Syzygies of monomial curves and a linear diophantine problem of Frobenius

by

Marcel Morales

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

West Germany

Institut Fourier
Univ. Grenoble I
B.P. 74
38402-St. Martin d'Hères

France

MPI/87- 48

Introduction

Consider $n_1,\dots,n_d \in \mathbb{N}$ without common factor, then for any $n \in \mathbb{N}$ large enough $n \in \Gamma = < n_1,\dots,n_d >$ the semigroup generated by $n_1,\dots,n_d$. The Frobenius's change money problem consists to find the biggest $g \in \mathbb{N} - \Gamma$ in function of $n_1,\dots,n_d$. This problem has been studied from the combinatorial point of view and only are known the solutions for $d = 2$ (Sylvester) and $d = 3$ (Rödseth). In fact using the terms of index of regularity one can remark that g is the index of regularity of the Hilbert function of the ring $k[t^{n_1},\dots,t^{n_d}]$ and can be interpreted by using syzygies, see [Mo], in fact the reader can see all the results in this like one generalisation paper of the Frobenius's problem: namely describe using combinatorics the syzygies of a monomial curve. Here we give positive solutions for affine and projective monomial curves in k^3 and $\mathbb{P}^3$ respectively and an easy criterium for a projective monomial curve in $\mathbb{P}^3$ to be arithmetically Cohen-Macaulay in terms of its semigroup.

I thank the Max-Planck-Institut for its generous hospitality, Monique Lejeune for her encouragement and S. Sertöz, A. Özlük to write me about this problem and G. Valla for interesting discussions.

§ 0.

0.1 - Consider a monomial curve $A = k[t^{a_1},\dots,t^{a_d}]$ given parametrically by $X_1 = t^{a_1},\dots,X_d = t^{a_d}$, this is a graded ring and the Hilbert function $H(n) = \dim_k A_n$ takes values 0 or 1. Further if $a_1,\dots,a_n$ are relatively prime, then $H(n) = 1$ for all n large. The smallest number g such that $H(n) = 1$ for all $n > g$ is called the index of regularity of the Hilbert function. Then it is clear that g is the biggest integer $g \notin \Gamma$ where Γ is the semigroup generated by $a_1,\dots,a_d$. Sometimes it is called the Frobenius number.

0.2 - Now consider A as a quotient of the weighted polynomial ring $R = k[X_1,\dots,X_d]$ where weight $(X_i) = a_i$ then as an R-module A has a syzygies (i.e. free resolution)

$$0 \to \bigoplus_i R[-n_{d-1,i}] \to \bigoplus_i R[-n_{d-2,i}] \to \dots \to R \to A \to 0$$

and we have the relation

$$g = \max_i \left\{u_{d-1,i}\right\} - \sum_{i=1}^{d} a_i .$$

For more about index of regularity see also [Mo].

§ 1. Some generalities about equations and relations of curves

1.1. - Let C a curve given parametrically in

$$k[X_1,\dots,X_d] \text{ by } X_1 = t^m,\ X_2 = \varphi_2(t),\dots,X_d = \varphi_d(t) .$$

Call P the ideal of this curve in $k[X_1,\dots,X_d]$. Consider $\alpha \in \mathbb{N}$ such that $(\alpha,m) = 1$. And the curve $\tilde{C}$ given parametrically in $k[X_1,\dots,X_d]$ by $X_1 = t^m$, $X_2 = \varphi_2(t^\alpha),\dots,X_d = \varphi_d(t^\alpha)$, call $\tilde{P}$ the ideal of this curve in $k[X_1,\dots,X_d]$. Also for one element $f \in k[X_1,\dots,X_d]$ we put $\tilde{f}$ the element $\tilde{f} := f(X_1^\alpha,X_2,\dots,X_d)$.

1.1.1 - Remark. We can also replace the ring of polynomials by the ring of convergent power series or the ring of formal power series.

1.2 - Lemma. $f \in P \Leftrightarrow \tilde{f} \in \tilde{P}$. In particular if $f_1,\dots,f_s$ is a minimal system of generators of P , then $\tilde{f}_1,\dots,\tilde{f}_s$ is a minimal system of generators of $\tilde{P}$. As a consequence P is a complete intersection if and only if $\tilde{P}$ is.

1.2.1 - Remark. The proof of all the facts in this section comes from the following trivial claim, where R is the ring of polynomials. Any $f \in R$ can be written as

$$f = \tilde{f}_0 + X_1\tilde{f}_1 + \dots + X_1^{\alpha-1}\tilde{f}_{\alpha-1}$$

where $f_i \in R$.

