HEIGHTS FOR p–ADIC HOLOMORPHIC FUNCTIONS OF SEVERAL VARIABLES

by

Ha Huy Khoai

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Straße 26
D–5300 Bonn 3

Federal Republic of Germany

Institute of Mathematics
P.O. Box 631 Bo Ho
10000 Hanoi

Vietnam

MPI/89– 83

HEIGHTS FOR p–ADIC HOLOMORPHIC FUNCTIONS OF SEVERAL VARIABLES

by Ha Huy Khoai

§ 1. Introduction.

The relation between the growth of an entire function and its zeroes was studied early by Hadamard. R. Nevanlinna considered this problem for meromorphic functions and constructed a famous theory well–known as the value distribution theory. Nevanlinna theory in higher dimensions is constructed by Griffiths, King, Stoll, Carlson and others. In [2], [3], [5] we give a p–adic version of value distribution theory in one–dimensional case. In the present paper we consider the relation between the growth of a p–adic holomorphic function of several variables and the distribution of its zeros. This problem is a part of our plan to construct p–adic analog of Nevanlinna theory in higher dimensions. As we have mentioned in [4] this study is stimulated by the papers about the relation between Nevanlinna theory and number theory (see [6], [7]).

To generalize p–adic Nevanlinna theory to higher dimensions in [4] we introduced the notion of heights for p–adic meromorphic functions of one variable. In the present paper this notion is defined for p–adic holomorphic functions of several variables and used to prove an analog of the Poisson–Jensen formula. It is well–known that in the higher dimensional case the set of zeros of a holomorphic functions is not discrete. This makes it difficult to use analytical arguments. Here the Poisson–Jensen formula is described in terms of relations of global and local heights. Almost all of the arguments in this paper are easy "geometrically" but require longer proofs using the analytic definitions, which are often omitted.

The author would like to thank the Max–Planck–Institut für Mathematik in Bonn for hospitality and financial support.

§ 2. Heights for holomorphic functions of several variables.

2.1. Let p be a prime number, $\mathbb{Q}_p$ the field of p–adic numbers, and $\mathbb{C}_p$ the p–adic completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value in $\mathbb{Q}_p$ is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation in $\mathbb{C}_p$ which extends ord_p. Let D_1 be the open unit disk in $\mathbb{C}_p$:

$$D_1 = \{z \in \mathbb{C}_p;\ |z| < 1\} \ ,$$

and $D = D_1 \times \dots \times D_1$ be the "unit polydisk" in $\mathbb{C}_p^k$. Let $f(z_1,\dots,z_k)$ be a p–adic holomorphic function on D represented by a convergent series

$$f(z_1,\dots,z_k) = \sum_{|m|=0}^{\infty} a_{m_1 \dots m_k} z_1^{m_1} \dots z_k^{m_k} \ . \tag{1}$$

This means that for all $(z_1,\dots,z_k) \in D$ we have

$$\lim_{|m| \longrightarrow \infty} |a_m z^m| = 0 \ ,$$

where $a_m = a_{m_1 \dots m_k}$, $z^m = z_1^{m_1} \dots z_k^{m_k}$ and $|m| = m_1 + \dots + m_k$. For every $(t_1,\dots,t_k) \in \mathbb{R}_+^k$ we have

$$\lim_{|m| \to \infty} \{v(a_m) + m_1 t_1 + \dots + m_k t_k\} = \infty .$$

From this it follows that for every $(t_1,\dots,t_k) \in \mathbb{R}_+^k$ there exists $(m_1,\dots,m_k) \in \mathbb{N}^k$ for which $v(a_m) + \sum_{i=1}^{k} m_i t_i$ is minimal.

