Positive λ-harmonic functions and conformal densities on homogeneous trees

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by

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§0.—Introduction

In this paper, we study some asymptotic aspects, from a geometric point of view, of the positive eigenfunctions of the combinatorial Laplacian associated to a homogeneous tree. The results are inspired by the paper [Sul] of Dennis Sullivan, which concerns the hyperbolic spaces $\mathbb{H}^n$.

Let k be an integer ≥ 3 and X the homogeneous tree of degree k, that is, the unique simply connected simplicial complex of dimension 1 in which every vertex belongs to exactly k edges. X is equipped with the length metric in which every edge is isometric to the unit interval $[0,1]$. The distance in X between two points x and y is denoted by $|x-y|$. We denote by ∂X the boundary (at infinity) of X, that is, the set of ends of X. Recall that the set $X \cup \partial X$ has a natural topology which makes it a compact space in which X sits as a dense open subspace.

For each $x \in X$, the *visual metric* $|\ |_x$ on ∂X is defined by the formula

$$|\xi - \eta|_x = e^{-L},$$

for each ξ and η in ∂X, where L is the length of the common path between the geodesic rays $[x,\xi[$ and $[x,\eta[$. We consider the function $j : X \times X \times (X \cup \partial X) \to \mathbb{R}$ defined by

$$j(x,y,z) = e^{|x-p|-|p-y|},$$

with p being the projection of z on the geodesic segment $[x,y]$ (see Figure 1).

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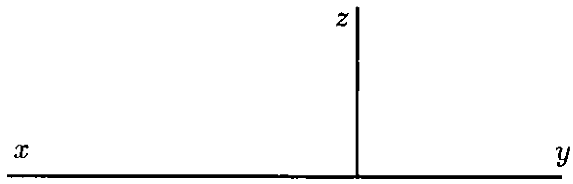

Figure 1

We have the following formula (which we shall refer to as the "formula for the change of point of view "):

$$|\xi - \eta|_y^2 = j(x, y, \xi) j(x, y, \eta) |\xi - \eta|_x^2. \tag{0.1}$$

All the measures considered in this paper are non-negative Radon measures. Let S denote the set of vertices of X and let d be a real number. A *conformal density* of dimension d on ∂X is a family $\mu = (\mu_x)_{x \in S}$ of non-trivial measures on ∂X which are absolutely continuous with respect to one another and such that, for every x and $y \in S$, we have

$$\frac{d\mu_y}{d\mu_x}(\xi) = j^d(x, y, \xi) \quad \forall \xi \in \partial X.$$

We note that a conformal density is entirely determined by its dimension and its value at a given vertex, which can be an arbitrary non-trivial measure on ∂X.

The *Laplace operator* Δ is defined on the space of functions on S by the formula:

$$\Delta f(x) = f(x) - \frac{1}{k} \sum_{y \sim x} f(y)$$

for every function $f : S \to \mathbb{R}$, where the notation $y \sim x$ means that x and y are the two vertices of the same edge. Given $\lambda \in \mathbb{R}$, a function $f : S \to \mathbb{R}$ is called λ*−harmonic* if it satisfies $\Delta f = \lambda f$.

We will be mainly interested in *positive* λ−harmonic functions, i.e. λ−harmonic functions ϕ such that $\phi(x) > 0$ for all $x \in S$. It is well-known that positive λ−harmonic functions exist if and only if $\lambda \leq \lambda_0$, where

$$\lambda_0 = 1 - 2\frac{\sqrt{k-1}}{k}.$$

(We refer to the papers [Dod] and [MW] for surveys and bibliographical references.)

Here is a fundamental example of a positive λ−harmonic function. Fix some real number d, and points $x \in S$ and $\xi \in \partial X$. Then the function

$$y \mapsto j^d(x, y, \xi)$$

is a positive λ−harmonic function on S (*cf.* [CP2]), with

$$\lambda = \frac{1}{k}(1 - e^{-d})(k - 1 - e^{d}). \tag{0.2}$$

We shall often refer to the fact that for a given $\lambda < \lambda_0$, equation (0.2) has two solutions, $d_- < d_+$, satisfying

$$d_- + d_+ = \log(k-1), \tag{0.3}$$

and for $\lambda = \lambda_0$, it has only one solution, $d_- = d_+ = \frac{1}{2}\log(k-1)$.

Let μ be a conformal density of dimension d on ∂X. Consider the *total mass* function $\phi_\mu : S \to \mathbb{R}$, defined by

$$\phi_\mu(x) = \mu_x(\partial X).$$

By the definition of a conformal density, we can write, for every $y \in S$,

$$\phi_\mu(y) = \int_{\partial X} j^d(x, y, \xi)\, d\mu_x(\xi). \tag{0.4}$$

Therefore, ϕ_μ is a positive λ−harmonic function on S, with λ given again by (0.2).

The plan of the paper is the following:

In section 1, we collect a few well-known results about spherical λ−harmonic functions which will be used in the rest of the paper.

Let $\mu = (\mu_x)_{x \in S}$ be a conformal density of dimension $\geq \frac{1}{2}\log(k-1)$. We show in section 2 that, for each $x \in S$, the measure μ_x is the weak limit, as $n \to \infty$, of the measure

$$\sum_{y : |x-y| = n} \phi_\mu(y)\delta_y,$$

suitably normalized to have total mass $\phi_\mu(x)$. (Here, δ_y is the Dirac measure at y.) Thus, in particular, a conformal density of dimension $\geq \frac{1}{2}\log(k-1)$ can be recovered from its total mass function.

In section 3, we prove a representation theorem for positive λ−harmonic functions. More precisely, we follow Martin's method (as explained in the paper [Sul]) to prove that if ϕ is a positive λ−harmonic function, then there exists a unique conformal density μ of dimension $\geq \frac{1}{2}\log(k-1)$ on ∂X whose total mass function is ϕ. We conclude that the map $\mu \mapsto \phi_\mu$ is a bijection from the set of conformal densities of dimension $\geq \frac{1}{2}\log(k-1)$ to the set of positive eigenfunctions of the Laplacian.

Let μ be now a conformal density of dimension d with $d < \frac{1}{2}\log(k-1)$. We know that its total mass function ϕ_μ is a positive λ−harmonic function, and *via* the representation theorem above, we have an associated conformal density $\mu^+ = (\mu^+_x)_{x \in S}$ of dimension $d_+ > \frac{1}{2}\log(k-1)$. In section 4, we study the correspondence $\mu \mapsto \mu^+$ and we give, for each $x \in S$, an explicit formula for μ_x^+ in terms of μ_x. We see in particular that each measure μ_x^+ is absolutely continuous with respect to the $\log(k-1)$−dimensional Hausdorff measure $\mathcal{H}_x$ associated with the visual metric $|\ |_x$, and we give a formula for the Radon-Nikodym derivative $\frac{d\mu_x^+}{d\mathcal{H}_x}$. The map $\mu \mapsto \mu^+$ from the set of conformal densities of dimension $< \frac{1}{2}\log(k-1)$ to the set of conformal densities of dimension $> \frac{1}{2}\log(k-1)$ is neither surjective nor injective.

