2(1+(\tfrac{-3}{p}))$$

$$x = \begin{cases} X_1(a,o,o) & \text{with } a^2-a+1 \equiv o,\ n\text{:arbitrary}, \quad i_o(x) = 1 \\ X_1(2,3b,3) & \text{with } b^2+b+1 \equiv o,\ 3n+1\equiv o \pmod p, \quad i_o(x) = 1 \end{cases}$$

$$(5.33)\quad \gamma = \pm\hat{\gamma}_7(n) = \begin{bmatrix} 1 & -1 & -n & -2n \\ 1 & -1 & n+2 & -n \\ o & o & -1 & -1 \\ o & o & 1 & o \end{bmatrix} \sim \hat{\gamma}(2\pi/3,n+2/3) \quad n\in\mathbb{Z}$$

$$d(\gamma) = 2(1+(\tfrac{-3}{p}))$$

$$x = \begin{cases} X_1(a,o,o) & \text{with } a^2-a+1 \equiv o,\ n\text{:arbitrary}, \quad i_o(x) = 1 \\ X_1(2,3b/2,3/2) & \text{with } b^2+b+1\equiv o,\ 3n+2\equiv o \pmod p, \quad i_o(x) = \end{cases}$$

$$(5.34)\quad \gamma = \hat{\delta}_1(m,n) = \begin{bmatrix} 1 & o & m & o \\ o & -1 & o & n \\ o & o & 1 & o \\ o & o & o & -1 \end{bmatrix} \qquad m,n\in\mathbb{Z}-\{o\}$$

$$d(\gamma) = 4$$

$$x = \begin{cases} \left.\begin{matrix} X_1(o,o,o) \\ X_2(o,o) \end{matrix}\right\} & m,n\text{:arbitrary} \qquad i_o(x) = 1 \\ X_1(o,1,o) & \ldots\ m \equiv o \pmod p \qquad i_o(x) = p \\ X_2(o,1) & \ldots\ n \equiv o \pmod p \qquad i_o(x) = p \end{cases}$$

$$(5.35)\quad \gamma = \hat{\hat{\delta}}_2(m,n) = \begin{bmatrix} 1 & 0 & m & -1 \\ 0 & -1 & 1 & n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad m,n \in \mathbb{Z} - \{0\}$$

$d(\gamma) = 4$

$$x = \begin{cases} X_1(0,0,0) \\ X_2(0,0) \end{cases} \; m,n\text{:arbitrary}, \qquad i_0(x) = 1$$

$$\begin{array}{lll} X_2(2,0) & \ldots\ m \equiv 0 \pmod p & i_0(x) = p \\ X_1(0,0,2) & \ldots\ n \equiv 0 \pmod p & i_0(x) = p \end{array}$$

$$(5.36)\quad \gamma = \hat{\hat{\delta}}_3(m,n) = \begin{bmatrix} 1 & 0 & 2m & m+2 \\ 1 & -1 & m-2 & n \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad m,n \in \mathbb{Z},\ m, 2n-m \neq 0$$

$d(\gamma) = 4$

$$x = \begin{cases} X_1(0,0,0) \\ X_2(0,0) \end{cases} \ldots\ m,n\text{:arbitrary}, \qquad i_0(x) = 1$$

$$\begin{array}{lll} X_2(2,0) & \ldots\ m \equiv 0 \pmod p & i_0(x) = p \\ X_1(0,0,2) & \ldots\ n \equiv 0 \pmod p & i_0(x) = p \end{array}$$

$$(5.37)\quad \gamma = \hat{\hat{\delta}}_4(m,n) = \begin{bmatrix} 1 & 0 & 2m-1 & m \\ 1 & -1 & m-1 & n \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad m,n \in \mathbb{Z},$$

$d(\gamma) = 4$

$$x = \begin{cases} X_1(0,0,0) \\ X_2(0,0) \end{cases} \ldots\ m,n\text{:arbitrary}, \qquad i_0(x) = 1$$

$$\begin{array}{lll} X_1(0,2,0) & \ldots\ 2m-1 \equiv 0 \pmod p & i_0(x) = p \\ X_1(0,2,-4) & \ldots\ 4n-2m+1 \equiv 0 \pmod p & i_0(x) = p \end{array}$$

$$(5.38)\quad \gamma = \pm \hat{\delta}_1(m) = \begin{bmatrix} 1 & 0 & m & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad m \in \mathbb{Z} - \{0\}$$

$d(\gamma) = 3$

$$x = \begin{cases} X_1(o,o,o) & \ldots \text{ m:arbitrary}, \quad i_o(x) = 1 \\ X_2(o,o) & \ldots \text{ m:arbitrary}, \quad i_o(x) = p+1 \\ X_1(o,1,o) & \ldots \ m \equiv o \pmod p, \quad i_o(x) = p \end{cases}$$

(5.39) $\gamma = \pm \hat{\delta}_2(m) = \begin{bmatrix} 1 & o & m & -1 \\ o & -1 & 1 & o \\ o & o & 1 & o \\ o & o & o & -1 \end{bmatrix} \quad m \in \mathbb{Z} - \{o\}$

$d(\gamma) = 3$

$$x = \begin{cases} X_1(o,o,o) & \ldots \text{ m:arbitrary}, \quad i_o(x) = 1 \\ X_2(o,o) & \ldots \text{ m:arbitrary}, \quad i_o(x) = p+1 \\ X_2(o,2) & \ldots \ m \equiv o \pmod p, \quad i_o(x) = p \end{cases}$$

(5.40) $\gamma = \pm \mathcal{E}_1(S), \mathcal{E}_3(S) = \begin{bmatrix} 1 & o & s_1 & s_{12} \\ o & 1 & s_{12} & s_2 \\ o & o & 1 & o \\ o & o & o & 1 \end{bmatrix}$; $S = \begin{pmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{pmatrix}$ det $S \neq o$

$d(\gamma) = 2$

$$x = \begin{cases} X_1(o,o,o) & \ldots \text{ S:arbitrary}; \ L = SM_2(\mathbb{Z}), \ (P_M \cap \Gamma)_o \cong G\Gamma_o(p) \\ X_1(o,1,o) & \ldots \ s_1, s_{12} \equiv o \pmod p; \ L = \begin{pmatrix} p\mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap SM_2(\mathbb{Z}) \\ & \quad (P_M \cap \Gamma)_o \cong G\Gamma_o^*(p) \end{cases}$$

where

$$G\Gamma_o(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}); \quad c \equiv o \pmod p \right\}$$

$$G\Gamma_o^*(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}); \quad b \equiv o \pmod p \right\}.$$

Note that $G\Gamma_o(p)$, $G\Gamma_o^*(p)$ both have two cusps o, $i\infty$. The invariants described in (4.68) are given as follows:

(i) For $x = X_1(o,o,o) = 1_4$,

$\beta = i\infty$: $L_\beta = \begin{pmatrix} p\mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & 0 \end{pmatrix} \cap SM_2(\mathbb{Z})$, $B_\beta = \pm \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$; $(c,d) = (2,1)$

$\beta = o$: $L_\beta = \begin{pmatrix} 0 & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap SM_2(\mathbb{Z})$, $B_\beta = \pm \begin{pmatrix} 1 & 0 \\ p\mathbb{Z} & 1 \end{pmatrix}$; $(c,d) = (2p,1)$