1.2.2 Proof of 1.2. - By definition P (resp. $\tilde{P}$) is the Kernel of the morphism $R \longrightarrow k[t]$ sending X_1 to t^m and X_i to $\varphi_2(t)$ (resp. X_i to $\varphi_i(t^\alpha)$) and this proves the first assertion in 1.2.

In order to prove the second assertion, we take $f \in \tilde{P}$ and we write it like in 1.2.1

$$f = \tilde{f}_0 + X_1\tilde{f}_1 + \dots + X_1^{\alpha-1}\tilde{f}_{\alpha-1}$$

then we claim that $\tilde{f}_i$ are such that $f_i \in P$, we replace the parametrisation of $\tilde{C}$ in f , but then the powers of t in $\tilde{f}_0(X_1(t),\dots,X_d(t))$ are in $\alpha\mathbb{Z}$, those of $X_1^i\tilde{f}_i$ in $mi + \alpha\mathbb{Z}$ $1 \leq i \leq \alpha-1$, this implies that $\tilde{f}_i \in \tilde{P}$ for any i , in particular after the first assertion in 1.2 $f_i \in P$, and this is enough to prove the second assertion in 1.2.

1.3 Proposition. - We consider a syzygies of the ideal $P \subset k[X_1,\dots,X_d]$ then a syzygies of $\tilde{P}$ is obtained by changing X_1 by X_1^α in any matrix syzygies of P . In particular if the resolution for P is minimal so it is for $\tilde{P}$ and the Betti numbers of P and $\tilde{P}$ are the same. If P is the ideal of a monomial curve then the shifts in the graded syzygies of $\tilde{P}$ are obtained from those of P .

<u>Proof</u>. - 1.2 says us that if P is generated by $f_1,\dots,f_s$ then $\tilde{P}$ is generated by $\tilde{f}_1,\dots,\tilde{f}_s$, that means that the proposition is true at the first step of syzygies. Now suppose that there is a relation

$$\sum_{i=1}^{s} g_i\tilde{f}_i = 0 \quad \text{with} \quad g_i \in R$$

then we can use 2.1.1 to write

$$g_i = \tilde{g}_{0,i} + X_1\tilde{g}_{1,i} + \dots + X_1^{\alpha-1}\tilde{g}_{\alpha-1,i}$$

this implies that

$$\sum_{i=1}^{s}\tilde{g}_{0,i}\tilde{f}_i + X_1 \sum_{i=1}^{s}\tilde{g}_{1,i}\tilde{f}_i + \dots + X_1^{\alpha-1} \sum_{i=1}^{s}\tilde{g}_{\alpha_1-1,i}\tilde{f}_i = 0$$

in $R = k[X_1,\dots,X_d]$

but monomials in each sum are different from other sums because the powers of X_1 are not in the same set. In particular we conclude that

$$\sum_{i=1}^{s}\tilde{g}_{j,i}\tilde{f}_i = 0 \ \forall\, j \quad \text{in} \quad R$$

but this implies $\sum_{i=1}^{s} g_{ji}f_i = 0$ in R, that means that the relations in the $\{\tilde{f}_1,\dots,\tilde{f}_s\}$ are generated by the relations in $\{f_1,\dots,f_s\}$ just by changing X_1 by X_1^{α}.

Now if K_n is the n-syzygies of P (resp. $\tilde{K}_n$ the n-syzygies of $\tilde{P}$ and if we suppose that $\tilde{K}_n$ is generated by a set of generators of K_n just by changing X_1 by X_1^{α}, then the same method apply to show that $\widetilde{K}_{n+1}$ is generated by K_{n+1} just by changing X_1 by X_1^{α}. This gives the proof of the Proposition.

1.4 Definition. - Let $C = \{C_q, \partial_q\}$ and $C' = \{C'_q, \partial'_q\}$ be two cochain complex over R. The tensor product $(C \otimes C')$ is the cochain complex defined by

$$(C \otimes C')^n = \bigoplus_{i+j=n} (C^i \otimes C'^j)$$

with the differential ε^n

$$\varepsilon^n : C^i \otimes C'^j \longrightarrow (C \otimes C')^{n-1}$$

$$(a \otimes b) \longmapsto \partial_i\, a \otimes b + (-1)^i\, a \otimes \partial'_j\, b \ .$$

The following lemma is well known, also it will be useful in the next section.