Section 5 contains different kinds of estimates on the growth of positive λ−harmonic functions along geodesic rays. These estimates, for ϕ positive λ−harmonic, are obtained in terms of the conformal density of dimension $\geq \frac{1}{2}\log(k-1)$ whose total mass function is ϕ.

All the results, with the exception of those of section 4, are discrete analogs of results contained in the paper [Sul] of Sullivan which concerns the case of hyperbolic space $\mathbb{H}^n$. The results of section 4 have also an analog for $\mathbf{H}^n$ and other rank one Riemannian symmetric spaces (cf. [CP3]).

Let us note finally that, as a general rule, the "infinite negative curvature" geometry of trees, reflected for example in the ultrametric property of the visual metrics on the boundary, makes the proofs simpler than in $\mathbb{H}^n$. On the other hand, the statements are often stronger than their analogs for hyperbolic spaces.

§1.—Preliminaries

We begin by recalling the definition of the spherical functions $S_\lambda(n)$, and we give some of their elementary and basic properties (see for example [Bro], [Car] and [FN]). Given a real number λ, it is easy to see that there exists a unique function $S_\lambda : \mathbf{N} \to \mathbb{R}$ such that, for every $x \in S$, the function $y \mapsto S_\lambda(|x-y|)$ is λ−harmonic on S and takes the value 1 at x. Indeed, for a fixed λ, the sequence $S_\lambda(n)$ is determined by the order two linear recurrence relation

$$\frac{k-1}{k}S_\lambda(n+2) - (1-\lambda)S_\lambda(n+1) + \frac{1}{k}S_\lambda(n) = 0 \tag{1.1}$$

with initial conditions

$$S_\lambda(0) = 1 \quad \text{and} \quad S_\lambda(1) = 1-\lambda.$$

For each $x \in S$ and $n \in \mathbf{N}$, let $S(x,n)$ denote the sphere in X of radius n centered at x, and let w_n denote the number of points in $S(x,n)$. We have $w_0 = 1$ and, for all $n \geq 1$, $w_n = k(k-1)^{n-1}$.

Proposition 1.1.— *Let $f : S \to \mathbb{R}$ be a λ−harmonic function. Then:*

$$\frac{1}{w_n}\sum_{y \in S(x,n)} f(y) = f(x)S_\lambda(n) \tag{1.2}$$

for every $x \in S$ *and* $n \in \mathbf{N}$.

Proof.—It is clear that the function which to every point at distance n from x associates the left hand side of equation (1.2) is λ–harmonic and takes the value $f(x)$ at x. ∎

By applying the proposition to the function $y \mapsto j^d(x, y, \xi)$, we obtain:

Corollary 1.2.— *Let* λ *and* d *be real numbers satisfying equation (0.2). Then:*

$$\frac{1}{w_n} \sum_{y \in S(x,n)} j^d(x, y, \xi) = S_\lambda(n)$$

for every $x \in S$, $\xi \in \partial X$ *and* $n \in \mathbf{N}$. ∎

Corollary 1.3.— (cf. [Bro], Theorem 1.1) *For* $\lambda \leq \lambda_0$, *we have* $S_\lambda(n) > 0$ *for all* $n \in \mathbf{N}$. ∎

We shall need the following estimate on spherical functions:

Proposition 1.4.— (cf. [Bro], Theorem 1.1) *For* $\lambda < \lambda_0$, *we have* $S_\lambda(n) \sim Ce^{-nd_-}$ *as* $n \to \infty$, *where* $C = C(k, \lambda) > 0$ *is a constant and* d_- *is, as before, the smallest of the two solutions of equation (0.2). For* $\lambda = \lambda_0$, *we have* $S_\lambda(n) \sim Cne^{-nd}$ *where* $C = C(k) > 0$ *is a constant and where* d *is the unique solution of equation (0.2).*

Proof.— $S_\lambda(n)$ satisfies the recurrence equation (1.1), whose associated characteristic equation is:

$$\frac{k-1}{k}\beta^2 - (1-\lambda)\beta + \frac{1}{k} = 0, \tag{1.3}$$

which is equation (0.2) with $\beta = e^{-d}$.

Therefore, for $\lambda < \lambda_0$, the general solution of (1.1) is of the form

$$S_\lambda(n) = c_1 e^{-nd_+} + c_2 e^{-nd_-}, \quad n \geq 0.$$

The initial conditions give $c_1 + c_2 = 1$ and $c_1 e^{-d_+} + c_2 e^{-d_-} = 1 - \lambda$, hence

$$c_2 = \frac{1 - \lambda - e^{-d_+}}{e^{-d_-} - e^{-d_+}}.$$

We have $e^{-d_+} < 1 - \lambda$, using $e^{-d_-} + e^{-d_+} = (1-\lambda)\frac{k}{k-1}$, which implies $c_2 > 0$.

Thus, $S_\lambda(n) \sim c_2 e^{-nd_-}$ as $n \to \infty$.

For $\lambda = \lambda_0$, equation (1.3) has one double solution $\beta = e^{-d}$, and the general solution of (1.1) is of the form

$$S_\lambda(n) = (c_1 + c_2 n)e^{-nd}.$$

Using the initial conditions, we can see as before that the constant c_2 is also positive in this case, and we have $S_\lambda(n) \sim c_2 n e^{-nd}$ as $n \to \infty$, which proves the proposition. ∎

§2.—Spherical approach to conformal densities

Proposition 2.1.— *Let $\lambda \leq \lambda_0$, and let d_+ be the largest solution of equation* (0.2). *Let x be a fixed vertex of X and $f : X \cup \partial X \to \mathbb{R}$ a continuous function. For every $n \in \mathbb{N}$, consider the function $g_n : \partial X \to \mathbb{R}$ defined by*

$$g_n(\xi) = \frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n)} j^{d_+}(x,y,\xi) f(y).$$

Then the sequence (g_n) converges uniformly to f on ∂X.