2.2 Definition. The height of the function $f(z_1,\dots,z_k)$ is defined by

$$H_f(t_1,\dots,t_k) = \min_{0 \le |m| < \infty} \{v(a_m) + \sum_{i=1}^{k} m_i t_i\} .$$

2.3 Definition. The group of functions mod $0(1)$ on D denoted by $\mathscr{H}(D)$ is defined by

$$\mathscr{H}(D) = \{\text{function } g : D \longrightarrow \mathbb{R}\} / \{\text{bounded functions}\}$$

2.4. Let f be a holomorphic function on D, the relative height associated to f is defined by the equivalent class of the following function in the group $\mathscr{H}(D)$:

$$\tilde{H}_f : D \longrightarrow \mathbb{R} \text{, where } \tilde{H}_f(z_1,\dots,z_k) = H_f(v(z_1),\dots,v(z_k))$$

2.5. We set

$$I_f(t_1,\dots,t_k) = \{(m_1,\dots,m_k) \in \mathbb{N}^k \ , \quad v(a_m) + \sum_{i=1}^{k} m_i t_i = H_f(t_1,\dots,t_k)\}$$

$$n_i^+(t_1,\dots,t_k) = \min\{m_i \mid \exists (m_1,\dots,m_i,\dots,m_k) \in I_f(t_1,\dots,t_k)\}$$

$$n_i^-(t_1,\dots,t_k) = \max\{m_i \mid \exists (m_1,\dots,m_i,\dots,m_k) \in I_f(t_1,\dots,t_k)\}$$

$$h_i^+(t_1,\dots,t_k) = n_i^+(t_1,\dots,t_k)t_i \ ,$$

$$h_i^-(t_1,\dots,t_k) = n_i^-(t_1,\dots,t_k)t_i$$

$$h_i(t_1,\dots,t_k) = h_i^-(t_1,\dots,t_k) - h_i^+(t_1,\dots,t_k)$$

$$h_f(t_1,\dots,t_k) = \sum_{i=1}^{k} h_i(t_1,\dots,t_k)$$

2.6. Definition. $h_f(t_1,\dots,t_k)$ is called the local height of the function $f(z_1,\dots,z_k)$ at $(t_1,\dots,t_k) = (v(z_1),\dots,v(z_k))$.

2.7. Remark. The local height induces a function on $\mathbb{C}_p^k$: $\tilde{h}_f : \mathbb{C}_p^k \longrightarrow \mathbb{R}$

$$\tilde{h}_f(z_1,\dots,z_k) = h_f(v(z_1),\dots,v(z_k))$$

2.8. Proposition. 1) If $\tilde{h}_f(z_1^0,\dots,z_k^0) \neq 0$ then $f(z_1,\dots,z_k)$ has zeros at $v(z_i) = v(z_i^0)$, $i = 1,\dots,k$.

2) If $\tilde{h}_f(z_1,\dots,z_k) = 0$ then $f(z_1,\dots,z_k) \neq 0$ and we have

$$|f(z_1,\dots,z_k)| = p^{-\tilde{H}_f(z_1,\dots,z_k)} \ .$$

2.9. We now give a geometrical interpretation of heights. Consider the function represented by the series (1). For ecah $(m_1,\dots,m_k)$ we draw the graph $\Gamma_{m_1\dots m_k}$ which depicts $v(a_m z^m)$ as a function of $(t_1,\dots,t_k)$. We obtain a hyperplane in $\mathbb{R}^{k+1}$:

$$\Gamma_{m_1\dots m_k} : t_{k+1} = v(a_{m_1\dots m_k}) + \sum_{i=1}^{k} m_i t_i \ .$$

Since $\lim_{|m|\to\infty} \{v(a_m) + \sum_{i=1}^{k} m_i t_i\} = \infty$ for each $(t_1,\dots,t_k) \in \mathbb{R}_+^k$ there exists a hyperplane which lies below any other one at $(t_1,\dots,t_k) \in \mathbb{R}_+^k$, i.e.

$$t_{k+1}(\Gamma_{m_1\dots m_k}) \le t_{k+1}(\Gamma_{m_1'\dots m_k'})$$

for all $\Gamma_{m_1'\dots m_k'}$. Let H be the boundary of the intersection of all parts in $\mathbb{R}_+^k$ of the half–spaces lying below the hyperplanes $\Gamma_{m_1\dots m_k}$. It is easy to see that if $(t_1,\dots,t_k, t_{k+1})$ is a point of H then we have

$$t_{k+1} = H_f(t_1,\dots,t_k)$$

2.10. Proposition. H is the boundary of a convex polyedron in $\mathbb{R}_+^k \times \mathbb{R}$.

2.11. Proposition. In the one–dimensional case $(k=1)$ the local height $h_f(t)$ is equals to the sum of valuations of zeros of $f(z)$ at $v(z) = t$.