(ii) for $x = X_1(o,1,o)$,

$$\beta = i\infty : L_\beta = \begin{pmatrix} p\mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & 0 \end{pmatrix} \cap SM_2(\mathbb{Z}),\ B_\beta = \pm \begin{pmatrix} 1 & p\mathbb{Z} \\ 0 & 1 \end{pmatrix};\ (c,d) = (2p^2,p)$$

$$\beta = o : L_\beta = \begin{pmatrix} 0 & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap SM_2(\mathbb{Z}),\ B_\beta = \pm \begin{pmatrix} 1 & 0 \\ \mathbb{Z} & 1 \end{pmatrix};\ (c,d) = (2p,p)$$

Remark: In the case (ii) above, the Levi-component P_M should be chosen carefully, so that (4.65) holds: namely

$$P_M = x^{-1} \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} ;\ A \in GL_2(\mathbb{Q}) \right\} x, \quad x = \begin{pmatrix} 1_2 & 1_2 \\ 0 & 1_2 \end{pmatrix} \cdot X_1(o,1,o) \quad .$$

(5.41) $\gamma = \pm\, \varepsilon_4(n) = \begin{pmatrix} 1 & o & o & o \\ o & 1 & o & n \\ o & o & 1 & o \\ o & o & o & 1 \end{pmatrix} \qquad n \in \mathbb{Z} - \{o\}$

$d(\gamma) = 3$

$$x = \begin{cases} X_1(o,o,o) & \ldots\ n:\text{arbitrary}, \quad i_o(x) = p(p+1) \\ X_2(o,o) & \ldots\ n:\text{arbitrary}, \quad i_o(x) = 1 \\ X_4 & \ldots\ n \equiv o \pmod p, \quad i_o(x) = p^3 \end{cases} \quad .$$

5-3. <u>Conjugacy Classes of</u> $B(p)$ (p = prime $\neq 2,3$)

We first recall, for the convenience of readers, that the coset space $Sp(2,\mathbb{Z})/\Gamma_o(p)$ has the following complete set of representatives:

$$Y_1(a,b) = \begin{pmatrix} a & -b & -1 & o \\ o & 1 & o & o \\ 1 & o & o & o \\ b & o & o & 1 \end{pmatrix} \quad \left(\text{or} \quad Y_1'(a,b) = \begin{pmatrix} -b & a & o & -1 \\ 1 & o & o & o \\ o & 1 & o & o \\ o & b & 1 & o \end{pmatrix}\right)$$

$$Y_2(a,b,c) = \begin{pmatrix} a & b & -1 & o \\ b & c & o & -1 \\ 1 & o & o & o \\ o & 1 & o & o \end{pmatrix} \qquad Y_3(a) = \begin{pmatrix} 1 & o & o & o \\ o & a & o & -1 \\ o & o & 1 & o \\ o & 1 & o & o \end{pmatrix}$$

$$Y_4 = 1_4 \qquad \left(\left[Sp(2,\mathbb{Z}) : \Gamma_o(p)\right] = (p+1)(p^2+1) \right)$$

where a,b,c runs over the set of integers modulo p. We shall make full use of the results of [16], § 7 where the decomposition of $Sp(2,\mathbb{Z})$-conjugacy classes into $\Gamma_o(p)$-conjugacy classes is described. In the following list, $d_y(\gamma)$ is the number of $B(p)$-conjugacy clsses contained in the set $\{y^{-1}\gamma\, y\}_{\Gamma_o(p)} \cap B(p)$, where y is as above; and $d(\gamma)$ denotes the sum of $d_y(\gamma)$.

(5.42) $\gamma = \pm\alpha_o$: $d(\gamma) = 1$, $x = Z_1(o)$ $(y = Y_4)$

$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = (p+1)^2(p^2+1)$

In the next two cases, θ denotes the integer mod. p, which generates $F_p^{\times}$.

(5.43) $\gamma = \alpha_1$ $\qquad A = \{\theta^j;\ 1 \leq j \leq \frac{p-1}{4},\ j \neq \frac{p-1}{8}\}/\sim$

$u \sim v \Longleftrightarrow (uv)^4 = 1$

y	p(mod 8)	$d_y(\alpha_1)$ ... x	order of centralizer in B(p)
$Y_2(i,o,i)$ and $Y_2(-i,o,-i)$ $i=\theta^{(p-1)/4}$	1	$(p-9)/8$ ... $Z_1(t), t \in A$	4
		1 ... $Z_1(\theta^{(p-1)/8})$	8
		1 ... $Z_1(1)$	8
		1 ... $Z_1(o)$	16
	5	$(p-5)/8$ ... $Z_1(t), t \in A$	4
		1 ... $Z_1(1)$	8
		1 ... $Z_1(o)$	16
	3,7	o	
$Y_2(i,o,-i)$	1,5	2 ... $Z_1(o)$, Z_2	16
	3,7	o	
$Y_2(a,b,-a)$ with $b \not\equiv o$ $a^2+b^2+1 \equiv o$	1	2 (a=o) $\quad Z_1(t)$:	8
		2 (a=1) $\quad (bt+a)^2+1 \equiv o$	8
		2 (otherwise; there are $\frac{p-9}{8}$ such pairs (a,b))	4
	5	2 (a=o) $\quad Z_1(t)$: $(bt+a)^2+1 \equiv o$	8
		2 (otherwise; there are $\frac{p-5}{8}$ such pairs (a,b))	4
	3,7	o	

From this table, we get (see also Remark (5.45)):

$$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = 2(p+1)(1+(\tfrac{-1}{p}))$$

(5.44) $\gamma = \alpha_2, \alpha_3$ $\qquad$ $B = \{\theta^j;\ 1 \le j \le \frac{p-1}{6},\ j \ne \frac{p-1}{12}\}/\sim$

$u \sim v \Longleftrightarrow (uv)^6 = 1$

y	p(mod 12)	$d_y(\alpha_2)$ ... x	order of centralizer in B(p)
$Y_2(\omega,0,\omega)$ and $Y_2(\bar{\omega},0,\bar{\omega})$, $\omega = \theta^{(p-1)/6}$	1	$(p-13)/12$ ... $Z_1(t)$, $t \in B$	6
		1 ... $Z_1(\theta^{(p-1)/12})$	12
		1 ... $Z_1(1)$	12
		1 ... $Z_1(0)$	36
	7	$(p-7)/12$ ... $Z_1(t)$, $t \in B$	6
		1 ... $Z_1(1)$	12
		1 ... $Z_1(0)$	36
	5,11	0	
$Y_2(\omega,0,\bar{\omega})$	1,7	2 ... $Z_1(0)$, Z_2	36
	5,11	0	
$Y_2(a,b,-1-a)$ with $b \not\equiv 0$, $a^2+a+1+b^2 \equiv 0$	1	2 (a=0) $\quad Z_1(t)$:	12
		2 (a=-2) $\quad bt^2+(2a+1)t-b \equiv 0$	12
		2 (otherwise; there are $\frac{p-13}{12}$ such pairs (a,b))	6
	7	2 (a=-2)	12
		2 (otherwise; there are $\frac{p-7}{12}$ such pairs (a,b))	6
	5,11	0	

From this table, we get (see also Remark (5.45)):

$$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = 2(p+1)\left(1+\left(\frac{-3}{p}\right)\right) \quad .$$

Remark 5.45. The above lists for $B(p)$-conjugacy classes belonging to α_1, α_2, and α_3 are obtained after somewhat complicated calculations. We gave these lists in order to make our description consistent. Indeed, it is worth noting that the relative Maß of α_1 (resp. α_2, α_3) coincides with that of γ_1, γ_2 (resp. γ_3), for which the calculation is quite easy. This fact can be proved without computing the former, by the method described in § 4-1 (see footnote to (5.3)). Of course similar observation can be made for the group $\Gamma_0(p)$. The complicated situation for $\gamma = \alpha_1$, α_2, and α_3 comes from the fact that the quaternion algebras $Z_0(\gamma)$ attached to their centralizers are definite, so that the class numbers $h(\Lambda;G)$ of their $\mathbb{Z}$-orders are big (c.f. Theorem (4.28)), while for $\gamma = \gamma_1, \gamma_2, \gamma_3$, $Z_0(\gamma)$ are indefinite, and we have $h(\Lambda;G) = 1$ by strong approximation theorem [32].