Proof.—By Corollary 1.2, we have

$$\frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n)} j^{d_+}(x,y,\xi) = 1, \tag{2.1}$$

for all $\xi \in \partial X, n \in \mathbb{N}$. Therefore,

$$g_n(\xi) - f(\xi) = \frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n)} j^{d_+}(x,y,\xi)\big(f(y) - f(\xi)\big).$$

Let us fix now an $\epsilon > 0$. The function f is uniformly continuous on the compact set $X \cup \partial X$. Therefore, we can find an integer $K \geq 0$ such that $|f(y) - f(\xi)| \leq \frac{\epsilon}{2}$ for every y in the set

$$W = \{y \in S \mid (y.\xi)_x \geq K\},$$

where $(y.\xi)_x$ denotes the *Gromov product* of the points y and ξ with respect to x, that is, the length of the common part of the geodesics $[x,y]$ and $[x,\xi[$. We have, by the triangle inequality:

$$\begin{aligned}
|g_n(\xi) - f(\xi)| &\leq \frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n)} j^{d_+}(x,y,\xi) \mid f(y) - f(\xi) \mid \\
&= \frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n) \cap W} j^{d_+}(x,y,\xi) \mid f(y) - f(\xi) \mid \\
&+ \frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n) \setminus W} j^{d_+}(x,y,\xi) \mid f(y) - f(\xi) \mid \\
&\leq \frac{\epsilon}{2} \frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n) \cap W} j^{d_+}(x,y,\xi) \\
&+ \frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n) \setminus W} j^{d_+}(x,y,\xi) \mid f(y) - f(\xi) \mid .
\end{aligned}$$

Using equation (2.1), we obtain

$$\frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n) \cap W} j^{d_+}(x,y,\xi) \le \frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n)} j^{d_+}(x,y,\xi) = 1,$$

which gives

$$|g_n(\xi) - f(\xi)| \le \frac{\epsilon}{2} + \frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n) \setminus W} j^{d_+}(x,y,\xi) \mid f(y) - f(\xi) \mid . \tag{2.2}$$

We remark now that $j(x,y,\xi) \le e^{2K-n}$ for all $y \in S(x,n) \setminus W$, and that

$$|f(y) - f(\xi)| \le 2||f||_\infty,$$

where

$$||f||_\infty = \sup_{X \cup \partial X} |f|.$$

Thus, inequality (2.2) implies

$$|g_n(\xi) - f(\xi)| \le \frac{\epsilon}{2} + 2||f||_\infty \frac{e^{(2K-n)d_+}}{S_\lambda(n)}. \tag{2.3}$$

For $\lambda < \lambda_0$, we have, by Proposition 1.4, $S_\lambda(n) \sim Ce^{-nd_-}$ as n tends to ∞, where $C > 0$ is some constant and where d_- is the smallest solution of (0.2). For $\lambda = \lambda_0$, we have $S_\lambda(n) \sim Cne^{-nd_+}$. Therefore, (2.3) shows that there exists an integer N such that, for all $\xi \in \partial X$ and for all $n \ge N$, we have

$$|g_n(\xi) - f(\xi)| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof of Proposition 2.1. ∎

We can now prove the following

Theorem 2.2.—*Let $\mu = (\mu_x)_{x \in S}$ be a conformal density of dimension d on ∂X, with $d \ge \frac{1}{2}\log(k-1)$. For every vertex $x \in X$ and every $n \in \mathbb{N}$, define the measure $\mu_{n,x}$ on X by the formula*

$$\mu_{n,x} = \frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n)} \phi_\mu(y) \delta_y \tag{2.4}$$

where δ_y is the Dirac measure at y. Then the sequence $(\mu_{n,x})_{n \in \mathbb{N}}$ converges weakly to μ_x in the space of measures on $X \cup \partial X$.

Proof.—Consider a continuous function f on $X \cup \partial X$ and, for $n \in \mathbf{N}$, let g_n be the function on ∂X defined in Proposition 2.1 (note that $d_+ = d$ here). We have, using (0.4),

$$\begin{aligned}
\mu_{n,x}(f) &= \frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n)} \phi_\mu(y) f(y) \\
&= \frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n)} \int_{\partial X} j^d(x,y,\xi)\, d\mu_x(\xi) f(y) \\
&= \int_{\partial X} \frac{1}{w_n S_\lambda(n)} \sum_{y \in S(x,n)} j^d(x,y,\xi) f(y)\, d\mu_x(\xi) \\
&= \int_{\partial X} g_n(\xi)\, d\mu_x(\xi) \\
&= \mu_x(g_n).
\end{aligned}$$

Now since (g_n) converges uniformly to f on ∂X (Proposition 2.1), we conclude that $\mu_{n,x}(f)$ converges to $\mu_x(f)$ as $n \to \infty$. Therefore, the sequence $(\mu_{n,x})$ converges weakly to μ_x. ∎

We note the following corollary which will be useful in the next section:

Corollary 2.3.—*A conformal density of dimension* $\geq \frac{1}{2}\log(k-1)$ *is uniquely determined by its total mass function.* ∎

§3.—Conformal representation at infinity of positive λ–harmonic functions

In this section, we show, by following Martin's classical method, that for every positive λ–harmonic function ϕ on S, there exists a unique conformal density μ of dimension $\geq \frac{1}{2}\log(k-1)$ on ∂X whose total mass function is ϕ. The dimension of μ is equal to the largest solution of equation (0.2). The uniqueness of μ will be a consequence of Corollary 2.3.

We shall follow the lines of the proof of the corresponding theorem of Sullivan (Theorem 2.11 of [Sul]), adapted to the discrete setting. For this purpose, we need to recall a few facts from discrete potential theory. For further details, we refer the reader to ([Mey], chapter 9). From now on, we suppose $\lambda \leq \lambda_0$.

The *transition kernel* $P : S \times S \to \{0, \frac{1}{k}\}$ is defined by

$$P(x,y) = \frac{1}{k} \text{ if } |x-y| = 1$$

$$\text{and} \quad P(x,y) = 0 \text{ otherwise.}$$

The $\lambda-$*Green kernel* $G_\lambda : S \times S \to [0,\infty]$ is defined by

$$G_\lambda(x,y) = \sum_{n=0}^{\infty} (1-\lambda)^{-n-1} P^n(x,y),$$

where $P^0 = I$ is the identity kernel, defined by $I(x,y) = 1$ if $x = y$ and $I(x,y) = 0$ otherwise, and where P^n is the matrix product, defined by induction on n by the formula

$$P^{n+1}(x,y) = \sum_{z \in S} P^n(x,z)P(z,y).$$

Let us note that by a result of Kesten (see for example [CP2],§5), we have an explicit formula for $\lambda \leq \lambda_0$,

$$G_\lambda(x,y) = \alpha e^{-|x-y|d_+}, \tag{3.1}$$

where d_+ is, as before, the largest solution of equation (0.2), and

$$\alpha = \frac{1}{1 - \lambda - e^{-d_+}}.$$

Let us recall also that any kernel $K : S \times S \to [0, \infty]$ acts on the set of positive functions on S by the formula

$$Kf(x) = \sum_{y \in S} K(x,y)f(y).$$

for every $f : S \to [0, \infty]$.