1.5 Lemma. - Consider a prime ideal P in a regular ring R (here $R = k[X_1,\dots,X_d]$) and suppose that we add a new variable W and a new equation $W-\varphi(X_1,\dots,X_d)$. Call P_1 the ideal in $R_1 = k[X_1,\dots,X_d,W]$ generated by a basis of P and $W-\varphi(X_1,\dots,X_d)$. Let S° be the syzygies of P over $k[X_1,\dots,X_d]$, we also call S° the syzygies of P over $k[X_1,\dots,X_d,W]$. Then the syzygies S_1° of P_1 is $S_1^\circ := S^\circ \otimes (0 \longrightarrow R_1 e \xrightarrow{G} R_1 \longrightarrow 0)$ where $G = W-\varphi(X_1,\dots,X_d)$.

1.6 Example. - Consider the monomial curve given parametrically in the fourth dimensional space by $X = t^4$, $Y = t^6$, $Z = t^7$, $U = t^9$. Call P the prime ideal defining this curve in $k[X,Y,Z,U]$. It is well known that P is generated by the following elements:

$$
\begin{aligned}
a &= X^3 - Y^2 \\
b &= XU - YZ \\
c &= X^2Y - Z^2 \\
d &= X^2Z - YU \\
e &= XY^2 - ZU \\
f &= Y^3 - U^2 \quad .
\end{aligned}
$$

Let $\varphi(X,Y,Z,U) = YZ = t^{13}$, then after 1.3 and 1.5 we can say that the prime ideal $\tilde{P}$ of the curve given parametrically by $X = t^{4\alpha}$, $Y = t^{6\alpha}$, $Z = t^{7\alpha}$, $U = t^{9\alpha}$, $W = t^{13}$ is generated by

$$\tilde{P} = (a,b,c,d,e,f,g = W^{\alpha} - XY)$$

for any α natural number prime with 13.

This is a powerful method to find syzygies without complicated calculations.

1.7 Example 2. - Consider a monomial curve C in k^3, given parametrically by $X = t^a$, $Y = t^b$, $Z = t^c$, a,b,c being three natural numbers. In order to study equations of this curve we can assume that a,b,c are relatively prime. Secondly by using 1.3 we can assume that a,b,c are coprime two by two. Incidentally we see that if one of the three numbers say c belongs to the semigroup generated by a,b, then $c = ma + nb$, with $m,n \in \mathbb{N}$, and we can write $Z = X^mY^n$. 1.5 says us that C is a complete intersection (in fact a plane curve).

§ 2. Monomial curves in k^3

2.0 - Let $a,b,c \in \mathbb{N}$ such that $(a,b,c) = 1$ and that $k[t^a,t^b,t^c]$ be a curve C of embedding dimension 3. Let $R = k[X,Y,Z]$ with grading given by weight $(X) = a$, weight $(Y) = b$, weight $(Z) = c$. After J. Herzog ([He], [K]) we know that if $k[t^a,t^b,t^c]$ is not a complete intersection, his syzygies are like follows

$$0 \longrightarrow R^2 \xrightarrow{M} R^3 \longrightarrow R \longrightarrow k[t^a,t^b,t^c] \longrightarrow 0$$

$$X \longmapsto t^a$$

$$Y \longmapsto t^b$$

$$Z \longmapsto t^c \quad .$$

Now we improve this result giving explicitely the matrix M . The equations of the curve C in k^3 being the 2×2 minors of the matrix M . The problem to find M is in fact equivalent to the solution of the Frobenius's change money problem in dimension 3 (cf. [Rö]):

<u>Problem</u>. - Find the biggest $g \in \mathbb{N}$ who can't be written

$$g = \alpha a + \beta b + \gamma c \ , \text{ with } \alpha,\beta,\gamma \in \mathbb{N} \ .$$

2.1 - In order to relate both problems we introduce the Apery sequences. Let Γ be the semigroup generated by a,b,c and

$s \in \Gamma$, then the Apery sequence $\Gamma(s)$ is by the definition

$$\Gamma(s) = \{l \in \Gamma / l - s \notin \Gamma\} .$$

2.1.1 - Remark. - Let $(a,b) = d$ and $\Gamma' = \langle a/d,\ b/d \rangle$ the semigroup generated by a/d and b/d . Suppose that

$$\Gamma'(a/d) = \{y\ b/d + zc \ /\ (y,z) \in \mathcal{D}\}$$

then

$$\Gamma(a) = \{y\ b + izc \ /\ (y,z) \in \mathcal{D},\ 1 \leq i \leq d\} .$$

2.1.2 - It follows from remark 2.1.1 that in order to describe the Apery sequence $\Gamma(a)$ we can consider only the case where $(a,b) = 1$.