(5.46) $\gamma = \alpha_4, \alpha_5$

$$d_y(\gamma) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod 8 \\ o & \text{otherwise} \end{cases}$$

$$x = \begin{cases} Z_1(t) & \ldots\ t^2+a \equiv o \quad \text{for } y = Y_2(c,a,a):\ a^2+1 \equiv o \\ Z_1(t) & \ldots\ t^2+bt-1 \equiv o \quad \text{for } y = Y_2(-1,b,1): b^2+2 \equiv o \end{cases}$$

$$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = \begin{cases} 8 & \text{if } p \equiv 1 \pmod 8 \\ 1 & \text{otherwise .} \end{cases}$$

(5.47) $\gamma = \pm\alpha_6$

$$d_y(\gamma) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod 8 \\ o & \text{otherwise} \end{cases}$$

$$x = \begin{cases} Z_1(t) \ \dots \ t^2+t+\frac{a}{a+1} \equiv o \quad \text{for} \quad y = Y_2(2a+1,a,2a+1): \\ \qquad\qquad 3a^2+2a+1 \equiv o \\ Z_1(t) \ \dots \ t^2+(b+1)t+\frac{b-1}{2} \equiv o \quad \text{for} \quad y = Y_2(b,b,1): \\ \qquad\qquad b^2+1 \equiv o \end{cases}$$

$$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = \begin{cases} 8 & \text{if} \quad p \equiv 1 \pmod 8 \\ o & \text{otherwise} \end{cases}$$

(5.48) $\gamma = \pm\alpha_7$

$$d_y(\gamma) = 1 + (\tfrac{-3}{p})$$

$$x = \begin{cases} Z_1(1),\ Z_1(-1) \ \dots \qquad \text{for} \quad y = Y_1(o,b):\ b^2+b+1 \equiv o \\ Z_1(t) \ \dots \ t^2+t+1 \equiv o \quad \text{for} \quad y = Y_2(2a,a,2a):\ 3a^2+1 \equiv o \end{cases}$$

$$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = (1+(\tfrac{-3}{p}))^3$$

(5.49) $\gamma = \pm\alpha_8$

$$d_y(\gamma) = 1 + (\tfrac{-3}{p})$$

$$x = Z_1(o),\ Z_2 \quad \text{for} \quad y = Y_2(a,o,c):\ a^2+a+1 \equiv c^2+c+1 \equiv o$$

$$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = (1+(\tfrac{-3}{p}))^3$$

(5.50) $\gamma = \alpha_9, \alpha_{10}$

$$d_y(\gamma) = 1 + (\tfrac{-3}{p})$$

$$x = \begin{cases} Z_1(t) \ \dots \ at^2-1 \equiv o \quad \text{for} \quad y = Y_2(a,o,o):\ a^2+a+1 \equiv o \\ Z_1(t) \ \dots \ t \equiv (-b+1)/2 \quad \text{for} \quad y = Y_2(-2,b,1): b^2+3 \equiv o \end{cases}$$

$$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = (1+(\tfrac{-3}{p}))^3$$

(5.51) $\gamma = \alpha_{11}, \alpha_{12}$

$$d_y(\gamma) = 1 + \left(\frac{-3}{p}\right)$$

$$x = Z_1(o),\ Z_2 \quad \text{for} \quad y = Y_2(a,o,c):\ a^2+a+1 \equiv c^2+c+1 \equiv o$$

$$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = \left(1+\left(\frac{-3}{p}\right)\right)^3$$

(5.52) $\gamma = \alpha_{13}, \alpha_{14}$

$$d_y(\gamma) = \begin{cases} 2 & \text{if} \quad p \equiv 1 \pmod{12} \\ o & \text{otherwise} \end{cases}$$

$$x = \begin{cases} Z_1(t) \ \ldots \ t^2 \equiv \dfrac{1}{a+1} & \text{for} \quad y = Y_2(a,o,a):\ a^2+a+1 \equiv o \\ Z_1(t) \ \ldots \ t^2+bt-1 \equiv o & \text{for} \quad y = Y_2(-1,b,o):\ b^2+1 \equiv o \end{cases}$$

$$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = 2\left(1+\left(\frac{-1}{p}\right)\right)\left(1+\left(\frac{-3}{p}\right)\right)$$

(5.53) $\gamma = \pm\alpha_{15}, \ldots, \alpha_{18}$

$$d_y(\gamma) = \begin{cases} 2 & \text{if} \quad p \equiv 1 \pmod 5, \text{or} \quad p = 5 \\ o & \text{otherwise} \end{cases}$$

$$x = Z_1(t) \ \ldots \ t^2 + \frac{a}{a+1}\, t + a \equiv o \quad \text{for} \quad y = Y_2(a,b,c):$$

$$a^4+a^3+a^2+a+1 \equiv o,\ b \equiv \frac{-1}{a+1},\ c \equiv \frac{-1}{a}$$

$$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = \begin{cases} 8 & \text{if} \quad p \equiv 1 \pmod 5 \\ 1 & \text{if} \quad p = 5 \\ o & \text{otherwise} \end{cases}$$

(5.54) $\gamma = \pm\alpha_{19}, \ldots, \alpha_{22}$

$$d_y(\gamma) = \begin{cases} 2 & \text{if } p \equiv 1 \pmod{12} \\ 0 & \text{otherwise} \end{cases}$$

$x = Z_1(0),\ Z_2$ for $y = Y_2(a,0,c)$: $a^2+a+1 \equiv c^2+1 \equiv 0$

$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = 2(1+(\frac{-1}{p}))(1+(\frac{-3}{p}))$

(5.55) $\gamma = \pm\beta_1, \ldots, \beta_4$

$d_y(\gamma) = 1 + (\frac{-3}{p})$

$x = Z_1(0),\ Z_2$ for $y = Y_1(a,0)$: $a^2+a+1 \equiv 0$

$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = 2(p+1)(1+(\frac{-3}{p}))$

(5.56) $\gamma = \pm\beta_5,\ \beta_6$

$d_y(\gamma) = 1 + (\frac{-1}{p})$

$x = Z_1(0),\ Z_2$ for $y = Y_1(a,0)$: $a^2+1 \equiv 0$

$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = 2(p+1)(1+(\frac{-1}{p}))$

(5.57) $\gamma = \gamma_1$ (resp. γ_2)

$$d_y(\gamma) = \begin{cases} 1+(\frac{-1}{p}) & \text{for } y = Y_4 \\ 1 & \text{for } y = Y_1'(0,b) \text{ (resp. } Y_1'(b,b)\text{)}: b^2+1 \equiv 0 \end{cases}$$

$$x = \begin{cases} Z_1(t) \ \ldots\ t^2+1 \equiv 0 & \text{for } y = Y_4 \\ Z_1(0) & \text{for } y = Y_1'(0,b) \text{ (resp. } Y_1'(b,b)\text{)} \end{cases}$$

