INTRODUCTION TO THE THEORY OF WEIGHTED PROJECTIVE SPACES

by

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INTRODUCTION. As the title suggests, this paper is devoted to the description of the main features of the so called weighted projective spaces (shortly w.p.s.). A w.p.s. is a special projective scheme, which arizes as the projective spectrum of the polynomial ring over a field k, with the extra condition that the degrees (weights) of the variables are arbitrary positive integers. If $Q = (q_0, \ldots, q_r)$ is the set of weights, we denote by $\mathbb{P}(Q)$ the associated w.p.s. and of course if $Q = (1, 1, \ldots, 1)$ we recover the usual projective spaces; clearly this generalization leads to a great deal of questions, but first of all let us try to explain what are good motivations for the study of such varieties.

For, it may be interesting to give some historical hints. While the origin of the theory is not fixed with absolute precision, it shouldn't be far from the truth to say that the first time w.p.s. appear in the literature is in [A]. Since then a great deal of interest was put on these varieties, which are shown in [Mo] to be the natural ambient where some problems of complete intersections have a natural solution (see Remark 1 after theorem 7.3) and which are fully investigate in [De], mainly in connection with the theory of duality and the purpose of giving good classes of Gorenstein rings. More or less at the same time a manuscript of Dolgachev, which was published in 1982, see [D], provides a main source for general informations on w.p.s.. It also gives some generalizations to w.p.s. of classical theorems on $\mathbb{P}^r$, such as a version of the Lefschetz theorem for complete intersections and the Hodge structure on cohomology of weighted hypersurfaces. Since then there was a growth of interest on this subject from several points of view. For instance we want to mention

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the papers [C] and [D1], where w.p.s. are used as a strong tool in the classification of algebraic surfaces, and [Fu] where the problem of classifying some polarized manifolds with given sectional genus gets the solution in a w.p.s. Other kind of applications are given in [R2], namely it is shown how to use w.p.s. in order to produce good classes of factorial and almost factorial rings (we refer to this paper for a more detailed explanation of the connection between these algebraic properties and the theory of w.p.s.). Another recent source of informations is [Am] where for instance some computations of the divisor class group and the Picard group of $\mathbb{P}(Q)$ are given.

Many other applications are pointed out in the text; however what is essential to say is the following: w.p.s. share with $\mathbb{P}^r$ a lot of good properties, therefore if one recognizes that a projective variety is for instance a hypersurface or a complete intersection in a w.p.s., then he may draw a lot of consequences from the general theorems which are the subject of the present paper.

To draw an organized picture of this beautiful theory we have divided our paper in seven sections plus an appendix which deals with some elementary facts connecting reflexive sheaves and Weil divisors. The first section is devoted to recalling some fundamental things on the action of a finite group on an algebraic variety and then we explain the relationship between invariants and quotient varieties. In the second section we put our attention on the link between the graduations on a ring A and the actions of $\mathbb{G}_m$ on Spec(A); this leads to the geometric notion of quasicone and a theorem of Flenner (see Th. 2.6) allows us to relate properties of quasicones associated to a graded ring A and properties of Proj(A) (see 2.7). As a special case of this picture we have the w.p.s. $\mathbb{P}(Q)$, for which the associated quasicone is the usual affine space, but a different graduation gives rise to a different action of $\mathbb{G}_m$ on Spec(A), hence to a different quotient. This is explained in the third section, where some properties of $\mathbb{P}(Q)$ are examined; for instance it is shown that $\mathbb{P}(Q)$ can be considered as a quotient of $\mathbb{P}^r$ by the action of a finite group and the associated projection is studied (see 3A.5). Moreover some basic cohomological properties and a technique of reduction and normalization of weights are explained. Properties of the sheaves $\mathcal{O}_{\mathbb{P}(Q)}(n)$, which first appear in section 3, are better investigated in section 4, where also ampleness criteria of Delorme are given. Section five is entirely devoted to the study of an open subset $\mathbb{P}^o(Q)$ of $\mathbb{P}(Q)$, which was introduced by Mori and is, in some sense, the "true analogous" to $\mathbb{P}^r$. Namely on $\mathbb{P}^o(Q)$ the sheaves $\mathcal{O}_{\mathbb{P}(Q)}(n)$ behave well, while in the third section it was shown that on $\mathbb{P}(Q)$ the sheaves $\mathcal{O}_{\mathbb{P}(Q)}(n)$ have a lot of "pathologies". Section six deals with differentials and dualizing sheaves on $\mathbb{P}(Q)$ and some ideas of Dolgachev and Delorme are used to give a proof that the dualizing sheaf of a complete intersection X in $\mathbb{P}(Q)$ is $\mathcal{O}_X(\sum_i d_i - |Q|)$, where the d_i's are the degrees of the hypersurfaces which define X and $|Q|$ is the sum of the weights. In the last section we first prove a theorem which gives a full description of the Divisor Class Group and the Picard Group of $\mathbb{P}(Q)$; then we recall the weighted version, given by Mori of the classical Lefschetz theorem on complete intersections and then we use it to give a generalization to w.p.s. of an old result of Robbiano (see [R1]).

With some exceptions the treated topics are contained in the literature, but we want to point out that our main purpose was that of producing a unified treatment of the theory and of providing almost everywhere full proofs and good connections among the various sources. Further it may be worth to mention that along the paper the reader can find a good deal of examples, remarks and questions.

Throughout the paper, k denotes an algebraically closed field of characteristic 0.

The content of this paper was the subject of several talks given by the second author in Bonn, Köln , Bochum, Osnabrück while he was Visiting Professor at the Max-Planck-Institut Für Mathematik (Bonn) during the winter semester 1984/85.

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§ 1. Quotient varieties by the action of a finite group .

DEFINITION 1.1. Let S be a set, G a group with operation denoted by $\cdot$, and identity element denoted by 1.

A map $G \times S \longrightarrow S$, $(g, s) \longmapsto gs$, such that $(g_1 \cdot g_2)\, s = g_1 \cdot (g_2 s)$ and $1s = s$ for every $g_1, g_2 \in G$, $s \in S$ is said to be an action of G on S; it is also said that G acts on S and if $gs \neq s$ for every $g \neq 1$ and every s, then G is said to act freely on S.

DEFINITION 1.2. If G is a group acting on S, the bijective map $T_g : S \longrightarrow S$, $s \longrightarrow gs$, is called a translation and it is easy to see that the map $G \longrightarrow \mathrm{Aut}(S)$, $g \longmapsto T_g$ is a homomorphism, which gives a representation of G inside the group Aut(S) of the permutations of S.

DEFINITION 1.3. If G is a group acting on S, and $s \in S$ we denote by G_s the set $\{gs \; ; \; g \in G\}$ and we call it the orbit of s. It is clear that orbits make a partition on S.

DEFINITION 1.4. Let G be a variety (not necessarily irreducible) which is a group. Then G is said to be an algebraic group if the two maps

$$G \times G \longrightarrow G \qquad\qquad G \longrightarrow G$$

$$(x, y) \longmapsto x \cdot y \qquad\qquad x \longmapsto x^{-1}$$

are morphisms of varieties.

DEFINITION 1.5. Let G be an algebraic group and X a variety. If G acts on X in such a way that the map $G \times X \longrightarrow X$ is a morphism of varieties, then we say that G acts morphically on X, or if no confusion arises, that G acts on X.

REMARKS. An algebraic group acts morphically on itself, hence translations are easily seen to be isomorphisms of varieties. Therefore algebraic groups are non singular. It is well-known that complete algebraic groups are commutative and usually referred to as abelian varieties.

DEFINITION 1.6. Let X, Y be varieties, $\varphi : X \longrightarrow Y$ a morphism and G a group acting on both of them. Then φ is said to be G-equivariant if $\varphi(gx) = g\,\varphi(x)$ for every $g \in G$, $x \in X$.

EXAMPLES. $\mathbb{G}_a = (\mathbb{A}^1, +)$, $\mathbb{G}_m = (k^*, \cdot)$ where $k^* = \mathbb{A}^1 - \{0\}$, $\mu_m = \operatorname{Spec}(k[X]/(X^n - 1))$, $Gl(n, k) = \mathbb{A}^{n^2} - Z(D)$ where D is the determinant of the generic $n \times n$ matrix are algebraic groups. An action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1} - \{0\}$ is the following: $(\lambda, (x_0, \dots, x_n)) \longmapsto (\lambda x_0, \dots, \lambda x_n)$ and the orbits are the punctured lines through the origin.

Let now G be a group acting on an affine variety X, whose coordinate ring is denoted by $k[X]$ and assume that the translations are morphisms (this is automatically true if G is algebraic and acting morphically on X).

DEFINITION 1.7. We call translation of functions given by g the k-automorphism

$$\tau_g : k[X] \longrightarrow k[X], \qquad f \longmapsto f \circ T_{g^{-1}}$$

LEMMA 1.8. The map $G \longrightarrow \operatorname{Aut}_k(k[X])$ given by $g \longmapsto \tau_g$ is a homomorphism, hence it induces an action of G on $k[X]$.

Proof. $\tau_{g\cdot h}(f) = f \circ T_{(g\cdot h)^{-1}} = f \circ T_{h^{-1}\cdot g^{-1}} = f \circ T_{h^{-1}} \circ T_{g^{-1}} =$

$= \tau_g(f \circ T_{h^{-1}}) = \tau_g(\tau_h(f)) = (\tau_g \circ \tau_h)(f)$.

REMARK. If we define $\tau_g(f)$ to be $f \circ T_g$, then in general the map $g \longmapsto \tau_g$ is no more a homomorphism, hence it does not induce an action of G on $k[X]$.

If G is an affine algebraic group acting (morphically) on the affine variety X, then an action of G on X is a morphism $\mu : G \times X \longrightarrow X$, which corresponds to a k-homomorphism $\varphi : k[X] \longrightarrow k[X] \otimes k[G]$ (where $k[X]$ and $k[G]$ denote the coordinate rings of X and G respectively).

Now let $g \in G$ and let M be the maximal ideal of $k[G]$ corresponding to g^{-1} and consider the composition of the following k-homomorphisms

$$k[X] \longrightarrow k[X] \otimes k[G] \xrightarrow{(id,\ p)} k[X] \otimes k[G]/M \xrightarrow{\sim} k[X]$$

It is easy to check that the composition sends $f(X)$ to $f(g^{-1}X)$ i.e. it sends f to $f \circ T_{g^{-1}} = \tau_g(f)$. This means that φ not only has the information of the action of G on X, but it has also the information of the action of G on $k[X]$.

Now we want to study the following situation: X is an affine irreducible variety whose coordinate ring is denoted by $k[X]$ and G is a finite group of order m acting on X in such a way that the translations are morphisms. As we know, the action of G on X gives rise to an action of G on $k[X]$ which obviously extends to an action of G on $k(X)$, the quotient field of $k[X]$.

DEFINITION 1.9. We denote by $k[X]^G$ the subring of $k[X]$ of the invariants under the action of G i.e.

$$k[X]^G = \{ f \in k[X];\ gf = \tau_g(f) = f \circ T_{g^{-1}} = f \text{ for every } g \in G \}$$
$$= \{ f \in k[X];\ f \text{ is constant on the orbits} \}.$$

In the same way, we denote by $k(X)^G$ the subring of invariants of $k(X)$.

THEOREM 1.10. The quotient X/G has a natural structure of affine variety, whose coordinate ring is $k[X]^G$, and whose field of rational functions is $k(X)^G$.

Proof. Step 1. $k[X]^G$ is a finitely generated k-algebra and a domain.

For every $f \in k[X]$ let us consider the finite set $\{gf\ ;\ g \in G\} = \{\tau_g(f)\ ;\ g \in G\}$ i.e. the orbit of f under the action of G on $k[X]$ and denote by $\sigma_r(f)$ the elementary symmetric function of degree r on the mentioned orbit. If $k[X] = k[x_1, \ldots, x_n]$, then denote by S the k-algebra generated by $\sigma_r(x_i)$, $r = 1, \ldots, m$; $i = 1, \ldots, n$.

We have the following chain of inclusions

$$k \subseteq S \subseteq k[X]^G \subseteq k[X]$$

Now the polynomial equations

$$\prod_{g \in G} (X - gx_i) = X^m - \sigma_1(x_i)\, X^{m-1} + \dots + (-1)^m \sigma_m(x_i) = 0$$

are satysfied by the x_i's, therefore $k[X]$ is integral over S, hence it is a finitely generated S-module. This implies that also $k[X]^G$ is a finitely generated S-module (S is a noetherian ring), generated, say, by $b_1, \dots, b_t$. Therefore $k[X]^G$ is generated over k by the $\sigma_r(x_i)$'s and the b_j's.

Step.2. The affine variety Y corresponding to $k[X]^G$ can be identified with the topological quotient X/G.

The inclusion $k[X]^G \hookrightarrow k[X]$ is a finite homomorphism, which corresponds to a finite surjective morphism $\pi : X \longrightarrow Y$. Now the diagram

$$\begin{array}{ccc} k[X] & \xrightarrow{\ \tau_g^{-1}\ } & k[X] \\ & \nwarrow \quad \nearrow & \\ & k[X]^G & \end{array}$$

is clearly commutative and it corresponds to the diagram

$$\begin{array}{ccc} X & \xleftarrow{\ g\ } & X \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & Y & \end{array}$$

But this means that π is constant on the orbits hence we get a factorization

$$\begin{array}{ccc} X & \xrightarrow{\ \pi\ } & Y \\ & {\scriptstyle\pi'}\searrow \quad \nearrow{\scriptstyle\xi} & \\ & X/G & \end{array}$$

Let now $x, x' \in X$ be elements with different orbits. Since G_x is finite and $x' \notin G_x$, there exists a function $f \in k[X]$ such that $f_{|G_x} = 1$, $f(x') = 0$. Let Φ denote $\sigma_m(f)$; then $\Phi_{|G_x} = 1$, $\Phi(x') = 0$, hence

$\Phi(\pi(x)) = 1, \quad \Phi(\pi(x')) = 0$, which implies that $\pi(x) \neq \pi(x')$. Therefore ξ is injective.

Now we have that Y can be identified with X/G as a set; but since π is continuous and closed (it is finite), then Y can be identified with X/G as a topological space.

Step 3. $k(X)^G = K(k[X]^G)$ the quotient field of $k[X]^G$.

The inclusion $\supseteq$ is clear.

Let $a/b \in k(X)^G$ and let $c = \prod_{g\neq 1} \tau_g(b)$. Since $a/b = ac/bc = ac/\sigma_m(b)$ we get that $ac = ac/\sigma_m(b) \cdot \sigma_m(b) \in k(X)^G \cap k[X] = k[X]^G$ and the other inclusion is proved.

REMARK. If, in addition to the hypotheses of Theorem 1.10, we assume that G acts freely on X, then π turns out to be étale (see [M] p. 66).

COROLLARY 1.11. If X is normal, then X/G is normal.

Proof. If $\overline{k[X]^G}$ denotes the integral closure of $k[X]^G$, then we have

$$\overline{k[X]^G} = \overline{k[X]} \cap k(X)^G = \overline{k[X]} \cap k(X)^G = k[X] \cap k(X)^G = k[X]^G.$$

PROPOSITION - DEFINITION 1.12. The cyclic group of order d is an algebraic subgroup of $\mathbb{G}_m$, which is denoted by μ_d.

Proof. Namely $\mathbb{Z}_d \cong \mathrm{Spec}(k[X]/(X^d - 1)) \cong \mathrm{Spec}\, k[X, X^{-1}]/(X^d - 1)$ (remember that k is assumed to be algebraically closed).

Let us now extend Theorem 1.10 to a more general situation; X denotes a variety (not necessarily affine), G is a finite group of order m acting on X in such a way that the translations are morphisms.

As before we denote by X/G the topological quotient and by $\pi : X \longrightarrow X/G$ the canonical projection. We observe that for every open set U of X/G, the translations of functions T_g operate on $\pi^{-1}(U)$, which is stable under the action of G. Therefore G acts naturally on $\pi_* \mathcal{O}_X$ in the following way

$$\begin{array}{ccc} G \times \Gamma(U, \pi_* \mathcal{O}_X) & \longrightarrow & \Gamma(U, \pi_* \mathcal{O}_X) \\ \| & & \| \\ G \times \Gamma(\pi^{-1}(U), \mathcal{O}_X) & \longrightarrow & \Gamma(\pi^{-1}(U), \mathcal{O}_X) \\ (g, \quad f) & \longmapsto & \tau_g(f) = f \circ T_{g^{-1}} \end{array}$$

So we can talk about the sheaf of invariants $(\pi_* \mathcal{O}_X)^G$, meaning the sheaf associated to the presheaf defined in the following way

$$\Gamma(U, (\pi_* \mathcal{O}_X)^G) = \Gamma(U, \pi_* \mathcal{O}_X)^G$$

THEOREM 1.13. If for every $x \in X$, the orbit G_x is contained in an affine open set, then X/G has a natural structure of algebraic variety, whose structure sheaf is $(\pi_* \mathcal{O}_X)^G$.

Proof. If X is affine, the theorem is already proved (Theorem 1.10). Let now $x \in X$, and let U' be an affine open set containing G_x; then $U = \bigcap_{g \in G} g\, U'$ is an open set with the following properties:

a) it is affine as a finite intersection of affine open sets in a separated variety;

b) it is stable under the action of G;

c) it contains G_x.

Then X can be covered by affine open sets U, which are G-stable; for every such U, $\pi(U)$ is open and $\pi : U = \pi^{-1}(\pi(U)) \longrightarrow \pi(U)$ is again the situation of Theorem 1.10; therefore $(\pi(U), \widetilde{\Gamma(U, \mathcal{O}_X)^G})$ is an affine variety; since the open sets $\pi(U)$ cover X/G, we are done.

REMARK. The condition that for every $x \in X$, the finite orbit G_x is contained in an affine open set is, for instance, satysfied if X is projective.

REMARK. In Definition 1.7 it is essential to use g^{-1} in order to get an action (Lemma 1.8). However if G is commutative it is possible to use g and then to define the action in the following way: $(g, f) \longmapsto f \circ T_g$. We shall say that this is the natural action of G on $k[X]$.

Since in the following we are going to use commutative groups, we shall mainly use the natural action.

EXAMPLE. Let us consider the natural action of μ_d on $\mathbb{A}^2$ given by the following morphism

$$\varphi: k[X, Y] \longrightarrow k[X, Y] \otimes_k k[T]/(T^d - 1) \simeq k[X, Y, t]$$

defined by $\varphi(X) = Xt$, $\varphi(Y) = Yt$.

We get the following action of μ_d on $k[X, Y]$

$$(\varepsilon, X) \longmapsto \varepsilon X, \qquad (\varepsilon, Y) \longmapsto \varepsilon Y,$$

(here ε denotes a primitive d-th root of unity) and we get the following action on $\mathbb{A}^2$

$$(\varepsilon, (x_o, y_o)) \longmapsto (\varepsilon x_o, \varepsilon y_o)$$

The orbit of (x_o, y_o) is $\{(\varepsilon^i x_o, \varepsilon^i y_o), i = 0, \ldots, d - 1\}$. Therefore the subring of invariants $k[X, Y]^{\mu_d}$ is the subring

$$\{f(X, Y); f(\varepsilon^i x_o, \varepsilon^i y_o) = f(x_o, y_o) \text{ for every } i\} = k[X^d, X^{d-1}Y, \ldots, Y^d].$$

§ 2. Graded rings.

Let A be a finitely generated k-algebra and denote by $\mathcal{G}$ the set of the $\mathbb{Z}$-graduations of A and by $\mathcal{A}$ the set of the actions of $\mathbb{G}_m$ on Spec(A).

PROPOSITION 2.1. There is a natural injective application of $\mathcal{G}$ in $\mathcal{A}$.

Proof. An action of $\mathbb{G}_m$ on Spec(A) corresponds to a k-homomorphism

$$A \longrightarrow A \otimes_k k[X, X^{-1}] \simeq A[X, X^{-1}]$$

Let $A = k[x_0, \ldots, x_r]$, where $\{x_0, \ldots, x_r\}$ is a minimal set of homogeneous generators of A of degrees $q_0, \ldots, q_r$ respectively. Then, to the given graduation we may associate the k-homomorphism

$$\deg : A \longrightarrow A[X, X^{-1}]$$

$$x_i \longmapsto x_i X^{q_i}$$

REMARK. There is also a converse to Proposition 2.1. (see [EGA] II, p. 167).

Let us now study the action of $\mathbb{G}_m$ on A corresponding to a $\mathbb{Z}$-graduation of A. A point of $\mathbb{G}_m$ corresponds to a maximal ideal $(X - t)$, $t \neq 0$ of $k[X, X^{-1}]$. We consider the composition

$$\rho : A \longrightarrow A \otimes_k k[X, X^{-1}] \xrightarrow{(id,\, p_t)} A \otimes_k k[X, X^{-1}]/(X - t) \simeq A$$

whence we deduce the natural (see the remark at the end of § 1) action of $\mathbb{G}_m$ on A

$$\mathbb{G}_m \times A \longrightarrow A$$

$$(t, x_i) \longmapsto t^{q_i} x_i$$

To get the action of $\mathbb{G}_m$ on $V = \mathrm{Max}(A) \subset \mathbb{A}^{n+1}$, let $t \in \mathbb{G}_m$ and $p = (a_0, \ldots, a_r) \in V^m$; to p it corresponds the maximal ideal $M_p =$ $= (x_0 - a_0, \ldots, x_r - a_r)$ and it is easy to see that $\rho^{-1}(M_p) = M_{p'}$, where

$p' = (t^{q_0}a_0, \ldots, t^{q_r}a_r)$. Therefore the action of $\mathbb{G}_m$ on V is defined in this way

$$\mathbb{G}_m \times V \longrightarrow V$$

$$(t, (a_0, \ldots, a_r)) \longmapsto (t^{q_0}a_0, \ldots, t^{q_r}a_r)$$

and the orbit of p is given parametrically by

$$\begin{cases} x_0 = t^{q_0}a_0 \\ \vdots \\ x_r = t^{q_r}a_r \end{cases} \qquad t \neq 0$$

hence it is a <u>monomial curve</u>.

LEMMA 2.2. <u>The subring of invariants</u> $A^{\mathbb{G}_m}$ <u>coincides with</u> A_0.