The function $f : S \to \mathbb{R}$ is said to be $\lambda-$*superharmonic* if it satisfies $\Delta f \geq \lambda f$, or, equivalently, $f \geq (1-\lambda)^{-1}Pf$, where Pf is defined by

$$Pf(x) = \sum_{y \in S} P(x,y)f(y).$$

Proposition 3.1.—*Let $\lambda \leq \lambda_0$. Consider a function $f : S \to [0, \infty[$. Then, the following two properties are equivalent:*

(i) f is $\lambda-$superharmonic and satisfies

$$\lim_{n \to \infty} (1-\lambda)^{-n} P^n f = 0,$$

in the sense of pointwise convergence.

(ii) There exists a function $g : S \to [0, \infty[$ such that $f = G_\lambda g$.

Proof.— Suppose that property (i) is satisfied. Let $g = (1-\lambda)f - Pf$. Since f is $\lambda-$superharmonic, we have $g \geq 0$. On the other hand, the identity

$$f = \sum_{i=0}^{n-1} (1-\lambda)^{-i-1} P^i g + (1-\lambda)^{-n} P^n f$$

shows that $f = G_\lambda g$ since, by hypothesis, we have $(1-\lambda)^{-n}P^n f \to 0$ as $n \to \infty$. Therefore, f satisfies property (ii).

Conversely, suppose that f satisfies property (*ii*). From the relation $f = G_\lambda g$, we deduce $f = (1-\lambda)^{-1}g + (1-\lambda)^{-1}Pf$, which implies $f \geq (1-\lambda)^{-1}Pf$, and f is $\lambda-$superharmonic. Furthermore, we can write

$$f = (1-\lambda)^{-1}g + (1-\lambda)^{-1}P\big((1-\lambda)^{-1}g + (1-\lambda)^{-1}Pf\big)$$

$$= (1-\lambda)^{-1}g + (1-\lambda)^{-2}Pg + ... + (1-\lambda)^{-n}P^{n-1}g + (1-\lambda)^{-n}P^n f,$$

hence $(1-\lambda)^{-n}P^n f$ appears as the $n-$th remainder of a convergent series, and therefore tends to 0. ∎

Proposition 3.2.—*Let $\lambda \leq \lambda_0$, and let $f : S \to [0,\infty[$ be $\lambda-$superharmonic. Then, there exists a sequence of functions $g_n : S \to [0,\infty[$ such that $f = \lim G_\lambda g_n$ in the sense of pointwise convergence.*

Proof.— Let us fix a basepoint $x \in S$, and for all $n = 0,1,2,...$, let χ_n denote the characteristic function of the ball $B(x,n)$ of radius n centered at x. Consider the sequence of functions $f_n = \min\{f, nG_\lambda\chi_n\}$. Being the minimum of two $\lambda-$superharmonic functions, f_n is itself $\lambda-$superharmonic.

For a given point $y \in S$, let us take n large enough so that y belongs to the ball $B(x,n)$. We can write in this case:

$$G_\lambda\chi_n(y) = \sum_{z\in S} G_\lambda(y,z)\chi_n(z) \geq G_\lambda(y,y) \geq (1-\lambda)^{-1}.$$

Therefore, there exists an integer n_0 such that for every $n \geq n_0$, we have

$$nG_\lambda\chi_n(y) \geq n(1-\lambda)^{-1},$$

which implies $f_n(y) = f(y)$ for all n large enough. Thus, f is the increasing limit of (f_n).

We prove finally that for every $n = 0,1,2,...$, the function f_n is of the form $G_\lambda g_n$, with $g_n \geq 0$. By proposition 3.1, it suffices to show that we have $\lim_{j\to\infty}(1-\lambda)^{-j}P^j f_n = 0$ pointwise.

Let us fix the integer $n \geq 0$. It is easy to see, by induction, that for every $j = 0,1,2,...$, we have

$$(1-\lambda)^{-j}P^j f_n \leq \min\{(1-\lambda)^{-j}P^j f, (1-\lambda)^{-j}P^j nG_\lambda\chi_n\}.$$

Now using Proposition 3.1, we have

$$\lim_{j\to\infty}(1-\lambda)^{-j}P^j f = \lim_{j\to\infty}(1-\lambda)^{-j}P^j nG_\lambda\chi_n = 0.$$

This completes the proof of Proposition 3.2. ∎

Now we are ready to prove the following

Theorem 3.3.—*Let ϕ be a positive λ−harmonic function on S. Then there exists a unique conformal density μ of dimension $\geq \frac{1}{2}\log(k-1)$ on ∂X whose total mass function is ϕ. The dimension of μ is equal to d_+, the largest solution of equation (0.2).*

Proof.— Let x be again a basepoint in S. By Proposition 3.2, there exists a sequence of functions $g_n : S \to [0, \infty[$ such that the sequence $f_n = G_\lambda g_n$ converges pointwise to ϕ.

For every $n \geq 0$, we define the measure $\nu_{n,x}$ on S by

$$\nu_{n,x} = \sum_{z \in S} G_\lambda(x, z) g_n(z) \delta_z,$$

where δ_z denotes the Dirac measure at z.

The total mass of $\nu_{n,x}$ is equal to

$$\sum_{z \in S} G_\lambda(x, z) g_n(z) = G_\lambda g_n(x) = f_n(x).$$

As $f_n(x)$ converges to $\phi(x)$, the total mass of $\nu_{n,x}$ is bounded, and we can find a subsequence $\nu_{n_i,x}$ which converges weakly to a measure μ_x on the compact set $S \cup \partial X$. Let us show now that the support of μ_x is necessarily contained in ∂X.

From the relation $f_n = G_\lambda g_n$, we deduce that $g_n = (\Delta - \lambda I) f_n$. Therefore, for every $z \in S$, we have

$$g_{n_i}(z) = (\Delta - \lambda I) f_{n_i}(z).$$

As $n_i \to \infty$, we have $f_{n_i}(z) \to \phi(z)$, and

$$\lim_{n_i \to \infty} g_{n_i}(z) = \lim_{n_i \to \infty} (\Delta - \lambda I) f_{n_i}(z) = (\Delta - \lambda I)\phi(z) = 0.$$

Therefore, for every vertex $z \in S$, we have

$$\mu_x(\{z\}) = \lim_{n_i \to \infty} \nu_{n_i,x}(\{z\}) = \lim_{n_i \to \infty} G_\lambda(x, z) g_{n_i}(z) = 0.$$

The support of μ_x is therefore contained in ∂X.