$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = 2(p+1)(1+(\frac{-1}{p}))$

(5.58) $\gamma = \pm\gamma_3$

$$d_y(\gamma) = \begin{cases} 1+(\frac{-3}{p}) & \text{for } y = Y_4 \\ 1 & \text{for } y = Y_1'(o,b): \quad b^2+b+1 \equiv o \end{cases}$$

$$x = \begin{cases} Z_1(t) \ \ldots \ t^2-t+1 \equiv o & \text{for } y = Y_4 \\ Z_1(o) \ \ldots & \text{for } y = Y_1'(o,b) \end{cases}$$

$$m(\gamma;Sp(2,\mathbb{Z})/B(p)) = 2(p+1)(1+(\tfrac{-3}{p}))$$

(5.59) $\gamma = \delta_1, \delta_2$

$$d_y(\gamma) = 2 \quad \text{for } y = Y_4$$

$$x = Z_1(o), Z_2$$

$$m(\gamma;Sp(2,\mathbb{Z})/B(p)) = 2(p+1)^2$$

P-unipotent classes

The group $B(p)$ has four point cusps and four one-dimensional cusps; the parabolic subgroups corresponding to these cusps are given as follows:

Point cusps: $\quad P_o{}^j = x_j^{-1}\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} x_j$,

$x_1 = Y_4$, $x_2 = Y_2(o,o,o)$, $x_3 = Y_1(o,o)$, and

$x_4 = Y_3(o)\cdot Z_2$

One dimensional cusps : $\quad P_1{}^j = x_j^{-1}\begin{pmatrix} * & o & * & * \\ * & * & * & * \\ * & o & * & * \\ o & o & o & * \end{pmatrix} x_j$,

$x_1 = Y_4$, $x_2 = Y_2(o,o,o)$, $x_3 = Z_2$, and

$x_4 = Y_2(o,o,o)\cdot Z_2$

We put, for each element γ' of $B(p)$,

$$i_o = i_o(\gamma') = \left[C_o(\gamma'; Sp(2,\mathbb{Z})) : C_o(\gamma'; B(p)) \right] .$$

(5.60) $\gamma = \pm \hat{B}_1(n), \hat{B}_2(n)$

$d_y(\gamma) = 1 + (\frac{-3}{p})$; $i_o = 1$

$x = Z_1(o),\ Z_2$ for $y = Y_1(c,o),\ Y_2(a,o,o)$:

$a^2 - a + 1 \equiv c^2 - c + 1 \equiv o$

$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = 2(1 + (\frac{-3}{p}))^2$

(5.61) $\gamma = \pm \hat{B}_3(n), \hat{B}_4(n)$

$d_y(\gamma) = 1 + (\frac{-3}{p})$; $i_o = 1$

$x = Z_1(o),\ Z_2$ for $y = Y_1(c,o),\ Y_2(a,o,o)$:

$a^2 - a + 1 \equiv c^2 - c + 1 \equiv o$

$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = 2(1 + (\frac{-3}{p}))^2$

(5.62) $\gamma = \pm \hat{B}_5(n), \hat{B}_6(n)$

$d_y(\gamma) = 1 + (\frac{-3}{p})$; $i_o = 1$

$$x = \begin{cases} Z_1(\frac{c+1}{3}),\ Z_2 & \text{for } y = Y_1(c,o):\ c^2 - c + 1 \equiv o \\ Z_1(o),\ Z_1(-3) & \text{for } y = Y_2(3a-1,a,o):\ 3a^2 - 3a + 1 \equiv o \end{cases}$$

$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = 2(1 + (\frac{-3}{p}))^2$

(5.63) $\gamma = \pm \hat{B}_7(n), \hat{B}_8(n)$

$d_y(\gamma) = 1 + (\frac{-1}{p})$; $i_o = 1$

$$x = Z_1(o),\ Z_2 \quad \text{for} \quad y = Y_1(c,o),\ Y_2(a,o,o): \quad c^2+1 \equiv a^2+1 \equiv o$$

$$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = 2(1+(\tfrac{-1}{p}))^2$$

(5.64) $\gamma = \pm\, \hat{\beta}_9(n), \hat{\beta}_{10}(n)$

$$d_y(\gamma) = 1+(\tfrac{-1}{p}) \quad ; \quad i_o = 1$$

$$x = \begin{cases} Z_1(\frac{c+1}{2}),\ Z_2 & \text{for} \quad y = Y_1(c,o): \quad c^2+1 \equiv o \\ Z_1(o),\ Z_1(2) & \text{for} \quad y = Y_2(a,o,o): \quad a^2+1 \equiv o \end{cases}$$

$$m(\gamma; Sp(2,\mathbb{Z})/B(p)) = 2(1+(\tfrac{-1}{p}))^2$$

(5.65) $\gamma = \hat{\gamma}_1(n)$

$$d_y(\gamma) = 1+(\tfrac{-1}{p})$$

$$x = \begin{cases} Z_1(t) \quad \text{with} \quad t^2+1 \equiv o \quad \text{for } y = Y_4, Y_2(o,o,o)\ ;\ i_o = 1 \\ \qquad\qquad\qquad n\text{:arbitrary,} \\ Z_1(o) \quad \text{for} \quad y = Y_1'(o,b): b^2+1 \equiv o,\ i_o = 1,\ n\text{:arbitrary} \\ Z_1(1) \quad \text{for} \quad y = Y_1'(o,b):\ i_o = p,\ n \equiv o \pmod p \end{cases}$$

(5.66) $\gamma = \hat{\gamma}_2(n)$

$$d_y(\gamma) = 1+(\tfrac{-1}{p})$$

$$x = \begin{cases} Z_1(t) \quad \text{with} \quad t^2+1 \equiv o \quad \text{for} \quad y = Y_4,\ Y_2(o,o,1/2);\ i_o = 1 \\ \qquad\qquad\qquad n\text{:arbitrary} \\ Z_1(o) \quad \text{for} \quad y = Y_1'(-1/2,b)\ : b^2+1 \equiv o,\ i_o = 1 \\ \qquad\qquad\qquad n\text{: arbitrary} \\ Z_1(1) \quad \text{for} \quad y = Y_1'(-1/2,b)\ :\ i_o = p,\ 2n+1 \equiv o \pmod p \end{cases}$$

(5.67) $\gamma = \hat{\gamma}_3(n)$

$d_y(\gamma) = 1+(\frac{-1}{p})$

$$x = \begin{cases} Z_1(t) \text{ with } t^2+1 \equiv o \text{ for } y = Y_4, Y_2(o,1/2,o),\ i_o=1 \\ \qquad\qquad n\text{:arbitrary} \\ Z_1(o) \text{ for } y = Y_1'(b,b): b^2+1 \equiv o,\ i_o = 1 \\ \qquad\qquad n\text{:arbitrary} \\ Z_1(1) \text{ for } y = Y_1'(b,b):\ i_o = p,\ n \equiv o \pmod p \end{cases}$$

(5.68) $\gamma = \hat{\gamma}_4(n)$

$d_y(\gamma) = 1+(\frac{-1}{p})$

$$x = \begin{cases} Z_1(t) \text{ with } t^2+1 \equiv o \text{ for } y = Y_4,\ Y_2(o,1/2,1/2), i_o=1 \\ Z_1(o) \text{ for } y = Y_1'(b-1/2,b): b^2+1 \equiv o,\ i_o = 1 \\ \qquad\qquad n\text{:arbitrary} \\ Z_1(1) \text{ for } y = Y_1'(b-1/2,b),\ i_o = p,\ 2n+1 \equiv o \pmod p \end{cases}$$