<u>Proof</u>. Let $a = \sum a_d \in A$ be an invariant. Then $a = ta = \sum t^d a_d$; hence $\sum a_d(t^d - 1) = 0$ which implies $a_d(t^d - 1) = 0$ for every $t \in k^*$ and we conclude.

LEMMA 2.3. <u>Let</u> $\{x_0, \ldots, x_r\}$ <u>be a minimal set of homogeneous generators of the graded ring</u> A, <u>of degrees</u> $q_0, \ldots, q_r$ <u>respectively and assume that some of the</u> q_i's <u>are positive. Then the following conditions are equivalent</u>

a) $\deg x_i > 0$, $i = 0, \ldots, r$;

b) $\deg x_i \geqslant 0$, $i = 0, \ldots, r$ <u>and</u> $A_0 = k$

c) <u>The closures of the orbits in</u> $\mathbb{A}^{r+1}$ <u>only meet at the origin</u>.

<u>Proof</u>. The equivalence a) and b) is clear.

a) $\Longrightarrow$ c) If p is the origin, then the orbit is the origin. The other orbits have no common points and they are punctured monomial curves; therefore their closures pass through the origin.

c) $\Longrightarrow$ a) If some $q_i = 0$, then the orbit of the point $(0, \ldots, a_i, \ldots, 0)$ is contained in the hyperplane $X_i - a_i = 0$, hence it closure does not pass through the origin, unless all the points of V have $i-^{th}$ coordinate 0; but this means that x_i can be dropped from the set of generators of A. If some $q_i < 0$, then it is again clear from the parametric equations of the orbits that the origin is not in the closure, unless all the points of V have $i-^{th}$ coordinate 0.

DEFINITION 2.4. Let A be as in Lemma 2.3 and V = Spec(A). Then V is said to be a <u>quasicone</u> if it satysfies the equivalent conditions of Lemma 2.3.

Therefore the quasicones are the closed subschemes of the affine spaces, which are invariant under the action of $\mathbb{G}_m$ corresponding to a "positive degree". As we have seen, if V is a quasicone associate to a ring A, the "degree" action of $\mathbb{G}_m$ on A induces an action of $\mathbb{G}_m$ on V, which is described by

$$\mathbb{G}_m \times V \longrightarrow V$$

$$(t, a_0, \ldots, a_r)) \longmapsto (t^{q_0}a_0, \ldots, t^{q_r}a_r) \ , \quad 0 < q_i = \deg x_i .$$

<u>If all the q_i's are 1 then the quasicone is an ordinary cone.</u>

Now, let us take a homogeneous element $f \in A_d$; it is clear that the graduation of A can be extended in a natural way, by extending the action of $\mathbb{G}_m$ on Spec(A_f) ,

$$A_f \longrightarrow A_f[X, X^{-1}]$$

$$a_n/f^s \longmapsto (a_n/f^s) \cdot X^{n-ds}$$

By Lemma 2.2 we have $A_f^{\mathbb{G}_m} = A_{(f)}$. Now let us denote by V* the punctured quasicone $V - \{0\}$ and by $\pi : V^* \longrightarrow$ Proj(A) the canonical projection; since π is locally described by the canonical inclusions $A_{(f)} \hookrightarrow A_f$ and since $A_{(f)} = A_f^{\mathbb{G}_m}$, we can write

$$\text{Proj}(A) \simeq V^*/\mathbb{G}_m$$

and say that Proj(A) is the <u>geometric quotient</u> of V* under the action of $\mathbb{G}_m$ (for a complete theory see [M1]).

But of course, being $\mathbb{G}_m$ infinite, we cannot use Theorems 1.10, 1.13 directly; therefore we must push our investigation a little further.

The graduation on A induces an action of μ_d on Spec(A) in the following natural way

$$A \xrightarrow{\deg} A \otimes_k k[X, X^{-1}] \longrightarrow A \otimes_k k[X, X^{-1}]/(X^d - 1) \simeq A[x, x^{-1}]$$
$$a_n \in A_n \longmapsto a_n x^n$$

Therefore, arguing as in the example at the end of § 1, we obtain that the natural action of μ_d on A is given by

$$\mu_d \times A \longrightarrow A$$
$$(\varepsilon, a_n) \longmapsto \varepsilon^n a_n$$

LEMMA 2.5. <u>The subring of invariants</u> A^{μ_d} <u>coincides with</u> $\oplus_n A_{nd}$.

<u>Proof</u>. The inclusion $\oplus_n A_{nd} \subseteq A^{\mu_d}$ is trivial. Let $a = \sum_m a_m$ be an invariant element; then $(\varepsilon, a) \longmapsto \sum_m a_m \varepsilon^m = \sum_m a_m$. Then $a_m(1 - \varepsilon^m) = 0$ for every m and every $\varepsilon \in \mu_d$; if $a_m \neq 0$, m must be divisible by the order of every element of μ_d and we are done.

So let us go on with our investigation. If $f \in A_d$, the ideal $(f - 1)$ is an invariant ideal hence we get a homomorphism

(1) $A/(f - 1) \longrightarrow A/(f - 1)[X, X^{-1}]/(X^d - 1)$

which gives rise to an action of μ_d on Spec(A/(f - 1)) which can be extended to an action of μ_d on Spec(A/(f - 1)[u, u^{-1}]) (where u is an indeterminate of degree 1), by putting

(2) $A/(f - 1)[u, u^{-1}] \longrightarrow A/(f - 1)[u, u^{-1}][X, X^{-1}]/(X^d - 1)$

$$\begin{array}{ccc} \| & & \| \\ \bar{A}[u, u^{-1}] & \longrightarrow & \bar{A}[u, u^{-1}][x, x^{-1}] \\ u & \longmapsto & ux^{-1} \\ \bar{a}_n & \longmapsto & \bar{a}_n x^n \end{array}$$

Let us now study the subring of invariants of $A/(f-1)$ and $A/(f-1)[u, u^{-1}]$ with respect to the action of μ_d given by (1) and (2); for, let us consider the homomorphisms

$$\alpha : A_{(f)} \longrightarrow A/(f-1) = \overline{A}$$

$$a_n/f^s \longmapsto \overline{a}_n \qquad (sd = n)$$

$$\beta : A_f \longrightarrow A/(f-1)[u, u^{-1}] = \overline{A}[u, u^{-1}]$$

$$a_n/f^s \longmapsto \overline{a}_n\, u^{n-sd}$$

where β is clearly an extension of α. Then we have the following

THEOREM 2.6. (Flenner)

a) α, β <u>are injective</u> ;

b) $A_{(f)} = A/(f-1)^{\mu_d}$ (via (1) and α);

c) $A_f = (A/(f-1)[u, u^{-1}])^{\mu_d}$ (via (2) and β);

d) β <u>is étale</u>.

<u>Proof</u>. a) is clear and b) follows from c).

c) It is easy to see that it is sufficient to take in account the elements of $\overline{A}[u, u^{-1}]$ of the type $\overline{a}_n u^r$. If $\overline{a}_n u^r$ is invariant, then $\overline{a}_n u^r = \overline{a}_n x^n u^r x^{-r} = \overline{a}_n u^r x^{n-r}$ and this is possible iff $n - r = \lambda d$ iff $r = n - \lambda d$ iff $\overline{a}_n u^r = \beta(a_n/f^{\lambda})$.

d) See [F] lemma 2.1., lemma 2.2.

REMARK. It is important to note that, while β is étale α need not be such; namely it may happen that $A/(f-1)$ <u>is non singular and</u> $A_{(f)}$ <u>is singular</u>. For, let $A = k[X_0, X_1, X_2]$ with $q_0 = q_1 = 1$, $q_2 = 2$ ($q_i = \deg X_i$) and $f = X_2$.

Then $A_{(f)} \simeq A^{(2)}_{(f)} = k\left[X_0^2, X_0X_1, X_1^2, X_2\right]_{(X_2)} \simeq k\left[X^2, XY, Y^2\right]$; on the other hand $A/(f-1) \simeq k\left[X_0, X_1\right]$.

COROLLARY 2.7. Let V be the quasicone associated to a graded ring A and let $V^* = V - \{0\}$. If V^* has one of the following properties: irreducible, normal, Cohen-Macaulay, rational singularities, then Proj(A) has the same property. Moreover, if V^* is non singular, then Proj(A) has only cyclic quotient singularities.

Proof. It is a consequence of Theorem 2.6 and standard results which say that all these properties are stable under extending by étale morphisms, suppressing indeterminates, taking quotients modulo the action of finite groups. Therefore if A_f has one of those properties, $A_{(f)}$ has the same property.

COROLLARY 2.8. Let V be the quasicone associated to a graded ring A, let $V^* = V - \{0\}$ and assume dim V = 2. If V^* is non singular, then Proj (A) is a non singular curve.

Proof. Proj(A) is 1-dimensional and normal by Corollary 2.7.

§ 3. The weighted projective space.

Let $Q = (q_0, \ldots, q_r)$ be a $r + 1$ - uple of positive integers, denote by $|Q|$ the integer $\sum q_i$ and by $S(Q)$ the polynomial ring $k[T_0, \ldots, T_r]$ graded by $\deg T_i = q_i$.

DEFINITION 3.1. We denote by $\mathbb{P}(Q)$ the scheme $\mathrm{Proj}(S(Q))$ and we call it the weighted projective space (w.p.s.) of type Q.

Let $U = \mathbb{A}^{r+1} - \{0\} = \mathrm{Spec}(S(Q)) - \{m\}$ be the punctured quasicone and denote, as usual, by $D_+(T_i)$ the standard affine open set $\mathrm{Spec}\, S(Q)_{(T_i)}$ of $\mathbb{P}(Q)$. If $S(Q)(n)$ means, as usual, the $S(Q)$-graded module which is defined by $S(Q)(n)_t = S(Q)_{n+t}$ we denote by $\mathcal{O}_{\mathbb{P}(Q)}(n)$ the sheaf $\widetilde{S(Q)(n)}$.
If f is a homogeneous element of $S(Q)$, we may consider the natural k-linear map

$$S(Q)_n \longrightarrow S(Q)(n)_{(f)}$$

$$a_n \longmapsto \frac{a_n}{1}$$

which yields a natural k-linear map

$$\alpha_n : S(Q)_n \longrightarrow H^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(n)),$$

the so called Serre homomorphism.
We shall denote by $\mathbb{P}^r$ the usual projective space i.e. $\mathbb{P}(Q)$ where $Q = (1, 1, \ldots, 1)$.

3A. First properties of $\mathbb{P}(Q)$.

THEOREM 3A.1. The w.p.s. $\mathbb{P}(Q)$ has the following properties :

a) $\mathbb{P}(Q)$ is the geometric quotient of U modulo the action of $\mathbb{C}_m$ given by the graduation of $S(Q)$.

b) $D_+(T_i)$ is isomorphic to V_i/μ_{q_i} where $V_i = \mathrm{Spec}(k[T_0, \ldots, \hat{T}_i, \ldots \ldots, T_r])$.

c) $\mathbb{P}(Q)$ is irreducible, normal, Cohen-Macaulay (C - M), and their singularities are cyclic quotient singularities hence they are rational.

d) $\mathbb{P}(Q)$ is isomorphic to $\mathbb{P}^r/\mu_Q$ where $\mu_Q = \mu_{q_0} \times \ldots \times \mu_{q_r}$ and the canonical projection $\pi : \mathbb{P}^r \longrightarrow \mathbb{P}(Q)$ corresponds to the canonical homomorphism

$$S(Q) \cong k[X_0^{q_0}, \ldots, X_r^{q_r}] \hookrightarrow S = k[X_0, \ldots, X_r].$$

Proof. a) See the discussion following Definition 2.4.

b) Follows from Theorem 2.6 and the fact that

$$k[T_0, \ldots, T_i, \ldots, T_r]/(T_i - 1) \cong k[T_0, \ldots, \hat{T}_i, \ldots, T_r]$$

c) Follows from Corollary 2.7.

d) Consider the action of μ_Q on S given in the following way: if ε_i is a q_i^{th} root of unity we define

$$\mu_Q \times S \longrightarrow S$$

by $((\varepsilon_0, \ldots, \varepsilon_r), f(X_0, \ldots, X_r)) \longmapsto f(\varepsilon_0 X_0, \ldots, \varepsilon_r X_r)$.

Then the subring of invariants is easily seen to be $k[X_0^{q_0}, \ldots, X_r^{q_r}]$ and if we extend the action to $S_{(X_i)}$, then the subring of invariants is $k[X_0^{q_0}, \ldots, X_r^{q_r}]_{(X_i^{q_i})}$. Since $k[X_0^{q_0}, \ldots, X_r^{q_r}]$ is equivariantly isomorphic to $S(Q)$, we are done.

COROLLARY 3A.2. The corresponding morphism to $\pi : \mathbb{P}^r \longrightarrow \mathbb{P}(Q)$ between the associated quasicones is $p : \mathbb{A}^{r+1} \longrightarrow \mathbb{A}^{r+1}$; it is given by the inclusion

$$k[X_0^{q_0}, \ldots, X_r^{q_r}] \hookrightarrow k[X_0, \ldots, X_r].$$

It is free and a base is given by the monomials $\prod X_i^{n_i}$ where $0 \leq n_i < q_i$, $i = 0, \ldots, r$.

Consequently $\pi_* \mathcal{O}_{\mathbb{P}^r} = \oplus \mathcal{O}_{\mathbb{P}(Q)}(-\Sigma n_i)$

Proof. It follows directly from the preceding theorem.

LEMMA 3A.3. Let a be a positive integer $Q' = (aq_0, \ldots, aq_r)$. Then there is a natural isomorphism $\mathbb{P}(Q) \cong \mathbb{P}(Q')$.

Proof. There is an obvious natural graded isomorphism between $S(Q')^{(a)}$ and $S(Q)$, whence one gets

$$\mathbb{P}(Q') = \operatorname{Proj} S(Q') \cong \operatorname{Proj}(S(Q')^{(a)}) \cong \operatorname{Proj} S(Q) = \mathbb{P}(Q)$$

PROPOSITION 3A.4. a) If $Q = (a, a, \ldots, a), a \geq 1$ then $\pi : \mathbb{P}^r \longrightarrow \mathbb{P}(Q)$ is flat.

b) If there exists a pair of indexes i, j such that $q_i \neq q_j$ and if H_i denotes the hyperplane $x_i = 0$ in $\mathbb{P}^r$, then π is not flat and if G.C.D. $(q_0, \ldots, q_r) = 1$ the ramification locus R has the following property:

$$\bigcup_{i,\, q_i > 1} H_i \subseteq R \subseteq \bigcup_i H_i$$

c) π is étale if and only if it is the identity map and if and only if $Q = (1, \ldots, 1)$.

Proof. a) Under the assumption that $Q = (a, a, \ldots, a)$, $\mathbb{P}(Q) \cong \mathbb{P}^r$ by Lemma 3A.3, hence π is a finite morphism between regular schemes, hence it is flat (see for instance [H] 10.9, p. 276).

b) We use again the fact that the action of μ_Q on $\mathbb{P}^r$ is given by the map

$$\mu_Q \times \mathbb{P}^r \longrightarrow \mathbb{P}^r$$

$$((\varepsilon_0, \ldots, \varepsilon_r), (x_0, \ldots, x_r)) \longrightarrow (\varepsilon_0 x_0, \ldots, \varepsilon_r x_r)$$

To prove that π is not flat, if $q_i \neq q_j$ for some $i \neq j$, we need knowing that in this case $\mathbb{P}(Q)$ is singular (this will be proved later, see Prop. 4A.6, c)) and then we use the following standard fact of local algebra: if $\varphi : A \longrightarrow B$ is a local homomorphism of local rings, such that φ is flat and B is regular, then A is regular (see [Ma], p. 155).

If $p = (x_0, \ldots, x_r)$, $g = (\varepsilon_0, \ldots, \varepsilon_r)$, $\varepsilon_i^{q_i} = 1$ then there exist $\varepsilon_i \neq 1$, $i = 0, \ldots, r$ with

$$\rho\begin{pmatrix} x_0 \dots\dots x_r \\ \varepsilon_0 x_0 \dots\dots \varepsilon_r x_r \end{pmatrix} = 1$$

iff there exist $\varepsilon_i \neq 1$ with $(\varepsilon_\alpha - \varepsilon_\beta) x_\alpha x_\beta = 0$ for every α, β.

Now, if all the x_i's are different from zero, then $\varepsilon_\alpha = \varepsilon_\beta = \varepsilon$ for every α, β; but then $\varepsilon^{q_i} = 1$ for every i and since G.C.D.$(q_0, \dots, q_r) = 1$, $\varepsilon = 1$. So we have proved that the generic fiber has $q_1 \times \dots \times q_r$ points and $R \subsetneq \bigcup_i H_i$.

If now $x_i = 0$ and $q_i > 1$, let ε be a q_i^{th} root of unity different from 1 and let $g = (1, \dots, 1, \varepsilon, \dots, 1)$. Of course $gp = p$, hence $\bigcup_{i, q_i > 1} H_i \subseteq R$.

c) By arguing as in b) one can easily see that in case a) $R = \bigcup_{i=0}^{r} H_i$ if $a > 1$ and of course $R = \emptyset$ if $a = 1$. So the conclusion follows.

REMARK. The interpretation of $\mathbb{P}(Q)$ given by Theorem 3A.1.a) shows that, while $\mathbb{P}^r$ is obtained by quotienting U with respect to the partition given by the straight lines through the origin, $\mathbb{P}(Q)$ is obtained by quotienting U with respect to the partition given by the monoidal curves of parametric equations

$$x_0 = a_0 t^{q_0}, \dots, x_r = a_r t^{q_r}.$$

EXAMPLES. 1) Let $Q = (1, \dots, 1, n)$, $n \neq 1$. Then $\mathbb{P}(Q)$ can be identified with the cone X_n^r of vertex $(0, \dots, 0, 1)$ which projects the Veronese variety $v_n(\mathbb{P}^{r-1})$. Namely there are canonical isomorphisms

$$\mathbb{P}(1, \dots, 1, n) = \mathrm{Proj}(S(Q)) \cong \mathrm{Proj}(S(Q)^{(n)}) \cong$$

$$\cong \mathrm{Proj}(k[T_0, \dots, T_{r-1}]^{(n)}[T_r]) = X_n^r.$$

2) As in the example 1) it easy to see that if $Q = (1, \dots, 1, n, n)$, $n \neq 1$, then $\mathbb{P}(Q)$ can be identified with the cone, whose vertex is the line in $\mathbb{P}^r$ given by $x_0 = \dots = x_{r-2} = 0$ and projecting the Veronese variety $v_n(\mathbb{P}^{r-2})$.

3) Let $Q = (1, q_1, \dots, q_r)$, $q_i > 1$, $i = 1, \dots, n$. Then $\mathbb{P}(Q)$ can be thought of as a compactification of $\mathbb{A}^{r-1}$, given by adjoining $\mathbb{P}(q_1, \dots, q_r)$. Namely $D_+(T_0) = \mathrm{Spec}(S(Q)_{(T_0)})$ and it is clear that

$$S(Q)_{(T_0)} = k\left[T_1/T_0^{q_1}, \ldots, T_r/T_0^{q_r}\right]$$

whence $D_+(T_0) \cong \mathbb{A}^r$ and of course $\mathbb{P}(Q) - D_+(T_0)$ coincides with $\mathbb{P}(q_1, \ldots, q_r)$.

4) Let us consider the action of μ_Q on $\mathbb{P}^r$ described in theorem 3A.1, d); let $m = \text{l.c.m.}(q_0, \ldots, q_r)$, $a \in \mathbb{N}^+$ and let $\alpha_0, \ldots, \alpha_r$ be positive integers such that $\alpha_i q_i = am$. Let F be the weighted hypersurface of $\mathbb{P}(Q)$ given by the equation (homogeneous of degree am).

$$F : \sum_i T_i^{\alpha_i} = 0$$

and let $\tilde{F}$ be the hypersurface of $\mathbb{P}^r$ given by the equation

$$\tilde{F} : \sum_i X_i^{am} = 0$$

Of course $\tilde{F}$ is invariant under the action of μ_Q and it is clear that F is the quotient of $\tilde{F}$ under the action of μ_Q.

REMARK. If we assume that $(q_i, q_j) = 1$ for every i, j, $i \neq j$, then of course $m = \prod q_i$ and the Chinese Remainder Theorem gives an isomorphism $\mathbb{Z}_m \cong \mathbb{Z}_{q_0} \oplus \ldots \oplus \mathbb{Z}_{q_r}$. Moreover one gets an isomorphism of algebraic groups

$$\mu_m \cong \mu_{q_0} \times \ldots \times \mu_{q_r}$$

which is described by the isomorphism

$$k[T]/(T^m - 1) \xrightarrow{\sim} k[X_0]/(X_0^{q_0} - 1) \otimes \ldots \otimes k[X_r]/(X_r^{q_r} - 1)$$

$$t \longmapsto t^{m/q_0} \otimes \ldots \otimes t^{m/q_r}$$

therefore the action of μ_Q on $\mathbb{P}^r$ described in Theorem 3A.1, d) can be expressed in the following way

$$\mu_m \times \mathbb{P}^r \longrightarrow \mathbb{P}^r$$

$$(\varepsilon^{\alpha}, (x_0, \ldots, x_r)) \longmapsto (\varepsilon^{\alpha m/q_0} x_0, \ldots, \varepsilon^{\alpha m/q_r} x_r)$$

where ε is a primitive m^{th} root of 1.

For instance if $Q = (2, 3, 5)$, $m = 30$ and we have the above described action of μ_{30} on $\mathbb{P}^2$. The quotient is $\mathbb{P}(2, 3, 5)$; if moreover $\widetilde{F}$ is the curve of $\mathbb{P}^2$ of equation $X^{30} + Y^{30} + Z^{30} = 0$, it is invariant under the action of μ_{30} and its quotient is the weighted curve of $\mathbb{P}(2, 3, 5)$ of equation $T_0^{15} + T_1^{10} + T_2^{6} = 0$.