We can write

$$\begin{aligned} f_{n_i}(y) &= \sum_{z \in S} G_\lambda(y, z) g_{n_i}(z) \\ &= \sum_{z \in S} \frac{G_\lambda(y, z)}{G_\lambda(x, z)} G_\lambda(x, z) g_{n_i}(z) \\ &= \int_S \frac{G_\lambda(y, z)}{G_\lambda(x, z)} \, d\nu_{n_i,x}(z). \end{aligned}$$

By (3.1), we have

$$\frac{G_\lambda(y,z)}{G_\lambda(x,z)} = \frac{\alpha e^{-|y-z|d_+}}{\alpha e^{-|x-z|d_+}} = j^{d_+}(x,y,z).$$

Therefore, we can write

$$f_{n_i}(y) = \int_{S\cup\partial X} j^{d_+}(x,y,z)\, d\nu_{n_i,x}(z).$$

Letting $n_i \to \infty$, we obtain

$$\phi(y) = \int_{S\cup\partial X} j^{d_+}(x,y,z)\, d\mu_x(z) = \int_{\partial X} j^{d_+}(x,y,\xi)\, d\mu_x(\xi).$$

Therefore, the uniquely defined conformal density of dimension d_+ which takes the value μ_x at x, has ϕ as total mass function.

Uniqueness of μ follows from Corollary 2.3. This completes the proof of Theorem 3.3. ∎

Remark. In [CP2], we give a probabilistic interpretation of the conformal density μ in terms of the random walk on S with transition probabilities

$$P_\phi(x,y) = (1-\lambda)^{-1}\frac{\phi(y)}{\phi(x)}P(x,y)$$

for every $x, y \in S$. More precisely, μ_x is $\phi(x)$ times the hitting probability at infinity of the random walk starting at x, with probability 1.

§4.—On the correspondence $\mu \mapsto \mu^+$

For each $x \in S$, let $\mathcal{H}_x$ denote the $\log(k-1)$–dimensional Hausdorff measure on ∂X associated to the visual metric $|\ |_x$, and normalized so that $\mathcal{H}_x(\partial X) = 1$. We recall a few basic properties of $\mathcal{H}_x$. First, it is the only probability measure on ∂X which is invariant by the full isometry group of X fixing the vertex x. From this symmetry property, we see that all the closed balls of a given radius (for the metric $|\ |_x$) have the same $\mathcal{H}_x$–mass. In fact the mass of a closed ball of radius e^{-n} centered at a point in ∂X is equal to $\frac{1}{w_n}$, where w_n is defined, as before, as the number of points on a sphere of radius n in X centered at a vertex. Let us note also that $\mathcal{H} = (\mathcal{H}_x)_{x\in S}$ is a conformal density on ∂X of dimension $\log(k-1)$. This can be deduced from the fact that $\mathcal{H}$ is the conformal density which is associated by Theorem 3.3 to the constant function $\phi = 1$ (which is 0–harmonic).

Proposition 4.1.—*Let $\xi \in \partial X$ and $x \in S$, and let $\alpha < \log(k-1)$. Then the function h_α on ∂X defined by*

$$h_\alpha(\eta) = \frac{1}{|\xi - \eta|_x^\alpha}$$

belongs to $L^1(\mathcal{H}_x)$ *and satisfies*

$$\int_{\partial X} h_\alpha \, d\mathcal{H}_x = \frac{1}{k} + \frac{k-2}{k}\left(1 - \frac{e^\alpha}{k-1}\right)^{-1}.$$

We shall denote this value of $\mathcal{H}_x(h_\alpha)$ *by* I_α.

Proof.— For each $n \in \mathbf{N}$, let E_n be the set of points $\eta \in \partial X$ such that the projection of η on the geodesic ray $[x, \xi[$ is at distance n from x. The function h_α is constant and equal to $e^{\alpha n}$ on E_n. Therefore, $\mathcal{H}_x(h_\alpha)$ is the limit as N tends to ∞ of

$$I_\alpha(N) = \sum_{n=0}^{N} e^{\alpha n} \mathcal{H}_x(E_n).$$

We have $\mathcal{H}_x(E_0) = \frac{k-1}{k}$. For every $n \geq 1$, we note that $E_n = B_n \setminus B_{n+1}$, where B_n is the closed $|\ |_x$−ball of radius e^{-n} centered at ξ. Therefore

$$\mathcal{H}_x(E_n) = \frac{1}{w_n} - \frac{1}{w_{n+1}} = \frac{1}{k(k-1)^{n-1}} - \frac{1}{k(k-1)^n} = \frac{k-2}{k(k-1)^n}.$$

This gives

$$I_\alpha(N) = \frac{k-1}{k} + \frac{k-2}{k} \sum_{n=1}^{N} \frac{e^{n\alpha}}{(k-1)^n},$$

which converges since $\alpha < \log(k-1)$, with limit I_α. ∎

Corollary 4.2.— *Let* m *be a measure on* ∂X *and* $\alpha < \log(k-1)$. *Then the function*

$$\eta \mapsto \int_{\partial X} \frac{dm(\xi)}{|\xi - \eta|_x^\alpha}$$

belongs to $L^1(\mathcal{H}_x)$.

Proof.— This is an immediate consequence of the preceding proposition and Fubini's theorem. ∎

Theorem 4.3.—*Let* $\mu = (\mu_x)_{x \in S}$ *be a conformal density on* ∂X *of dimension* $d_- < \frac{1}{2}\log(k-1)$, *and for each* $x \in S$, *let* μ_x^+ *be the measure on* ∂X *which is absolutely continuous with respect to* $\mathcal{H}_x$ *with Radon-Nikodym derivative given by the formula*

$$\frac{d\mu_x^+}{d\mathcal{H}_x}(\xi) = \frac{1}{C} \int_{\eta \in \partial X} \frac{d\mu_x(\eta)}{|\xi - \eta|_x^{2d_-}},$$

where $C = I_{2d_-}$, *using the notations of Proposition 4.1. Note that this function is in* $L^1(\mathcal{H}_x)$, *by Corollary 4.2.*

Then, $\mu^+ = (\mu_x^+)_{x \in S}$ *is a conformal density on* ∂X *of of dimension* $d_+ = \log(k-1) - d_-$. *Furthermore,* μ_+ *is the unique conformal density of dimension* $\geq \frac{1}{2}\log(k-1)$ *having the same total mass function as* μ.

Proof.— Let us show first that μ^+ is a conformal density of dimension d_+.