(5.69) $\gamma = \pm\hat{\gamma}_5(n)$

$d_y(\gamma) = 1+(\frac{-3}{p})$

$$x = \begin{cases} Z_1(t) \text{ with } t^2-t+1 \equiv o \text{ for } y = Y_4,\ i_o = 1 \\ \qquad\qquad n\text{:arbitrary} \\ Z_1(t) \text{ with } t^2+t+1 \equiv o \text{ for } y = Y_2(o,o,n),\ i_o = 1 \\ \qquad\qquad n\text{:arbitrary} \\ Z_1(o) \text{ for } y = Y_1'(o,b): b^2+b+1 \equiv o,\ i_o= 1,\ n\text{:arbitrary} \\ Z_1(1) \text{ for } y = Y_1'(o,b),\ i_o= p,\ n \equiv o \pmod p \end{cases}$$

(5.70) $\gamma = \pm \hat{\gamma}_6(n)$

$d_y(\gamma) = 1+\left(\frac{-3}{p}\right)$

$$x = \begin{cases} Z_1(t) \text{ with } t^2-t+1 \equiv o \text{ for } y = Y_4,\ i_o = 1 \\ \qquad n\text{:arbitrary} \\ Z_1(t) \text{ with } t^2+t+1 \equiv o \text{ for } y = Y_2(-1/3,o,1/3), \\ \qquad i_o = 1,\ n\text{:arbitrary} \\ Z_1(o) \text{ for } y = Y_1'\left(\frac{-b}{2b+1},b\right) : b^2+b+1 \equiv o,\ i_o = 1, \\ \qquad n\text{:arbitrary} \\ Z_1(1) \text{ for } y = Y_1'\left(\frac{-b}{2b+1},b\right),\ i_o = p,\ 3n \equiv \frac{2b}{2b+1} \pmod p \end{cases}$$

(5.71) $\gamma = \pm \hat{\gamma}_7(n)$

$d_y(\gamma) = 1+\left(\frac{-3}{p}\right)$

$$x = \begin{cases} Z_1(t) \text{ with } t^2-t+1 \equiv o \text{ for } y = Y_4,\ i_o = 1 \\ \qquad n\text{:arbitrary} \\ Z_1(t) \text{ with } t^2+t+1 \equiv o \text{ for } y = Y_2(-2/3,o,2/3), \\ \qquad i_o = 1,\ n\text{:arbitrary} \\ Z_1(o) \text{ for } y = Y_1'\left(\frac{-2b}{2b+1},\ b\right) : b^2+b+1 \equiv o,\ i_o = 1, \\ \qquad n\text{:arbitrary} \\ Z_1(1) \text{ for } y = Y_1'\left(\frac{-2b}{2b+1},\ b\right),\ i_o = 1,\ 3n \equiv \frac{4b}{2b+1} \pmod p \end{cases}$$

(5.72) $\gamma = \hat{\hat{\delta}}_1(m,n)$ (resp. $\hat{\hat{\delta}}_2(m,n)$)

$d_y(\gamma) = 2$ for $y = Y_4,\ Y_2(o,o,o),\ Y_3(o),\ Y_1'(o,o)$

(resp. $Y_4,\ Y_2(o,1/2,o),\ Y_3(o),\ Y_1'(o,o)$)

$x = Z_1(o),\ Z_2$ (resp. $Z_1(o),\ Z_1(2)$) $\quad i_o = 1$

condition on (m,n):

Y_4	$Y_2(o,o,o)$, $Y_2(o,\frac{1}{2},o)$	$Y_3(o)$	$Y_1'(o,o)$
arbitrary	$m \equiv n \equiv o \pmod p$	$n \equiv o$	$m \equiv o$

(5.73) $\gamma = \hat{\hat{\delta}}_3(m,n)$ (resp. $\hat{\hat{\delta}}_4(m,n)$)

$d_y(\gamma) = 2$ for $y = Y_4, Y_2(2,o,o), Y_1'(o,o), Y_1'(o,-2)$
(resp. $Y_4, Y_2(1/2,o,o), Y_1'(o,o), Y_1'(o,-2)$)

$x = Z_1(o), Z_1(-1)$ (resp. $Z_1(o), Z_1(-4)$), $i_o = 1$

condition on (m,n):

Y_4	$Y_2(2,o,o), Y_2(\frac{1}{2},o,o)$	$Y_1'(o,o)$	$Y_1'(o,-2)$
arbitrary	$m \equiv n \equiv o \pmod p$	$m \equiv o$ resp. $2m \equiv 1$	$m \equiv 2n$ resp. $2m-4n \equiv 1$

(5.74) $\gamma = \pm\hat{\delta}_1(n)$ (resp. $\hat{\delta}_2(n)$)

$d_y(\gamma) = 2$ for $y = Y_4, Y_2(o,o,o)$ (resp. $Y_4, Y_1'(o,o)$)

$$x = \begin{cases} Z_1(o),\ Z_2 & \text{for } Y_4,\ Y_2(o,o,o),\ i_o = 1 \\ Z_1(o),\ Z_1(2) & \text{for } Y_1'(o,o),\ i_o = 1 \end{cases}$$

Y_4	$Y_2(o,o,o), Y_1'(o,o)$
n:arbitrary	$n \equiv o \pmod p$

(5.75) $\gamma = \pm\,\varepsilon_1(S), \varepsilon_3(S)$

$$d_y(\gamma) = \begin{cases} 1 & \text{for } y = Y_4,\ Y_2(o,o) \\ 2 & \text{for } y = Y_3(o) \end{cases}$$

$$x = \begin{cases} Z_1(o) & \text{for } Y_4,\ Y_2(o,o) \\ Z_1(o),\ Z_2 & \text{for } Y_3(o) \end{cases}$$

y x	Y_4 $Z_1(o)$	$Y_2(o,o)$ $Z_1(o)$	$Y_3(o)$ $Z_1(o)$	$Y_3(o)$ $Z_1(o)$
L	$\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ & \mathbb{Z} \end{pmatrix}$	$\begin{pmatrix} p\mathbb{Z} & p\mathbb{Z} \\ & p\mathbb{Z} \end{pmatrix}$	$\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ & p\mathbb{Z} \end{pmatrix}$	$\begin{pmatrix} \mathbb{Z} & p\mathbb{Z} \\ & p\mathbb{Z} \end{pmatrix}$
i_o	1	p^3	p	p^2

For each family, the corresponding Levi-component is isomorphic to

$$P_M \cap B(p) \cong G\Gamma_o(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) ;\ c \equiv o \pmod p \right\}.$$

Therefore we have the following invariants (c,d) for each cusp of $G\Gamma'_o(p)$ (c.f. Theorem P-6, (4.68)).