3 B. Structure and homological properties of the projecting quasicones.

Let now A be a finitely generated k-algebra and let $\{t_0, \ldots, t_r\}$ be a minimal set of homogeneous generators of positive degrees $q_0, \ldots, q_r$. Then of course there exists a homogeneous ideal $\mathfrak{a}$ of $S(Q)$ such that $A \cong S(Q)/\mathfrak{a}$. Write $X = \mathrm{Proj}(A)$ and denote by p the canonical projection $p : U = \mathbb{A}^{r+1} - \{0\} \longrightarrow \mathbb{P}(Q)$, by C_X the inverse image $p^{-1}(X)$.
If $\mathcal{J}_{C_X}$ is the sheaf of ideals defining the embedding of C_X in U and if i is the embedding of U in $\mathbb{A}^{r+1}$, we get the following commutative diagram

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}(Q) \\ \uparrow & & \uparrow p \\ C_X & \hookrightarrow & U \\ \downarrow & & \downarrow i \\ C_X^+ & \hookrightarrow & \mathbb{A}^{r+1} \end{array}$$

where C_X^+ is the schematic closure of C_X in $\mathbb{A}^{r+1}$ i.e. $i_* \mathcal{J}_{C_X} = \mathcal{J}_{C_X^+}$ (see [H], p. 92).

DEFINITION 3B.1. We say that C_X^+ is the projecting quasicone of X in $\mathbb{P}(Q)$.

Notation: in the following we denote as usual by $\mathcal{O}_{\mathbb{P}(Q)}(n)$ the sheaf $\widetilde{S(Q)(n)}$.

LEMMA 3B.2 a) There is a canonical isomorphism of schemes

$$U \cong \underset{\sim}{\mathrm{Spec}}\left(\bigoplus_{n \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}(Q)}(n)\right);$$

b) The maps p^*, p_* are isomorphisms between the sheaves $\mathcal{O}_U$ and $\bigoplus_{n \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}(Q)}(n)$, which are inverse to each other.

c) The Serre homomorphism $S(Q) \longrightarrow \bigoplus_n H^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(n))$ is a graded isomorphism.

d) If $\mathcal{J}(n)$ is the kernel of the morphism of schemes

$$\mathcal{O}_{\mathbb{P}(Q)}(n) \longrightarrow \mathcal{O}_X(n)$$

then

$$\mathcal{J}_{C_X^+} = i_*(\bigoplus_{n \in \mathbb{Z}} \mathcal{J}(n))$$

e) If I is the ideal of $S(Q)$ corresponding to $\bigoplus_n H^0(\mathbb{P}(Q), \mathcal{J}(n))$, via the Serre isomorphism, then $\mathcal{J}_{C_X^+} = \widetilde{I}$;

f) I is a saturated ideal, hence $\operatorname{depth}(S(Q)/I) \geq 1$.

Proof. a), b) For every $i = 0, \ldots, r$ there is a canonical isomorphism

$$S(Q)_{T_i} \cong \bigoplus_{n \in \mathbb{Z}} S(Q)(n)_{(T_i)}$$

Since p is locally defined from the inclusion

$$S(Q)_{(T_i)} \hookrightarrow S(Q)_{T_i}$$

and since

$$U = \coprod_i \operatorname{Spec}(S(Q)_{T_i})$$

we get the conclusions.

c) Since $\mathbb{A}^{r+1}$ clearly verifies the property S_2 of Serre, we get

$$\Gamma(U, \mathcal{O}_U) \cong \Gamma(\mathbb{A}^{r+1}, \mathcal{O}_{\mathbb{A}^{r+1}}) = S(Q)$$

Combining with a) we are done.

d) We have the following exact sequence

$$0 \longrightarrow \bigoplus_{n \in \mathbb{Z}} \mathcal{J}(n) \longrightarrow \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_{\mathbb{P}(Q)}(n) \longrightarrow \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n) \longrightarrow 0$$

By using the isomorphism of a) one gets that $\bigoplus_{n \in \mathbb{Z}} \mathcal{J}(n)$ is the sheaf of ideals defining the inclusion of $p^{-1}(X)$ in U. Therefore

$$i_*(\bigoplus_{n \in \mathbb{Z}} \mathcal{J}(n)) = i_* \mathcal{J}_{p^{-1}(X)} = \mathcal{J}_{C_X^+}$$

e) We use the Serre isomorphism to identify $i_*(\bigoplus_{n \in \mathbb{Z}} \mathcal{J}(n))$ with the sheaf associated to a well defined ideal I of S(Q). Therefore

$$I = H^0(\mathbb{A}^{r+1}, i_*(\bigoplus_{n \in \mathbb{Z}} \mathcal{J}(n)) = H^0(\mathbb{P}(Q), \bigoplus_{n \in \mathbb{Z}} \mathcal{J}(n)) =$$

$$= \bigoplus_{n \in \mathbb{Z}} H^0(\mathbb{P}(Q), \mathcal{J}(n)).$$

f) Follows from the preceding description.

DEFINITION 3B.3. We say that I is the ideal of the projecting quasicone C_X^+, with respect to the inclusion $X \subset \mathbb{P}(Q)$.

PROPOSITION 3B.4. Let $X \subset \mathbb{P}(Q)$ be a closed subscheme, C_X^+ its projecting quasicone, $I \subset S(Q)$ the ideal of C_X^+. Then

a) There is a canonical injective graded homomorphism

$$\alpha : S(Q)/I \longrightarrow \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n))$$

b) If depth$(S(Q)/I) = t > 1$ then α is an isomorphism and $H^i(X, \mathcal{O}_X(n)) = (0)$ for every n if $0 \leqslant i \leqslant t - 2$.

c) If $X = \mathbb{P}(Q)$, then $H^i(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(n)) = (0)$ for every n if $i \neq 0$, r and

$$H^r(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(n)) \cong S(Q)_{-n-|Q|}$$

Proof. We consider the standard exact sequence of local cohomology

$$0 \longrightarrow H^0_{\{\mathfrak{m}\}}(S(Q)/I) \longrightarrow S(Q)/I \longrightarrow H^0(C_X, \mathcal{O}_{C_X}) \longrightarrow H^1_{\{\mathfrak{m}\}}(S(Q)/I) \longrightarrow 0$$

and the isomorphisms

$$H^i_{\{\mathfrak{m}\}}(S(Q)/I) \cong H^{i-1}(C_X, \mathcal{O}_{C_X}) \text{ for } i > 1$$

(see for instance [H], p. 212, 217). Since depth$(S(Q)/I) = t > 1$ by Lemma 3B.2.f) one gets $H^0_{\{\mathfrak{m}\}}(S(Q)/I) = (0)$. Reasoning as in 3B.2, we have an isomorphism of sheaves $\mathcal{O}_{C_X} \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)$. So we get a).

If $t > 1$, $H^i_{\{\mathfrak{m}\}}(S(Q)/I) = (0)$, $i < t$, hence $H^i(C_X, \mathcal{O}_{C_X}) = (0)$, $i = 0, \ldots, t - 2$.

This proves b) and the first part of c) since if $X = \mathbb{P}(Q)$ one has $t = r + 1$. For the second part of c) see [D], 1.41.

DEFINITION 3B.5. If X is a closed subscheme of $\mathbb{P}(Q)$, we say that X is quasi-smooth with respect to the inclusion $X \hookrightarrow \mathbb{P}(Q)$ if the projecting quasicone C_X^+ is smooth outside the vertex.

REMARK. If $X \subset \mathbb{P}(Q)$ is quasismooth, then the only possible singularities of X are cyclic quotient singularities. This comes directly from 2.7.

3C. Reduction of the weights

We have already seen (see 3A.3) that if $Q' = (aq_0, \ldots, aq_r)$, then $\mathbb{P}(Q') \cong \mathbb{P}(Q)$. Now we can say a little more

PROPOSITION 3C.1. If $v : \mathbb{P}(Q') \xrightarrow{\sim} \mathbb{P}(Q)$ is the isomorphism described in 3A.3, then

$$v^* \mathcal{O}_{\mathbb{P}(Q)}(n) = \mathcal{O}_{\mathbb{P}(Q')}(an)$$

Proof. $\mathcal{O}_{\mathbb{P}(Q)}(n) = (S(Q)(n))^{\sim} \cong (S(Q')^{(a)}(n))^{\sim} \cong (S(Q')(an))^{\sim} =$
$= \mathcal{O}_{\mathbb{P}(Q')}(an)$.

DEFINITION 3C.2. If $G.C.D.(q_0, \ldots, q_r) = 1$ we say that Q is reduced.

After 3C.1 we may assume that Q is reduced and we fix the following notations

$$d_i = G.C.D.(q_0, \ldots, q_{i-1}, q_{i+1}, \ldots, q_r)$$

$$a_i = l.c.m.(d_0, \ldots, d_{i-1}, d_{i+1}, \ldots, d_r)$$

$$a = l.c.m.(d_0, \ldots, d_r).$$

LEMMA 3C.3. The following relations hold

a) $a_i | q_i$ for $i = 0, \ldots, r$; ("|" means "divides")

b) $(q_i, d_i) = 1$ for $i = 0, \ldots, r$;

c) $(d_j, d_i) = 1$ for $j \neq i$;

d) $(a_i, d_i) = 1$ for $i = 0, \ldots, r$;

e) $a_i d_i = a$ for $i = 0, \ldots, r$;

f) $d_j | a_i$ for $j \neq i$.

Proof. Easy exercise.

Therefore we may associate to Q the new $r + 1$ - ple

$$\overline{Q} = (q_0/a_0, \ldots, q_r/a_r)$$

LEMMA - DEFINITION 3C.4. The following conditions are equivalent

a) $d_i = 1$ for every $i = 0, \ldots, r$;

b) $Q = \overline{Q}$. (in this case we say that Q is normalized).

In particular, for every given reduced Q, $\overline{Q}$ is normalized i.e. $\overline{Q} = \overline{\overline{Q}}$.

Proof. a) $\Longleftrightarrow$ b) Clear.

To prove that $\overline{Q} = \overline{\overline{Q}}$ we have to show that

$$\text{G.C.D.}(q_0/a_0, \ldots, q_{i-1}/a_{i-1}, q_{i+1}/a_{i+1}, \ldots, q_r/a_r) = 1.$$

Since the G.C.D. of the numerators is d_i, it is sufficient to show that d_i divides the denominators. But this is exactly 3C.3.f).

PROPOSITION 3C.5. There exists a natural isomorphism of $\mathbb{P}(Q)$ and $\mathbb{P}(\overline{Q})$.

Proof. Let S' be the graded subring of $S(Q)$ defined by

$$S' = \bigoplus_{n \in \mathbb{Z}} S(Q)_{an}$$

It is well known that Proj S' = Proj $S(Q)$. We are going to show that

$S' = k\left[T_0^{d_0}, \ldots, T_r^{d_r}\right]$. Namely deg $T_i^{d_i} = q_i d_i = q_i a_i d_i / a_i = a q_i / a_i$ by

3C.3. e), and q_i/a_i is an integer by 3C.3.a).

Conversely if a monomial $T_0^{s_0} \cdots T_r^{s_r}$ belongs to S', then $s_0q_0 + \cdots$

$\cdots + s_rq_r = \lambda a$, whence

$$s_iq_i = -\sum_{j \neq i} s_jq_j + \lambda a_id_i, \quad \lambda \in \mathbb{Z}.$$

But $d_i | q_j$ if $i \neq j$ hence $s_iq_i \in (d_i)$. Since $(q_i, d_i) = 1$ this implies that $s_i \in (d_i)$. In conclusion $S' = S(Q')$ where $Q' = (a\, q_0/a_0, \ldots, aq_r/a_r)$ hence $\mathrm{Proj}\ (S') \cong \mathrm{Proj}\ (S(\overline{Q}))$ by 3A.3 and we are done.

EXAMPLE. $\mathbb{P}(6, 10, 15) \cong \mathbb{P}^2$. Namely $(d_0, d_1, d_2) = (5, 3, 2)$; $(a_0, a_1, a_2) = (6, 10, 15)$, $\overline{Q} = (1, 1, 1)$.

COROLLARY 3C.6 <u>For every</u> $Q = (q_0, q_1)$ <u>there is an isomorphism</u> $\mathbb{P}(Q) \cong \mathbb{P}^1$.

<u>Proof</u>. We may assume Q to be reduced. Then $(d_0, d_1) = (q_1, q_0)$; $(a_0, a_1) = (q_0, q_1)$; $\overline{Q} = (1, 1)$.

Now we recall that $(q_i, d_i) = 1$, therefore for every integer n we may write

$$n = b_i(n)q_i + c_i(n)d_i$$

where $b_i(n)$, $c_i(n)$ are <u>uniquely determined by the condition</u>

$$0 \leqslant b_i(n) < d_i$$

In this way to every n we have associated two integers $b_i(n)$, $c_i(n)$.

PROPOSITION 3C.7. a) <u>The number</u> $\varphi(n) = (n - \sum_{i=0}^{r} b_i(n)\, q_i)/a$ <u>is an integer</u>;

b) <u>There is an isomorphism of sheaves</u> $\mathcal{O}_{\mathbb{P}(Q)}(n) \cong \mathcal{O}_{\mathbb{P}(Q)}(n - \sum_{i=0}^{r} b_i(n)q_i)$ <u>and the isomorphism</u> $\mathbb{P}(Q) \cong \mathbb{P}(\overline{Q})$ <u>of</u> 3C.5 <u>induces an isomorphism of sheaves</u>

$$\mathcal{O}_{\mathbb{P}(Q)}(n) \cong \mathcal{O}_{\mathbb{P}(\overline{Q})}(\varphi(n))$$

<u>Proof</u>. a) It is sufficient to show that $n - \sum b_i(n)q_i$ is a multiple of d_j

for every $j = 0, \ldots, r$. Now $n - \sum b_i(n)q_i = n - b_j(n)q_j - \sum_{i \neq j} b_i(n)q_i =$ $c_j(n)d_j - \sum_{i\neq j} b_i(n)q_i$; but $d_j | q_i$ if $j \neq i$ and the conclusion follows.

b) Let us show that a monomial $T_0^{s_0} \ldots T_r^{s_r}$ of degree $n + hd_i$ is divided by $T_i^{b_i(n)}$. Namely the relation

$$s_0q_0 + \ldots + s_rq_r = n + hd_i$$

implies that $n = s_iq_i + \lambda d_i$ since $d_j | q_i$, $j \neq i$.

By definition of $b_i(n)$ it follows that $s_i \geqslant b_i(n)$. Therefore, if p is a multiple of a, it is also a multiple of every d_i, whence a monomial of degree $n + p$ is a multiple of $T_0^{b_0(n)} \ldots T_r^{b_r(n)}$. Therefore

$$\bigoplus_{p,\, a/p} (S(Q)(n))_p = \bigoplus_{p,\, a/p} (T_0^{b_0(n)} \ldots T_r^{b_r(n)})\, S(Q)(n - \sum b_i(n)q_i)_p$$

and this equality between $S' = \oplus_n S(Q)_{an}$-graded modules induces an isomorphism between

$$\mathcal{O}_{\mathbb{P}(Q)}(n) \quad \text{and} \quad \mathcal{O}_{\mathbb{P}(Q)}(n - \sum b_i(n)q_i)$$

(remember that $\mathbb{P}(Q) = \operatorname{Proj} S'$). But we have already seen (see 3C.5) that $S'^{(a)} \cong S(\overline{Q})$, hence in the isomorphism $\mathbb{P}(Q) \cong \mathbb{P}(\overline{Q})$, the sheaf $\mathcal{O}_{\mathbb{P}(Q)}(n - \sum b_i(n)q_i)$ corresponds to $\mathcal{O}_{\mathbb{P}(\overline{Q})}(\varphi(n))$.

3D. <u>Examples and pathologies.</u>

1. It may happen that $H^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(1)) = (0)$ and it happens exactly if $q_i > 1$, $i = 0, \ldots, r$.

This follows from 3B.2c).

2. It may happen that $\mathcal{O}_{\mathbb{P}(Q)}(n) \cong \mathcal{O}_{\mathbb{P}(Q)}(n')$, $n \neq n'$.

For example if $Q = (2, 4, 5)$, $n = 5$, $(d_0, d_1, d_2) = (1, 1, 2)$ and let us compute $5 - b_0(5) \cdot 2 - b_1(5) \cdot 4 - b_2(5) \cdot 5$.

$$5 = 0 \cdot 2 + 5 \cdot 1; \quad 5 = 0 \cdot 4 + 5 \cdot 1; \quad 5 = 1 \cdot 5 + 0 \cdot 2$$

hence $b_0(5) = 0$, $b_1(5) = 0$, $b_2(5) = 1$ and

$$\mathcal{O}_{\mathbb{P}(Q)}(5) \cong \mathcal{O}_{\mathbb{P}(Q)}(0) \cong \mathcal{O}_{\mathbb{P}(Q)}$$

after 3C.7.b).

REMARK. If $Q = \overline{Q}$ this pathology does not occur because $d_i = 1$, $i = 0, \ldots, r$ hence $b_i(n) = 0$, $i = 0, \ldots, r$.

3. Even if $Q = \overline{Q}$ it may happen that $\mathcal{O}_{\mathbb{P}(Q)}(n)$ is not invertible. For example let $Q = (1, 1, 2)$ and consider the open set $D_+(T_2)$. Then

$$\Gamma(D_+(T_2), \mathcal{O}_{\mathbb{P}(Q)}(1)) = S(Q)(1)_{(T_2)} = \left\{ a/T_2^h,\ a \in S(Q)_{2h+1} \right\},$$

$$\Gamma(D_+(T_2), \mathcal{O}_{\mathbb{P}(Q)}) = S(Q)_{(T_2)} = \left\{ a/T_2^h,\ a \in S(Q)_{2h} \right\}.$$

It is easy to see that $S(Q)(1)_{(T_2)}$ is minimally generated over $S(Q)_{(T_2)}$ by T_0, T_1 and it is not free as the following relation shows

$$T_1^2/T_2 \cdot T_0 - T_0T_1/T_2 \cdot T_1 = 0.$$

It may be worthwhile to observe that in the case $Q = (1, 1, 1)$ the module $S(Q)(1)_{(T_2)}$ is minimally generated over $S(Q)_{(T_2)}$ by T_2.

4. A sheaf $\mathcal{O}_{\mathbb{P}(Q)}(n)$, $n > 0$, may be invertible but not ample. For example, let $Q = (3, 5)$; then $\overline{Q} = (1, 1)$ and the isomorphism $\mathbb{P}(Q) \cong \mathbb{P}^1$ induces an isomorphism of sheaves $\mathcal{O}_{\mathbb{P}(Q)}(n) \cong \mathcal{O}_{\mathbb{P}^1}(\varphi(n))$ by 3C.7. So let $n = 2$; then $\varphi(2) = -1$, hence $\mathcal{O}_{\mathbb{P}(Q)}(2) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ which is invertible but not ample.

5. The canonical homomorphism

$$\mathcal{O}_{\mathbb{P}(Q)}(n) \otimes \mathcal{O}_{\mathbb{P}(Q)}(m) \longrightarrow \mathcal{O}_{\mathbb{P}(Q)}(n+m)$$

may be not an isomorphism.

For, let $Q = (2, 3)$; the isomorphism $\mathbb{P}(Q) \cong \mathbb{P}^1$ yields the following isomorphisms of sheaves (see 3C.7)

$$\mathcal{O}_{\mathbb{P}(Q)}(2) \cong \mathcal{O}_{\mathbb{P}^1}; \quad \mathcal{O}_{\mathbb{P}(Q)}(4) \cong \mathcal{O}_{\mathbb{P}^1}; \quad {}_{\mathbb{P}(Q)}(6) \cong \mathcal{O}_{\mathbb{P}^1}(1).$$

§4. Properties of the sheaves $\mathcal{O}_{\mathbb{P}(Q)}(n)$.

We have just seen that the behavior of the sheaves $\mathcal{O}_{\mathbb{P}(Q)}(n)$ is not the same as in the usual projective space, so that we are led to analize the situation in a more accurate way.

4A. Sections of $\mathcal{O}_{\mathbb{P}(Q)}(n)$ over standard open sets.

Let us fix some notations; let $Q = (q_0, \ldots., q_r)$ and let A be a commutative ring with identity. Let L be a subset of $\{0, \ldots., r\}$, let $A[T_l]_{l \in L}$ be the polynomial ring over A generated by $\{T_l, l \in L\}$ and choose $n \in \mathbb{Z}$, $d \in \mathbb{N}^+$.

DEFINITION 4A.1. We define $A[L, n, d]$ to be the sub-A-module of $A[T_l]_{l \in L}$, generated by all the monomials

$$T^b = \prod_{l \in L} T_l^{b_l} \text{ such that } \deg T^b = \sum_{l \in L} b_l q_l \equiv n \text{ mod. } d$$

It is clear that $A[L, 0, d]$ is an A-algebra, $A[L, n, d]$ is an $A[L, 0, d]$-module and there are natural homomorphisms of $A[L, 0, d]$-modules

$$A[L, n, d] \otimes A[L, n', d] \longrightarrow A[L, n + n', d]$$
$$A[L, n, d] \longrightarrow \mathrm{Hom}(A[L, n', d], A[L, n + n', d]).$$

REMARKS. 1) If $d = 1$, $A[L, n, 1] = A[T_l]_{l \in L}$.
2) If $d|n$, $A[L, n, d] = A[L, 0, d]$.

DEFINITION 4A.2. Let $I = \{0, \ldots., r\}$; if $J \subsetneq I$ we define d_J to be the G.C.D. of the q_i's, $i \in J$. We put

$$D_J = \bigcap_{i \in J} D_+(T_i) = \mathrm{Spec}(S(Q)_{(T_J)})$$

where $T_J = \prod_{i \in J} T_i$ and we call G_J the group of the rational monomials $T^c = \prod_{i \in J} T^{c_i}$ in $S(Q)_{(T_J)}$ i.e. such that $\sum_{i \in J} c_i q_i = 0$.

REMARKS. 1) The group G_J is free of rank $\# J - 1$ since it is isomorphic to the group of the integral solutions of $\sum_{i\in J} q_i x_i = 0$.

2) $A[G_J] = (A[T_i]_{i\in J})_{(T_J)}$.

EXAMPLE. Let $Q = (1, 2, 3, 4, 5)$, $J = \{1, 2, 3\}$. In this case G_J is the free group of rank 2 whose elements are the rational monomials $T^c = T_2^{c_2} T_3^{c_3} T_4^{c_4}$ such that $2c_2 + 3c_3 + 4c_4 = 0$. Two generators of G_J are for instance $T_2^{-3} T_3^2$, $T_2^{-2} T_4$.