For x and $y \in S$, and for every continuous function f on ∂X, we have:

$$\int_{\xi\in\partial X} f(\xi) j^{d_+}(x,y,\xi)\, d\mu_x^+(\xi)$$
$$= \int_{\xi\in\partial X} f(\xi) j^{d_+}(x,y,\xi) \frac{1}{C} \int_{\eta\in\partial X} \frac{d\mu_x(\eta)}{|\xi-\eta|_x^{2d_-}}\, d\mathcal{H}_x(\xi)$$
$$= \frac{1}{C} \int_{\xi\in\partial X} f(\xi) \int_{\eta\in\partial X} \frac{j^{d_+}(x,y,\xi)\, d\mu_x(\eta)}{|\xi-\eta|_x^{2d_-}}\, d\mathcal{H}_x(\xi)$$
$$= \frac{1}{C} \int_{\xi\in\partial X} f(\xi) \int_{\eta\in\partial X} \frac{j^{d_++d_-}(x,y,\xi) j^{d_-}(x,y,\eta)\, d\mu_x(\eta)}{|\xi-\eta|_x^{2d_-} j^{d_-}(x,y,\xi) j^{d_-}(x,y,\eta)}\, d\mathcal{H}_x(\xi)$$
$$= \frac{1}{C} \int_{\xi\in\partial X} f(\xi) \int_{\eta\in\partial X} \frac{j^{d_++d_-}(x,y,\xi) j^{d_-}(x,y,\eta)\, d\mu_x(\eta)}{|\xi-\eta|_y^{2d_-}}\, d\mathcal{H}_x(\xi)$$

(using formula (0.1) of "change of point of view")

$$= \frac{1}{C} \int_{\xi\in\partial X} f(\xi) \int_{\eta\in\partial X} \frac{d\mu_y(\eta)}{|\xi-\eta|_y^{2d_-}} j^{d_++d_-}(x,y,\xi)\, d\mathcal{H}_x(\xi)$$

(since the conformal density (μ_x) is of dimension d_-)

$$= \frac{1}{C} \int_{\xi\in\partial X} f(\xi) \int_{\eta\in\partial X} \frac{d\mu_y(\eta)}{|\xi-\eta|_y^{2d_-}}\, d\mathcal{H}_y(\xi)$$

(using (0.3) and since $\mathcal{H}_x$ is conformal of dimension $\log(k-1)$).

This proves that μ^+ is a conformal density of dimension d_+.

Now, with ϕ_μ and ϕ_{μ^+} denoting respectively the total mass functions of the conformal densities μ and μ^+ respectively, we have, for every $x \in S$,

$$\begin{aligned}\phi_{\mu^+}(x) &= \frac{1}{C} \int_\eta \int_\xi \frac{d\mu_x(\xi)}{|\xi-\eta|_x^{2d_-}}\, d\mathcal{H}_x(\eta) \\ &= \frac{1}{C} \int_\xi \Big(\int_\eta \frac{d\mathcal{H}_x(\eta)}{|\xi-\eta|_x^{2d_-}} \Big)\, d\mu_x(\xi) \\ &= \int_\xi d\mu_x(\xi) \\ &= \phi_\mu(x).\end{aligned}$$

This completes the proof of Theorem 4.3. ∎

Example.—Let $\mu = (\mu_x)_{x\in S}$ be a conformal density of dimension $d = 0$. In this case, μ_x does not depend on x, and the total mass function of μ is constant. Then, $\mu^+ = (\alpha\mathcal{H}_x)_{x\in S}$ where the constant α is the common mass of the μ_x's.

Corollary 4.4.—*Let $\nu = (\nu_x)_{x\in S}$ be a conformal density of dimension $\geq \frac{1}{2}\log(k-1)$. Assume that ν_x is not absolutely continuous with respect to $\mathcal{H}_x$ for some (or, equivalently,* *for all) $x \in S$. Then, there is no conformal density of dimension $< \frac{1}{2}\log(k-1)$ having the same total mass function as ν.* ∎

§5.—On the radial growth of positive λ−harmonic functions

In this section, we study the asymptotic growth of a positive λ−harmonic function along geodesic rays.

Let us first fix some notation. Consider a positive λ−harmonic function ϕ on S. We denote, as before, by $d_- \leq d_+$ the solutions of equation (0.2). For $x \in S$, $n \in \mathbf{N}$ and $\xi \in \partial X$, we denote by (x, n, ξ) the point in S situated on the geodesic ray $[x, \xi[$ at distance n from x. Define the *exponential growth coefficient* of ϕ in the direction ξ as

$$\mathrm{gr}_\phi(\xi) = \limsup_{n\to\infty} \frac{1}{n} \log \phi(x, n, \xi).$$

It is clear that $\mathrm{gr}_\phi(\xi)$ does not depend on x, since two geodesic rays anding at ξ eventually coincide.

Proposition 5.1.—*There exists a constant $C = C(k, \lambda) > 0$ such that, for all points $x, y \in S$, we have*

$$\phi(y) \leq C\phi(x)e^{nd_+}$$

where $n = |x - y|$.

Proof.— Proposition 1.1 shows that $\phi(y) \leq \phi(x)w_n S_\lambda(n)$. Using proposition, 1.4, we have $w_n S_\lambda(n) \leq \mathrm{Const}.e^{nd_+}$. This proves the formula. ∎

Corollary 5.2.—*For all $\xi \in \partial X$, we have $-d_+ \leq gr_\phi(\xi) \leq d_+$.* ∎

Examples.
1.— Fix $x_0 \in S$, $d_0 \in \mathbb{R}$, $\xi_O \in \partial X$ and take $\phi(x) = j^{d_0}(x_0, x, \xi_0)$. Then, $\mathrm{gr}_\phi(\xi) = -d_0$ for $\xi \neq \xi_0$ and $\mathrm{gr}_\phi(\xi_0) = d_0$.
2.— For $\phi(x) = S_\lambda(|x - x_0|)$ with $\lambda \leq \lambda_0$, we have, for every $\xi \in \partial X$, $\mathrm{gr}_\phi(\xi) = -d_-$.

Given the positive λ−harmonic function ϕ, let $\mu = (\mu_x)_{x\in S}$ be the unique conformal density of dimension $\geq \frac{1}{2}\log(k-1)$ with total mass function ϕ. We recall that μ is of dimension d_+. We shall see now how to use μ to get estimates on the radial growth of ϕ.

Theorem 5.3.—*Let $x \in S$ and let ξ be a point in ∂X which is not in the support of μ. Then, there exist $C > 0$ and $N \in \mathbf{N}$ such that, for all integers $n \geq N$,*

$$\phi(x, n, \xi) = Ce^{-nd_+}.$$

In particular, we have $c_\phi(\xi) = -d_+$.