L	$\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ & \mathbb{Z} \end{pmatrix}$	$\begin{pmatrix} p\mathbb{Z} & p\mathbb{Z} \\ & p\mathbb{Z} \end{pmatrix}$	$\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ & p\mathbb{Z} \end{pmatrix}$	$\begin{pmatrix} \mathbb{Z} & p\mathbb{Z} \\ & p\mathbb{Z} \end{pmatrix}$
$\beta = i\infty$ $B_\beta = \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$	$t_1 = 1$ $t_2 = o$ $d = 1$ $c = 2$	p o p 2	1 o 1 2	1 o p 2p
$\beta = o$ $B_\beta = \begin{pmatrix} 1 & 0 \\ p\mathbb{Z} & 1 \end{pmatrix}$	$t_1 = 1$ $t_2 = o$ $d = 1$ $c = 2p$	p o p 2p	p o 1 2	p o p 2p

(5.76) $\gamma = \pm \varepsilon_4(m)$

$d_y(\gamma) = 2$ for $y = Y_4,\ Y_2(o,o,o)$

$x = Z_1(o),\ Z_2$

	Y_4	$Y_2(o,o,o)$
$Z_1(o)$	$i_o = p$ m:arbitrary	$i_o = 1$ $m \equiv o \pmod p$
Z_2	$i_o = 1$ m:arbitrary	$i_o = p$ $m \equiv o \pmod p$

§6. Local data for $B'(p)$.

In this section, we shall give local data which are necessary to calculate $\dim M_\nu(B'(p))$. The local data for $q = p$ have been given in 19, so we shall calculate $c_p(g, R_p, \Lambda_p)$ and masses, where g is a torsion element of G_p^* and

$$R_p = \begin{pmatrix} O_p & O_p \\ \pi O_p & O_p \end{pmatrix}.$$

Throughout this section, we assume that $p \neq 2, 3$.

Proposition 6.1

Put $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Then,

$$c_p(g, R_p, \Lambda_p) = \begin{cases} 1 \ \dots \text{ if } \Lambda_p \sim R_p, \\ 0 \ \dots \text{ otherwise.} \end{cases}$$

Let Λ be the order of $M_2(O)$ such that $\Lambda_p = R_p$ and $\Lambda_q = M_2(O_q)$ $(q \neq p)$. Then,

$$M_G(\Lambda) = (p^4-1)/2^7 3^2 5.$$

Proof. This is obvious, because $[P_1' : B'] = p+1$. (cf. 19 (I) Prop.9)

Proposition 6.2 If the principal polynomial of $g \in G_p^*$ is $f_i(x)$ or $f_i(-x)$ for some $i = 2, 3, 4, 5, 8,$ or 9, then $c_p(g, R_p, \Lambda_p) = 0$ for all orders of $Z(g)_p$.

Proof. We assumed $p \nmid 2, 3$, so it is known that

$$c_p(g, \begin{pmatrix} o_p & \pi^{-1}o_p \\ \pi o_p & o_p \end{pmatrix}, \Lambda_p) = 0 \text{ for any above } g \text{ and any } \Lambda_p.$$

([19](III)). Thus, our Prop. is obvious. q.e.d.

Next, we treat elements $g \in G_p^*$ such that $f_6(g) = 0$ or $f_7(\pm g) = 0$. Put $Z_0(g)_p = Z_o(g) \underset{Q}{\otimes} Q_p$, where $Z_o(g)$ is the quaternion algebra over Q defined as in [19] I(12)(p.562). For $\Lambda_p \subset Z(g)_p$, we define $d_p(\Lambda_p)$ and $e_p(\Lambda_p)$ as in [19](I) Prop.12(p.572). Put $F = Q[g]$ and $o = Z[g]$. As we have assumed $p \nmid 2,3$, $(\frac{F}{p}) \neq 0$, where $(\frac{F}{p})$ is the Legendre symbol. By definition, we have $Z_o(g)_p \underset{Q}{\otimes} F \cong D_p \underset{Q}{\otimes} F$, so $(\frac{F}{p}) = -1$, if $Z_o(g)_p$ is not division.

<u>Proposition</u> 6.3. <u>If</u> $Z_o(g)_p$ <u>is</u> <u>split</u> <u>and</u> $(\frac{F}{p}) = -1$, <u>we</u> <u>get</u>

$$c_p(g, R_p, \Lambda_p) = \begin{cases} 2 \ \ldots \text{ if } \Lambda_p \sim \begin{pmatrix} o_p & o_p \\ po_p & o_p \end{pmatrix} = \Lambda, \\ o \ \ldots \ \underline{\text{otherwise}}, \end{cases}$$

<u>where</u> $o_p = o \underset{Z}{\otimes} Z_p$. <u>We</u> <u>have</u> $e_p(\Lambda) = 1$ <u>and</u> $d_p(\Lambda) = p+1$.

Proof. If $Z_o(g)_p$ is split and $(\frac{F}{p}) = -1$, then g is G_p^*-conjugate to $\begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$, where $\omega \in O_p$ is of order 3, 4, or 6. So, we put $g = \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}$. By virtueof [19](III) Prop.2.5(i), if $x^{-1}gx \in B' \subset P_0'$, then $x \in Z_{G_p^*}(g)\cdot P_0'$. We can put $O_p = Z_p + Z_p\varepsilon + Z_p\pi + Z_p\pi\varepsilon$, where $F_p = F \otimes_Q Q_p = Q_p$, $\varepsilon^2 \in Q_p^\times$, $\pi^2 \in Q_p^\times$, and $\varepsilon\pi = -\pi\varepsilon$. Then,

$$P_0' = \coprod_a \begin{pmatrix} 1 & \pi^{-1}a \\ 0 & 1 \end{pmatrix} B' \amalg \begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix} B', \qquad \text{where}$$

$a \in Z_p[\varepsilon]$ runs through a set of complete representatives of $Z_p[\varepsilon]/pZ_p[\varepsilon]$. If $x \in Z_{G_p^*}(g)\begin{pmatrix} 1 & \pi^{-1}a \\ 0 & 1 \end{pmatrix}B'$, then

$$\begin{pmatrix} 1 & -\pi^{-1}a \\ 0 & 1 \end{pmatrix} g \begin{pmatrix} 1 & \pi^{-1}a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \omega, & (\omega-\bar{\omega})\pi^{-1}a \\ 0 & \omega \end{pmatrix} \in B',$$

so $a \in po_p$. Thus, we get $x \in Z_{G_p^*}(g)B'$ in this case.

For $x = \begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}$, we get $x^{-1}gx = {}^t g \in B'$. Now, assume that $\begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix} = hk$, where $h \in Z_{G_p^*}(g)$ and $k \in B'$.

Then, $h \in Z_{G_p^*}(g) \cap P_0' \subset \begin{pmatrix} o_p & o_p \\ po_p & o_p \end{pmatrix}$, which is a contradiction, because $h \in B'$. Thus, we get $c_p = 2$. We have

$\Lambda_0 = \begin{pmatrix} Z_p & \varepsilon Z_p \\ p\varepsilon Z_p & Z_p \end{pmatrix}$ and we get $d_p(\Lambda) = p+1$. q.e.d.

<u>Proposition</u> 6.4 <u>If</u> $Z_o(g)$ <u>is division, we get</u>

(i) if $(\frac{F}{p}) = 1$, <u>then</u>

$$c_p(g, R_p, \Lambda_p) = \begin{cases} 2 & \ldots \text{ if } \Lambda_p \sim Z(g)_p \cap R_p = \Lambda, \\ 0 & \ldots \text{ otherwise} \end{cases}$$

<u>and</u> $d_p(\Lambda) = e_p(\Lambda) = 1$,

(ii) <u>if</u> $(\frac{F}{p}) = -1$, <u>then</u>

$c_p(g, R_p, \Lambda_p) = 0$ <u>for any</u> Λ_p.