PROPOSITION 4A.3. <u>Let J be a non empty subset of</u> $I = \{0, \ldots, r\}$ <u>and fix a rational monomial</u> $M_J = \prod_{i\in J} T_i^{t_i}$ <u>of degree</u> d_J. <u>Then there are isomorphisms of A-modules (depending on</u> M_J)

$$\Gamma(D_J, \mathcal{O}_{\mathbb{P}(Q)}(n)) \cong A[G_J][I - J, n, d_J],$$

<u>which are compatible with the homomorphisms described after</u> 4A.1.

<u>Proof</u>. Of course $\Gamma(D_J, \mathcal{O}_{\mathbb{P}(Q)}(n)) = S(Q)(n)_{(T_J)}$ is generated over k by the rational monomials T^a such that $a_i \geq 0$ if $i \in I - J$ and $\sum_{i\in I} a_i q_i = n$. Such a monomial T^a can be written in the form $T^a = \prod_{i\in J} T_i^{a_i} \cdot \prod_{i\in I-J} T_i^{a_i}$. Now, $\deg(\prod_{i\in J} T_i^{a_i}) = e\, d_J$ for a suitable e and

$$\deg(\prod_{i\in I-J} T_i^{a_i}) = n - e\, d_J \equiv n \bmod(d_J).$$

Dividing by M_J^e we get the following map

$$\Gamma(D_J, \mathcal{O}_{\mathbb{P}(Q)}(n)) = S(Q)(n)_{(T_J)} \longrightarrow A[G_J][I - J, n, d_J]$$

which sends $T^a = \prod_{i\in I} T_i^{a_i}$ to $(\prod_{i\in J} T_i^{a_i}/M_J^e) \cdot \prod_{i\in I-J} T_i^{a_i}$. This is clearly an injective homomorphism of A-modules. Let us show that it is surjective. A rational monomial of $A[G_J][I - J, n, d_J]$ is a product of $T^c \cdot T^b$ where

$$T^c = \prod_{i\in J} T^{c_i}, \quad T^b = \prod_{i\in I-J} T^{b_i}$$

and $\deg T^c = \sum_{i\in J} c_i q_i = 0$, $\deg T^b = \sum_{i\in I-J} b_i q_i = n + \lambda d_J$, $\lambda \in \mathbb{Z}$.

The element $T^a = T^c \cdot T^b/M_J^\lambda$ has degree n, hence it is a section of $\Gamma(D_J, \mathcal{O}_{\mathbb{P}(Q)}(n))$ and it may be written as $(T^c/M_J^\lambda) \cdot T^b$. But $\deg(T^c/M_J^\lambda) = -\lambda d_J$; hence, dividing by $M_J^{-\lambda}$ we get $T^c \cdot T^b$.

EXAMPLES. 1. Let $Q = (1, 2, 3, 5, 7)$, $J = \{4\}$.
Then G_J = identity, $d_J = 7$, $M_J = T_4$. Given a monomial $T^a = \prod T_i^{a_i}$ the number e is a_4 hence the isomorphism

$$\Gamma(D_+(T_4), \mathcal{O}_{\mathbb{P}(Q)}(n)) \cong A[G_J][I - J, n, 7] = A[I - J, n, 7]$$

is obtained by deleting the variable T_4.
2. Let $Q = (1, 2, 3, 5, 7)$, $J = \{2, 3, 4\}$.
Now G_J is the group of monomials $T^c = T_2^{c_2} T_3^{c_3} T_4^{c_4}$ such that $3c_2 + 5c_3 + 7c_4 = 0$, hence it is freely generated by $T_2^{-4} T_3 T_4$ and $T_2 T_3^{-2} T_4$. We have $d_J = 1$ and we may choose M_J to be $T_3^3 T_4^{-2}$. Therefore

$$\Gamma(D_J, \mathcal{O}_{\mathbb{P}(Q)}(n)) \cong A[G_J][I - J, n, 1] = A[G_J, T_0, T_1] =$$

$$= A[T_2^{-4} T_3 T_4, T_2 T_3^{-2} T_4, T_0, T_1]$$

3. Let $Q = (1, 1, 1)$, $J = \{1, 2\}$.
Then $A[G_J] = A[T_1 \cdot T_2^{-1}]$, $M_J = T_1$, $d_J = 1$. Therefore

$$\Gamma(D_J, \mathcal{O}_{\mathbb{P}^2}(n)) \cong A[T_1/T_2, T_0]$$

PROPOSITION 4A.4. Let J be a non empty subset of I.
a) If $n = h \cdot d_J$, then $\Gamma(D_J, \mathcal{O}_{\mathbb{P}(Q)}(n))$ is free of rank 1 over $\Gamma(D_J, \mathcal{O}_{\mathbb{P}(Q)})$ and a generator is M_J^h.
b) If Q is normalized and n is not a multiple of d_J, then $\Gamma(D_J, \mathcal{O}_{\mathbb{P}(Q)}(n))$ is not free over $\Gamma(D_J, \mathcal{O}_{\mathbb{P}(Q)})$.

Proof. a) Since $n = h \cdot d_J$ there is a rational monomial of degree n, namely M_J^h, and the multiplication by M_J^h yields the isomorphism

$$S(Q)_{(T_J)} \longrightarrow S(Q)(n)_{(T_J)}$$

b) If $d_J \nmid n$ then $\#(I - J) \geq 2$ (otherwise $d_J = 1$). Now, for every $i \in I - J$, G.C.D.$(q_0, \ldots, \hat{q}_i, \ldots, q_r) = 1$, therefore we may choose solutions of the integral equation $\sum_i a_i q_i = n$ with i^{th} coordinate zero. Hence we get rational monomials of degree n, in which the variable T_i does not appear. If $S(Q)(n)_{(T_J)}$ has a unique generator over $S(Q)_{(T_J)}$, this must be of the type $T^b = \prod_{i \in J} T_i^{b_i}$ because if the variable T_i $(i \in I - J)$ appears in T^b then it has necessarily a positive exponent and so the particular monomials where T_i does not appear cannot be a multiple of T^b with coefficient in $S(Q)_{(T_J)}$. However, if $T^b = \prod_{i \in J} T_i^{b_i}$ is a generator, then every monomial of $S(Q)(n)_{(T_J)}$ has degree $n = \sum_{i \in J} q_i b_i$ which is a multiple of d_J. This is a contradiction. On the other hand $\Gamma(D_J, \mathcal{O}_{\mathbb{P}(Q)}(n)) = S(Q)(n)_{(T_J)}$ and $\Gamma(D_I, \mathcal{O}_{\mathbb{P}(Q)}(n)) =$ $= S(Q)(n)_{(T_I)}$ which is a localization of $S(Q)(n)_{(T_J)}$ and it is free of rank 1 over $S(Q)_{(T_I)}$ by a). Thus the rank of $\Gamma(D_J, \mathcal{O}_{\mathbb{P}(Q)}(n))$ is 1 and the proof is complete.

COROLLARY 4A.5. <u>Let</u> $m = \text{l.c.m.}(q_0, \ldots, q_r)$, <u>then</u>

a) $\mathcal{O}_{\mathbb{P}(Q)}(\alpha m)$ <u>is invertible for every</u> $\alpha \in \mathbb{Z}$;

b) <u>The canonical morphisms</u>

$$\mathcal{O}_{\mathbb{P}(Q)}(\alpha m) \otimes \mathcal{O}_{\mathbb{P}(Q)}(p) \longrightarrow \mathcal{O}_{\mathbb{P}(Q)}(\alpha m + p),$$

$$\mathcal{O}_{\mathbb{P}(Q)}(p) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathbb{P}(Q)}}(\mathcal{O}_{\mathbb{P}(Q)}(\alpha m), \mathcal{O}_{\mathbb{P}(Q)}(\alpha m + p))$$

<u>are isomorphisms</u>.

<u>Proof</u>. a) Of course αm is a multiple of every d_J, hence $\mathcal{O}_{\mathbb{P}(Q)}(\alpha m)$ is invertible on every standard open set.

b) It can be checked by standard computations on the open sets $D_+(T_i)$ and by means of 4A.3.

REMARK. If $Q = (1, 1, 2)$, $\mathcal{O}_{\mathbb{P}(Q)}(1)$ is not invertible, while $\mathcal{O}_{\mathbb{P}(Q)}(2)$ is invertible and it easy to see that $\mathcal{O}_{\mathbb{P}(Q)}(1) \otimes \mathcal{O}_{\mathbb{P}(Q)}(1) \longrightarrow \mathcal{O}_{\mathbb{P}(Q)}(2)$ is not an isomorphism. Therefore the first part of b) cannot be generalized even in the case $Q = \overline{Q}$. On the other hand, if $Q = \overline{Q}$, then the natural morphism

$$\mathcal{O}_{\mathbb{P}(Q)}(a) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathbb{P}(Q)}}(\mathcal{O}_{\mathbb{P}(Q)}(b), \mathcal{O}_{\mathbb{P}(Q)}(a + b))$$

is an isomorphism for every integers a, b (see [De], 4.1 p. 210).

PROPOSITION 4A.6. a) Let J be a non empty subset of I such that $d_J = 1$; then $\Gamma(D_J, \mathcal{O}_{\mathbb{P}(Q)})$ is a localization of a polynomial ring hence D_J is non singular. Moreover the canonical projection $p : U \longrightarrow \mathbb{P}(Q)$ restricted to D_J is isomorphic to the canonical projection

$$D_J \times \mathrm{Spec}(k[X, X^{-1}]) \longrightarrow D_J;$$

b) If Q is normalized and $d_J \neq 1$, then D_J is singular;

c) If Q is normalized, then $\mathbb{P}(Q)$ is non singular if and only if $Q = (1, 1, \ldots 1)$.

Proof. a) By 4A.3 we get that $S(Q)_{(T_J)}$ is isomorphic to $k[G_J][T_i]_{i \in I-J}$. Since G_J is a free group, $S(Q)_{(T_J)}$ turns out to be a localization of a polynomial ring. Let now M_J be a rational monomial of degree 1. Then of course

$$S(Q)_{T_J} = S(Q)_{(T_J)}[M_J, M_J^{-1}] .$$

b) For semplicity, let us prove b) in the case $J = \{i\}$, so that $D_J = D_+(T_i)$. We have already seen (3A.1) that $D_+(T_i)$ can be identified with V_i/μ_{q_i}, where $V_i = \mathrm{Spec}(k[T_0, \ldots, \hat{T}_i, \ldots, T_r])$ and the action of μ_{q_i} is given in the following way

$$T_j \longmapsto g^{q_j} T_j \text{ where } g^{q_i} = 1.$$

Therefore the action of μ_{q_i} on $V_i \cong \mathbb{A}^r$ can be described by the matrix

$$\begin{bmatrix} g^{q_1} & & & & & & & \\ & \cdot & & & & 0 & & \\ & & \cdot & & & & & \\ & & & g^{q_{i-1}} & & & & \\ & & & & g^{q_{i+1}} & & & \\ & & 0 & & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & & g^{q_r} \end{bmatrix} = A_g$$

More precisely the action of μ_{q_i} on $\mathbb{A}^r$ gives rise to a representation

$$\mu_{q_i} \longrightarrow GL_r(k)$$

which is clearly injective. Now, the quotient $D_+(T_i)$ is non singular iff the representation of μ_{q_i} is generated by pseudoreflections (see [D], 1·3.2) and g yields a pseudoreflection iff $\rho(A_g - I) \leqslant 1$, i.e. iff $g^{q_j} = 1$ for all the indexes except i and possibly another one. If ε is a primitive q_i^{th} root of unity, g is a pseudoreflectioniff $\varepsilon^{q_j} = 1$ for all the indexes except one at most. But this happens iff $q_i | q_j$ for every j except one at most. Since Q is normalized this is possible iff $q_i = 1$.

c) Follows from b).

REMARK. If Q is normalized and $J = I - \{i\}$ then $d_J = 1$ hence 4A.4, a), 4A.6, a) can be applied to the open set $D_J = \bigcap_{j \neq i} D_+(T_j)$.

4B. Ampleness criteria.

In the following let $Q = (q_0, \ldots, q_r)$, $|Q| = \sum_{i=0}^{r} q_i$ and $m =$ $=$ l.c.m.$(q_0, \ldots, q_r)$. For every non empty subset J of $I = \{0, \ldots, r\}$, let $m_J = \text{l.c.m.}\{q_i\}_{i \in J}$.

DEFINITION 4B.1. We let G(Q) be the rational number defined in the following way

$$G(Q) = -q_0 \quad \text{if} \quad r = 0$$

$$G(Q) = -|Q| + 1/r \cdot \sum_{2 \leq \nu \leq r+1} \left[\binom{r-1}{\nu-2}^{-1} \sum_{\# J = \nu} m_J \right] \quad \text{if} \quad r > 0.$$

DEFINITION 4B.2. We say that an integer n <u>satisfies the condition</u> D(n) if equivalently

(A) given the relation $\sum_{i=0}^{r} B_i q_i = n + hm$ with $h \in \mathbb{N}^+$ and $B_0, \ldots, B_r$ natural numbers, then there exist $b_0, \ldots, b_r$ natural numbers with $B_i \geq b_i$, $i = 0, \ldots, r$ and $\sum b_i q_i = hm$.

(B) every integral monomial $T^B = \prod_{i=0}^{r} T_i^{B_i}$ of degree n + hm is divisible by an integral monomial $T^b = \prod_{i=0}^{r} T_i^{b_i}$ of degree hm.

DEFINITION 4B.3. We define F(Q) or simply F to be the smallest integer such that $n > F$ implies that D(n) holds. We define E(Q) or simply E to be the smallest integer such that $n > E$ implies that D(mn) holds. In particular $m E \leq F$.

We have the following

LEMMA 4B.4. <u>If</u> Q_i <u>denotes the r-ple</u> $(q_0, \ldots, \hat{q}_i, \ldots, q_r)$ <u>then</u>

$$\sum_{i=0}^{r} G(Q_i) = r \cdot G(Q) - m .$$

<u>Proof</u>. If $s_i = \sum_{j \neq i} q_j$, then by definition

$$G(Q_i) = -s_i + 1/(r-1) \sum_{2 \leq \nu \leq r} \left[\binom{r-2}{\nu-2}^{-1} \sum_{\# J = \nu} m_J \right]$$

hence

$$\sum_{i=0}^{r} G(Q_i) = -r|Q| + 1/(r-1) \sum_{2 \leq \nu \leq r} \left[\binom{r-2}{\nu-2}^{-1} (r+1-\nu) \sum_{\# J = \nu} m_J \right]$$

Now, since

$$(r + 1 - \nu)/(r - 1) \cdot \binom{r-2}{\nu-2}^{-1} = \binom{r-1}{\nu-2}^{-1}$$

we get

$$\sum_{2 \leqslant \nu \leqslant r+1} \left[\binom{r-1}{\nu-2}^{-1} \sum_{\# J = \nu} m_J \right] = \sum_{2 \leqslant \nu \leqslant r} \left[\binom{r-1}{\nu-2}^{-1} \sum_{\# J = \nu} m_J \right] + m$$

Therefore

$$\sum_{i=0}^{r} G(Q_i) = -r|Q| + \sum_{2 \leqslant \nu \leqslant r+1} \left[\binom{r-1}{\nu-2}^{-1} \sum_{\# J = \nu} m_J \right] - m =$$

$$= r\left(-|Q| + (1/r) \sum_{2 \leqslant \nu \leqslant r+1} \left[\binom{r-1}{\nu-2}^{-1} \sum_{\# J = \nu} m_J \right]\right) - m =$$

$$= r \cdot G(Q) - m.$$

PROPOSITION 4B.5. <u>For every</u> $n > G(Q)$ <u>the condition</u> $D(n)$ <u>holds, hence</u> $F \leqslant G(Q)$.

<u>Proof</u>. We use a double induction on the pair (h, r) with $h \geqslant 0$, $r \geqslant 0$ (remind that h is the integer appearing in the definition of $D(n)$).
Of course the statement is true for $h = 0$ and also for $r = 0$. It is also easy to show that if the statement is proved for $h = 1$, then it is true in general; for, it is sufficient to replace n with $n + (h - 1)\, m$. Therefore it is sufficient to prove the statement for $(1, r)$ assuming that it holds for $(1, r - 1)$, hence for $(h, r - 1)$, every h. Let

$$\sum_{i=0}^{r} B_i q_i > G(Q) + m$$

then

$$r \cdot \sum_{i=0}^{r} B_i q_i > r \cdot G(Q) + rm,$$

hence, by 4B.4.:

$$r \cdot \sum_{i=0}^{r} B_i q_i > \sum_{i=0}^{r} G(Q_i) + m(r + 1)$$

But $r \cdot \sum_{i=0}^{r} B_i q_i = \sum_i \left(\sum_{j \neq i} B_j q_j \right)$, hence for at least one index i we have

$\sum_{j \neq i} B_j q_j > G(Q_i) + m.$

Now, if $J = I - \{i\}$, $m = \lambda_i m_J$ and by induction the statement is true for $(\lambda_i, r - 1)$; hence we get an r-uple $(B_0, \ldots, \hat{i}, \ldots, B_r)$ such that $\sum_{j \neq i} B_j q_j = m$. Therefore the $(r + 1)$-uple $(B_0, \ldots, B_i = 0, \ldots, B_r)$ solves the problem.

LEMMA 4B.6. Let X be a projective k-scheme, Y a Cartier divisor on X; let $A = \oplus_p A_p$ where $A_p = \Gamma(X, \mathcal{O}_X(pY))$ and let φ_Y be the rational map associated to the complete linear system $|Y|$. Then, if A is generated by A_1 as a k-algebra,

$\varphi_Y(X) = \mathrm{Proj}(A)$.

Moreover if Y is ample then the following conditions are equivalent

a) A is generated by A_1 as a k-algebra;

b) Y is very ample.

Proof. Let

$$\varphi_Y : X - - - \to \mathbb{P}^N = \mathbb{P}(\Gamma(X, \mathcal{O}_X(Y)) = \mathrm{Proj}(\mathrm{Symm}(A_1)),$$

and let L_p be the image of $\Gamma(X, \mathcal{O}_X(Y))^{\otimes p}$ in $\Gamma(X, \mathcal{O}_X(pY))$. If L is the graded algebra $\oplus_p L_p$, then it is known that $\varphi_Y(X) = \mathrm{Proj}(L)$. But clearly A is generated by A_1 implies that $L_p = A_p$ for every p, hence $\varphi_Y(X) = \mathrm{Proj}(A)$.

Assume now that Y is ample and let q be a natural number such that Z = qY is very ample.

a) $\Longrightarrow$ b) We have

$$X \cong \varphi_Z(X) \cong \mathrm{Proj}(A^{(q)}) \cong \mathrm{Proj}(A) \cong \varphi_Y(X)$$

hence φ_Y is an isomorphism and Y is very ample.

b) $\Longrightarrow$ a) If Y is very ample, then it corresponds to an hyperplane section in a suitable embedding of X in $\mathbb{P}^N$, then $A_1 \cong \Gamma(X, \mathcal{O}_X(1))$ and the conclusion follows.

THEOREM 4B.7. a) <u>The sheaf</u> $\mathcal{O}_{\mathbb{P}(Q)}(m)$ <u>is ample</u>;

b) <u>If</u> $n > F$, <u>the sheaf</u> $\mathcal{O}_{\mathbb{P}(Q)}(n)$ <u>is generated by global sections</u>.

c) <u>If</u> $n > 0$, $n > E$, <u>then</u> $\mathcal{O}_{\mathbb{P}(Q)}(nm)$ <u>is very ample</u>;

d) <u>For every</u> $p \in \mathbb{Z}$, <u>the sheaf</u> $\mathcal{O}_{\mathbb{P}(Q)}(p)$ <u>is coherent</u>;

e) <u>For every</u> $p \in \mathbb{Z}$, <u>the sheaf</u> $\mathcal{O}_{\mathbb{P}(Q)}(p)$ <u>is Cohen Macaulay</u> (C - M).

<u>Proof</u>. b) Let $u \in \Gamma(D_+(T_i), \mathcal{O}_{\mathbb{P}(Q)}(n))$; then $u = U/T_i^s$ with $s > 0$ and $U \in S(Q)_{sq_i+n}$. Assume that U is a monomial. Therefore $u = U\,T_i^{sm-s}/T_i^{sm}$ with

$$\deg(U\,T_i^{sm-s}) = sq_i + n + q_i(sm - s) = sq_i m + n.$$

Since $n > F$ the condition D(n) holds hence we may write $UT_i^{sm-s} = A \cdot B$ where A is a monomial of degree n and B a monomial of degree $sq_i m$. In conclusion $u = AB/T_i^{sm}$ and A defines a global section of $\mathcal{O}_{\mathbb{P}(Q)}(n)$ while $B/T_i^{sm} \in \Gamma(D_+(T_i), \mathcal{O}_{\mathbb{P}(Q)})$ and we are done.

c) By assumption $n > E$, hence the condition D(mn) holds, which implies that every monomial of degree $pmn = (p - 1)\,mn + mn$ is divisible by a monomial of degree $(p - 1)\,mn$; thus a monomial of degree pmn is the product of monomials of degree mn and this implies that $S(Q)^{(mn)}$ is generated by $S(Q)^{(mn)}_1$ as a k-algebra. Now $\mathcal{O}_{\mathbb{P}(Q)}(mn)$ is invertible by 4A.5.a) and $\Gamma(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(mn)) =$
$= S(Q)_{mn} \neq \{0\}$, hence if we choose a non trivial section of $\mathcal{O}_{\mathbb{P}(Q)}(mn)$ we get a divisor Y such that $\mathcal{O}_{\mathbb{P}(Q)}(mn) \cong \mathcal{O}_{\mathbb{P}(Q)}(Y)$. By lemma 4B.6 we have that $\varphi_Y(\mathbb{P}(Q))$ is isomorphic to $\mathrm{Proj}(S(Q)^{(mn)})$ which is isomorphic to $\mathrm{Proj}(S(Q))$.

d) We know that $\mathcal{O}_{\mathbb{P}(Q)}(\alpha m)$ are invertible by 4A.5. Since the property of being coherent is local, it is sufficient to show that $\mathcal{O}_{\mathbb{P}(Q)}(p) \otimes \mathcal{O}_{\mathbb{P}(Q)}(\alpha m) \cong \mathcal{O}_{\mathbb{P}(Q)}(\alpha m + p)$ (see 4A.5) is coherent for some α. On the other hand $\mathcal{O}_{\mathbb{P}(Q)}(\alpha m + p)$ is generated by $H^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(\alpha m + p))$ by b) and this is isomorphic to $S(Q)_{\alpha m+p}$, which is a finitely generated vector space.

e) After c) we may choose an integer n such that $\mathcal{O}_{\mathbb{P}(Q)}(nm)$ is very ample; according to [SGA2], Exp. XII, 1.4 it is therefore sufficient to show that $H^i(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(p) \otimes \mathcal{O}_{\mathbb{P}(Q)}(\alpha nm)) = (0)$ for $i < r$ and $\alpha \gg 0$. But

$$\mathcal{O}_{\mathbb{P}(Q)}(p) \otimes \mathcal{O}_{\mathbb{P}(Q)}(\alpha mn) \cong \mathcal{O}_{\mathbb{P}(Q)}(p + \alpha nm)$$

by 4A.5 and $H^i(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(p + \alpha nm)) = (0)$ for $i < r$ by 3B.4, b).

a) The sheaf $\mathcal{O}_{\mathbb{P}(Q)}(m)$ is invertible by 4A.5, a) and $\mathcal{O}_{\mathbb{P}(Q)}(m)^{\otimes n} \cong \mathcal{O}_{\mathbb{P}(Q)}(nm)$ (see 4A.5,b)) is very ample. Therefore $\mathcal{O}_{\mathbb{P}(Q)}(m)$ is ample.