Let s_0 be the unique integer such that

$$s_0 b \equiv c \bmod a \quad \text{and} \quad 0 < s_0 < a$$

put $s_{-1} := a$ and consider the Jung-Hirzebruch continuous fraction

$$\begin{aligned} s_{-1} := a &= q_1 s_0 - s_1 \qquad\qquad q_i \geq 2,\ s_i \geq 0 \\ s_0 &= q_2 s_1 - s_2 \\ &\dots \\ s_{m-1} &= q_{m+1}\ s_m \end{aligned}$$

and the sequences P_i, R_i defined by

$$P_{-1} = 0,\ P_0 = 1,\ P_{i+1} = P_iQ_i - P_{i-1}$$
$$R_i = \frac{s_ib - P_ic}{a},\ R_{-1} = b,\ R_0 = \frac{s_0b-c}{a}\ .$$

Then $\{s_i\}$ and $\{R_i\}$ are strictly decreasing sequences, P_i is a strictly increasing sequence of integers; R_{m+1} is a negative integer, and for any i we have $s_iP_{i+1} - s_{i+1}P_i = a$.

2.2 - Definition. - Let ν the unique integer number s.t.

$$R_\nu+1 \leq 0 < R_\nu\ .$$

This is equivalent to saying that $\frac{s_{\nu+1}}{P_{\nu+1}} \leq \frac{c}{b} < \frac{s_\nu}{P_\nu}$. Now we can state the nice theorem of Rödseth.

2.3 - Theorem ([Rö]). - With the above notations, any element $\theta \in \Gamma(a)$ can be written $\theta = yb + zc$, we suppose that z is the minimal with this property and when this is the case the pair (y,z) is unique. Now let

$$A = \{(y,z)/0 \leqq y < s_\nu - s_{\nu+1},\ 0 \leqq z < P_{\nu+1}\}$$

$$B = \{(y,z)/s_\nu - s_{\nu+1} \leq y < s_\nu,\ 0 \leq z < P_{\nu+1} - P_\nu\}$$

then $\Gamma(a) = \{yb + zc/(y,z) \in A \cup B\}$.

And now we can describe the syzygies of a monomial curve C given by $X = t^a,\ Y = t^b,\ Z = t^c$.

2.4 - Theorem. - Let a,b,c three natural numbers if a,b are not coprime by using 1.3 we can suppose $(a,b) = 1$; then with the above notations the matrix syzygies for the curve C in k^3 is:

$$M = \begin{pmatrix} X^{R_\nu} & Y^{s_\nu - s_{\nu+1}} \\ Y^{s_{\nu+1}} & Z^{P_\nu} \\ Z^{P_{\nu+1}-P_\nu} & X^{-R_{\nu+1}} \end{pmatrix} .$$

Moreover the curve C is a complete intersection if one of $R_{\nu+1}$, P_ν or $s_{\nu+1}$ is null.

Proof. - By definition of the sequences s_i, P_i, R_i we have the relations

(1) $P_{\nu+1}c = bs_{\nu+1} + (-R_{\nu+1})a$

(2) $s_\nu b = R_\nu a + P_\nu c$

(3) $(R_\nu - R_{\nu+1})a = (s_\nu - s_{\nu+1})b + (P_{\nu+1} - P_\nu)c$

the third relation is a consequence of the first two. Also we use the relation

(4) $s_\nu P_{\nu+1} - s_{\nu+1}P_\nu = a$.

Now by [He] p.10 it is enough to prove the following claims:

Claim I. - $P_{\nu+1}c$ is the least multiple of c in $\langle a,b \rangle$.

<u>Claim II</u>. - i) $s_\nu b$ is the least multiple of b in $<a,c>$

or ii) $(R_\nu - R_{\nu+1})a$ is the least multiple of a in $<b,c>$.

<u>Proof of Claim I</u>. - Suppose that there exists γ, $0<\gamma<P_{\nu+1}$ such that $\gamma c \in <a,b>$, then $(0,\gamma) \in A \cup B$ by 2.3 and $\gamma c \in \Gamma(a)$, i.e. $\gamma c = \lambda b$ this contradicts the minimality condition on the coefficient given to define A and B (see 2.3).