<u>Proof.</u> By virtue of Prop.2.6(ii) in [19](III)(p.398), the above (ii) is obvious. So, assume that $(\frac{F}{p}) = 1$. We can assume that $g = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G_p^*$, where $a, b \in Q_p$ are different roots of $f_6(x) = 0$, $f_7(x) = 0$, or $f_7(-x) = 0$. If $x^{-1}gx \in B'$, then $x \in Z_{G_p^*}(g)P'$ by virtue of [19](III)Prop.2.6.(i). In the similar way as in the proof of Prop.6.3, we can show that $x \in Z_{G_p^*}(g)B'$ or $Z_{G_p^*}(g)\begin{pmatrix} 0 & -\pi^{-1} \\ \pi & 0 \end{pmatrix}B'$, and these two double cosets are disjoint. q.e.d.

For $g \in G_p^*$ such that $f_i(\pm g) = 0$ ($i = 10, 11$, or 12), it is obvious that $c_p(g, R_p, \Lambda_p) = 0$, unless $\Lambda_p = Z_p[g]$. From now on, we put $c_p(g) = c_p(g, R_p, Z_p[g])$. For a fixed $i = 10, 11$, or 12, denote by t the number of G_p^*-conjugacy classes in $\{ g \in G_p^*;\ f_i(g) = o \}$.

<u>Proposition</u> 6.5. <u>Let</u> $g \in G_p^*$ <u>be of order</u> 5 <u>or</u> 1o.

(i) <u>If</u> $p = 5$, <u>then</u> $c_p(g) = 1$, <u>and</u> $t = 1$ <u>for</u> $i = 10$.

(ii) <u>If</u> $p \neq 5$, <u>then</u> $c_p(g) = 0$.

<u>Proof</u>. By virtue of (I)Prop.19(ii) and (III)Prop.2.8(ii), the above (ii) is obvious. Assume $p = 5$. Then, we can put $g = \begin{pmatrix} o & -\varepsilon \\ \varepsilon^{-1} & \omega \end{pmatrix}$, where ω is an element of O_p such that $\omega^2 - \omega - 1 = 0$, and $\varepsilon \in O_p$, $\varepsilon^2 = -3$, $\varepsilon\omega = \bar{\omega}\varepsilon$. If $x^{-1}gx \in B'$, then $x \in Q_p(g)P_1'$, by virtue of [19] Prop.19(iv). It is easy to see $x \in Q_p(g)\begin{pmatrix} 1 & o \\ -2\varepsilon & 1 \end{pmatrix}B'$. q.e.d.

<u>Proposition</u> 6.6. <u>Let</u> g <u>be of order</u> 8. <u>Then</u>,

(i) <u>if</u> $p \equiv \pm 1$ mod. 8, <u>then</u> $c_p(g) = 0$,

(ii) <u>if</u> $p \equiv 3$, <u>or</u> 5 mod. 8, <u>then</u> $c_p(g) = 4$ <u>and</u> $t = 1$.

<u>Proposition</u> 6.7. <u>Let</u> g <u>be an element of</u> G_p^* <u>such that</u> $f_{12}(g) = 0$. <u>Then</u>,

(i) <u>if</u> $p \equiv \pm 1$ mod.12, <u>then</u> $c_p(g) = 0$,

(ii) <u>if</u> $p \equiv 5$ mod.12, <u>then</u> $c_p(g) = 4$, $t = 1$, <u>and</u>

$Z_o(g^2)_p$ = <u>split</u>,

(iii) <u>if</u> $p \equiv 7$ mod.12, <u>then</u> $c_p(g) = 4$, $t = 1$, <u>and</u>

$Z_o(g^2)_p$ = <u>division</u>.

Proof of Prop.6.6 and 6.7. By virtue of [19](I)Prop.20, 21, and (III) Prop.2.9, 2.1o, the above (i) of Prop.6.6 and 6.7 are obvious. If $p \equiv 3, 5 \bmod. 8$ (resp. $p \equiv 5, 7 \bmod. 12$), we can write $f_{11}(x)$ (resp. $f_{12}(x)$) as a product of quadratic polynomials in $Q_p[x]$:

$f_i(x) = (x^2+ax+b)(x^2+ab^{-1}x+b^{-1})$ (i=11 or 12),

where $b \neq 1$. We can take $\omega \in O_p$ such that $\omega^2+a\omega+b = 0$. Put $\omega_1 = \omega$ and $\omega_2 = b^{-1}\omega$. Then, g is G_p^*-conjugate to $\begin{pmatrix} \omega_1 & o \\ o & \omega_2 \end{pmatrix}$. If $x^{-1}gx \in B'$ for some $x \in G_p^*$, then

$x \in Z_{G_p^*}(g)P_1'$ or $Z_{G_p^*}(g)\begin{pmatrix} \pi & o \\ o & \pi \end{pmatrix}P_1'$, by virtue of [19](I)Prop.21.

We have $P_1' = \coprod_a \begin{pmatrix} 1 & o \\ a & 1 \end{pmatrix}B' \amalg \begin{pmatrix} o & 1 \\ 1 & o \end{pmatrix}B'$, where a runs through a set of representatives of

$\{x \in O_p;\ tr(x) = 0\} \ / \ \{x \in \pi O_p;\ tr(x) = 0\}$.

In the similar way as in the proof of Prop.6.4(ii), we have $x \in Z_{G_p^*}(g)y_iB'$ (i = 1,2,3,or 4), where

$y_1 = \begin{pmatrix} 1 & o \\ o & 1 \end{pmatrix}$, $y_2 = \begin{pmatrix} \pi & o \\ o & \pi \end{pmatrix}$, $y_3 = \begin{pmatrix} o & 1 \\ 1 & 0 \end{pmatrix}$, and $y_4 = \begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix}$.

These four double cosets are disjoint. q.e.d.

§7. Related topics

Here, we would like to take this opportunity to write briefly on some related topics.

(1) <u>Ihara lifting</u> For $n = 2$, Ihara[28] has shown that there exists a kind of lifting of automorphic forms of $S_{2\nu+4}(\Gamma_0(p))$ to $M_\nu(U_1'(p))$, where

$$\Gamma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z);\ c \equiv 0 \bmod. p \right\} .$$

(Actually, he did not assume that the discriminant of D is a prime. As for this , see his paper.)

More exactly, we can take the representation space V_ν of ρ_ν (for $n = 2$) as follows: We identify $\mathbb{H}$ (the Hamilton quaternions) with R^4. V_ν is the set of real valued homogeneous polynomials $f(x,y)$ on $\mathbb{H}^2 \cong R^8$ such that

1) $f(ax,ay) = N(a)f(x,y)$ for all $a \in \mathbb{H}^\times$, and

2) $\Delta f = 0$,

where $N(a)$ is the reduced norm of a and Δ is the usual Laplacian. $Sp(2)$ acts on V_ν by

$$f(x,y) \longrightarrow f((x,y)g) \quad \text{for all } g \in Sp(2).$$

For the sake of simplicity, we assume here that the class number of $U_1(p)$ is one, i.e. $\dim M_\nu(U_1(p)) = 1$ for $\nu = 0$, although, as Ihara has kindly shown us, his theory works completely in the same way without any such restriction.