REMARK 1. We have already seen that

$$mE \leqslant F \leqslant G(Q).$$

Thus $\mathcal{O}_{\mathbb{P}(Q)}(mn)$ is very ample for every $n > 0$ such that $n > G(Q)/m$.

COROLLARY 4B.8. <u>The scheme $\mathbb{P}(Q)$ is projective.</u>

<u>Proof.</u> $\mathbb{P}(Q) = \mathrm{Proj}(S(Q)) \cong \mathrm{Proj}(S(Q)^{(mn)})$. But $S(Q)^{(mn)}$ is generated by its part of degree 1. This gives an embedding of $\mathrm{Proj}(S(Q)^{(mn)})$ in $\mathrm{Proj}(\mathrm{Symm}(S(Q)^{(mn)}_1)) = \mathbb{P}^N$.

REMARK 2. The same result can be obtained by considering $\mathbb{P}(Q)$ as a quotient of $\mathbb{P}^r$ by the action of the finite group μ_Q (see 3A.1.d) and then applying numerical criteria of ampleness (see [H1], p. 30).

REMARK 3. Let $Q = (q_0, q_1, q_2)$ and assume that Q is normalized. Then

$m = q_0 q_1 q_2$ and

$$G(Q) = \frac{1}{2}(q_0 q_1 q_2 + q_0 q_1 + q_0 q_2 + q_1 q_2) - (q_0 + q_1 + q_2)$$

But this number is easily seen to be strictly smaller than m, so by Remark 1 we get that

$\mathcal{O}_{\mathbb{P}(Q)}(m)$ is <u>very ample</u>.

However, if $r = 3$, this fact is no longer true in general. For, let $Q =$ $= (1, 6, 10, 15)$. In this case G(Q) is between 1 and 2. Therefore $\mathcal{O}_{\mathbb{P}(Q)}(m \cdot 30)$ is very ample for $m \geq 2$. But we are going to show that $\mathcal{O}_{\mathbb{P}(Q)}(30)$ is only ample without being very ample.

Namely the monomial $T_0 T_1^4 T_2^2 T_3$ has degree 60 but it cannot be expressed as the product of two monomials of degree 30. Thus $S(Q)^{(30)}$ is not generated by its part of degree 1, hence $\mathcal{O}_{\mathbb{P}(Q)}(30)$ is not very ample by 4B.6.

§ 5. The regular locus of Mori.

Let $Q = (q_0, \dots, q_r)$ and put as usual $d = \text{G.C.D.}(q_0, \dots, q_r)$, $m = \text{l.c.m.}(q_0, \dots, q_r)$. For every prime number p let us denote by ν_p the number of indexes i such that $p \nmid q_i$ and denote by

$$\nu(Q) = \min_{p,\ p \text{ prime}} \{ \nu_p \}$$

From the definition of $\nu(Q)$ we have immediately

PROPOSITION 5.1. a) $\nu(Q) > 0$ *iff* $d = 1$, i.e. Q *is reduced*;

b) $\nu(Q) > 1$ *iff* Q *is normalized*;

c) $\nu(Q) > s$ *iff for every choice of* s *weights the* G.C.D. *of the others is* 1;

d) $\nu(Q) > r - 1$ *iff the weights are pairwise coprime*;

e) $\nu(Q) = r + 1$ *iff* $Q = (1, \dots, 1)$.

DEFINITION 5.2. For every integer h we denote by S_h the closed subscheme of $\mathbb{P}(Q)$ which is defined by the ideal $I(S_h)$ generated by those indeterminates T_i such that $h \nmid q_i$.

With the notation $V_+(S(Q)_n)$ we indicate, as usual, the set of relevant primes P such that $P \supseteq S(Q)_n$. Then we can prove the following

LEMMA 5.3. a) $\bigcup_{h>1} S_h = \bigcup_{\substack{h \text{ prime} \\ h|m}} S_h$;

b) $\nu(Q) = \text{codim } (\bigcup_h S_h)$;

c) *For every integer* h, S_h *is a set theoretic complete intersection for every immersion of* $\mathbb{P}(Q)$ *in* $\mathbb{P}^N$ *given by a sheaf* $\mathcal{O}_{\mathbb{P}(Q)}(n)$;

d) $\bigcup_h S_h = \bigcap_{a \geq 0} V_+(S(Q)_{am+1})$.

Proof. a) Let a be a natural number such that $p \nmid a$. Therefore $S_{p^n a} \subseteq S_{p^n} \cup S_a$; namely if $p^n a \nmid q_i$ then either $p^n \nmid q_i$ or $a \nmid q_i$; moreover if $p \nmid m$ then

$S_p = \emptyset$. In conclusion if $h = p_1^{a_1} \cdots\cdots p_s^{a_s}$ then $S_h \subseteq S_{p_1^{a_1}} \cup \cdots\cdots \cup S_{p_s^{a_s}}$ $\subseteq S_{p_1} \cup \cdots \cup S_{p_s}$

b) It follows from a), after remarking that if p is a prime number, then by definition $\nu(p) = \operatorname{codim} S_p$.

c) Namely S_h is defined set-theoretically by $\left\{ T_i^{n/q_i},\ h \nmid q_i \right\}_i$; and T_i^{n/q_i} define hyperplane sections of $\varphi(\mathbb{P}(Q))$ if $\varphi : \mathbb{P}(Q) \longrightarrow \mathbb{P}^N$ is given by a very ample $\mathcal{O}_{\mathbb{P}(Q)}(n)$.

d) Let P be a relevant prime ideal such that $P \notin \bigcup_{h>1} S_h$. Then for every prime h, h dividing m, one has $P \notin S_h$ by a); hence there exists a homogeneous element F_h (actually an indeterminate) of degree d_h such that $h \nmid d_h$ with $F_h \notin P$. Therefore the ideal generated in $\mathbb{Z}$ by the d_h's and m is the unit ideal, hence there exist positive integers a_h such that $\sum a_h d_h = 1 + am$ and $\prod F_h^{a_h} \in S(Q)_{am+1} - P$.

Conversely assume that $P \in S_h$ for a prime h such that $h | m$. This means that P contains all the T_i's such that $h \nmid q_i$. But of course every monomial of $S(Q)_{am+1}$ must contain such an indeterminate (whatever a is chosen). Then $P \supset S(Q)_{am+1}$.

EXAMPLE. Let $Q = (1, 2, 3, 4)$. Then $Q = \overline{Q}$, $m = 12$. Therefore $\bigcup_h S_h =$ $= S_2 \cup S_3$, $I(S_2) = (T_0, T_2)$, $I(S_3) = (T_0, T_1, T_3)$ so that <u>the locus $\cup S_h$ is not of pure codimension</u>.

DEFINITION 5.4. We denote by $\mathbb{P}^o(Q)$ the open set $\mathbb{P}(Q) - \bigcup_{h>1} S_h$ and we call it the M-<u>regular locus of $\mathbb{P}(Q)$</u>.

PROPOSITION 5.5. a) $\nu(Q) = 0$ *iff* $\mathbb{P}^o(Q) = \emptyset$;

b) $\nu(Q) = 1$ *iff* $\mathbb{P}^o(Q)$ *is quasi affine and non empty;*

c) *If* $\nu(Q) > 1$ *then* $\mathbb{P}^o(Q)$ *contains a complete subscheme of dimension* $\nu(Q) - 1$ *and does not contain any complete subscheme of dimension greater than or equal to* $\nu(Q)$.

Proof. a) It follows from the definitions.

b) By b) of 5.3 (see also the proof), if $\nu(Q) = 1$ there exists an indeterminate T_i which defines a component of $\bigcup_{h>1} S_h$. Therefore the complement is quasi affine ad if $\bigcup_{h>1} S_h$ is pure then $\mathbb{P}^o(Q)$ is affine. The converse follows from c).

c) It follows from the description given in Lemma 5.3.

PROPOSITION 5.6. a) *The following conditions are equivalent*

i) $\mathcal{O}_{\mathbb{P}(Q)}(1) \neq 0$; ii) $d = 1$; iii) $\mathbb{P}^o(Q) \neq \emptyset$

b) *For every* $n \in \mathbb{Z}$, *the sheaf* $\mathcal{O}_{\mathbb{P}^o(Q)}(n)$ *is invertible;*

c) $(\mathcal{O}_{\mathbb{P}^o(Q)}(1))^{\otimes n} \cong \mathcal{O}_{\mathbb{P}^o(Q)}(n)$ *for every* $n \in \mathbb{Z}$;

d) $\mathbb{P}^o(Q)$ *is the largest open subset of* $\mathbb{P}(Q)$ *with the properties* b), c). *Moreover if* $\nu(Q) > 1$, *then* $\mathbb{P}^o(Q)$ *is the largest open subset of* $\mathbb{P}(Q)$ *with the property* b).

Proof. a) $\Gamma(D_+(T_i), \mathcal{O}_{\mathbb{P}(Q)}(1)) = S(Q)(1)_{(T_i)}$ and the latter is clearly reduced to 0 if and only if d is greater than 1. The equivalence between ii) and iii) follows from 5.1, a) and 5.5, a).

b) Let P be a relevant homogeneous prime of S(Q), $P \notin \bigcup_{h>1} S_h$. Then by 5.3, d) there exists a homogeneous element F, $F \notin P$, $F \in S(Q)_{am+1}$ for a suitable a. Therefore $P \in D_+(F)$ and there is an isomorphism of $S(Q)_{(F)}$-modules

$$S(Q)(am+1)_{(F)} \cong (S(Q)_F)_{am+1} \cong S(Q)_{(F)} \cdot F$$

This shows that $\mathcal{O}_{\mathbb{P}^o(Q)}(am + 1)$ is invertible in the neighborhood of every $P \in \mathbb{P}^o(Q)$, hence it is invertible. But

$$\mathcal{O}_{\mathbb{P}^o(Q)}(n) = \left(\mathcal{O}_{\mathbb{P}^o(Q)}(am + 1) \otimes \mathcal{O}_{\mathbb{P}(Q)}(- m)^{\otimes a} \right)^{\otimes n}$$

by 4A.5 and all of them are invertible.

c) We use again the same technique as in b); we first observe that it suffices to show that

$$\mathcal{O}_{\mathbb{P}^o(Q)}(a) \otimes \mathcal{O}_{\mathbb{P}^o(Q)}(b) \longrightarrow \mathcal{O}_{\mathbb{P}^o(Q)}(a + b)$$

is an isomorphism for every a, b. This natural morphism is induced by the natural $S(Q)$-module homomorphism

$$S(Q)(a) \otimes S(Q)(b) \longrightarrow S(Q)(a + b).$$

Now let $P \in \mathbb{P}^o(Q)$. Then, as in the proof of b) there exists a homogeneous element $F \notin P$, $F \in S(Q)_{am+1}$. Moreover we can choose an indeterminate T_i of degree q_i such that $T_i \notin P$. Then of course it is sufficient to show that

$$S(Q)(a)_{(FT_i)} \otimes S(Q)(b)_{(FT_i)} \longrightarrow S(Q)(a + b)_{(FT_i)}$$

is an isomorphism. But

$$S(Q)(a)_{(FT_i)} = S(Q)_{(FT_i)}(F/T_i^{c_i})^a$$

where $c_i = am/q_i$ and similary

$$S(Q)(b)_{(FT_i)} = S(Q)_{(FT_i)}(F/T_i^{c_i})^b$$

and

$$S(Q)(a + b)_{(FT_i)} = S(Q)_{(FT_i)}(F/T_i^{c_i})^{a+b}$$

hence the conclusion follows immediately.

d) Let U be an open set with the properties b), c) and let x be a closed point of U; we must show that $x \in \mathbb{P}^o(Q)$; by 5.3, d) it is sufficient to show that if P is the homogeneous prime ideal corresponding to x, then there exists a homogeneous element $F \in S(Q)_{am+1} - P$ for a suitable a.

Since $\mathcal{O}_{\mathbb{P}(Q)}(m)$ is ample by 4B.7 and b), c) hold, there exists a suitable a such that

$$\mathcal{O}_{\mathbb{P}(Q)}(am + 1) \cong \mathcal{O}_{\mathbb{P}(Q)}(1) \otimes \mathcal{O}_{\mathbb{P}(Q)}(m)^{\otimes a}$$

is generated by global sections. Therefore there exists an element $F \in S(Q)_{am+1} = H^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(am + 1))$ generating the stalk of $\mathcal{O}_{\mathbb{P}(Q)}(am + 1)$ at x, i.e.

$$(\mathcal{O}_{\mathbb{P}(Q)}(am + 1))_x = \mathcal{O}_{\mathbb{P}(Q),x} \cdot F$$

Let now G be an element of $S(Q)_m$ such that $G \notin P$. Again we may assume that

$$\mathcal{O}_{\mathbb{P}(Q)}(m(am + 1))_x = \mathcal{O}_{\mathbb{P}(Q),x} \cdot G^{am+1}$$

and by c) we get that G^{am+1} and F^m both generate $\mathcal{O}_{\mathbb{P}(Q)}(m(am + 1))_x$. Therefore G^{am+1}/F^m is invertible in $\mathcal{O}_{\mathbb{P}(Q),x}$ which means that also $F^m \notin P$. So we have got an element $F \in S(Q)_{am+1}-P$.

Let now U be an open set with the property b). We first note that property c) is equivalent to

c') $(\mathcal{O}_{\mathbb{P}(Q)}(1)|_U)^{\otimes m} \xrightarrow{\sim} \mathcal{O}_{\mathbb{P}(Q)}(m)|_U$

by 4A.5. Therefore $U - \mathbb{P}^o(Q)$ is the closed subset of U which is the locus of the points of U where c') does not hold. Let

$$(\mathcal{O}_{\mathbb{P}(Q)}(1)|_U)^{\otimes m} \longrightarrow \mathcal{O}_{\mathbb{P}(Q)}(m)|_U$$

be the canonical morphism; it corresponds to a section of the invertible sheaf $\mathcal{O}_{\mathbb{P}(Q)}(m)|_U \otimes (\mathcal{O}_{\mathbb{P}(Q)}(1)|_U)^{-m}$ and $U - \mathbb{P}^o(Q)$ is defined by the vanishing of such a section hence either $\operatorname{codim}_U(U - \mathbb{P}^o(Q)) = 1$ or the section does not vanish. But, if $\gamma(Q) > 1$, then $\operatorname{codim}_{\mathbb{P}(Q)}(\mathbb{P}(Q) - \mathbb{P}^o(Q)) = \gamma(Q) > 1$. Therefore $U \subseteq \mathbb{P}^o(Q)$ and we are done.

PROPOSITION 5.7. <u>Let</u> p <u>be the canonical projection</u> $p : U \longrightarrow \mathbb{P}(Q)$ <u>and let</u> $U^o = p^{-1}(\mathbb{P}^o(Q))$, <u>then</u>

a) $U^{\circ} \xrightarrow{p} \mathbb{P}^{\circ}(Q)$ <u>is a $\mathbb{G}_m$-bundle, hence</u> $\mathbb{P}^{\circ}(Q) \subseteq \mathrm{Reg}(\mathbb{P}(Q))$;

b) $\mathcal{O}_{\mathbb{P}^{\circ}(Q)}(1)$ <u>generates</u> $\mathrm{Pic}(\mathbb{P}^{\circ}(Q))$. <u>Moreover if</u> Q <u>is normalized then</u>

$\mathrm{Pic}(\mathbb{P}^{\circ}(Q)) = \mathbb{Z}$.

<u>Proof.</u> a) The morphism p is locally defined by the inclusions $S(Q)_{(F)} \hookrightarrow S(Q)_F$. If Q is not reduced, then $\mathbb{P}^{\circ}(Q) = \emptyset$ (5.5, a)) hence we may assume Q to be reduced. Since G.C.D.$(q_0, \ldots, q_r) = 1$, there exist b_i's such that $\sum b_i q_i = 1$ hence there exist a_i's, $a_i \in \mathbb{N}$, $a \in \mathbb{N}^+$ such that $\sum_{i=0}^{r} a_i q_i = am + 1$. Let G be a nonzero element of $S(Q)_{am+1}$ and F a nonzero element of $S(Q)_{am}$. Then $\deg(G/F) = 1$ and it is easy to see (see also 4A.6.a)) that

$$S_{FG} = S_{(FG)}\left[G/F, F/G\right] \cong S_{(FG)}\left[X, X^{-1}\right]$$

It is therefore sufficient to prove that the open sets of the type $D_+(FG)$ cover $\mathbb{P}^{\circ}(Q)$. For, let $P \in \mathbb{P}^{\circ}(Q)$; by 5.3, d) there exists an integer a such that $P \notin V_+(S(Q)_{am+1})$, hence there exists $G \in S(Q)_{am+1}$ and $P \in D_+(G)$. On the other hand $P \notin V_+(S(Q)_{am})$ for every a, hence there exists $F \in S(Q)_{am}$, $P \in D_+(F)$. Therefore $P \in D_+(FG)$.

b) Since $\mathbb{P}^{\circ}(Q)$ is smooth, $\mathrm{Pic}(\mathbb{P}^{\circ}(Q)) = \mathrm{Cl}(\mathbb{P}^{\circ}(Q))$; moreover $\mathcal{O}_{\mathbb{P}^{\circ}(Q)}(n) \cong \mathcal{O}_{\mathbb{P}^{\circ}(Q)}(1)^{\otimes n}$ by 5.6, c) hence, to show that $\mathcal{O}_{\mathbb{P}^{\circ}(Q)}(1)$ generates $\mathrm{Pic}(\mathbb{P}^{\circ}(Q))$ it suffices to prove that for every subvariety D of codimension 1 of $\mathbb{P}^{\circ}(Q)$ there is a homogeneous prime element F of $S(Q)$ such that $\mathrm{Supp}\, D = V_+(F)$. If D is such a subvariety , then $p^{-1}(D)$ is a subvariety of codimension 1 of $U^{\circ} \subseteq \mathbb{A}^{r+1}$ hence there exists a homogeneous prime element $F \in S(Q)$ such that $\mathrm{Supp}(p^{-1}(D)) = V(F)$. Therefore $\mathrm{Supp}\, D = V_+(F)$.

Now, if Q is normalized, then $\nu(Q) \geq 2$ and there is a positively dimensional complete variety X contained in $\mathbb{P}^{\circ}(Q)$ (see 5.1, b) and 5.5, c)). Since

$\mathcal{O}_{\mathbb{P}^{o}(Q)}(1)$ is ample by 4B.7 and 5.6, d), also its restriction to X is ample, hence no power of it can be trivial.

COROLLARY 5.8. *If* $\mathcal{O}_{\mathbb{P}(Q)}(n) \neq 0$, *then it is reflexive of rank* 1.

Proof. If $\mathcal{O}_{\mathbb{P}(Q)}(n) \neq 0$ then n is a multiple of G.C.D.$(q_0, \ldots, q_r)$. Hence by 3C.1 we may assume that Q is reduced and by 3C.5 and 3C.7 we may assume that Q is normalized. We know that $\mathbb{P}^{o}(Q) \subseteq \mathrm{Reg}(\mathbb{P}(Q)) \subseteq \mathbb{P}(Q)$ (see 5.7). Let $j : \mathbb{P}^{o}(Q) \rightarrow \mathbb{P}(Q)$ denote the canonical inclusion; then $\mathcal{O}_{\mathbb{P}(Q)}(n) = j_* \mathcal{O}_{\mathbb{P}^{o}(Q)}(n)$ and $\mathrm{codim}_{\mathbb{P}(Q)}(\mathbb{P}(Q) - \mathbb{P}^{o}(Q)) \geqslant 2$ by 5.1, b). On the other hand $\mathcal{O}_{\mathbb{P}(Q)}(n)$ is invertible by 5.6, so the conclusion follows by Appendix Theorem 17.
Another way of getting this result is by using the remark following Corollary 4A.5.

COROLLARY 5.9. *Let* X *be a closed subscheme of* $\mathbb{P}(Q)$ *and assume that* $X \subset \mathbb{P}^{o}(Q)$. *Then* X *is non singular if and only if* X *is quasi-smooth* (see 3B.5).

Proof. It follows directly from 5.7, a).

EXAMPLES 1. If $X \not\subset \mathbb{P}^{o}(Q)$, X may be quasi-smooth and singular. Take for instance $X = \mathbb{P}(1, 1, 2)$.

2. If $X \not\subset \mathbb{P}^{o}(Q)$, X may be not quasi-smooth and non singular. For instance let $Q = (2, 2, 1)$, $A = k[T_0, T_1, T_2]/(T_0^2 - T_1T_2^2)$ and $X = \mathrm{Proj}(A)$. Then $X \cong \mathrm{Proj}(A^{(2)})$, but $A^{(2)} = k[X_0, X_1, X_2]/(X_0^2 - X_1X_2)$ with $\deg X_i = 1$, hence X is non singular. On the other hand Spec(A) has the line $T_0 = T_2 = 0$ as the singular locus. Now $\nu(Q) = 1$ and $\mathbb{P}^{o}(Q) = \mathbb{P}(Q) - \{T_2 = 0\}$, so that $X \not\subset \mathbb{P}^{o}(Q)$.