2.12. Definition. A point $(t_1,\dots,t_k) \in \mathbb{R}_+^k$ is called a critical point of $f(z_1,\dots,z_k)$ if $h_f(t_1,\dots,t_k) \neq 0$.

2.13. <u>Proposition</u>. The set of critical points of the function $f(z_1,\dots,z_k)$ denotes by $\Delta(H)$ consists the sides of the polyedron H .

2.14. <u>Remark</u>. It is easy to see that for every finite parallelpiped of $\mathbb{R}_+^k$ $P = \{0 < r_i < t_i < s_i < +\infty , i = 1,\dots,k\}$, $H \cap P \times \mathbb{R}$ is consisted of the parts of a finite number of hyperplanes $\Gamma_{m_1\dots m_k}$. In fact, these are hyperplanes $\Gamma_{m_1\dots m_k}$ for which there exists at least one index i such that $m_i = n_i^{\pm}(t_1,\dots,t_k)$ for some $(t_1,\dots,t_k,t_{k+1}) \in P \times \mathbb{R}$.

2.15. <u>Example</u>. Consider the function

$$f(z_1,z_2) = \log(1+z_1) - \log(1 + z_2)$$

The simple computation gives us:

$$h_1(t_1,t_2) = \begin{cases} 0 & \text{if } t_1<t_2 \quad \text{or} \quad t_1>t_2\neq 1/\varphi(p^i) \ \forall i \\ 1 & \text{if } t_1>t_2=1/\varphi(p^i) \quad \text{for some} \quad i \\ p/p-1 & \text{if } t_1=t_2=1/\varphi(p^i) \quad \text{for some} \quad i \\ p^{i-1}t & \text{if } t_1=t_2=t \ , \quad 1/\varphi(p^i)<t<1/\varphi(p^{i-1}) \end{cases}$$

where $\varphi(n)$ is the Euler function, $\varphi(p^i) = p^i - p^{i-1}$

$$h_2(t_1,t_2) = \begin{cases} 0 & \text{if } t_2<t_1 \text{ or } t_2>t_1\neq 1/\varphi(p^i)\ \forall i \\ 1 & \text{if } t_2>t_1=1/\varphi(p^i) \text{ for some } i \\ p/p-1 & \text{if } t_2=t_1=1/\varphi(p^i) \text{ for some } i \\ p^{i-1}t & \text{if } t_1=t_2=t,\ 1/\varphi(p^i)<t<1/\varphi(p^{i-1}) \end{cases}$$

$$h(t_1,t_2) = \begin{cases} 0 & \text{if } t_2>t_1\neq 1/\varphi(p^i)\forall i \text{ or } t_1>t_2\neq 1/\varphi(p^i)\forall i \\ 1 & \text{if } t_1>t_2=1/\varphi(p^i) \text{ for some } i \text{ or} \\ & \quad t_2>t_1=1/\varphi(p^i) \text{ for some } i \\ 2p/p-1 & \text{if } t_1=t_2=1/\varphi(p^i) \text{ for some } i \\ 2p^{i-1}t & \text{if } t_1=t_2=t,\ 1/\varphi(p^i)<t<1/\varphi(p^{i-1}) \end{cases}$$

$$H_f(t_1,t_2) = \begin{cases} 1/p-1 + [\log_p(p-1)t_1] & \text{if } t_1\leq t_2 \\ \frac{1}{p-1} + [\log_p(p-1)t_2] & \text{if } t_1>t_2 \end{cases}$$

where $[x]$ denotes the largest integer being equals or less than x .

The set of critical points is

$$\left\{ \left[\frac{1}{\varphi(p^i)}, \frac{1}{\varphi(p^i)} + t_2\right],\ t_2 \in \mathbb{R}_+,\ i = 0,1,\dots \right\}$$

$$\cup \left\{ \left[t_1 + \frac{1}{\varphi(p^i)}, \frac{1}{\varphi(p^i)}\right],\ t_1 \in \mathbb{R}_+,\ i = 0,1,\dots \right\}$$

$$\cup \left\{ (t,t),\ t \in \mathbb{R}_+ \right\} .$$

§ 3. The Poisson–Jensen formula.