Proof.— Let us set $y = (x, n, \xi)$. For each $i \in \mathbf{N}$, let E_i be the set of points in ∂X whose projection on the geodesic ray $[x, \xi[$ is at distance i from x. As ξ is not in the support of

μ, there is an integer N such that every point in the support of μ is in E_i for some $i \leq N$. Therefore,

$$\phi(y) = \int_{\partial X} j^{d_+}(x, y, \eta)\, d\mu_x(\eta)$$
$$= \int_{E_0 \cup E_1 \cup \ldots \cup E_N} j^{d_+}(x, y, \eta)\, d\mu_x(\eta).$$

We remark now that $j(x, y, \eta) = e^{2i-n}$ for all $\eta \in E_i$ and $i \leq n$. Therefore we have, for all $n \geq N$,

$$\phi(y) = \sum_{i=0}^{N} e^{(2i-n)d_+} \mu_x(E_i)$$
$$= Ce^{-nd_+}$$

with $C = \sum_{i=0}^{N} e^{2id_+} \mu_x(E_i)$. ∎

In the next lemma, we prove a general property of the visual metric $|\ |_x$ on ∂X which will be useful in the proof of the subsequent theorem.

Lemma 5.4.—*Let m be a measure on ∂X. Then, for $m-$almost all $\xi \in \partial X$, we have*

$$\inf_{0<r<1} \frac{m(B(\xi, r))}{r^{\log(k-1)}} > 0,$$

where $B(\xi, r)$ denotes the closed ball of center ξ and radius r, with respect to the visual metric $|\ |_x$.

Proof.— Let

$$A = \{\xi \in \partial X \text{ such that } \inf_{0<r<1} \frac{m(B(\xi, r))}{r^{\log(k-1)}} = 0\}.$$

We prove that the $m-$measure of A is zero.

Let us fix $\epsilon > 0$. For each $\xi \in A$, we can find a real number r_ξ with $0 < r_\xi < 1$ such that

$$m(B(\xi, r_\xi)) \leq \epsilon r_\xi^{\log(k-1)}.$$

We use now the fact that given any two closed balls in the ultrametric space $(\partial X, |\ |_x)$, then either one of them contains the other or they are disjoint. Therefore, we can find a countable family of points $\{\xi_i\} \subset A$, and associated real numbers $\{r_i\}$ such that the family of closed balls $\{B(\xi_i, r_i)\}$ centered at ξ_i and of radii r_i covers A, with these balls being two by two disjoint. We deduce that

$$m(A) \leq \sum_i m(B(\xi_i, r_i)) \leq \epsilon \sum_i r_i^{\log(k-1)}.$$

Now from the definition of the visual metric, we see that we can suppose without loss of generality that each of the radii r_i is of the form e^{-n_i}, with $n_i \in \mathbf{N}^*$, and we recall the fact that the $\mathcal{H}_x-$measure of a closed ball of radius e^{-n_i} is equal to

$$\frac{1}{w_{n_i}} = \frac{1}{k(k-1)^{n_i-1}} = \frac{k-1}{k} e^{-n_i \log(k-1)} = \frac{k-1}{k} r_i^{\log(k-1)}.$$

Therefore, we have

$$m(A) \leq \frac{k}{k-1} \epsilon \sum_i \mathcal{H}_x\big(B(\xi_i, r_i)\big) \leq \frac{k}{k-1} \epsilon \mathcal{H}_x(\partial X) = \frac{k}{k-1} \epsilon.$$

We conclude that $m(A) = 0$, and the lemma is proved. ∎

The measures μ_x being absolutely continuous with respect to one another, they have the same sets of measure 0. Therefore, we can say that a certain property holds "μ−almost everywhere" if it holds μ_x−almost everywhere for some (or equivalently for all) $x \in S$.

Theorem 5.5.—*There exists $C > 0$ such that, for μ−almost all $\xi \in \partial X$,*

$$\phi(x, n, \xi) \geq C e^{-nd_-}.$$

In particular, we have $gr_\phi(\xi) \geq -d_-$ for μ−almost all $\xi \in \partial X$.

Proof.—Let ξ be an arbitrary point in ∂X. We set, as before, $y = (x, n, \xi)$ and we denote, for each $i \in \mathbf{N}$, by A_i the set of points in ∂X whose projection on $[x, y[$ is at distance i from x. Let B_i be the closed ball in $(\partial X, |\ |_x)$ of radius e^{-i} centered at ξ. We have $A_i = B_i \setminus B_{i+1}$ for $i = 0, 1, ..., n-1$ and $A_n = B_n$. We note also that $j(x, y, \eta) = e^{2i-n}$ for all $\eta \in A_i$. Therefore,

$$\begin{aligned}
\phi(y) &= \int_{\partial X} j^{d_+}(x, y, \eta)\, d\mu_x(\eta) \\
&= \sum_{i=0}^{n} e^{(2i-n)d_+} \mu_x(A_i) \\
&= e^{-nd_+} \sum_{i=0}^{n-1} e^{2id_+} \big(\mu_x(B_i) - \mu_x(B_{i+1})\big) + e^{nd_+} \mu_x(B_n).
\end{aligned}$$

Using Abel's summation formula, we obtain

$$\phi(y) = e^{-nd_+} \big(\mu_x(B_0) + \sum_{i=1}^{n} (e^{2id_+} - e^{2(i-1)d_+}) \mu_x(B_i)\big),$$

which implies

(5.1)
$$\phi(y) \geq e^{-nd_+} \sum_{i=1}^{n} (e^{2id_+} - e^{2(i-1)d_+}) \mu_x(B_i).$$

By Lemma 5.4, there is a constant $C_0 > 0$ such that, for μ−almost all $\xi \in \partial X$ and all $i \in \mathbf{N}$, we have

(5.2)
$$\mu_x(B_i) \geq C_0 e^{-i\log(k-1)}.$$

Inequalities (5.1) and (5.2) imply that, for μ-almost all ξ, we have

$$\begin{aligned}\phi(y) &\geq e^{-nd_+}\sum_{i=1}^{n}(e^{2id_+} - e^{2(i-1)d_+})C_0 e^{-i\log(k-1)} \\ &= C_0 e^{-nd_+}(1-e^{-2d_+})\sum_{i=1}^{n} e^{i(2d_+-\log(k-1))} \\ &\geq Ce^{-nd_+}e^{n(2d_+-\log(k-1))} \\ &= Ce^{n(d_+-\log(k-1))},\end{aligned}$$

where $C>0$ is some constant. This proves Theorem 5.5 since we have $d_+ - \log(k-1) = d_-$, by equation (0.3). ∎

Theorem 5.6.—*Let A be a Borel subset of ∂X. Assume that $\mu_x(A) > 0$ for some (or, equivalently, for any) $x \in S$. Assume furthermore that there exists some real number σ such that $gr_\phi(\xi) \leq \sigma$ for all $\xi \in A$. Then, the Hausdorff dimension of A (with respect to the visual metrics) is $\geq d_+ - \sigma$.*

Proof.—As before, the proof relies on the existence of the following representation:

$$\phi(y) = \int_{\partial X} j^{d_+}(x,y,\xi)\, d\mu_x(\xi).$$

Let $\epsilon > 0$ be a fixed real number. For each $N \in \mathbf{N}$, define the set

(5.3)
$$A(N) = \{\xi \in A \mid \phi(x,n,\xi) \leq e^{n(\sigma+\epsilon)}\ \forall n \geq N\}.$$

Then A is the increasing union of the $A(N)$'s and therefore we can find an integer N such that $\mu_x\big(A(N)\big) > 0$. We fix such an integer N.