Let us make some computations on the last example:

$$A_{(T_1)} \cong k[T_0/T_1, T_2^2/T_1]/((T_0/T_1)^2 - T_2^2/T_1) \cong k[X, Y]/(X^2 - Y) \cong k[X]$$

$$A/(T_1 - 1) \cong k[T_0, T_2]/(T_0^2 - T_2^2).$$

Over $A/(T_1 - 1)$ the action of μ_2 is described by $T_0 \longmapsto T_0$, $T_2 \longmapsto - T_2$ therefore (see 2.6):

$$(A/(T_1 - 1))^{\mu_2} \cong k[T_0, T_2^2]/(T_0^2 - T_2^2) \cong k[X, Y]/(X^2 - Y)$$

Moreover $A_{(T_1)} \hookrightarrow A/(T_1 - 1)$ is a flat and finite morphism, $A_{(T_1)}$ is regular, $A/(T_1 - 1)$ is not regular.

REMARK (see 3A.4). The canonical projection $\pi : \mathbb{P}^r \longrightarrow \mathbb{P}(Q)$ induces $\pi^o : U^o \longrightarrow \mathbb{P}^o(Q)$ which is flat. Namely we have seen that $\pi_* \mathcal{O}_{\mathbb{P}^r} = \oplus \mathcal{O}_{\mathbb{P}(Q)}(- \sum n_i)$ (see 3A.2). But over $\mathbb{P}^o(Q)$ the sheaves $\mathcal{O}_{\mathbb{P}(Q)}(n)$ are invertible, hence the morphism π^o is flat. For instance, if $Q = (1, 1, 2)$, we know that $\mathbb{P}(Q)$ is isomorphic to the cone of $\mathbb{P}^3$ which projects a smooth conic and $\mathbb{P}^o(Q)$ is the cone except its vertex. All the fibres of π have 2 points (not necessarily distinct), while on the vertex there is only <u>one</u> point and at this point the morphism is not flat. The corresponding local rings are

$$(k[T_0, T_1, T_2]_{(T_0, T_1)})_0 \cong (k[X_0, X_1, X_2^2]_{(X_0, X_1)})_0 \cong$$

$$\cong (k[X_0^2, X_0X_1, X_1^2, X_2^2]_{(X_0^2, X_0X_1, X_1^2)})_0 \cong k[X^2, XY, Y^2]_{(X^2, XY, Y^2)} \cong$$

$$\cong k[X, Y, Z]_{loc}/(XY - Z^2)$$

and $(k[X_0, X_1, X_2]_{(X_0, X_1)})_0 \cong k[X, Y]_{loc}$. Clearly the morphism

$$k[X^2, XY, Y^2]_{loc.} \longrightarrow k[X, Y]$$

is <u>not flat</u>.

§6. Differentials and dualizing sheaves on $\mathbb{P}(Q)$.

In the first part of this section we deal with some generalities on regular differential forms on $\mathbb{P}(Q)$, while the second part is mainly devoted to compute the dualizing sheaf of a complete intersection in $\mathbb{P}(Q)$.

6A. Regular differentials.

Let A be a ring (commutative with identity), B an A-algebra and M a B-module. An A-derivation of B into M is an A-linear map $d : B \longrightarrow M$ such that $d(bb') = bd(b') + b'd(b)$ and $d(a) = 0$ for every $a \in A$.

DEFINITION 6A.1. We define the module of relative differentials of B over A to be a B-module $\Omega_{B/A}$ together with an A-derivation $d : B \longrightarrow \Omega_{B/A}$ which satysfies the following universal property: for every B-module M, and for every A-derivation $d' : B \longrightarrow M$, there exists a unique B-module homomorphism $f : \Omega_{B/A} \longrightarrow M$ such that $d' = f \circ d$.

It is well-known and easy to see that $(\Omega_{B/A}, d)$ exists and it is unique up to isomorphism (see [Ma]).

PROPOSITION 6A.2. (First exact sequence). Let $A \longrightarrow B \longrightarrow C$ be homomorphisms of rings. Then there is a natural exact sequence of C-modules

$$\Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow \Omega_{C/B} \longrightarrow 0$$

Proof. See [Ma], p. 186.

PROPOSITION 6A.3. (Second exact sequence). Let B an A-algebra, I an ideal of B and $C = B/I$. Then there is a natural exact sequence of C-modules

$$I/I^2 \xrightarrow{\delta} \Omega_{B/A} \otimes_B C \longrightarrow \Omega_{C/A} \longrightarrow 0$$

where for every $b \in I$, if $\bar{b}$ denotes its image in I/I^2, then $\delta\bar{b} = db \otimes 1$.

<u>Proof</u>. See [Ma], p. 187.

Let now $f : X \longrightarrow Y$ be a morphism of schemes and let us consider the associate diagonal morphism $\Delta : X \longrightarrow X \times_Y X$. Then X is isomorphic to $\Delta(X)$ (see [H], II, 4) which is a closed subscheme of an open subset W of $X \times_Y X$.

DEFINITION 6A.4. Let $\mathfrak{I}$ be the sheaf of ideals of $\Delta(X)$ in W. Then we define the <u>sheaf of relative differentials</u> of X over Y to be the sheaf $\Omega_{X/Y} = \Delta^*(\mathfrak{I}/\mathfrak{I}^2)$ on X.

PROPOSITION 6A.5. <u>If X and Y are affine, then Definitions 6A.4 and 6A.1 agree</u>.

<u>Proof</u>. See [Ma], p. 182.

Let now consider the usual projective space $X = \mathbb{P}^r_k$.

THEOREM 6A.6. <u>There is an exact sequence of sheaves on X</u>

$$0 \longrightarrow \Omega_{X/k} \longrightarrow \mathcal{O}_X(-1)^{r+1} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

(The exponent r + 1 in the middle means a direct sum of r + 1 copies of $\mathcal{O}_X(-1)$).

<u>Proof</u>. See [H], p. 176.

DEFINITION 6A.7. If X is a <u>smooth</u>, r-dimensional scheme over k, then we define the <u>sheaf of</u> i^{th} <u>regular differentials</u> of X (over k) to be

$$\Omega^i_X = \Omega^i_{X/k} = \bigwedge^i \Omega_{X/k}$$

Furthermore we say that $\omega_X = \Omega^r_X$ is the <u>canonical sheaf</u>.

So now let us compute Ω^i_X when $X = \mathbb{P}^r_k$. To this end let us consider a slightly more general situation, which will be useful in the following.

Let as usual $Q = (q_0, \ldots, q_r)$, $S(Q) = k[T_0, \ldots, T_r]$ graded by $q_i = \deg T_i$. Since $S(Q)$ is a polynomial ring it is easy to see that $\Omega_{S(Q)}$ is a free module generated by $\{dT_0, \ldots, dT_r\}$ hence it can be given a structure of graded $S(Q)$-module by putting $\deg(dT_i) = q_i$; consequently $\Omega_{S(Q)} \simeq \bigoplus_{i=0}^{r} S(Q)(-q_i)$.

Therefore $\bigwedge^i \Omega_{S(Q)}$ becomes a free graded module with basis given by $\{dT_{j_1} \wedge \ldots \wedge dT_{j_i} ; 0 \leqslant j_1 < \ldots < j_i \leqslant r\}$, hence

$$\bigwedge^i \Omega_{S(Q)} \cong \bigoplus_{0 \leqslant j_1 < \ldots < j_i \leqslant r} S(Q)(-q_{j_1} - \ldots - q_{j_i})$$

In particular

$$\bigwedge^{r+1} \Omega_{S(Q)} \cong S(Q)(-|Q|).$$

Let now $f_0, \ldots, f_r$ be elements of degree $q_0, \ldots, q_r$ respectively (e.g. $f_i = T_i$) and consider the graded homomorphisms

$$\Delta_i : \bigwedge^i \Omega_{S(Q)} \longrightarrow \bigwedge^{i-1} \Omega_{S(Q)}$$

$$dy_{j_1} \wedge \ldots \wedge dy_{j_i} \longmapsto \sum_{h=1}^{i} (-1)^{h+1} f_{j_h} \, dy_{j_1} \wedge \ldots \widehat{j_h} \ldots \wedge dy_{j_i}$$

We get the complex

$$0 \longrightarrow \bigwedge^{r+1} \Omega_{S(Q)} \xrightarrow{\Delta_{r+1}} \bigwedge^{r} \Omega_{S(Q)} \longrightarrow \ldots \longrightarrow \Omega_{S(Q)} \xrightarrow{\Delta_1} S(Q)$$

which is isomorphic to the Koszul complex $K(f_0, \ldots, f_r; S(Q))$ and such that $\operatorname{Coker} \Delta_1 = S(Q)/(f_0, \ldots, f_r)$.

Assume now that $f_0, \ldots, f_r$ is a regular $S(Q)$-sequence. Then the complex is <u>exact</u> and we may consider the graded modules

$$\mathrm{Syz}_i = \ker\left(\bigwedge^i \Omega_{S(Q)} \longrightarrow \bigwedge^{i-1} \Omega_{S(Q)}\right) = \mathrm{Im}\left(\bigwedge^{i+1} \Omega_{S(Q)} \longrightarrow \bigwedge^{i} \Omega_{S(Q)}\right)$$

In particular we have exact sequences for every n

$$(*) \qquad 0 \longrightarrow \mathrm{Syz}_i(n) \longrightarrow \overset{i}{\wedge}\Omega_{S(Q)}(n) \longrightarrow \mathrm{Syz}_{i-1}(n) \longrightarrow 0$$

Let us go back to the usual projective space, hence assume that $Q = (1, \dots, 1)$ and $S(Q) = S = k[X_0, \dots, X_r]$ with $\deg X_i = 1$. Then we have the following

THEOREM 6A.8. <u>If</u> Syz_i <u>denotes the graded module</u> $\ker(\overset{i}{\wedge}\Omega_S \longrightarrow \overset{i-1}{\wedge}\Omega_S)$ <u>and if</u> $(\mathrm{Syz}_i)^{\sim}$ <u>is the associated sheaf on</u> $\mathbb{P}^r$, <u>then there is an isomorphism</u>

$$\Omega^i_{\mathbb{P}^r} \cong (\mathrm{Syz}_i)^{\sim}$$

<u>In particular</u> $\omega_{\mathbb{P}^r} \cong \mathcal{O}_{\mathbb{P}^r}(-r-1)$.

<u>Proof</u>. Let us consider the exact sequence of Theorem 6A.6. Now

$$\mathcal{O}_{\mathbb{P}^r}(-1)^{r+1} = (S(-1)^{r+1}) \cong (\Omega_S)^{\sim}$$

Therefore

$$\Omega_{\mathbb{P}^r} \cong (\ker\{\Omega_S \longrightarrow S\})^{\sim} = (\mathrm{Syz}_1)^{\sim}$$

and the theorem is proved for $i = 1$. So we make induction on i.

Let now $p : U \longrightarrow \mathbb{P}^r$ be the canonical projection. From the globalization of the first exact sequence (6A.2) we get the exact sequence

$$p^*\Omega_{\mathbb{P}^r} \xrightarrow{\alpha} \Omega_U \longrightarrow \Omega_{U/\mathbb{P}^r} \longrightarrow 0$$

Now the standard map $\Delta_1 : \Omega_S \longrightarrow S$ given by $dx_i \longmapsto x_i$ induces a morphism of sheaves $\tilde{\Delta}_1 : (\Omega_S)^{\sim} \longrightarrow \mathcal{O}_{\mathbb{A}^{r+1}}$ which is clearly surjective if restricted to U. Therefore we get a surjective morphism

$$\Delta_U : \Omega_U \longrightarrow \mathcal{O}_U$$

Let us prove that $\Delta_U \circ \alpha = 0$. For, we know that the morphism p is locally given by the inclusions $S_{(X_i)} \hookrightarrow S_{X_i}$, hence it is sufficient to show that $\Delta_U \circ \alpha$ applied to the differential of a monomial of degree 0 (X_i may have negative exponents) is zero. Let $\prod_{j=0}^{r} X_j^{s_j}$, $s_j \geq 0$, $j \neq i$ be such a monomial.

Then, recalling that $\sum_{n=0}^{r} s_n = 0$,

$$(\Delta_U \circ \alpha)(d(\prod X_j^{s_j})) = \Delta_U(\sum_n (\prod_{j \neq n} X_j^{s_j} \cdot s_n X_n^{s_n - 1}) dX_n =$$

$$= \sum_n (\prod_{j \neq n} X_j^{s_j} \cdot s_n X_n^{s_n}) = (\sum_n s_n) \cdot \prod X_j^{s_j} = 0.$$

Therefore Δ_U factors through a surjective morphism $\Delta : \Omega_{U/\mathbb{P}^r} \longrightarrow \mathcal{O}_U$.

On the other hand $\Omega_{U/\mathbb{P}^r}$ is invertible (see [H], III, 10.4), hence Δ is an

<u>isomorphism</u>.

Note that the sheaves $\Omega_{U/\mathbb{P}^r}$, $\mathcal{O}_U$ are clearly isomorphic since $\Omega_{U/\mathbb{P}^r}$ is trivial because it is invertible and $\mathrm{Pic}(U) = (0)$. Indeed the argument above shows that just the canonical map Δ is an isomorphism. Now we have an exact sequence

$$0 \longrightarrow K \longrightarrow p^*\Omega_{\mathbb{P}^r} \xrightarrow{\alpha} \Omega_U \longrightarrow \mathcal{O}_U \longrightarrow 0$$

Since $\mathcal{O}_U$, Ω_U, $p^*\Omega_{\mathbb{P}^r}$ are locally free sheaves of ranks 1, r + 1, r respectively, also K is a locally free sheaf of rank 0 hence $K = (0)$ and we get the exact sequence of locally free sheaves

$$0 \longrightarrow p^*\Omega_{\mathbb{P}^r} \xrightarrow{\alpha} \Omega_U \longrightarrow \mathcal{O}_U \longrightarrow 0$$

For every i = 1, ..., r one has the exact sequence

$$0 \longrightarrow \bigwedge^i p^*\Omega_{\mathbb{P}^r} \longrightarrow \bigwedge^i \Omega_U \longrightarrow \bigwedge^{i-1} p^*\Omega_{\mathbb{P}^r} \otimes \mathcal{O}_U \longrightarrow 0$$

i.e.

$$0 \longrightarrow p^*\Omega^i_{\mathbb{P}^r} \longrightarrow \Omega^i_U \longrightarrow p^*\Omega^{i-1}_{\mathbb{P}^r} \longrightarrow 0$$

Now we apply p_* and we get again an exact sequence since we are dealing with locally free sheaves. On the other hand the projection formula (see [H], p. 124) yields

$$p_*p^*\Omega^i_{\mathbb{P}^r} \cong \Omega^i_{\mathbb{P}^r} \otimes p_*\mathcal{O}_U = \Omega^i_{\mathbb{P}^r} \otimes (\bigoplus_n \mathcal{O}_{\mathbb{P}^r}(n)) = \bigoplus_n \Omega^i_{\mathbb{P}^r}(n)$$

while $p_* \Omega^i_U$ can be easily computed to be $(\oplus_n \bigwedge^i \Omega_S(n))^\sim$ (as in the proof of 3.B. 2 a), b)). Therefore we get exact sequences

$$0 \longrightarrow \oplus_n \Omega^i_{\mathbb{P}^r}(n) \longrightarrow (\oplus_n \bigwedge^i \Omega_S(n))^\sim \longrightarrow \oplus_n \Omega^{i-1}_{\mathbb{P}^r}(n) \longrightarrow 0$$

By comparing with the exact sequences (*) we get the diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \oplus_n \Omega^i_{\mathbb{P}^r}(n) & \longrightarrow & (\oplus_n \bigwedge^i \Omega_S(n))^\sim & \longrightarrow & \oplus \Omega^{i-1}_{\mathbb{P}^r}(n) & \longrightarrow & 0 \\ & & \downarrow \alpha_i & & \| \ \mathrm{id} & & \downarrow \alpha_{i-1} & & \\ 0 & \longrightarrow & (\oplus_n \mathrm{Syz}_i(n))^\sim & \longrightarrow & (\oplus_n \bigwedge^i \Omega_S(n))^\sim & \longrightarrow & (\oplus_n \mathrm{Syz}_{i-1}(n))^\sim & \longrightarrow & 0 \end{array}$$

where in the middle we have the identity, the right-side morphism α_{i-1} is an isomorphism by the induction and the morphism α_{i-1} is induced by the commutativity of the right-side square. Clearly α_i turns out to be an isomorphism, so the proof is complete.

At this point we are naturally led to make the following consideration. Let $Q = (q_0, \ldots, q_r)$, $S(Q) = k[T_0, \ldots, T_r]$ graded by deg $T_i = q_i$ and consider the exterior algebra complex associated to the elements $q_0T_0, \ldots, q_rT_r$ as in the discussion preceding Theorem 6A.8. Denote, as before, by Syz_i the graded module of i-th syzygies of the complex, which is isomorphic to the Koszul complex $\mathbb{K}(q_0T_0, \ldots, q_rT_r; S(Q))$.

DEFINITION 6A.9. We define $\Omega^i_{\mathbb{P}(Q)}$ to be $(\mathrm{Syz}_i)^\sim$ and call it the sheaf of the i-th <u>regular differential forms</u> on $\mathbb{P}(Q)$.

REMARK. It is easy to see that the proof of 6A.8 cannot be extended to $\mathbb{P}(Q)$. Moreover even the proof of 6A.6 cannot be extended, hence over $\mathbb{P}(Q)$ it is not even true that $\Omega_{\mathbb{P}(Q)} = \Omega^1_{\mathbb{P}(Q)}$ in the new sense.

So now the most important step is to show that the new sheaves $\Omega^i_{\mathbb{P}(Q)}$ behave well and this will be achieved in the next theorem, for which we need the two following results.

LEMMA 6A.10. *Let G be a finite group whose order is invertible in* k *and let it act on a polynomial ring* $B = k[X_1, \ldots, X_n]$. *If* G *is generated by pseudo-reflexions and* $A = B^G$ *then there is a canonical isomorphism of A-modules*

$$\bigwedge^i \Omega_{A/k} \cong \left(\bigwedge^i \Omega_{B/k}\right)^G$$

Proof. See [D], 2.2.2.

LEMMA 6A.11. *If* $\pi : \mathbb{P}^r \longrightarrow \mathbb{P}(Q)$ *is the canonical projection of* 3A.1, *then there is a canonical isomorphism*

$$\Omega^i_{\mathbb{P}(Q)} \cong \pi_*^{\mu_Q}(\Omega^i_{\mathbb{P}^r})$$

Proof. See [D], 2.2.3.

So we are ready to prove the following important

THEOREM 6A.12. *If* $W = \mathrm{Nonsing}(\mathbb{P}(Q))$ *and* $j : W \longrightarrow \mathbb{P}(Q)$ *is the inclusion, then there is a canonical isomorphism*

$$\Omega^i_{\mathbb{P}(Q)} \cong j_* \Omega^i_W$$

Proof. Let us consider the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(W) & \xrightarrow{j'} & \mathbb{P}^r \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ W & \xrightarrow{j} & \mathbb{P}(Q) \end{array}$$

Since W is non singular, then the action of μ_Q on $\pi^{-1}(W)$ coincides locally with that one of a group G generated by pseudoreflections(see [D], 1.3.2).

Hence by 6A.10 and 4A.6 we get

$$\Omega^i_W \cong \pi'^G_*(\Omega^i_{\pi^{-1}(W)})$$

Since $\mathbb{P}(Q)$ is normal (see 3A.1.c)), $\operatorname{codim}_{\mathbb{P}(Q)}(\mathbb{P}(Q) - W) \geqslant 2$ hence

$\operatorname{codim}_{\mathbb{P}^r}(\mathbb{P}^r - \pi^{-1}(W)) \geqslant 2$, so that

$$j'_*\Omega^i_{\pi^{-1}(W)} = \Omega^i_{\mathbb{P}^r} \quad \text{(they are locally free sheaves)}$$

Then

$$j_*\Omega^i_W = j_*\,\pi'^G_*(\Omega^i_{\pi^{-1}(W)}) = \pi^{\mu_Q}_*\, j'_*(\Omega^i_{\pi^{-1}(W)}) =$$

$$= \pi^{\mu'_Q}_*(\Omega^i_{\mathbb{P}^r}) \cong \Omega^i_{\mathbb{P}(Q)}.$$

6B. Duality.

First, let us recall some general facts on duality whose source are, for instance, [H], [G], [A-K], [R].

DEFINITION 6B.1. Let X be a proper scheme over k which is equidimensional of dimension d. A dualizing sheaf for X is a coherent sheaf ω^o_X on X together with a trace morphism $t : H^n(X, \omega^o_X) \longrightarrow k$ such that for all coherent sheaves $\mathcal{F}$ on X, the natural pairing

$$\operatorname{Hom}(\mathcal{F}, \omega^o_X) \times H^d(X, \mathcal{F}) \longrightarrow H^d(X, \omega^o_X)$$

followed by t gives an isomorphism

$$\operatorname{Hom}(\mathcal{F}, \omega^o_X) \xrightarrow{\sim} H^n(X, \mathcal{F})^\vee \quad (\text{"}\vee\text{" means dual})$$

Now it is well-known that for a proper scheme over k a dualizing sheaf exists and it is unique up to isomorphism and the natural pairing of 6A.13 can be extended to the so called Yoneda pairing

$$\mathrm{Ext}^{d-p}(\mathcal{F}, \omega_X^o) \times H^p(X,\mathcal{F}) \longrightarrow H^d(X, \omega_X^o)$$

which, composed with t, gives rise to the pairing

(Y) $$\mathrm{Ext}^{d-p}(\mathcal{F}, \omega_X^o) \times H^p(X,\mathcal{F}) \longrightarrow k$$

THEOREM 6B.2. *If, in addition,* X *is Cohen-Macaulay* (C-M) *then* (Y) *is non singular, hence it gives the isomorphisms*

$$\mathrm{Ext}^{d-p}(\mathcal{F}, \omega_X^o) \xrightarrow{\sim} H^p(X,\mathcal{F})^\vee, \quad p \geq 0$$

PROPOSITION 6B.3. *The following conditions are equivalent*

a) X *is Gorenstein,*

b) ω_X^o *is invertible.*

COROLLARY 6B.4. *If* X *is* C-M *and* $\mathcal{F}$ *is locally free of finite type then the isomorphisms of* 6B.1 *yield the isomorphisms*

$$H^{d-p}(X, \mathcal{F}^\vee \otimes \omega_X^o) \xrightarrow{\sim} H^p(X, \mathcal{F})^\vee, \quad p \geq 0$$

Proof. If $\mathcal{F}$ is locally free of finite type then the canonical morphism

$$\mathcal{F}^\vee \otimes \mathcal{G} \longrightarrow \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$$

is an isomorphism (see Bourbaki) for every $\mathcal{O}_X$-module $\mathcal{G}$ and (see [G], p. 265)

$$\mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) = H^i(X, \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G}))$$

COROLLARY 6B.5. *If* X *is non singular, then* $\omega_X^o = \omega_X$, *the canonical sheaf*.