3.1. For every $(t_1,\dots,t_k) \in \mathbb{R}^k_+$ we denote

$$h_f(t_1,\dots,t_i^-,\dots,t_k) = \lim_{\epsilon \to 0} h_f(t_1,\dots,t_i-\epsilon,\dots,t_k) \;.$$

Note that the function $h_f(t_1,\dots,t_k)$ is continuous only at the points $(t_1,\dots,t_k)$ such that

$$h_f(t_1,\dots,t_k) = 0 \;,$$

but can be continuous in some variables separately while $h_f(t_1,\dots,t_k) \neq 0$.

3.2. Let $(t_1^0,\dots,t_k^0)$ and $(t_1,\dots,t_k)$ be two points of $\mathbb{R}^k_+$. We set

$$\begin{aligned}\delta_i = {} & h^{-\epsilon_i}(t_1,\dots,t_{i-1},t_i^{0-},\dots,t_k^{0-}) - \\ & - h_i^{\epsilon_i}(t_1,\dots,t_{i-1},t_i,t_{i+1}^{0-},\dots,t_k^{0-}) + \\ & \sum_{s_i} h_f(t_1^-,\dots,t_{i-1}^-,s_i,t_{i+1}^{0-},\dots,t_k^{0-}) \;,\end{aligned}$$

where $\epsilon_i = \mathrm{sign}(t_i^0-t_i)$ and the sum in the right extends over all $s_i \in (t_i^0,t_i)$.

3.3. Theorem (the Poisson–Jensen formula)

$$H_f(t_1^0,\dots,t_k^0) - H_f(t_1,\dots,t_k) = \sum_{i=1}^{k} \epsilon_i \delta_i$$

3.4. Remark. When $k=1$ we have the Poisson–Jensen formula proved in [4].

3.5. Remark. Theorem 3.3 is not symmetric in variables $t_1,...,t_k$, and we obtain several formulas to express the global height in terms of local heights. From this one can deduce the equalities between local heights. This fact is similar to the one in the case of holomorphic functions of two complex variables (see H. Cartan [1]).

3.6. Remark. In view of Theorem 3.3 the relative height $\tilde{H}_f$ induced by f depends only on the local height.

3.7. Theorem 3.3 is proved by using the following remarks.

1) For every finite parallelpiped P (see 2.4) and every hyperplane L in general position $L \cap H \cap P$ is a hyperplane of $k-1$ dimension.

2) If the hyperplane $t_i = s_i = \text{const}$ is not in general position then the hyperplane $t_i = s_i - \epsilon$ is in general position with ϵ enough small. On the other hand we have

$$\lim_{\epsilon \to 0} H_f(\ldots,s_i-\epsilon,\ldots) = H_f(\ldots,s_i,\ldots) \; .$$

3) The set of critical points $\Delta(H)$ is a union of planes of dimensions equal or less than $k-1$ (see 2.12, 2.13).

4) Suppose $S = S_1 \cup \ldots \cup S_{k-1}$, where S_i is the hyperplane $t_i = s_i$, $i = 1,\ldots,k-1$. By replacing S_i by $S_i^- : t_i = s_i - \epsilon$ one can assume that S_i are in general position. Then the intersection $S \cap \Delta(H) \cap P$ is a finite set of points.

5) By the remarks above the proof of Theorem 3.3 is reduced to the proof of Poisson–Jensen formula on one–dimensional case (see [4]).

REFERENCES

[1] H.Cartan. Sur la fonction de croissance attachée à une fonction méromorphe de deux variables, et ses applications aux fonctions méromorphes d'une variable. C.R. Acad. Sc. Paris, 189 (1929), 521–523.

[2] Ha Huy Khoai. On p–adic meromorphic functions. Duke Math. J., Vol. 50, 1983, 695–711.

[3] Ha Huy Khoai. Sur la théorie de Névanlinna p–adique. Groupe d'Etude d'Analyse ultramétrique, Paris, 1988.

[4] Ha Huy Khoai. Heights for p–adic meromorphic functions and Value distribution theory. Max–Planck–Institut für Mathematik Bonn. MPI 89–76 (1989).

[5] Ha Huy Khoai and My Vinh Quang. On p–adic Nevanlinna Theory. Lecture Notes in Math. 1351, 1988, p.

[6] S. Lang. Introduction to Complex Hyperbolic Spaces. Springer 1986.

[7] P. Vojta. Diophantine Approximations and Value Distribution Theory. Lecture Notes in Math., 1239, 1986.