Let ξ be an element of $A(N)$, let y be a point on $[x,\xi[$ satisfying $|x-y| = n \geq N$ and let $B \subset A(N)$ be the closed ball (for the induced metric) of center ξ and radius e^{-n}. For every $\eta \in B$, we have $j(x,y,\eta) = e^n$, which implies

$$\phi(y) \geq e^{nd_+}\mu_x(B).$$

Using (5.3), we have also

$$\phi(y) \leq e^{(\sigma+\epsilon)n}.$$

Therefore, we have

$$\mu_x(B) \leq e^{(\sigma+\epsilon-d_+)n}.$$

Thus, for all $\xi \in A(N)$ and for any closed ball B of $A(N)$ of radius $r \leq e^{-N}$, we have

$$\mu_x(B) \leq r^{-\sigma-\epsilon+d_+}. \tag{5.4}$$

Consider now an arbitrary covering of $A(N)$ by closed balls of radii $\leq e^{-N}$. Using again the fact that given any two closed balls in ∂X, either one of them is contained in the other or they are disjoint, we can extract a countable subcover $\{B_i\}$ of closed balls which are two by two disjoint. Each of the balls B_i satisfies

$$\mu_x(B_i) \leq r_i^{-\sigma-\epsilon+d_+},$$

where r_i is the radius of B_i. Therefore ,we have

$$0 < \mu_x(A(N)) \leq \sum_i \mu_x(B_i) \leq \sum_i r_i^{-\sigma-\epsilon+d_+}.$$

We deduce that the $(d_+ - \sigma - \epsilon)-$dimensional Hausdorff measure of $A(N)$ is > 0, which implies that the Hausdorff dimension of $A(N)$ is $\geq d_+ - \sigma - \epsilon$. Making $\epsilon \to 0$, we conclude that this dimension is $\geq d_+ - \sigma$. Therefore, the Hausorff dimension of A itself is $\geq d_+ - \sigma$, which proves the theorem. ∎

Corollary 5.7.— *We have $gr_\phi(\xi) \geq -d_-$ for $\mu-$almost all $\xi \in \partial X$.* ∎

Proof.— Let $\sigma < -d$ and suppose that there exists a Borel subset $A \subset \partial X$ such that for every $\xi \in A$, we have $\mathrm{gr}_\phi(\xi) \leq \sigma$. By Theorem 5.6, the Hausdorff dimension of A is $\geq d_+ - \sigma > d_+ + d_- = \log(k-1)$, contradicting the fact that $\log(k-1)$ is the Hausdorff dimension of ∂X. We conclude that $\mathrm{gr}_\phi(\xi) > \sigma$ for $\mu-$almost all $\xi \in \partial X$.

Let For every $n \in \mathbf{N}^*$, let define $\sigma_n = -d_- - \frac{1}{n}$, and let $E(\sigma_n)$ be the set of points $\xi \in \partial X$ such that $\mathrm{gr}_\phi(\xi) \leq \sigma_n$. For all n, we have $\mu_x(E_n) = 0$. The set of points satisfying $\mathrm{gr}_\phi(\xi) < -d_-$ is the countable union of the $E(\sigma_n)$'s. The proof of the corollary follows. ∎

Now we use the Fatou-type theorem given in [CP1] for conformal densities of the same dimension, and the proof of Theorem 5.6, to obtain the following

Theorem 5.8.— *The following four statements are equivalent:*

(i) $gr_\phi(\xi) = -d_-$ for $\mu-$almost all $\xi \in \partial X$.

(ii) $gr_\phi(\xi) \leq -d_-$ for $\mu-$almost all $\xi \in \partial X$.

(iii) For all $x \in S$, the measure μ_x is absolutely continuous with respect to the $\log(k-1)-$dimensional Hausdorff measure $\mathcal{H}_x$ on ∂X.

(iv) There exists a point $x \in S$ such that he measure μ_x is absolutely continuous with respect to the $\log(k-1)-$dimensional Hausdorff measure $\mathcal{H}_x$ on ∂X.

Furthermore, if one of these conditions is satisfied, then, for $\mu-$almost all $\xi \in \partial X$, we have

$$(5.5) \qquad \lim_{n\to\infty} \frac{\phi(x,n,\xi)}{S_\lambda(n)} = \frac{d\mu_x}{d\mathcal{H}_x}(\xi).$$

In particular, there is a constant $C = C(k,\lambda) > O$ *such that, for* $\mu-$*almost all* $\xi \in \partial X$, *we have*

$$\phi(x,n,\xi) \sim C\frac{d\mu_x}{d\mathcal{H}_x}(\xi)e^{-nd_-}$$

as $n \to \infty$. *(Note that we already knew that* $gr_\phi(\xi) = -d_-$ *for* $\mu-$*almost all* ξ, *by Corollary 5.7).*

Proof.— $(i) \Rightarrow (ii)$ is trivial. Let us prove $(ii) \Rightarrow (iii)$.

Assume that $\mathrm{gr}_\phi(\xi) \le -d_-$ for $\mu-$almost all $\xi \in \partial X$. Let us fix $x \in S$. By the proof of Theorem 5.6, taking $\sigma = -d_-$, we have, for every Borel subset $A \in \partial X$, $\mathcal{H}_x(A) > 0$ if $\mu_x(A) > 0$. Therefore, μ_x is absolutely continuous with respect to $\mathcal{H}_x$. This proves (iii).

The equivalence $(iii) \Leftrightarrow (iv)$ follows from the definition of a conformal density.

Let us prove finally that $(iv) \Rightarrow (i)$ and that (iv) imples the relation (5.5). We suppose therefore that there is a point $x \in S$ such that the measure μ_x is absolutely continuous with respect to $\mathcal{H}_x$. Let ν be the (unique) conformal density of dimension d_+ such that $\nu_x = \mathcal{H}_x$. By symmetry, we have $\phi_\nu(x,n,\xi) = S_\lambda(n)$ for all $\xi \in \partial X$. Recall now that μ and ν have the same dimension d_+. Therefore we can apply to them the Fatou-type theorem of [CP1], and we obtain formula (5.5). By Proposition 1.4, we have $S_\lambda(n) \sim Ce^{-nd_-}$, which implies $\mathrm{gr}_\phi(\xi) = -d_-$. This proves (i). ∎

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