COROLLARY 6B.6. *Let* W = Nonsing(X) *and let* $j : W \longrightarrow X$ *be the canonical embedding. If* $\mathrm{codim}_X(X - W) \geq 2$, then $\omega_X^o = j_* \omega_W^o = j_* \omega_W$.

THEOREM 6B.7. *If* $Y \subset X$ *is a closed subscheme of codimension* c *and both* Y *and* X *are* C-M *then*

$$\omega_Y^o = \underline{\mathrm{Ext}}^c(\mathcal{O}_Y, \omega_X^o).$$

At this point we can draw some consequences for the weighted projective spaces.

COROLLARY 6B.8. <u>The following isomorphism holds true:</u>

$$\omega^{o}_{\mathbb{P}(Q)} \cong \mathcal{O}_{\mathbb{P}(Q)}(-|Q|).$$

<u>Proof.</u> $\mathbb{P}(Q)$ is a C-M, normal scheme (see 3A.1) of dimension r, therefore we can apply 6B.6 and we get $\omega^{o}_{\mathbb{P}(Q)} \cong j_* \omega^{o}_{W}$ where W = Nonsing $(\mathbb{P}(Q))$. On the other hand $\omega^{o}_{W} \cong \Omega^{r}_{W}$ by 6B.5 and $j_* \Omega_{W} = \Omega^{r}_{\mathbb{P}(Q)}$ by 6A.12.
So we have only to show that $\Omega^{r}_{\mathbb{P}(Q)} \cong \mathcal{O}_{\mathbb{P}(Q)}(-|Q|)$. We know that, by definition,

$$\Omega^{r}_{\mathbb{P}(Q)} = (Syz_r)^{\sim}$$

and

$$Syz_r \cong \bigwedge^{r+1}(\bigoplus_i S(Q)(-q_i)),$$

being the complex exact. Therefore $Syz_r \cong S(Q)(-|Q|)$ and we are done.

COROLLARY 6B.9. <u>Let</u> $Q = \overline{Q}$ <u>and let</u> $X \subset \mathbb{P}(Q)$ <u>be a complete intersection of multidegree</u> $(d_1, \ldots, d_c)$. <u>Then</u>

$$\omega^{o}_{X} \cong \mathcal{O}_X(\sum d_i - |Q|).$$

<u>Proof.</u> By definition there exist c forms $F_1, \ldots, F_c$ such that $F_1, \ldots, F_c$ is an S(Q)-sequence and if $I = (F_1, \ldots, F_c)$ then $X = \mathrm{Proj}(S(Q)/I)$. Let us consider the Koszul complex associated to $F_1, \ldots, F_c$ and call it $\mathbb{K}_\cdot$. Then $\mathbb{K}_\cdot$ resolves S(Q)/I and it is a complex of graded free S(Q)-modules. Applying $\sim$ to

$$\mathbb{K}_\cdot \longrightarrow S(Q)/I \longrightarrow 0$$

we get a resolution of $\mathcal{O}_X$ given by sheaves of the type $\bigoplus_i \mathcal{O}_{\mathbb{P}(Q)}(n_i)$. Let us denote by $\mathbb{H}_\cdot = \widetilde{\mathbb{K}}_\cdot$ this complex which resolves $\mathcal{O}_X$. By [De] Prop. 5.4,

$$\underline{\mathrm{Ext}}^i_{\mathcal{O}_{\mathbb{P}(Q)}}(\mathcal{O}_{\mathbb{P}(Q)}(n), \mathcal{O}_{\mathbb{P}(Q)}(-|Q|)) = 0$$

for every $i > 0$ and $n \in \mathbb{Z}$. Therefore $\mathbb{H}_{\bullet}$ is acyclic for the functor $\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathbb{P}(Q)}}(\ldots\ldots, \mathcal{O}_{\mathbb{P}(Q)}(-|Q|))$. By standard arguments of homological algebra we get that $\underline{\mathrm{Ext}}^{i}_{\mathcal{O}_{\mathbb{P}(Q)}}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}(Q)}(-|Q|))$ can be computed as the i-th homology of $\mathbb{H}_{\bullet}$. In particular $\underline{\mathrm{Ext}}^{c}_{\mathcal{O}_{\mathbb{P}(Q)}}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}(Q)}(-|Q|))$ is the homology of

$$\begin{array}{ccc}
\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathbb{P}(Q)}}(\bigoplus_i \mathcal{O}_{\mathbb{P}(Q)}(-\sum_{j\neq i} d_j), \mathcal{O}_{\mathbb{P}(Q)}(-|Q|)) & \longrightarrow & \underline{\mathrm{Hom}}(\mathcal{O}_{\mathbb{P}(Q)}(\sum_i d_i), \mathcal{O}_{\mathbb{P}(Q)}(-|Q|)) \to 0 \\
\downarrow\wr & & \downarrow\wr \\
\bigoplus_i \mathcal{O}_{\mathbb{P}(Q)}(\sum_{j\neq i} d_j - |Q|) & \longrightarrow & \mathcal{O}_{\mathbb{P}(Q)}(\sum_i d_i - |Q|) \longrightarrow 0
\end{array}$$

where the vertical isomorphisms follow from the remark after 4A.5. This homology is clearly $\mathcal{O}_X(\sum_i d_i - |Q|)$. On the other hand

$$\underline{\mathrm{Ext}}^{c}_{\mathcal{O}_{\mathbb{P}(Q)}}(\mathcal{O}_X, \mathcal{O}_{\mathbb{P}(Q)}(-|Q|)) \cong \omega^{\circ}_X$$

by ;6B.7 and 6B.8.

For more details on duality on $\mathbb{P}(Q)$ see [De] where some consequences are drawn, particularly in connection with the Gorenstein property. For instance we have the following

COROLLARY 6B.10. a) *The scheme* $\mathbb{P}(Q)$ *is Gorenstein if and only if* m *divides* $|Q|$;

b) *If* m *divides* $\sum_i d_i - |Q|$ *then the scheme* X *of* 6B.9 *is Gorenstein.*

Proof. a) It follows from 6B.3, 6B.8 and 4A.4.

b) It follows from 6B.3, 6B.9 and 4A.4.

COROLLARY 6B.11. *Let* $Q = \overline{Q}$ *and let* $X \subset \mathbb{P}(Q)$ *be a complete intersection of multidegree* $(d_1, \ldots\ldots, d_c)$. *Then the arithmetic genus of* X *is* $p_a(X) = \dim_k(S(Q)/I)_{\alpha}$, *where* $\alpha = \sum_i d_i - |Q|$.

Proof. By definition $p_a(X) = \sum_{i=0}^{d-1} (-1)^i H^i(X, \omega_X^o)$, $d = \dim X = r - c$, and by 6B.9

$$p_a(X) = \sum_{i=0}^{r-c-1} (-1)^i H^i(X, \mathcal{O}_X(\sum_i d_i - |Q|))$$

Since depth $S(Q)/I = r - c - 1$, all the H^i's vanish for $0 < i \leq d - 1$ by 3B.4, b) second part. Therefore

$$p_a(X) = H^0(X, \mathcal{O}_X(\sum_i d_i - |Q|)) = (S(Q)/I)_\alpha$$

by 3B.4, b) first part.

REMARK. The formula of 6B.9 is a typical adjunction formula, which has another formulation under different assumptions. Namely, if X is a quasismooth subscheme of $\mathbb{P}(Q)$ and $p^{-1}(X)$ is denoted by C_X ($p : U \longrightarrow \mathbb{P}(Q)$ is the canonical projection) and if $d = \dim X$ then $\omega_X^o = p_*^{\mathbb{G}_m}(\Omega_{C_X}^{d+1})$. As a consequence, it can be deduced the following adjunction formula:

If X is a quasismooth subscheme of $\mathbb{P}(Q)$, then

$$\omega_X^o \cong \Omega_{\mathbb{P}(Q)}^r \otimes \bigwedge^{r-d} \mathcal{N}_X^{\mathbb{P}(Q)}$$

where $\mathcal{N}_X^{\mathbb{P}(Q)}$ denotes the normal bundle of X in $\mathbb{P}(Q)$.

For details see [D], 3.3.

§7. On weighted complete intersections.

We have already seen some properties of the weighted complete intersections at the end of the last section. In this section we are going to make some computations of the Divisor Class Group and the Picard group of $\mathbb{P}(Q)$ and of weighted complete intersections.

Let us just recall that by m we denote the l.c.m.$(q_0, \ldots, q_r)$ and by a the l.c.m.$(d_0, \ldots, d_r)$. The description of the above mentioned groups for $\mathbb{P}(Q)$ is contained in the following

THEOREM 7.1. Assume Q to be reduced; then

a) $Cl(\mathbb{P}(Q)) = \mathbb{Z}$ generated by $[\mathcal{O}_{\mathbb{P}(Q)}(a)]$;

b) If $Q = \overline{Q}$ then $Cl(\mathbb{P}(Q)) = \mathbb{Z}$ generated by $[\mathcal{O}_{\mathbb{P}(Q)}(1)]$;

c) $Pic(\mathbb{P}(Q)) = \mathbb{Z}$ generated by $[\mathcal{O}_{\mathbb{P}(Q)}(m)]$;

d) $\mathbb{P}(Q)$ is locally almost factorial (i.e. the local class groups are torsion) and it is locally factorial if and only if $\overline{Q} = (1, \ldots, 1)$.

Proof. A proof can be found in [R2]. See also [Am] for another proof of a), b), c) with some generalizations. With regard to a), b) we only want to point out that the proof of [R2] is based on a more general theory of Demazure (see [Dem]), which describes the normal graded k-algebras by means of suitable Weil divisors with rational coefficients and on some consequences drawn by Watanabe (see [W]) on the description of the associated class groups. In any case an essential step in the proof of a), b) is to recognize that the sheaves $\mathcal{O}_{\mathbb{P}(Q)}(n)$ are reflexive of rank 1 (see 5.8).

As for the proof of c) it is clear that $Pic(\mathbb{P}(Q)) = \mathbb{Z}$, being a subgroup of $Cl(\mathbb{P}(Q))$ and we may assume $Q = \overline{Q}$ since if $Q \neq \overline{Q}$, then $\mathcal{O}_{\mathbb{P}(Q)}(m) \cong \mathcal{O}_{\mathbb{P}(\overline{Q})}(m/a)$ by 3C.7 and $m/a = $ l.c.m.$(q_0/a_0, \ldots, q_r/a_r)$. So we may use 4A.4. As we know, the sheaves $\mathcal{O}_{\mathbb{P}(Q)}(\alpha m)$ are invertible and if n is not a multiple of m then there exists a T_i such that $\Gamma(D_+(T_i), \mathcal{O}_{\mathbb{P}(Q)}(n))$ is not free over $\Gamma(D_+(T_i), \mathcal{O}_{\mathbb{P}(Q)})$. To conclude that $\mathcal{O}_{\mathbb{P}(Q)}(n)$ is not invertible we need showing that $\mathrm{Pic}(D_+(T_i)) = (0)$. For we know that $D_+(T_i) = \mathrm{Spec}\ k[T_0, \dots, \hat{T}_i, \dots, T_r]^{\mu_{q_i}}$ and

$$k[T_0, \dots, \hat{T}_i, \dots, T_r]^{\mu_{q_i}} = \bigoplus_n k[T_0, \dots, \hat{T}_i, \dots, T_r]_{nq_i}$$

Therefore $D_+(T_i)$ is the spectrum of a ring which can be naturally graded over $\mathbb{N}$ and whose part of degree 0 is k. By [Fo], 10.4, p. 43 its Pic is trivial. As for d), we observe that by [B-O], Prop. 2.1 and by a), b), c) it turns out that $\mathbb{P}(Q)$ is locally almost factorial and it is locally factorial iff m = a. But $\mathbb{P}(Q) \cong \mathbb{P}(\overline{Q})$ and $a(\overline{Q}) = 1$, so $\mathbb{P}(Q)$ is locally factorial iff $m(\overline{Q}) = 1$ that is iff $\overline{Q} = (1, 1, \dots, 1)$.

COROLLARY 7.2. Let $Q = (q_0, \dots, q_r)$, $Q' = (q'_0, \dots, q'_r)$, assume that $q_0 \leq \dots \leq q_r$, $q'_0 \leq \dots \leq q'_r$ and that $Q = \overline{Q}$, $Q' = \overline{Q}'$. If $\mathbb{P}(Q)$ is isomorphic to $\mathbb{P}(Q')$ then $Q = Q'$.

Proof. (see also [Am]). The isomorphism φ between $\mathbb{P}(Q)$ and $\mathbb{P}(Q')$ induces an isomorphism φ^* between $\mathrm{Cl}(\mathbb{P}(Q'))$ and $\mathrm{Cl}(\mathbb{P}(Q))$. Since $[\mathcal{O}_{\mathbb{P}(Q')}(1)]$ generates $\mathrm{Cl}(\mathbb{P}(Q'))$, the inverse image $\varphi^*[\mathcal{O}_{\mathbb{P}(Q')}(1)]$ is a generator of $\mathrm{Cl}(\mathbb{P}(Q))$ hence $\varphi^*\mathcal{O}_{\mathbb{P}(Q')}(1) = \mathcal{O}_{\mathbb{P}(Q)}(1)$ because no multiple of $\mathcal{O}(-1)$ has global sections. Consequently $\varphi^*\mathcal{O}_{\mathbb{P}(Q')}(s) = \mathcal{O}_{\mathbb{P}(Q)}(s)$ for every $s \in \mathbb{Z}$, therefore

$$\dim_k H^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(s)) = \dim_k H^0(\mathbb{P}(Q'), \mathcal{O}_{\mathbb{P}(Q')}(s))$$

But $H^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(s)) = S(Q)_s$ by 3B.2.c). This means that the Poincaré series of $S(Q)$ and $S(Q')$ coincide. Now it is sufficient to prove the following

CLAIM: If $Q = (q_0, \dots, q_r)$, $q_0 \leq \dots \leq q_r$ and $Q' = (q'_0, \dots, q'_r)$,

$q'_0 \leqslant \ldots\ldots \leqslant q'_r$ are such that $P_t(S(Q)) = P_t(S(Q'))$ then $Q = Q'$ (here P_t is the Poincaré serie).

We prove the Claim by induction on r. If $r = 0$, it is clear. Let $q_0 \leqslant q'_0$ and let us evaluate both series at q_0. Then we find

$$\dim(S(Q)_{q_0}) = \dim(S(Q')_{q_0}) \neq 0$$

whence $q'_0 = q_0$. But

$$P_t(S(Q)) = 1/(t^{q_0} - 1) \ldots\ldots (t^{q_r} - 1),$$

hence we find

$$1/(t^{q_1} - 1) \ldots\ldots (t^{q_r} - 1) = 1/(t^{q'_1} - 1) \ldots\ldots (t^{q'_r} - 1)$$

If we denote by $Q_0 = (q_1, \ldots\ldots, q_r)$, $Q'_0 = (q'_1, \ldots\ldots, q'_r)$ we get by induction that $Q_0 = Q'_0$ and the proof is complete.

Now, starting from the fundamental theorem 7.1, we may try to compute the divisor class group and the Picard group of suitable subschemes of $\mathbb{P}(Q)$.
First, let us recall the following result of Mori (see [Mo], 3.7) which yields a "weighted version" of the classical Lefschetz theorem on complete intersections

THEOREM 7.3. _Let_ X _be a projective variety which is a complete intersection in a weighted projective space_ $\mathbb{P}(Q)$ _and such that_ $X \subset \mathbb{P}^o(Q)$. _Then_

a) _If_ $\dim X \geqslant 3$ _then_ $\mathrm{Pic}(X) \simeq \mathbb{Z}$ _generated by_ $[\mathcal{O}_X(1)]$;

b) _If_ $\dim X = 2$ _then there is an exact sequence_

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathrm{Pic}(X) \longrightarrow K \longrightarrow 0$$

where 1 _goes to_ $[\mathcal{O}_X(1)]$ _under_ α _and_ K _is torsion free._

Proof. a) The assumption that $X \subset \mathbb{P}^o(Q)$ already implies that Q is normalized by 5.1, b) and 5.5. Now

$$X = \mathrm{Proj}(k[T_0, \ldots\ldots, T_r]/(F_1, \ldots\ldots, F_c))$$

where $\deg T_i = q_i$, $\deg F_j = d_j$ and $F_1, \ldots\ldots, F_c$ is a regular sequence in $k[T_0, \ldots\ldots, T_r]$. Let

$$\varphi : k[T_0, \dots, T_r] \longrightarrow k[X_0, \dots, X_r]$$

be defined by $\varphi(T_i) = X_i^{q_i}$. If $\deg X_i = 1$ for $i = 0, \dots, r$, φ is a graded homomorphism which is finite and free (see 3A.2). This implies that if we put $G_i = \varphi(F_i)$, then $G_1, \dots, G_c$ is a regular sequence in $k[X_0, \dots, X_r]$ Look at

$$\widetilde{X} = \mathrm{Proj}(k[X_0, \dots, X_r]/(G_1, \dots, G_c))$$

embedded in $\mathbb{P}^r$ and consider $\pi : \mathbb{P}^r \longrightarrow \mathbb{P}(Q)$ (see 3A.1, d)) which induces $\pi : \widetilde{X} \longrightarrow X$. Arguing as in 3A.2 we get that

$$\pi_* \mathcal{O}_{\widetilde{X}} = \bigoplus_{0 \leqslant \alpha_i < q_i} \mathcal{O}_X(-\sum_{i=0}^{r} \alpha_i)$$

But $X \subset \mathbb{P}^o(Q)$, hence the sheaves $\mathcal{O}_X(n)$ are invertible, whence $\pi_* \mathcal{O}_{\widetilde{X}}$ is locally free. Now π induces $\pi^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\widetilde{X})$ and since it comes from an equivariant homomorphism, one has $\pi^* \mathcal{O}_X(1) = \mathcal{O}_{\widetilde{X}}(1)$. We know that $\widetilde{X}$ is an usual complete intersection in $\mathbb{P}^r$ such that $\dim \widetilde{X} \geqslant 3$, hence the "usual" Lefschetz theorem applies to say that $\mathrm{Pic}(\widetilde{X}) = \mathbb{Z} \cdot [\mathcal{O}_{\widetilde{X}}(1)]$. Therefore we have only to show that π^* is injective. For, let $\mathcal{L} \in \mathrm{Pic}(X)$ be such that $\pi^*\mathcal{L} \cong \mathcal{O}_{\widetilde{X}}$. We deduce that

$$\pi_* \pi^* \mathcal{L} \cong \pi_* \mathcal{O}_{\widetilde{X}} = \bigoplus_{0 \leqslant \alpha_i < q_i} \mathcal{O}_X(-\sum_i \alpha_i)$$

and by using the projection formula

$$\pi_* \pi^* \mathcal{L} \cong \mathcal{L} \otimes \pi_* \mathcal{O}_{\widetilde{X}} \cong \mathcal{L} \otimes (\bigoplus_{0 \leqslant \alpha_i < q_i} \mathcal{O}_X(-\sum_i \alpha_i))$$

Since X is projective, the Krull-Schmidt theorem implies that $\mathcal{L} \cong \mathcal{O}_X(n)$ for some $n \in \mathbb{Z}$. If $n \neq 0$ we may assume $n > 0$ by interchanging $\mathcal{L}$ and $\mathcal{L}^{-1}$. Therefore $\mathcal{L}$ is ample and since π is finite also $\pi^*\mathcal{L} \cong \mathcal{O}_{\widetilde{X}}$ is ample.

This is a contradiction and the proof is complete.

b) As in the case above one has a morphism $\pi : \widetilde{X} \longrightarrow X$ which induces an injective morphism $\pi^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\widetilde{X})$. Then we get an exact diagram with commutative left square

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\alpha} & \mathrm{Pic}(X) & \longrightarrow & K & \longrightarrow & 0 \\ & & \big\downarrow \mathrm{id} & & \big\downarrow \pi^* & & & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\widetilde{\alpha}} & \mathrm{Pic}(\widetilde{X}) & \longrightarrow & \widetilde{K} & \longrightarrow & 0 \end{array}$$

where 1 goes to $[\mathcal{O}_X(1)]$, $[\mathcal{O}_{\widetilde{X}}(1)]$ under α, $\widetilde{\alpha}$ respectively and $\widetilde{K}$ is torsion free as proved in [R1]. The diagram induces an embedding $K \hookrightarrow \widetilde{K}$ so the assertion follows.

A consequence of Theorem 7.3 b) is the following (compare with statement 7.6):

COROLLARY 7.4. Let X be a normal projective surface which is a complete intersection in a weighted projective space $\mathbb{P}(Q)$ and such that $X \subset \mathbb{P}^o(Q)$. Then every prime Cartier divisor of X which is a set-theoretic complete intersection on X is actually a complete intersection on X.

Proof. It runs as in [R1], §3.

REMARK 1. It should be noticed that Theorem 7.3 was an essential tool to prove the following fact, which was the motovating point for introducing weighted complete intersections: let Y be a complete intersection of multidegree $(d_1, \ldots, d_c)$ in $\mathbb{P}^N$, $\dim Y \geqslant 3$ and assume that Y is an ample divisor in a smooth projective variety X. Then there exists $s \in \mathbb{N}$ such that X is a weighted complete intersection of multidegree $(d_1, \ldots, d_c)$ in $\mathbb{P}(1, \ldots, 1, s)$ and s divides d_i for every i (see [Mo]).