<u>Proof of Claim II</u>. - Suppose that there exists γ, $0<\gamma<s_\nu$ such that $\gamma b \in <a,c>$, γ minimal with this property by 2.3 $(\gamma,0) \in A \cup B$, and $\gamma b = \lambda c$, using the Claim I $\lambda \geqq P_{\nu+1}$ and we can write $\gamma b = (\lambda - P_{\nu+1})c + bs_{\nu+1} + (-R_{\nu+1})a$. But $\gamma b \in \Gamma(a)$ then $-R_{\nu+1} = 0$. If $R_{\nu+1} \neq 0$ we get a contradiction and we have proved the part i) of the Claim II. If $R_{\nu+1} = 0$, because γ is minimal we must have $s_{\nu+1} = 0$ and $P_{\nu+1} = 0$ or $\gamma = s_{\nu+1}$ and $\lambda = P_{\nu+1}$ the first relation is impossible by definition of s_i, P_i . From the relation $P_{\nu+1}c = s_{\nu+1}b$ we get that $s_{\nu+1} = \delta \frac{c}{\alpha}$, $P_{\nu+1} = \delta \frac{b}{\alpha}$, where $\alpha = (b,c)$ and $\delta = (s_{\nu+1}, P_{\nu+1})$ and by (4) that $(s_\nu - s_{\nu+1})\delta(b/\alpha) + (P_{\nu+1} - P_\nu)\delta(c/\alpha) = a$ in particular δ divides a . Now using (3) we obtain

$$\delta R_\nu = \alpha \text{ , i.e. } \delta \text{ divides } \alpha = (b,c)$$

this implies $\delta = 1$ by hypothesis. Now it is clear that if $\gamma a \in <b,c>$ then α divides γ ; this implies that $R_\nu a$ is the least multiple of a in $\langle b,c \rangle$ and finish the proof of 2.4. Note that $R_\nu = R_\nu - R_{\nu+1}$ because in this case $R_{\nu+1} = 0$.

The last affirmation in 2.4 is clear from the description of the matrix syzygies.

2.5 - Remark. - The results in this section are used in [Mo-1] to construct a large series of examples where the symbolic Rees ring $\oplus P^{(n)}$ is noetherian.

§ 3. Syzygies of projective monomial curves in $\mathbb{P}^3$

3.0 - We consider the projective curve given by the parametrisation $X = u^a$, $Y = u^b t^{a-b}$, $Z = u^c t^{a-c}$, $W = t^a$ when a,b,c are natural numbers $a>b>c>0$. Related with this projective curve we have two associated affine curves putting respectively $u = 1$ or $t = 1$

$$C : X = u^a,\ Y = u^b,\ Z = u^c$$
$$D : Y = t^{a-b},\ Z = t^{a-c},\ W = t^a .$$

Many people have studied the question of finding the syzygies of a projective monomial curve, specially Bresinsky and Renschuch, now using the results on section two it is possible to give equations and syzygies in function of the invariants introduced there. This algorithm is of special interest in cases where computer fails because of high number of computations. Using the algorithm in section 2 and then Bresinsky-Renschuch [B-R] and Bresinsky [B]; computation of syzygies needs only fourth operations on integer numbers.

3.1 <u>Proposition</u>. - Consider the morphism $\Phi : k[X,Y,Z,W] \to k[u,t]$ given by $X = u^a$, $Y = u^b t^{a-b}$, $Z = u^c t^{a-c}$, $W = t^a$ where $a>b>c$ and we assume $(a,b) = 1$. The notations are those introduced in section 2 and we put $R_j' = s_j - P_j - R_j$ for any j. Then the ideal $\mathrm{Ker}\Phi$ is generated by the polynomials

$$\begin{cases} Z^{P_{\nu+1}} - Y^{s_{\nu+1}} X^{-R_{\nu+1}} W^{-R'_{\nu+1}} \\ X^{R_\nu - R_{\nu+1}} W^{R'_\nu - R'_{\nu+1}} - Y^{s_\nu - s_{\nu+1}} Z^{P_{\nu+1} - P_\nu} \\ X^{R_\nu} Z^{P_\nu} - Y^{s_\nu} W^{-R'_\nu} \end{cases}$$

$$\begin{cases} Y^{2s_\nu - s_{\nu+1}} Z^{P_{\nu+1} - 2P_\nu} - X^{2R_\nu - R_{\nu+1}} W^{2R'_\nu - R'_{\nu+1}} \\ Y^{(q_{\nu+1}-1)s_\nu - s_{\nu+1}} Z^{P_{\nu+1} - (q_{\nu+1}-1)P_\nu} - X^{(q_{\nu+1}-1)R_\nu - R_{\nu+1}} \\ W^{(q_{\nu+1}-1)R'_\nu - R'_{\nu+1}} \\ \quad \cdots \\ W^{-R'_{\nu-1}} Y^{s_{\nu-1}} - X^{R_{\nu-1}} Z^{P_{\nu-1}} \quad . \end{cases}$$