Put $\Gamma = U_1(p) \cap G'$. Then, under the above assumption, we get

$$M_\nu(U_1'(p)) = \left\{ f \in V \ ;\ f((x,y)\gamma) = f(x,y) \text{ for all } \gamma \in \Gamma \right\} .$$

Let $f \in M_\nu(U_1'(p))$ be a common eigen form of all the Hecke operators $T(m)$. For such f, put

$$\vartheta_f(\tau) = \sum_{(x,y)\in O^2} f(x,y)e^{2\pi i(N(x)+N(y))\tau}, \quad \tau \in H_1.$$

Then, $\vartheta_f(\tau) \in S_{2\nu+4}(\Gamma_0(p))$.

Theorem 7.1 (Ihara[29]) Assume that $f(1,0) \neq 0$. Then, ϑ_f is also a common eigen form (of the Hecke operators of $\Gamma_0(p)$), and we get

$$L(s,f) = \zeta(s-\nu-1)\,\zeta(s-\nu-2)L(s,\vartheta_f)$$

up to Euler p-factors.

This Ihara's result was the first one among results on lifting obtained later by many mathematitians. For example, the Saito-Kurokawa lifting may be regarded as a similar version of Ihara lifting for the split group $Sp(2,\mathbf{R})$. The second named author has extended Th.7.1 to general n: under similar condition on f as $f(1,0) \neq 0$ for $n = 2$, he expressed the eigen values of f by some group theoretical numbers and coefficients of some one dimensional automorphic forms, and at least for $n = 3$, gave $L(s,f)$ explicitly.

Some examples of Th.7.1 have been given by Ihara(loc.cit.). We give here another example. We assume $n = 2$. Put $p = 2$ and $\nu = 2$. Then, $\dim M_\nu(U_1'(2)) = 1$ and this space is spanned by:

$$f(x,y) = N(x)^2 - 3N(x)N(y) + N(y)^2.$$

Then, by Th.7.1, we have

$$L(s,f) = \zeta(s-5)\,\zeta(s-6)L(s,h),$$

up to Euler 2 factors, where h is the unique normalized cusp form of $S_8(\Gamma_0(2))$. On the other hand, Maaß[38] has shown that

$$L(s,F) = \zeta(s-5)\,\zeta(s-6)L(s,h),$$

for some $F \in S_5(\Gamma_e(1))$ (unique up to constant), where $\Gamma_e(1)$ is the unique index two subgroup of $Sp(2,Z)$ which contains the level two principal congruence subgroup. So, (f,F) gives an example for Langlands philosophy. But, this example is less essential than the examples in[21], because this is a relation through one dimensional forms and does not satisfy the Ramanujan Conjecture. In our set up in Conjecture 1.11, there is no relation, at least apparently, between old forms and those forms obtained from lifting. As for another aspect between lifting and old forms, see[24].

(2) <u>Construction of automorphic forms</u> For $n = 1$, it is well known that we can construct the forms in $S_k(\Gamma_0(p))$ from $M_{k-2}(O_A)$ through the Weil representation: We can embed $Sp(1) \cong SU(2)$ to $SO(4)$, and roughly speaking, we can get forms in $S_k(\Gamma_0(p))$ through theta functions

$$\vartheta(f) = \sum_{n \in Z^4} P(n)e^{2\pi i Q(n)},$$

where Q are quadratic forms of four variables and P are spherical functions(i.e.automorphic forms of Sp(1)). In our case of $n = 2$, the situation is fairly different. We can embed Sp(2) to SO(8) for example, and get a Siegel modular form in a similar way, but the weight of this form cannot be $\nu+3$. On the other hand, we have $Sp(2)/\pm 1 = SO(5)$. By using this isomorphism, it has been shown in [25], that we can construct automorphic forms on $\widetilde{Sp(2,\mathbb{R})}$ (the non trivial double cover of $Sp(2,\mathbb{R})$) from forms belonging to Sp(2), and that this construction preserves L-functions. Let $\rho(f_1,f_2)$ be the representation of Sp(2) whose Young diagram is

$$\begin{array}{|c|c|c|c|c|}\hline 1 & \cdot & \cdot & \cdot & f_1 \\ \hline 1 & \cdot & \cdot & f_2 & \multicolumn{1}{c}{} \\ \cline{1-4}\end{array}\ .$$

Then, $\rho(f_1,f_2)$ factors through SO(5) if and only if f_1+f_2 = even. We assume this. Then, from any form $\varphi \in M_{\rho(f_1,f_2)}(U_1'(p))\,(n = 2)$, we can construct a vector valued Siegel modular form $\sigma(\varphi)$ of weight $\det^{(f_1-f_2+5)} \otimes Sym(f_2)$, where $Sym(f_2)$ is the symmetric tensor representation of GL(2) of degree f_2. We can develop the Hecke theory on $\widetilde{Sp(2,\mathbb{R})}$ and define L-series. By some local thoery similar to Yoshida[51], we can show that $L(s,\varphi) = L(s,\sigma(\varphi))$ up to finitely many Euler factors. It is very plausible that there exists a similar mapping from forms of $\widetilde{Sp(2,\mathbb{R})}$ to those of $Sp(2,\mathbb{R})$. So, the above results might be regarded as the first half of an explicit mapping of forms of $Sp(2)$ to those of $Sp(2,R)$.

(3) A relation to supersingular abelian varieties

We have some geometrical interpretation of $\dim M_0(U_i'(p))$ $(i = 0,1)$ and the Hecke operators. Let H_n be the class number of the principal genus of the definite quaternion hermitian space D^n with metric $N(x_1)+\ldots+N(x_n)$ for $(x_1,\ldots,x_n) \in \Gamma^n$ Put $U = G_\infty' \prod_q (GL_n(O_q) \cap G_q')$.

Then, $\dim M_0(U) = H_n$, so $\dim M_0(U_1'(p)) = H_2$ (cf.Shimura [44]).

For $n = 1$, it is known by Deuring [8] that H_1 is equal to the number of isomorphism classes of super singular elliptic curves E over fields of characteristic p.

It is clear that the Brandt matrices defined by Eichler[9] coincide with matrices which consists of numbers of isogenies between supersingular elliptic curves.

Now, we assume $n \geqq 2$. We have similar (but slightly different) relation also for these cases: H_n is equal to the number of principal polalizations of E^n up to $\mathrm{Aut}(E^n)$ (cf.Ibukiyama-Katsura-Oort [26], J.P.Serre [42]). Combining this fact for $n = 2$ with some geometrical consideration, the number of supersingular curves of genus two with prescribed automorphism groups have been counted (Ibukiyama-Katsura-Oort. loc.cit.) This gives an example of explicit descriptions of Γ_i in (1.3) up to isomorphisms. Next, let C_i $(i=1\ldots H_n)$

be the complete set of representatives of the principal polarizations of E^n up to $\mathrm{Aut}(E^n)$. For natural integers m, put

$$s_{ij}(m) = \#\{\varphi \in \mathrm{End}(E_n);\ \varphi^*(C_j) \equiv mC_i\} / \mathrm{Aut}(E_n, C_i) \quad ,$$

where $\equiv$ denotes the algebraic equivalence. Put $S(m) = (s_{ij}(m))$. On the other hand, denote by $H(m) = (h_{ij}(m))$ the Brandt matrix, i.e., the matrix induced from the Hecke operator T(m) on the right hand side of (1.4). Then, changing the numbering, if necessary, we get $H(m) = S(m)$.

The class number H_2' of the non-principal genus in D^2 is equal to $\dim M_0(U_0'(p))$ (cf. Shimura[44]). It is known by Katsura-Oort [31] that H_2' is equal to the number of irreducible components of the set of principally polarized supersingular abelian surfaces in the coarse moduli scheme $A_{2,1}$.

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