REMARK 2. In [D], it is proved another version of the "weighted Lefschetz theorem". Namely the following is true. Let X be a projective variety of dimension $\geqslant 3$ which is a complete intersection in a weighted projective space $\mathbb{P}(Q)$ and such that X is quasismooth. Then $\mathrm{Pic}(X) \cong \mathbb{Z}$.

Of course theorem 7.3 of Mori and the theorem above of Dolgachev lead naturally to the following

QUESTION 1. Is there a version of the Leschetz theorem for every complete intersection in $\mathbb{P}(Q)$?

REMARK 3. If we look more carefully at the proof of theorem 7.3 we see that what we need is that $\mathrm{Pic}(\widetilde{X}) \cong \mathbb{Z}$ generated by $[\mathcal{O}_{\widetilde{X}}(1)]$. Therefore, if $\dim X = 2$ and $\widetilde{X}$ is sufficiently general then we may use the classical Noether theorem to get the desired conclusion. However $\widetilde{X}$ is not a priori sufficiently general, because in the equations defining $\widetilde{X}$ the variable X_i only appears with exponents multiple of q_i.

The remark above leads naturally to the following

QUESTION 2. Is there any "Noether type" theorem for weighted complete intersections of dimension 2?

It should be noted that a partial answer to Question 2 was given recently by Steenbrink (see [S]), who proved that for a sufficiently general surface

X of $\mathbb{P}(1, 1, a, b)$ with G.C.D.$(a, b) = 1$, $\mathrm{Pic}(X) \cong \mathbb{Z}$.

For the next application we need the following

LEMMA 7.5. Let $Q = \overline{Q}$ and let n be such that $\mathcal{O}_{\mathbb{P}(Q)}(n)$ is invertible. Then $\mathcal{O}_{\mathbb{P}(Q)}(sn) \cong \mathcal{O}_{\mathbb{P}(Q)}(n)^{\otimes s}$ for every $s \in \mathbb{Z}$.

Proof. By 7.1, $n = am$ hence $\mathcal{O}_{\mathbb{P}(Q)}(sn) = \mathcal{O}_{\mathbb{P}(Q)}(sam)$ is invertible. Of course also $\mathcal{O}_{\mathbb{P}(Q)}(n)^{\otimes s}$ is invertible.
To show that they are isomorphic it is sufficient to prove that their restrictions to $\mathbb{P}^{o}(Q)$ are isomorphic and this follows from 5.6, d).

In the following, if X is a subscheme of $\mathbb{P}(Q)$ and I is the ideal of X, then we denote by $S(X) = S(Q)/I$ and by $\alpha_X : S(X) \longrightarrow \bigoplus_n H^0(X, \mathcal{O}_X(n))$ the Serre-homomorphism (see 3B.4).

THEOREM 7.6. Let $Q = \overline{Q}$ and let X be a normal closed subvariety of $\mathbb{P}(Q)$. Assume that the Serre homomorphism α_X is an isomorphism and that $\mathrm{Pic}(X)/\mathbb{Z}\cdot\left[\mathcal{O}_X(m)\right]$ is torsion free. Then every Cartier prime divisor D of X which is a set-theoretic complete intersection (s.t.c.i.) on X is actually a complete intersection (c.i.) on X.

Proof. Since D is s.t.c.i. on X there exists $d \in \mathbb{N}$ and a form $F \in S(Q)_d$ such that $D = Z(F) \cap X$ (set-theoretically). Let us consider $F^m \in S(Q)_{md} = H^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(md))$. Since $\mathcal{O}_X(md)$ is invertible by 4A.5, F^m defines an effective Cartier divisor Δ such that

$$\mathcal{O}_{\mathbb{P}(Q)}(\Delta) \cong \mathcal{O}_{\mathbb{P}(Q)}(md) \cong \mathcal{O}_{\mathbb{P}(Q)}(m)^{\otimes d} \qquad \text{(by 7.5)}$$

On the other hand, $Z(F^m) \cap X = D$ and the restriction of Δ to X is a Cartier divisor, having D as its support. Since D is prime it follows that

$$\Delta \cdot X = q D$$

hence

$$\mathcal{O}_X(D)^{\otimes q} \cong \mathcal{O}_X(qD) \cong \mathcal{O}_{\mathbb{P}(Q)}(\Delta) \otimes \mathcal{O}_X \cong \mathcal{O}_X(m)^{\otimes d}$$

The assumption on the torsion freeness of $\mathrm{Pic}(X)/\mathbb{Z}\cdot\mathcal{O}_X(m)$ implies that

$$\mathcal{O}_X(D) \cong \mathcal{O}_X(m)^{\otimes s} \cong \mathcal{O}_X(ms)$$

for a suitable s. The assumption on the Serre homomorphism implies that the canonical map

$$S(Q)_{ms} = H^0(\mathbb{P}(Q), \mathcal{O}_{\mathbb{P}(Q)}(ms)) \longrightarrow H^0(X, \mathcal{O}_X(ms))$$

is surjective. Then there exists a form $G \in S(Q)_{ms}$ which defines a Cartier divisor div(G) such that

$\mathrm{div}(G) \cdot X = D$ (scheme theoretically).

REMARK 1. From the arguments above it turns out that significant applications of Theorem 7.6 would follow from a generalized Lefschetz type theorem in which the assumption "$X \subset \mathbb{P}^o(Q)$" occurring in 7.3 was removed.

REMARK 2. While a global section of an invertible sheaf yields a Cartier divisor, it should be noted that if we take for instance Q = (1, 1, 2), the global equation $T_0 = 0$ yields a Weil divisor on $\mathbb{P}(Q)$ which is <u>not</u> a Cartier divisor. This corresponds to the fact that $\mathcal{O}_{\mathbb{P}(Q)}(1)$ is <u>not</u> invertible.

We conclude this section with some remarks on the following question. By means of 7.3 some examples are constructed in [R2] of <u>smooth</u> varieties X with dim $X \geqslant 3$ which are c.i. in $\mathbb{P}(Q)$ with $Q = \overline{Q}$ and whose projective coordinate ring is U.F.D. Moreover such varieties <u>do not have</u> immersions in any $\mathbb{P}^N$ such that the corresponding projective coordinate ring is U.F.D. Here we remark that examples of this kind can be constructed also in dimension 2. Namely, if we drop the assumption "X smooth" then $\mathbb{P}(1, 1, 2)$ is such an example, but if we want to keep the assumption, then we may consider for instance

$$A = k\left[T_0, T_1, T_2, T_3\right]/(T_0^3 + T_1^2 + T_2T_3),\ Q = (2, 3, 1, 5)$$

$$B = k\left[T_0, T_1, T_2, T_3, T_4\right]/(T_0^5 + T_1^3 + T_2^2,\ T_0^5 + bT_1^3 + T_3T_4),\ b \neq 1,\ Q = $$
$$= (6, 10, 15, 1, 29).$$

As it is proved in [I], A, B are U.F.D., hence the <u>smooth</u> surfaces $X = \mathrm{Proj}(A)$, $Y = \mathrm{Proj}(B)$ have a normalized U.F.D. immersion in $\mathbb{P}(2, 3, 1, 5)$ and $\mathbb{P}(6, 10, 15, 1, 29)$ respectively. Using the results of [R2], §§ 3, 4 we know that X, Y have no U.F.D. immersion in any $\mathbb{P}^N$.

APPENDIX: Reflexive modules and Weil divisors.

First, let us recall some results from algebra. All the rings we consider are noetherian and the modules are of finite type.

LEMMA 1. Let A be an integral domain and let M, N be A-modules with N torsion free. Let $\varphi : M \longrightarrow N$ be an A-homomorphism and let $\mathfrak{p} \in \mathrm{Spec}(A)$ be such that $\varphi_{\mathfrak{p}} = 0$. Then $\varphi = 0$.

Proof. Easy, left to the reader.

LEMMA 2. Let A be an integral domain, M, N submodules of the fraction field K(A) and let $\mathfrak{p} \in \mathrm{Spec}(A)$ be such that $M_{\mathfrak{p}} \simeq N_{\mathfrak{p}} \simeq A_{\mathfrak{p}}$. Then every A-homomorphism $\varphi : M \longrightarrow N$ is the multiplication by an element of K(A).

Proof. From the assumptions one sees that $\varphi_{\mathfrak{p}}$ is the multiplication by an element $a/b \in K(A)$. Consider the A-homomorphism $b\varphi - a$. One has $(b\varphi - a)_{\mathfrak{p}} = 0$, hence $b\varphi - a = 0$ by Lemma 1 and we are done.

PROPOSITION 3. Let A be a normal ring and let M be an A-module of finite type. Then the following are equivalent

a) $M \simeq M^{**}$;

b) $M = A : (A : M)$;

c) $M = \bigcap_{\mathfrak{p}} M_{\mathfrak{p}}$, $\mathrm{ht}(\mathfrak{p}) = 1$;

d) Every A-regular sequence of lenght two is a M-regular sequence too.

Proof. See [Fo], p. 23-24.

THEOREM 4. (The Approximation Theorem for Krull Domains). Let A be a Krull domain and denote by $v_{\mathfrak{p}}$ the valuation associated to the principal valuation ring $A_{\mathfrak{p}}$. For each $\mathfrak{p}$ in Spec(A) let $n(\mathfrak{p})$ be a given integer such that $n(\mathfrak{p}) = 0$ for almost all $\mathfrak{p}$. For every finite set $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ there is an f in K(A)* such that $v_{\mathfrak{p}_i}(f) = n(\mathfrak{p}_i)$ and $v_{\mathfrak{p}}(f) \geq 0$ otherwise.

Proof. See [Fo], p. 26-27.

From now on let $(X, \mathcal{O}_X)$ be a noetherian, irreducible, normal scheme of finite type over a field k, and let $\mathcal{K}(X)$ be the constant sheaf of rational functions.

PROPOSITION 5. Let $\mathcal{M}, \mathcal{M}'$ be submodules of finite type of $\mathcal{K}(X)$ and let $x \in X$ be such that $\mathcal{M}_x \simeq \mathcal{M}'_x \simeq \mathcal{O}_{X,x}$. Let $\varphi : \mathcal{M} \longrightarrow \mathcal{M}'$ be an $\mathcal{O}_X$-homomorphism. Then φ is the multiplication by an element of $\mathcal{K}(X)$.

Proof. Lemma 2 says that for every affine open subset U of X one has $\varphi_{|U} = F(U)$, F(U) rational function on U (hence on the whole X). Therefore the F(U)'s give the same rational function on X since they coincide on non empty open subsets.

Now we refer to W-div(X) as to the set of the Weil divisors on X and if $D \in$ W-div(X) the sheaf $\mathcal{O}_X(D)$ is defined in the following way: if U is an open set, then $\Gamma(U, \mathcal{O}_X(D)) = \{f \in \mathcal{K}(X)^* ; (\mathrm{div}(f) + D)_{|U} \geqslant 0$.

LEMMA 6. The sheaf $\mathcal{O}_X(D)$ is a submodule of finite type of $\mathcal{K}(X)$.

Proof. It suffices to give a local proof. Let $x \in X$ and put $A = \mathcal{O}_{X,x}$, $L = \mathcal{O}_X(D)_x$, $L' = \mathcal{O}_X(-D)_x$. Note that $L' \neq 0$ in view of Theorem 4. Let $0 \neq f \in L'$ and take $l \in L$. Then

$$\mathrm{div}(fl) = \mathrm{div}(f) + \mathrm{div}(l) = \mathrm{div}(f) + D + \mathrm{div}(l) - D \geqslant 0$$

that is $fL \subseteq A$ which implies $L \subseteq (1/f)A$ and we are done since A is noetherian.

LEMMA 7. Let $\mathcal{M}$ be a submodule of finite type of $\mathcal{K}(X)$. Then

$$\mathcal{O}_X : \mathcal{M} \cong \underset{\sim}{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X).$$

Proof. The inclusion "$\subseteq$" is clear. Let U be an affine open subset belonging to an affine covering of X and write $A = \Gamma(U, \mathcal{O}_X)$, $M = \Gamma(U, \mathcal{M})$. Let

$a_1/b_1, \ldots, a_r/b_r$ be generators of M over A, $a_i, b_i \in A$. Since a_i, b_i are not zero, there exists a point $\mathfrak{p}$ such that $a_i(\mathfrak{p}) \neq 0$, $b_i(\mathfrak{p}) \neq 0$, $i = 1, \ldots, r$. Therefore $M_{\mathfrak{p}} = A_{\mathfrak{p}}$ and Proposition 5 gives the result.

LEMMA 8. For every $D \in$ W-div(X), $\mathcal{O}_X : \mathcal{O}_X(D) = \underset{\sim}{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X(D), \mathcal{O}_X)$.

Proof. $\mathcal{O}_X(D)$ is a $\mathcal{O}_X$-module of finite type, so Lemma 7 applies.

LEMMA 9. Let $f \in \mathcal{K}(X)^*$ and let $D, D' \in$ W-div(X). Then we have

a) $\mathcal{O}_X(\mathrm{div}(f)) = (1/f)\,\mathcal{O}_X$;

b) $\mathcal{O}_X(D) \cdot \mathcal{O}_X(D') \subseteq \mathcal{O}_X(D + D')$;

c) $\mathcal{O}_X(D) \cdot \mathcal{O}_X(\mathrm{div}(f)) = \mathcal{O}_X(D + \mathrm{div}(f))$.

Proof. a) and b) are clear. To prove c), let $g \in \Gamma(U, \mathcal{O}_X(D + \mathrm{div}(f)))$. Then $(\mathrm{div}(g) + D + \mathrm{div}(f))_{|U} \geq 0$, that is $(\mathrm{div}(gf) + D)_{|U} \geq 0$, so $g = gf \cdot (1/f)$ where $gf \in \Gamma(U, \mathcal{O}_X(D))$, $1/f \in \Gamma(U, \mathcal{O}_X(\mathrm{div}(f)))$.

REMARK. The equality in b) is not true in general. Indeed, take $X = \mathrm{Spec}\, k[x, y, z]/(xy - z^2)$, $\mathfrak{p} = (x, z)$, $\mathfrak{p}' = (y, z)$, $D = [\mathfrak{p}]$, $D' = [\mathfrak{p}']$. Then $\mathcal{O}_X(D + D') = \mathcal{O}_X(\mathrm{div}(z)) \neq \mathcal{O}_X(D) \cdot \mathcal{O}_X(D')$.

LEMMA 10. Let $D, D' \in$ W-div(X) and let $\varphi: \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D')$ be an isomorphism. Then $D' = D - \mathrm{div}(f)$.

Proof. Proposition 5 and Lemma 9, c) yield $\mathcal{O}_X(D') = \varphi(\mathcal{O}_X(D)) = f \cdot \mathcal{O}_X(D) = \mathcal{O}_X(-\mathrm{div}(f)) \cdot \mathcal{O}_X(D) = \mathcal{O}_X(D - \mathrm{div}(f))$.

PROPOSITION 11. For every $D \in$ W-div(X), $\mathcal{O}_X(-D) = \mathcal{O}_X : \mathcal{O}_X(D)$.

Proof. The inclusion "$\subset$" is clear. The converse can be proved locally. Therefore we may assume X = Spec A. Let $f \in \mathcal{K}(X) = K(A)$ be such that

$f \in A : A(D)$ and write $\mathrm{div}(f) = D + T$. We have to prove that $T \geq 0$. In view of Theorem 4 there exists $g \in K(A)^*$ such that $\mathrm{div}(g) = -D + S$ where $\mathrm{Supp}\, S \cap \mathrm{Supp}\, T = \emptyset$ and $S \geq 0$. Hence $g \in A(D)$, so $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g) \geq 0$ Since $\mathrm{div}(f) + \mathrm{div}(g) = T + S$ and $\mathrm{Supp}\, S \cap \mathrm{Supp}\, T = \emptyset$ it follows that $T \geq 0$.

COROLLARY 12. *If* $D \in$ W-div(X), *then* $\mathcal{O}_X(D) \simeq \mathcal{O}_X : (\mathcal{O}_X : \mathcal{O}_X(D)) \simeq$
$\simeq \mathcal{O}_X(D)^{**}$.

Proof. It follows from Lemma 7 and Proposition 9.

As well as the sheaves $\mathcal{O}_X(D)$, all the invertible sheaves can be considered as submodules of $\mathcal{K}(X)$. In fact one has

LEMMA 13. *Let* $\mathcal{L}$ *be an invertible sheaf on an integral scheme* X. *Then there exists* $\mathcal{L}' \in \mathcal{K}(X)$ *such that* $\mathcal{L} \simeq \mathcal{L}'$.

Proof. See [H], p. 145.

THEOREM 14. *Let* $\mathcal{L}$ *be a submodule of* $\mathcal{K}(X)$. *Then the following are equivalent:*

a) $\mathcal{L}$ *is invertible;*

b) *There exists a submodule* $\mathcal{L}'$ *of* $\mathcal{K}(X)$ *such that* $\mathcal{L} \cdot \mathcal{L}' = \mathcal{O}_X$;

c) $\mathcal{L} = \mathcal{O}_X(D)$ *for some Cartier divisor on* X.

Proof. a) $\Longrightarrow$ c) There exists an open covering $\{U_i\}_i$ of X such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{X|U_i}$. Then Proposition 5 implies that $\mathcal{L}|_{U_i} \cong \mathcal{O}_{X|U_i}(\mathrm{div}(f_i))$ for some $f_i \in K(\Gamma(U_i, \mathcal{O}_X))$.

c) $\Longrightarrow$ b). Lemma 9, b) gives the inclusion $\mathcal{O}_X(D) \cdot \mathcal{O}_X(-D) \subseteq \mathcal{O}_X$. The equality can be proved locally, so it follows from Lemma 9, c).

b) $\Longrightarrow$ a). Let $x \in X$, $(A, \mathfrak{m}) = \mathcal{O}_{X,x}$, $L = \mathcal{L}_x$, $L' = \mathcal{L}'_x$. Then $L \cdot L' = A$, so that $L \subseteq (1/l')A$ for every $0 \neq l' \in L'$. Therefore L and L' are A-modules of finite type. Now $L \cdot L' = A$ can be read as $L \otimes L' = A$, hence $L/\mathfrak{m}L \otimes L'/\mathfrak{m}L' \simeq A/\mathfrak{m}$, so that Nakajama's Lemma implies that $\mu(L) = \mu(L') = 1$ where "$\mu(\cdot)$" means the minimal number of generators.

LEMMA 15. <u>Let</u> $D \in$ W-div(X) <u>and let</u> D' <u>be a Cartier divisor. Then</u>
$\mathcal{O}_X(D) \cdot \mathcal{O}_X(D') = \mathcal{O}_X(D + D')$.

<u>Proof.</u> Lemma 9, b) gives the inclusion "$\subset$". The converse follows from Lemma 9, c).

COROLLARY 16. <u>Let</u> $D \in$ W-div(X). <u>Then</u> D <u>is a Cartier divisor if and only if there esists a divisor</u> D' <u>such that</u> $\mathcal{O}_X(D) \cdot \mathcal{O}_X(D') = \mathcal{O}_X$.

<u>Proof.</u> The "only part" is clear by taking $D' = -D$. The converse follows from Theorem 14, b).

Now, let U be a nonsingular open subset of X such that $\operatorname{codim}_X(X\setminus U) \geq 2$ and let $j : U \longrightarrow X$ be the canonical embedding. We have the following

THEOREM 17. <u>Let</u> $\mathcal{M}$ <u>be a submodule of</u> $\mathcal{K}(X)$. <u>Then the following conditions are equivalent</u>

a) $\mathcal{M} = \mathcal{O}_X(D)$ <u>for some (uniquely determined)</u> $D \in$ W-div(X);

b) $\mathcal{M}$ <u>is a rank one, reflexive module of finite type</u>;

c) $j^*\mathcal{M}$ <u>is an invertible sheaf and</u> $\mathcal{M} = j_*j^*\mathcal{M}$.

<u>Proof.</u> a) $\Longrightarrow$ b). It follows from Lemma 6 that $\mathcal{M}$ is of finite type and from Corollary 12 that it is reflexive. To show that it has rank one it is sufficient to find a point $x \in X$ such that $\mathcal{M}_x \cong \mathcal{O}_{X,x}$. For this we just take a point x which does not belong to Supp(D).

b) $\Longrightarrow$ c) To prove that $j^*\mathcal{M}$ is invertible, take $x \in U$ and write $M = (j^*\mathcal{M})_x$, $A = \mathcal{O}_{X,x}$. Since M is of finite type there exists $f \in A$ such that $M \subseteq (1/f)A$. Hence $M \simeq fM = \mathfrak{a}$, $\mathfrak{a}$ ideal of A. Up to dividing by the greatest common divisor of the generators of $\mathfrak{a}$ (remember that A is regular hence U.F.D.), we may assume $\mathfrak{a} = (1)$ or $ht(\mathfrak{a}) \geqslant 2$. To conclude, it suffices to exclude the case $ht(\mathfrak{a}) \geqslant 2$. Therefore we have only to prove that $A : \mathfrak{a} = A$ if $ht(\mathfrak{a}) \geqslant 2$ (indeed it would follows $A : (A : \mathfrak{a}) = A \neq \mathfrak{a}$ and $A : \mathfrak{a} = \mathfrak{a}^*$, so $A : (A : \mathfrak{a}) = \mathfrak{a}^{**}$ by Lemma 6, contradiction). Thus assume $ht(\mathfrak{a}) \geqslant 2$ and let us prove that $A : \mathfrak{a} \subseteq A$. Let $(a/b)\mathfrak{a} \subseteq A$ and let $(a_1, \ldots, a_r)$ be a system of generators of $\mathfrak{a}$. Now we can take a, b to be coprime and we know that A is U.F.D. because it is regular, then $a_i \in (b)$ but $ht(\mathfrak{a}) \geqslant 2$, whence $b = 1$.

Moreover from Proposition 3, c) one deduces $M = j_*j^*M$.

c) $\Longrightarrow$ a) $j^*M = \mathcal{O}_U(D)$ for some Cartier divisor D on U. Then D gives rise to a uniquely determined Weil divisor on U, which we denote again by D. Clearly such a divisor extends to the whole X since $\operatorname{codim}_X(X\setminus U) \geqslant 2$. Therefore $M = j_*j^*M = \mathcal{O}_X(D)$.

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