We repeat this last group by changing ν by $\nu-1$ and so on. The process stops when we find $R'_i \geq 0$, then the last equation will be

$$Y^{s_i} - X^{R_i} Z^{P_i} W^{R'_i}$$

instead of $W^{-R'_i} Y^{s_i} - X^{R_i} Z^{P_i}$.

3.1.1 - The proof is a direct consequence of our algorithm described in section 2 and Bresinsky-Renschuch [B-R]. The syzygies follows from these equations by using Bresinsky [B].

3.2 - In the case where $(a,b) \neq 1$ we can give the following receapt to find the equations:

Let $\lambda := (a,b)$ and $\bar{a} = a/\lambda$, $\bar{b} = b/\lambda$. Let $\bar{s}_i$, $\bar{P}_i$, $\bar{R}_i$ be the sequences associated to $\bar{a},\bar{b},c$ using 2.1.2. Then we define the sequences s_i, P_i, R_i associated to a,b,c by:

$$s_i = \bar{s}_i,\ P_i = \lambda \bar{P}_i,\ R_i = \bar{R}_i \quad \text{for all } i .$$

Put $R_i' = s_i - P_i - R_i$ and the equations obtained with these values in 3.1 are a minimal basis defining the curve $X = u^a$, $Y = u^b t^{a-b}$, $Z = u^c t^{a-c}$, $W = t^a$.

3.3 - Corollary. - The ring $B = [u^a, u^b t^{a-b}, u^c t^{a-c}, t^a]$ is a Cohen-Macaulay ring if and only if $R_\nu' \geq 0$. In this case the matrix syzygies is given by

$$\begin{pmatrix} X^{R_\nu} W^{R'_\nu} & Y^{s_\nu - s_{\nu+1}} \\ Y^{s_{\nu+1}} & Z^{P_\nu} \\ Z^{P_{\nu+1} - P_\nu} & X^{-R_{\nu+1}} W^{-R'_{\nu+1}} \end{pmatrix} .$$

Proof. - This follows from the above proposition and the well known fact (see for example [S-V1] pp. 167).

3.3.1 - Lemma. - Let C be a monomial curve in $\mathbb{P}^3_k$, then C is arithmetically Cohen-Macaulay if and only if the ideal defining C in the ring k[X,Y,Z,W] has at most three generators.

3.3.2 - Remark. - Corollary 3.3 answers a question of Stückrad and Vogel ([S-V2] p. 101); namely say when the ring B is Cohen Macaulay in terms of a,b,c . Note also that this corollary make superflows case i of Lemma 2 [S-V2].

Bibliography

[B] H. Bresinsky - Minimal free resolutions of monomial curves in $\mathbb{P}_k^3$. Linear Algebra and its App. 59, 121-129 (1984).

[B-R] H. Bresinsky and B. Renschuck - Basisbestimmung Veronescher Projektionsideale mit allgemeiner Nullstelle $(t_0^m, t_0^{m-r}, t_1^r, t_0^{m-s}, t_1^s, t_1^m)$, Math. Nachr. 96: 257-269 (1980).

[He] J. Herzog - Generators and relations of Abelian semigroups and semigroups rings. Man. Math. 3: 23-26 (1970).

[K] E. Kunz - Introduction to Commutative Algebra and Algebraic Geometry. Birkhäuser 1985.

[Mo] M. Morales - Fonctions de Hilbert, genre géométrique d'une singularité quasi-homogène. CRAS Paris t 301, Série I, n° 14, 1985.

[Mo-1] M. Morales - Noetherian Symbolic blow up and examples in any dimension, Preprint.

[Rö] Ö.J. Rödseth - On a linear diophantine problem of Frobenius. J. für die Reine und Angew. Math. 301 (1978) 171-178.

[S-V1] J. Stückrad and W. Vogel - Buchsbaum Rings and Applications. Springer Verlag 1986.

[S-V2] J. Stückrad and W. Vogel - On the number of equations defining an algebraic set of zeros in n-space. In Seminar Eisenbud/Singh/Vogel vol.2 Band 48 Teubner-Texte zur Mathematik, 1982.